ON CONTINUITY AND INVARIANCE OF THE ÁLVAREZ CLASS UNDER DEFORMATION

HIRAKU NOZAWA

ABSTRACT. A manifold M with a foliation $\mathcal{F}$ is minimizable if there exists a Riemannian metric g on M such that every leaf of $\mathcal{F}$ is a minimal submanifold of (M, g). For a closed manifold M with a Riemannian foliation $\mathcal{F}$, Álvarez López [1] defined a cohomology class of degree 1 called the Álvarez class whose triviality characterizes the minimizability of $(M, \mathcal{F})$. In this paper, we show that the family of the Álvarez classes of a smooth family of Riemannian foliations on a closed manifold is continuous with respect to the parameter. Since the Álvarez class has algebraic rigidity under certain topological conditions on $(M, \mathcal{F})$ as the author showed in [19], we show that the minimizability of Riemannian foliations is invariant under deformation under the same topological conditions.

CONTENTS

1. Introduction

The minimizability of Riemannian foliations. The minimizability of general foliations is characterized in terms of dynamical tools, for example, foliation cycles (Sullivan [25]) or holonomy pseudogroups (Haefliger [11]). On the other hand, remarkably, the minimizability of Riemannian foliations has a strong relation with the topology of manifolds. For example,

- The minimizability of an orientable Riemannian foliation of codimension q on an orientable closed manifold is characterized by the nontriviality of the basic cohomology of degree q by a theorem of Masa [13].
- For a Riemannian foliation on a closed manifold, Álvarez López [1] defined the Álvarez class, which is a basic cohomology class of degree 1 whose triviality characterizes the minimizability.
- In particular, this characterization of the minimizability by Álvarez López implies that every Riemannian foliation on a closed manifold with zero first Betti number is minimizable. This is a generalization of a theorem of Ghys [8] on simply connected case.
- A Riemannian foliation $\mathcal{F}$ on a closed manifold M is minimizable if $\pi_1 M$ is of polynomial growth and $\mathcal{F}$ is developable by a result of the author [20].

In this paper, we show that the Álvarez classes of a smooth family of Riemannian foliations on a closed manifold is continuous with respect to the parameter. Combining this result with a rigidity theorem of the Álvarez class in Nozawa [19], we obtain the invariance of the minimizability of Riemannian foliations under deformation under the topological conditions in [19] which imply an algebraic rigidity of Álvarez class there (see Corollary 4 below).

Note that the invariance of the minimizability under deformation does not hold for general foliations. We present examples of families of foliations in Section 8 to describe the situation.

A continuity theorem of the Álvarez class. Our main result is as follows: Let M be a closed manifold. Let U be an open neighborhood of 0 in $\mathbb{R}^l$. Let $\{\mathcal{F}^t\}_{t \in U}$ be a smooth family of Riemannian foliations on M over U.

Theorem 1. *The Álvarez class $\xi(\mathcal{F}^t)$ of $(M, \mathcal{F}^t)$ is continuous in $H^1(M; \mathbb{R})$ with respect to t.*

We mention why Theorem 1 does not follow from the definition of the Álvarez class or the classical deformation theory. By the definition, the Álvarez classes of $(M, \mathcal{F}^t)$ is represented by the closed 1-form obtained by orthogonally projecting the mean curvature form of $(M, \mathcal{F}^t, g^t)$ to the space of basic 1-forms on $(M, \mathcal{F}^t)$ for any bundle-like metric g^t on $(M, \mathcal{F}^t)$. But the space of basic 1-form on $(M, \mathcal{F}^t)$ changes discontinuously with respect to t in the space of 1-forms on M, when the dimension of closures of generic leaves changes. Because of this discontinuity of the spaces of basic 1-forms, we cannot obtain a continuous family of closed 1-forms which represents the Álvarez classes directly from the definition. Furthermore, this discontinuity of the spaces of basic 1-forms breaks the continuity of the domains of the families of basic Laplacians. Thus we cannot apply the classical technique of deformation theory using smooth families of self-adjoint operators to show Theorem 1 at least directly.

We mention why Theorem 1 does not follow from the interpretation of the Álvarez class in terms of the holonomy homomorphism of the Molino's commuting sheaf of $(M,\mathcal{F}^t)$ by a theorem of Álvarez López [2]. If the dimension of closures of generic leaves changes, the ranks of family of Molino's commuting sheaves as flat vector bundles changes. Hence the family of Molino's commuting sheaves is not smooth as a family of flat vector bundles, and we cannot prove the continuity directly by the result of [2].

To prove Theorem 1, we will take a suitable representative of the Álvarez class at $t=0$. Then, we will approximate the Álvarez class by non-closed 1-forms (see Section 6.3). Some technical consideration on Riemannian foliations will be needed to take a suitable representative of the Álvarez class at $t=0$ in Section 5, which is the main part of this article.

Deformation of minimizable Riemannian foliations. Combining Theorem 1 with the characterization of the minimizability by the triviality of the Álvarez class by Álvarez López [1], we have

Corollary 2. *In parameter spaces of smooth families of Riemannian foliations on closed manifolds, the subsets consisting of parameters corresponding to minimizable Riemannian foliations are closed.*

This corollary is not true for general foliations as we will see in Example 8.2.

A foliation $\mathcal{F}$ is defined to be of polynomial growth if the fundamental group of every leaf of $\mathcal{F}$ is of polynomial growth. A group Γ is polycyclic if there exists a sequence $\{\Gamma_i\}_{i=1}^n$ of subgroups of Γ such that $\Gamma_0=\Gamma$, $\Gamma_n=\{1\}$, $\Gamma_{i-1}\rhd\Gamma_i$ and Γ_i/Γ_{i+1} is cyclic for every i. Let $(M,\mathcal{F})$ be a closed manifold with a Riemannian foliation. If $\pi_1 M$ is polycyclic or $\mathcal{F}$ is of polynomial growth, then the integration of the Álvarez class of $(M,\mathcal{F})$ along every closed path on M is exponential of an algebraic integer by a result of the author [19]. By the totally disconnectedness of the set of algebraic integers in $\mathbb{R}$, we have the following corollary of Theorem 1: Let M be a closed manifold. Let U be a connected open neighborhood of 0 in $\mathbb{R}^l$. Let $\{\mathcal{F}^t\}_{t\in U}$ be a smooth family of Riemannian foliations on M over U.

Corollary 3. *If $\pi_1 M$ is polycyclic or $\mathcal{F}^t$ is of polynomial growth for every t, then $\xi(\mathcal{F}^t)=\xi(\mathcal{F}^0)$ in $H^1(M,\mathbb{R})$ for every t.*

The Álvarez class changes nontrivially for examples of families of solvable Lie foliations constructed by Meigniez [15] (see also [16]) as we mentioned in [19]. Hence Corollary 3 is not true in general case. By the characterization of the minimizability by the triviality of the Álvarez class by Álvarez López [1] and Corollary 3, we have

Corollary 4. *If $\pi_1 M$ is polycyclic or $\mathcal{F}^t$ is of polynomial growth for every t, then one of the following holds:*

(i) *For every t in U, $(M,\mathcal{F}^t)$ is minimizable.*
(ii) *For every t in U, $(M,\mathcal{F}^t)$ is not minimizable.*

Note that $\mathcal{F}$ is always of polynomial growth if $\dim\mathcal{F}=1$. Hence for Riemannian flows, the minimizability is invariant under deformation. As noted above, the invariance of the minimizability under deformation is not true for general foliations. For Riemannian foliations, it is not clear if the minimizability is invariant under deformation in general. We ask

Question 5. *Is the minimizability of Riemannian foliations on closed manifolds invariant under deformation ?*

Let $(M, \mathcal{F})$ be a closed 4-manifold with a GA(1)-Lie foliation. By a theorem of Matsumoto and Tsuchiya [14], $(M, \mathcal{F})$ is a homogeneous GA(1)-Lie foliation up to a finite covering. As a corollary, $\pi_1 M$ is isomorphic to a lattice in a connected simply connected solvable Lie group up to finite index subgroups. It is well known that a lattice of connected simply connected solvable Lie group is polycyclic (see Raghunathan [21]). Therefore $\pi_1 M$ is polycyclic. By a generalization of a minimality theorem of Masa [13] by Álvarez López [1] to nonoriented cases, a transversely oriented Riemannian foliation on a closed manifold is minimizable if and only if the top degree component of the basic cohomology is nontrivial. In the case of a G-Lie foliation, it is easy to see that the nontriviality of the top degree component of basic cohomology is equivalent to the unimodularity of G. Hence a GA(1)-Lie foliation is not minimizable, and an $\mathbb{R}^2$-Lie foliation is minimizable. By Corollary 4, $(M, \mathcal{F})$ cannot be deformed to a minimizable Riemannian foliation. Thus, we obtained

Corollary 6. *A* GA(1)*-Lie foliation on a closed* 4*-manifold cannot be deformed into an* $\mathbb{R}^2$*-Lie foliation.*

In dimensions lower than 4, Corollary 6 is easily confirmed to be true is as a consequence of classification of Riemannian foliations. In higher dimensions, it is not clear if a similar result is true or not.

The invariance of basic cohomology of Riemannian foliations under deformation. Let $H^{\bullet}(M/\mathcal{F}^t)$ be the basic cohomology of $(M, \mathcal{F}^t)$. For a Riemannian foliation on a closed manifold, the dimension of $H_b^{\operatorname{cod} \mathcal{F}^t}(M/\mathcal{F}^t)$ is 1 or 0 (see El Kacimi, Sergiescu and Hector [10]). As remarked by Álvarez López in the proof of Corollary 6.2 of [1], the triviality of the Álvarez class directly implies the nontriviality of $H_b^{\operatorname{cod} \mathcal{F}}(M/\mathcal{F}^t)$. Hence Corollary 4 is paraphrased to

Corollary 7. *If* $\pi_1 M$ *is polycyclic or* $\mathcal{F}^t$ *is of polynomial growth for every* t*, then we have* $H_b^{\operatorname{cod} \mathcal{F}^t}(M/\mathcal{F}^t) \cong H_b^{\operatorname{cod} \mathcal{F}^0}(M/\mathcal{F}^0)$ *for every* t *in* U*.*

This corollary gives a partial positive answer to the following question asked by the author in VIII International Colloquium on Differential Geometry at Santiago de Compostela (see [3]):

Question 8. *Is basic cohomology of Riemannian foliations invariant under deformation ?*

The component of the basic cohomology of the degree equal to the codimension of Riemannian foliations is invariant under deformation if and only if the answer of Question 5 is true. We note that the answer of Question 8 is negative in degree lower than the codimension of Riemannian foliations. We have a simple counterexample as we present in Example 7.4.

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2. Basic definitions

2.1. **Families of foliations.** We use the terminology in Molino [18]. We recall the definition of some basic terminology here to avoid confusion.

Let $(M, \mathcal{F})$ be a foliated manifold. By the integrability of $\mathcal{F}$, the Lie bracket on $C^\infty(TM)$ induces the Lie derivative with respect to vector fields tangent to the leaves

(1)
$$C^\infty(T\mathcal{F}) \otimes C^\infty\left(\bigotimes^r (TM/T\mathcal{F}) \otimes \bigotimes^s (TM/T\mathcal{F})^*\right) \longrightarrow C^\infty\left(\bigotimes^r (TM/T\mathcal{F}) \otimes \bigotimes^s (TM/T\mathcal{F})^*\right)$$

for every nonnegative integer s and r.

Definition 9. (i) *An element X of $C^\infty(TM/T\mathcal{F})$ is called a transverse field on $(M, \mathcal{F})$ if $L_Y X = 0$ for every Y in $C^\infty(T\mathcal{F})$. A vector field Y on M is called a basic vector field if Y is mapped to a transverse field by the projection $C^\infty(TM) \longrightarrow C^\infty(TM/T\mathcal{F})$.*

(ii) *An element g of $C^\infty(\bigotimes^2 (TM/T\mathcal{F})^*)$ is called a transverse metric if the following three conditions are satisfied:*
  1. *$g(Y, Z) = g(Z, Y)$ for every Y and Z in $C^\infty(TM/T\mathcal{F})$,*
  2. *$L_X g = 0$ for every X in $C^\infty(T\mathcal{F})$ and*
  3. *$g_x(Z, Z) > 0$ for every point x on M for every nonzero vector Z in $T_x M/(T\mathcal{F})_x$.*

A Riemannian metric g on M is called a bundle-like metric if the restriction of g to $\bigotimes^2 (T\mathcal{F})^\perp$ is a transverse metric under the natural identification of $\bigotimes^2 (T\mathcal{F})^\perp$ with $\bigotimes^2 (TM/T\mathcal{F})$.

(iii) *Let q be the codimension of $(M, \mathcal{F})$. A transversal parallelism of $(M, \mathcal{F})$ is a q-tuple of transverse fields X^1, X^2, $\cdots$, X^q on $(M, \mathcal{F})$ such that $\{(X^1)_x, (X^2)_x, \cdots, (X^q)_x\}$ is a basis of $T_x M/(T\mathcal{F})_x$ at each point x on M.*

We recall the definition of smooth families of foliations with transverse structures. Let U be an open set in $\mathbb{R}^L$ which contains 0. Let M be a smooth manifold.

Definition 10. *A smooth family of p-dimensional foliations of M over U is defined by a p-dimensional smooth foliation $\mathcal{F}^{\mathrm{amb}}$ of $M \times U$ such that every leaf of $\mathcal{F}^{\mathrm{amb}}$ is contained in $M \times \{t\}$ for some t.*

For t in U, let $\mathcal{F}^t$ be the foliation of $M \times \{t\}$ defined by the collection of the leaves of $\mathcal{F}^{\mathrm{amb}}$ contained in $M \times \{t\}$. Families of foliations are written as $\{\mathcal{F}^t\}_{t \in U}$ throughout this paper. $(\nu\mathcal{F})^{\mathrm{amb}}$ denotes the vector bundle over $M \times U$ defined by the kernel of the map $T(M \times U)/T\mathcal{F}^{\mathrm{amb}} \longrightarrow TU$ induced by the differential map of the second projection $M \times U \longrightarrow U$. We call $(\nu\mathcal{F})^{\mathrm{amb}}$ the family of normal bundles of $\{\mathcal{F}^t\}_{t\in U}$. Note that $(\nu\mathcal{F})^{\mathrm{amb}}|_{M\times\{t\}}$ is the normal bundle of the foliation $\mathcal{F}^t$ of $M\times\{t\}$ for each t.

Definition 11. (i) *A smooth family of Riemannian foliations of M over U is a pair of a smooth family of foliation of $M\times U$ defined by $\mathcal{F}^{\mathrm{amb}}$ and a smooth metric g^{amb} on $(\nu\mathcal{F})^{\mathrm{amb}}$ such that the restriction of g^{amb} to the orthogonal normal bundle of $(M\times\{t\},\mathcal{F}^t)$ is a transverse metric on $(M\times\{t\},\mathcal{F}^t)$.*

(ii) *A smooth family of transversely parallelizable foliations of codimension q of M over U is a pair of a smooth family $\{\mathcal{F}^t\}_{t\in U}$ of foliations of M of codimension q and a q-tuple of global sections X^1_{amb}, X^2_{amb}, $\cdots$, X^q_{amb} of $(\nu\mathcal{F})^{\mathrm{amb}}$ such that $\{X^1_{\mathrm{amb}}|_{M\times\{t\}}, X^2_{\mathrm{amb}}|_{M\times\{t\}},\cdots,X^q_{\mathrm{amb}}|_{M\times\{t\}}\}$ is a transverse parallelism of $(M\times\{t\},\mathcal{F}^t)$ for each t.*

2.2. **The Álvarez class.** We recall the definition of the Álvarez class of a closed manifold with a Riemannian foliation by Álvarez López [1]. We restrict ourselves to the case of oriented manifolds. The definition in nonorientable case is done by lifting the foliation to the orientation cover as in [1].

Let $(M,\mathcal{F})$ be an oriented closed manifold with a Riemannian foliation. We fix a bundle-like metric g on $(M,\mathcal{F})$. We have a direct sum decomposition

$$C^\infty(\wedge^k T^*M) = C^\infty_b(\wedge^k T^*M)\oplus C^\infty_b(\wedge^k T^*M)^\perp \tag{2}$$

with respect to the metric induced by g where $C^\infty_b(\wedge^k T^*M)$ is the space of basic k-forms on $(M,\mathcal{F})$. Let $\rho_{\mathcal{F}}$ be the first projection

$$\rho_{\mathcal{F}}\colon C^\infty(\wedge^k T^*M)\longrightarrow C^\infty_b(\wedge^k T^*M). \tag{3}$$

We denote the mean curvature form of $(M,\mathcal{F},g)$ by κ (see, for example, Section 10.5 of Candel and Conlon [4] for the definition of the mean curvature form of $(M,\mathcal{F},g)$).

Definition 12. *For an oriented closed manifold M with a Riemannian foliation $\mathcal{F}$, we define a basic 1-form κ_b on $(M,\mathcal{F})$ by*

$$\kappa_b=\rho_{\mathcal{F}}(\kappa) \tag{4}$$

and call κ_b the Álvarez form of $(M,\mathcal{F})$. This κ_b is closed by Corollary 3.5 of Álvarez López [1]. We define the Álvarez class of $(M,\mathcal{F})$ by the cohomology class of κ_b in $H^1(M;\mathbb{R})$. We denote the Álvarez class of $(M,\mathcal{F})$ by $\xi(\mathcal{F})$.

Let $H^1_b(M/\mathcal{F})$ be the basic cohomology group of degree 1 of $(M,\mathcal{F})$ (see Section 2 of Reinhart [22] or Section 2.3 of Molino [18] for the definition of the basic cohomology). Álvarez López defined the Álvarez class as an element of $H^1_b(M/\mathcal{F})$ in [1]. Since the canonical map $H^1_b(M/\mathcal{F})\longrightarrow H^1(M/\mathcal{F})$ is injective as easily confirmed by the definition, Definition 12 gives the essentially same data as in [1].

The simple proof of the following lemma is due to a comment of Álvarez López to the author:

Lemma 13. *Let $(M_1,\mathcal{F}_1)$ be a closed manifold with a Riemannian foliation. Let $p\colon M_2\longrightarrow M_1$ be a finite covering. We define a Riemannian foliation $\mathcal{F}_2$ on M_2 by $\mathcal{F}_2=p^*\mathcal{F}_1$. Then we have $\xi(\mathcal{F}_2)=p^*\xi(\mathcal{F}_1)$.*

Proof. We take a bundle-like metric g_1 on $(M_1, \mathcal{F}_1)$. Then $p^* g_1$ is a bundle-like metric on $(M_1, \mathcal{F}_1)$. We consider orthogonal decompositions

$$\begin{aligned} \Omega^1(M_1) &= \Omega^1_b(M_1/\mathcal{F}_1) \oplus (\Omega^1_b(M_1/\mathcal{F}_1))^{\perp}, \\ \Omega^1(M_2) &= \Omega^1_b(M_2/\mathcal{F}_2) \oplus (\Omega^1_b(M_2/\mathcal{F}_2))^{\perp} \end{aligned} \tag{5}$$

with respect to the metric induced by g_1 and $p^* g_1$, respectively. By the definition of metrics, we have

$$\begin{aligned} p^*\Omega^1_b(M_1/\mathcal{F}_1) &= p^*\Omega^1(M_1) \cap \Omega^1_b(M_2/\mathcal{F}_2), \\ p^*(\Omega^1_b(M_1/\mathcal{F}_1))^{\perp} &= p^*\Omega^1(M_1) \cap (\Omega^1_b(M_2/\mathcal{F}_2))^{\perp}. \end{aligned} \tag{6}$$

Let $\rho_{\mathcal{F}_1}$ and $\rho_{\mathcal{F}_2}$ be the first projections on decompositions (5). These equalities imply

$$p^* \rho_{\mathcal{F}_1} = \rho_{\mathcal{F}_2} p^*. \tag{7}$$

Let κ_i be the mean curvature forms of $(M_i, \mathcal{F}_i)$ with respect to g_1 and $p^* g_1$, respectively. We have $\kappa_2 = p^* \kappa_1$. By (7), we have

$$p^*(\kappa_1)_b = p^* \rho_{\mathcal{F}_1}(\kappa_1) = \rho_{\mathcal{F}_2}(p^* \kappa_1) = \rho_{\mathcal{F}_2}(\kappa_2) = (\kappa_2)_b. \tag{8}$$

□

3. Fundamentals of Lie foliation theory

We summarize the fundamental facts of Lie foliation theory due to Fedida [6] and [7] (see also Section 4.2 of Molino [18] or Section 4.3.1 of Moerdijk and Mrčun [17]) to use in Sections 5 and 6.

Let G be a connected Lie group. Recall that a G-Lie foliation is a foliation with a transverse (G, G)-structure where G acts on G by the left multiplication. A G-Lie foliation has a structure of G'-Lie foliation for any covering group G' of G as easily confirmed. Thus we will assume the simply connectedness of G throughout this paper. We recall the following

Definition 14. *The Lie algebra of G is called the structural Lie algebra of the Lie foliation.*

Let $(M, \mathcal{F})$ be a G-Lie foliation. Let $p_M^{\mathrm{univ}} \colon M^{\mathrm{univ}} \longrightarrow M$ be the universal cover of M. Fix a point x_0^{univ} on M^{univ}. We put $x_0 = p_M^{\mathrm{univ}}(x_0^{\mathrm{univ}})$.

- **(I):** We have a fiber bundle $\mathrm{dev} \colon M^{\mathrm{univ}} \longrightarrow G$ which maps x_0^{univ} to the unit element e of G and whose fibers are the leaves of the foliation $(p_M^{\mathrm{univ}})^* \mathcal{F}$ of M^{univ}. This dev is called the developing map of $(M, \mathcal{F})$.
- **(II):** We have a group homomorphism $\mathrm{hol} \colon \pi_1(M, x_0) \longrightarrow G$ such that

$$\mathrm{dev}(\gamma \cdot x) = \mathrm{dev}(x) \cdot_G \mathrm{hol}(\gamma) \tag{9}$$

  for x in M^{univ} and γ in $\pi_1(M, x_0)$ where $\cdot_G$ is the multiplication of G. The image of hol is called the holonomy group of $(M, \mathcal{F})$. The holonomy group of $(M, \mathcal{F})$ is dense in G if and only if the leaves of $\mathcal{F}$ are dense in M.
- **(III):** Recall that the Maurer-Cartan form θ on G is a $\mathrm{Lie}(G)$-valued 1-form on G defined by $\theta_g(v) = (R_g)_* v$ for g in G and v in $T_g G$ where R_g is the right multiplication map of g. Since a $\mathrm{Lie}(G)$-valued 1-form $\mathrm{dev}^* \theta$ on M^{univ} is invariant under the $\pi_1(M, x_0)$-action, $\mathrm{dev}^* \theta$ induces a $\mathrm{Lie}(G)$-valued 1-form Ω on M. The structure of a G-Lie foliation $(M, \mathcal{F})$ is determined by this $\mathrm{Lie}(G)$-valued 1-form Ω on M. This Ω is called the Maurer-Cartan form of a G-Lie foliation $(M, \mathcal{F})$.

(IV): The Maurer-Cartan form Ω of a G-Lie foliation $(M,\mathcal{F})$ satisfies the equation

$$d\Omega + \frac{1}{2}[\Omega,\Omega] = 0. \tag{10}$$

Conversely, if a $\mathrm{Lie}(G)$-valued 1-form Ω on M satisfies (10) and $\Omega_x \colon T_xM \longrightarrow \mathrm{Lie}(G)$ is surjective for every x in M, then Ω is the Maurer-Cartan form of a G-Lie foliation of M.

(V): Let $\{\overline{X}^j\}_{j=1}^{\mathrm{cod}(M,\mathcal{F})}$ be a basis of $\mathrm{Lie}(G)$. Let $\{\overline{\omega}_i\}_{i=1}^{\mathrm{cod}(M,\mathcal{F})}$ be the dual basis of $\mathrm{Lie}(G)^*$. Let ω_i be the 1-form on M induced from a $\pi_1(M,x_0)$-invariant 1-form $\mathrm{dev}^*\,\overline{\omega}_i$ on M^{univ} by the quotient. We define a vector field X^j on M by $\omega_i(X^j) = \delta_{ij}$ for each i and j where δ_{ij} is the Kronecker's delta. Then $\{X^j\}_{j=1}^{\mathrm{cod}(M,\mathcal{F})}$ is clearly a transverse parallelism of $(M,\mathcal{F})$. Here, the Maurer-Cartan form Ω of $(M,\mathcal{F})$ is given by the equation $\Omega_x((X^j)_x) = \overline{X}^j$ for each point x on M.

4. Reduction to the orientable transversely parallelizable case

We reduce the proof of Theorem 1 to the case where $\{\mathcal{F}^t\}_{t\in U}$ is a family of transversely parallelizable foliations.

Let $\{\mathcal{F}^t\}_{t\in U}$ be a smooth family of Riemannian foliations of codimension q of a closed manifold M over U. Clearly we can assume that U is contractible without loss of generality. Let $\mathrm{Fr}(\nu\mathcal{F})^{\mathrm{amb}}$ be the family of the frame bundles associated with the family $(\nu\mathcal{F})^{\mathrm{amb}}$ of vector bundles on M. The metric g^{amb} on $(\nu\mathcal{F})^{\mathrm{amb}}$ determines an $O(q)$-reduction $O(\nu\mathcal{F})^{\mathrm{amb}}$ of $\mathrm{Fr}(\nu\mathcal{F})^{\mathrm{amb}}$. We denote $O(\nu\mathcal{F})^{\mathrm{amb}}|_{M\times\{0\}}$ by $O(\nu\mathcal{F})^0$. Since $O(\nu\mathcal{F})^{\mathrm{amb}}$ is the total space of a $O(\nu\mathcal{F})^0$-bundle over a contractible base space U, we can trivialize $O(\nu\mathcal{F})^{\mathrm{amb}}$ as $O(\nu\mathcal{F})^{\mathrm{amb}} \cong O(\nu\mathcal{F})^0 \times U$. By the standard construction of the Molino theory, we have the following lemma:

Lemma 15. *There exists a foliation $\mathcal{G}^{\mathrm{amb}}$ of $O(\nu\mathcal{F})^0\times U$ and a $\frac{q(q+1)}{2}$-tuple of transverse fields of $(O(\nu\mathcal{F})^0\times U, \mathcal{G}^{\mathrm{amb}})$ defining a smooth family $\{\mathcal{G}^t\}_{t\in U}$ of transversely parallelizable foliations of codimension $\frac{q(q+1)}{2}$ of $O(\nu\mathcal{F})^0$ over U.*

Proof. Let $\{(V_\lambda,\phi_\lambda)\}$ be a Haefliger cocycle defining a foliation $\mathcal{F}^{\mathrm{amb}}$ of $M\times U$. Then $\{(O(\nu\mathcal{F})^{\mathrm{amb}}|_{V_\lambda}, d\phi_\lambda)\}$ is a Haefliger cocycle on $O(\nu\mathcal{F})^{\mathrm{amb}}$ where $d\phi_\lambda$ is the map induced on the frame bundle by ϕ_λ. We define a foliation $\mathcal{G}^{\mathrm{amb}}$ of $O(\nu\mathcal{F})^0\times U$ pushing out the foliation of $O(\nu\mathcal{F})^{\mathrm{amb}}$ defined by the Haefliger cocycle $\{(O(\nu\mathcal{F})^{\mathrm{amb}}|_{V_\lambda}, d\phi_\lambda)\}$ by the trivialization $O(\nu\mathcal{F})^{\mathrm{amb}} \cong O(\nu\mathcal{F})^0\times U$.

We put $\mathcal{G}^t = \mathcal{G}^{\mathrm{amb}}|_{O(\nu\mathcal{F})^0\times\{t\}}$. For each t, we can construct a transversely parallelism of $(O(\nu\mathcal{F})^0\times\{t\},\mathcal{G}^t)$ from the transverse Levi-Civita connection on $O(\nu\mathcal{F})^{\mathrm{amb}}|_{M\times\{t\}}$ and the canonical 1-form on the frame bundle $O(\nu\mathcal{F})^{\mathrm{amb}}|_{M\times\{t\}}$ as in Section 5.1 of Molino [18] or Theorem 4.20 of Moerdijk and Mrčun [17]. Since the transverse Levi-Civita connections and the canonical 1-forms on $(O(\nu\mathcal{F})^0\times\{t\},\mathcal{G}^t)$ are smooth with respect to the parameter t, we have a smooth family of transverse parallelisms. □

Lemma 16. *If $\xi(\mathcal{G}^t)$ is continuous with respect to t, then $\xi(\mathcal{F}^t)$ is also continuous with respect to t.*

Proof. Let $\pi\colon O(\nu\mathcal{F})^0\times U \longrightarrow M\times U$ be the projection. By Lemma 7 of Nozawa [19], we have $(\pi|_{O(\nu\mathcal{F})^0\times\{t\}})^*\xi(\mathcal{F}^t) = \xi(\mathcal{G}^t)$ for each t. Since $(\pi|_{O(\nu\mathcal{F})^0\times\{t\}})^*\colon H^1(M\times$

$\{t\};\mathbb{R}) \longrightarrow H^1(O(\nu\mathcal{F})^0 \times \{t\};\mathbb{R})$ is injective, the continuity of $\xi(\mathcal{F}^t)$ follows from the continuity of $\xi(\mathcal{G}^t)$. □

By Lemmas 13 and 16, we have the following

Lemma 17. *The general case of Theorem 1 follows from the special case where*

(i) $\{\mathcal{F}^t\}_{t\in U}$ *is a smooth family of transversely parallelizable foliations of* M.

(ii) M *and the basic fibration of* $\mathcal{F}^0$ *are orientable. Moreover if we have a foliation* $\tilde{\mathcal{F}}$ *on* M, *we can assume that* $\tilde{\mathcal{F}}$ *is also orientable.*

Here, $\tilde{\mathcal{F}}$ in the statement of (ii) will be $\tilde{\mathcal{F}}^0$ which appears in Section 6.2. For the proof of Lemma 17, we note that the finite covering $p\colon M_1 \longrightarrow M_2$ induces an injection $p_*\colon H_1(M_1;\mathbb{R}) \longrightarrow H_1(M_2;\mathbb{R})$. Then $p^*\colon H^1(M_2;\mathbb{R}) \longrightarrow H^1(M_1;\mathbb{R})$ is injective by the Poincaré duality for closed manifolds M_1 and M_2.

5. A representative $\tilde{\kappa}_b$ of the Álvarez class

5.1. Definition of the $\tilde{\mathcal{F}}$-integrated component $\tilde{\kappa}_b$ of the mean curvature form. We will define the $\tilde{\mathcal{F}}$-integrated component $\tilde{\kappa}_b$ of the mean curvature form for transversely parallelizable foliations in a way similar to that of the definition of the Álvarez form κ_b for trasnversely parallelizable foliations in Álvarez López [1]. Let $(M,\mathcal{F})$ be a closed manifold with a transversely parallelizable foliation. We denote the codimension of $(M,\mathcal{F})$ by $\mathrm{cod}(M,\mathcal{F})$. We take a transverse parallelism $\{X^j\}_{j=1}^{\mathrm{cod}(M,\mathcal{F})}$ of $(M,\mathcal{F})$. Let $\{\omega_i\}_{i=1}^{\mathrm{cod}(M,\mathcal{F})}$ be the set of basic 1-forms on $(M,\mathcal{F})$ such that $\omega_i(X^j) = \delta_{ij}$ where δ_{ij} is the Kronecker's delta. We put

(11) $$\omega_I = \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k}$$

for a set $I = \{i_1, i_2, \cdots, i_k\}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq \mathrm{cod}(M,\mathcal{F})$. Assume that we have the following diagram:

(12) $$\begin{array}{ccc} M & \xrightarrow{\pi_b} & W \\ & \underset{\pi_{\tilde{\mathcal{F}}}}{\searrow} & \downarrow \\ & & V \end{array}$$

where $\pi_b\colon M \longrightarrow W$ is the basic fibration of $(M,\mathcal{F})$ and $\pi_{\tilde{\mathcal{F}}}$ is a submersion. Recall that the basic fibration of $(M,\mathcal{F})$ is a fiber bundle whose fibers are closures of leaves of $(M,\mathcal{F})$. We denote the foliation of M defined by the fibers of $\pi_{\tilde{\mathcal{F}}}$ by $\tilde{\mathcal{F}}$. We denote V by $M/\tilde{\mathcal{F}}$ in below.

We assume that M and the fiber bundle $\pi_{\tilde{\mathcal{F}}}$ are orientable. We fix a bundle-like metric g on $(M,\mathcal{F})$. Then we define a map $\rho_{\tilde{\mathcal{F}}}$ by

(13) $$\rho_{\tilde{\mathcal{F}}}\Big(\tau + \sum_{I\subset\{1,2,\cdots,\mathrm{cod}(M,\mathcal{F})\},|I|=k} f^I\omega_I\Big) = \frac{1}{\pi_{\tilde{\mathcal{F}}}^*\left(\int_{\tilde{\mathcal{F}}} \mathrm{vol}_{\tilde{\mathcal{F}}}\right)} \sum_{I\subset\{1,2,\cdots,\mathrm{cod}(M,\mathcal{F})\},|I|=k} \Big(\int_{\tilde{\mathcal{F}}} f^I \,\mathrm{vol}_{\tilde{\mathcal{F}}}\Big)\omega_I$$

for f^I in $C^\infty(M)$ and τ in $C^\infty\big(T^*\mathcal{F} \otimes (\wedge^{k-1}T^*M)\big)$ where $\int_{\tilde{\mathcal{F}}}$ is the integration along the fiber of $\pi_{\tilde{\mathcal{F}}}$ with respect to the fixed orientation and $\mathrm{vol}_{\tilde{\mathcal{F}}}$ is the fiberwise volume form of $\pi_{\tilde{\mathcal{F}}}$ determined by the metric g. Note that we have a direct sum decomposition

(14) $$C^\infty(\wedge^k T^*M) = (\ker \rho_{\tilde{\mathcal{F}}})^\perp \oplus \ker \rho_{\tilde{\mathcal{F}}}$$

and $\rho_{\tilde{\mathcal{F}}}$ is the first projection as the case of $\rho_{\mathcal{F}}$.

Definition 18. *We define the $\tilde{\mathcal{F}}$-integrated component $\tilde{\kappa}_b$ of the mean curvature form κ of $(M, \mathcal{F}, g)$ with respect to the transverse parallelism $\{\omega_i\}_{i=1}^{\mathrm{cod}(M,\mathcal{F})}$ by*

$$\tilde{\kappa}_b = \rho_{\tilde{\mathcal{F}}}(\kappa). \tag{15}$$

Note that $\rho_{\tilde{\mathcal{F}}}$ coincides with $\rho_{\mathcal{F}}$ if $\tilde{F}$ is the foliation whose leaves are closures of leaves of $\mathcal{F}$ according to according to Álvarez López [1] (see).In this case, $\tilde{\kappa}_b$ coincides with κ_b.

Note that $\rho_{\tilde{\mathcal{F}}}$ depends on the choice of the transverse parallelism $\{X^j\}_{j=1}^{\mathrm{cod}(M,\mathcal{F})}$. This is different from the case of $\rho_{\mathcal{F}}$, which is determined only by the metric g. We remark that there will be a natural choice of $\{\omega_i\}_{i=1}^{\mathrm{cod}(M,\mathcal{F})}$, when we apply this construction in Section 6. Note that

$$\rho_{\tilde{\mathcal{F}}}\rho_{\mathcal{F}} = \rho_{\tilde{\mathcal{F}}} \tag{16}$$

by the definition.

Note that $\tilde{\kappa}_b$ may not be closed. We do not define $\tilde{\kappa}_b$ for the case where M or $\pi_{\tilde{\mathcal{F}}}$ is not orientable. This is because it is not used in this paper.

5.2. **The statement of Proposition 20.** We will state Proposition 20 which asserts that the Álvarez class of $(M, \mathcal{F})$ is represented by $\tilde{\kappa}_b$ under certain conditions. These conditions will be naturally satisfied in our application in Section 6. The proof of the essential part of Proposition 20 occupies the rest of this section.

Let $(M, \mathcal{F})$ be a closed manifold with a transversely parallelizable foliation. We fix a bundle-like metric g on $(M, \mathcal{F})$. We assume that orientability of M, $\tilde{\mathcal{F}}$ and the basic fibration of $(M, \mathcal{F})$. Let $\mathrm{vol}_{\tilde{\mathcal{F}}}$ be the characteristic form of $(M, \tilde{\mathcal{F}}, g)$. Let $\kappa_{\tilde{\mathcal{F}}}$ be the mean curvature form of $(M, \tilde{\mathcal{F}}, g)$. Let $d_{1,0}$ be the composition of the de Rham differential and the projection $C^{\infty}(\wedge^{\bullet+1}T^*M) \longrightarrow C^{\infty}((T\mathcal{F}^{\perp})^* \otimes \wedge^{\bullet}T^*\mathcal{F})$ determined by g. We take a transverse parallelism $\{X^j\}_{j=1}^{\mathrm{cod}(M,\mathcal{F})}$ of $(M, \mathcal{F})$. Let $\{\omega_i\}_{i=1}^{\mathrm{cod}(M,\mathcal{F})}$ be the set of basic 1-forms on $(M, \mathcal{F})$ such that $\omega_i(X^j) = \delta_{ij}$ where δ_{ij} is the Kronecker's delta.

Lemma 19. *If each leaf of $(M, \tilde{\mathcal{F}})$ is minimal with respect to g, then the function $\int_{\tilde{\mathcal{F}}} \mathrm{vol}_{\tilde{\mathcal{F}}}$ on the leaf space $M/\tilde{\mathcal{F}}$ is constant.*

Proof. By the assumption, we have $\kappa_{\tilde{\mathcal{F}}} = 0$. By the Rummler's formula (see the second formula in the proof of Proposition 1 in Rummler [23] or Lemma 10.5.6 of Candel and Conlon [4]), we have $d_{1,0}\,\mathrm{vol}_{\tilde{\mathcal{F}}} = -\kappa_{\tilde{\mathcal{F}}} \wedge \mathrm{vol}_{\tilde{\mathcal{F}}}$. Hence we have $d_{1,0}\,\mathrm{vol}_{\tilde{\mathcal{F}}} = 0$. Then

$$d\left(\int_{\tilde{\mathcal{F}}} \mathrm{vol}_{\tilde{\mathcal{F}}}\right) = \int_{\tilde{\mathcal{F}}} d\,\mathrm{vol}_{\tilde{\mathcal{F}}} = \int_{\tilde{\mathcal{F}}} d_{1,0}\,\mathrm{vol}_{\tilde{\mathcal{F}}} = 0. \tag{17}$$

Here the second equality follows from the degree counting of the differential forms. □

Proposition 20. *We assume that M, $\tilde{\mathcal{F}}$ and the basic fibration π_b of $(M, \mathcal{F})$ are orientable. We assume that*

- **(a):** *The fixed bundle-like metric g on $(M, \mathcal{F})$ is bundle-like also with respect to $\tilde{\mathcal{F}}$.*
- **(b):** *Each leaf of $(M, \tilde{\mathcal{F}})$ is minimal with respect to g.*

(c): *We define functions* c_i^{jk} *on* M *by*

$$d\omega_i = \sum_{1\leq j<k\leq \mathrm{cod}(M,\mathcal{F})} c_i^{jk}\omega_j \wedge \omega_k. \tag{18}$$

Then $c_i^{jk}|_{\tilde{L}}$ *is a constant for each fiber* $\tilde{L}$ *of* $\pi_{\tilde{\mathcal{F}}}$.

(d): *Let* $\mathrm{proj}_{\tilde{\mathcal{F}}}\colon C^\infty(TM/T\mathcal{F}) \longrightarrow C^\infty(TM/T\tilde{\mathcal{F}})$ *be the canonical projection. Then* $\mathrm{proj}_{\tilde{\mathcal{F}}} X_i$ *is a transverse field on* $(M,\tilde{\mathcal{F}})$ *for each* i.

Then we have

(i) $\tilde{\kappa}_b$ *is a closed* 1*-form on* M *and*
(ii) $[\kappa_b] = [\tilde{\kappa}_b]$ *in* $H^1(M;\mathbb{R})$.

We will show (i) here. (ii) will be shown in the end of this section.

Proof of Proposition 20 (i). By Lemma 19, the function $\pi_{\tilde{\mathcal{F}}}^*\left(\int_{\tilde{\mathcal{F}}} \mathrm{vol}_{\tilde{\mathcal{F}}}\right)$ on M is constant. We put

$$C = \pi_{\tilde{\mathcal{F}}}^*\left(\int_{\tilde{\mathcal{F}}} \mathrm{vol}_{\tilde{\mathcal{F}}}\right). \tag{19}$$

We put

$$\kappa_b = \sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} h^i \omega_i. \tag{20}$$

By the condition (c), we have

$$\rho_{\tilde{\mathcal{F}}}\Big(\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} h^i d\omega_i\Big) = \sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \left(\int_{\tilde{\mathcal{F}}} h^i \,\mathrm{vol}_{\tilde{\mathcal{F}}}\right) d\omega_i. \tag{21}$$

Using (15), (16), (4) in this order, we have

$$\tilde{\kappa}_b = \rho_{\tilde{\mathcal{F}}}\kappa = \rho_{\tilde{\mathcal{F}}}\rho_{\mathcal{F}}\kappa = \rho_{\tilde{\mathcal{F}}}\kappa_b. \tag{22}$$

We have

$$\begin{aligned} & d\tilde{\kappa}_b \\ &= d\rho_{\tilde{\mathcal{F}}}\kappa_b \\ &= \tfrac{1}{C} d\Big(\textstyle\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i \,\mathrm{vol}_{\tilde{\mathcal{F}}}\big)\omega_i\Big) \\ &= \tfrac{1}{C}\textstyle\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} dh^i\wedge \mathrm{vol}_{\tilde{\mathcal{F}}}\big)\wedge\omega_i + \tfrac{1}{C}\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i d\,\mathrm{vol}_{\tilde{\mathcal{F}}}\big)\wedge\omega_i + \tfrac{1}{C}\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i \,\mathrm{vol}_{\tilde{\mathcal{F}}}\big) d\omega_i \\ &= \rho_{\tilde{\mathcal{F}}}\Big(\textstyle\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} dh^i\wedge\omega_i\Big) + \tfrac{1}{C}\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i d\,\mathrm{vol}_{\tilde{\mathcal{F}}}\big)\wedge\omega_i + \rho_{\tilde{\mathcal{F}}}\Big(\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} h^i d\omega_i\Big). \end{aligned} \tag{23}$$

Here, we used (22) in the first equality. The second equality follows from the combination of (13), (19) and (20). We used the commutativity of the integration along the fiber with d in the third equality. The fourth equality follows from the equations (13) and (21). We have

$$\begin{aligned} & \rho_{\tilde{\mathcal{F}}}\Big(\textstyle\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} dh^i\wedge\omega_i\Big) + \tfrac{1}{C}\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i d\,\mathrm{vol}_{\tilde{\mathcal{F}}}\big)\wedge\omega_i + \rho_{\tilde{\mathcal{F}}}\Big(\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} h^i d\omega_i\Big) \\ &= \rho_{\tilde{\mathcal{F}}}\Big(d\big(\textstyle\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} h^i\omega_i\big)\Big) + \tfrac{1}{C}\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i d\,\mathrm{vol}_{\tilde{\mathcal{F}}}\big)\wedge\omega_i \\ &= \rho_{\tilde{\mathcal{F}}}\Big(d\big(\textstyle\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} h^i\omega_i\big)\Big) + \tfrac{1}{C}\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i d_{1,0}\,\mathrm{vol}_{\tilde{\mathcal{F}}}\big)\wedge\omega_i \\ &= \rho_{\tilde{\mathcal{F}}}(d\kappa_b) - \tfrac{1}{C}\textstyle\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} \big(\int_{\tilde{\mathcal{F}}} h^i\kappa_{\tilde{\mathcal{F}}}\wedge \mathrm{vol}_{\tilde{\mathcal{F}}}\big)\wedge\omega_i. \end{aligned} \tag{24}$$

Here, in the first equality, we combined the first and the third terms. The second equality follows from the degree counting. The third equality follows from (20) and the Rummler's formula (see the second formula in the proof of Proposition 1 in Rummler [23] or Lemma 10.5.6 of Candel and Conlon [4]).

The first term of the last line is 0, because κ_b is closed by Corollary 3.5 of Álvarez López [1]. Since $\kappa_{\tilde{\mathcal{F}}}$ is 0 by the condition (b), the second term is also 0. Hence (i) is proved. □

5.3. $\xi(\mathcal{F}) = [\tilde{\kappa}_b]$ **on the fibers of** $\pi_{\tilde{\mathcal{F}}}$**.** We prove a lemma which will be used in the proof of Proposition 20 to show the restriction of $\xi(\mathcal{F})$ and $[\tilde{\kappa}_b]$ to the fibers of $\pi_{\tilde{\mathcal{F}}}$ are equal. Here, α will be considered to be $\xi(\mathcal{F}) - [\tilde{\kappa}_b]$ in the application in Section 5.6.

Lemma 21. *Assume that the conditions (a) and (b) in Proposition 20 are satisfied. Let M' be an orientable submanifold of M which is a union of fibers of $\pi_{\tilde{\mathcal{F}}}$. Let $\{\phi_t\}_{t\in[0,1]}$ be the flow on M' generated by a vector field X on M'. Assume that $\omega_i(X)$ is constant on each fiber of $\pi_{\tilde{\mathcal{F}}}$ for each i. Then we have*

$$\int_{M'} \Big(\int_{\gamma_x} \kappa_b \Big) \operatorname{vol}_{M'}(x) = \int_{M'} \Big(\int_{\gamma_x} \tilde{\kappa}_b \Big) \operatorname{vol}_{M'}(x) \tag{25}$$

where γ_x is the orbit of x of $\{\phi_t\}_{t\in[0,1]}$, and $\operatorname{vol}_{M'}$ is the volume form on M' determined by g.

Proof. The function $\pi^*_{\tilde{\mathcal{F}}}\left(\int_{\tilde{\mathcal{F}}} \operatorname{vol}_{\tilde{\mathcal{F}}}\right)$ on M' is constant by the condition (b) and Lemma 19. We take a real number C and functions h^i on M as (19) and (20). Then we have

$$\begin{aligned}
&\textstyle\int_{M'} \Big(\int_{\gamma_x} \rho_{\tilde{\mathcal{F}}}(\kappa) \Big) \operatorname{vol}_{M'}(x) \\
&= \textstyle\int_{M'} \Big(\int_{\gamma_x} \rho_{\tilde{\mathcal{F}}}(\kappa_b) \Big) \operatorname{vol}_{M'}(x) \\
&= \textstyle\frac{1}{C} \int_{M'} \Big(\int_{\gamma_x} \Big(\sum_{i=1}^{\operatorname{cod}(M,\mathcal{F})} \int_{\tilde{\mathcal{F}}} h^i \operatorname{vol}_{\tilde{\mathcal{F}}} \Big) \omega_i \Big) \operatorname{vol}_{M'}(x) \\
&= \textstyle\frac{1}{C} \int_{M'} \Big(\int_0^1 \Big(\sum_{i=1}^{\operatorname{cod}(M,\mathcal{F})} \int_{\tilde{\mathcal{F}}} h^i \operatorname{vol}_{\tilde{\mathcal{F}}} \Big)_{\gamma_x(t)} \omega_i(X)_{\gamma_x(t)} dt \Big) \operatorname{vol}_{M'}(x)
\end{aligned} \tag{26}$$

Here, we used (22) in the first equality. We used (13) and (20) in the second equality.

By the assumption, $\omega_i(X)$ is constant on the fibers of $\pi_{\tilde{\mathcal{F}}}$. Then we have

$$\begin{aligned}
&\textstyle\frac{1}{C} \int_{M'} \Big(\int_0^1 \sum_{i=1}^{\operatorname{cod}(M,\mathcal{F})} \Big(\int_{\tilde{\mathcal{F}}} h^i \operatorname{vol}_{\tilde{\mathcal{F}}} \Big)_{\gamma_x(t)} \omega_i(X)_{\gamma_x(t)} dt \Big) \operatorname{vol}_{M'}(x) \\
&= \textstyle\frac{1}{C} \int_{M'} \Big(\int_0^1 \Big(\sum_{i=1}^{\operatorname{cod}(M,\mathcal{F})} \int_{\tilde{\mathcal{F}}} h^i \omega_i(X)_{\gamma_x(t)} \operatorname{vol}_{\tilde{\mathcal{F}}} \Big) dt \Big) \operatorname{vol}_{M'}(x) \\
&= \textstyle\frac{1}{C} \int_{M'} \Big(\int_0^1 \Big(\int_{\tilde{\mathcal{F}}} \gamma_x^* \kappa_b(X) \operatorname{vol}_{\tilde{\mathcal{F}}} \Big) dt \Big) \operatorname{vol}_{M'}(x) \\
&= \textstyle\frac{1}{C} \int_{M'} \Big(\int_{\tilde{\mathcal{F}}} \Big(\int_0^1 \gamma_x^* \kappa_b(X) dt \Big) \operatorname{vol}_{\tilde{\mathcal{F}}} \Big) \operatorname{vol}_{M'}(x) \\
&= \textstyle\frac{1}{C} \int_{M'} \Big(\int_{\tilde{\mathcal{F}}} \Big(\int_{\gamma_x} \kappa_b \Big) \operatorname{vol}_{\tilde{\mathcal{F}}} \Big) \operatorname{vol}_{M'}(x).
\end{aligned} \tag{27}$$

By the condition (a), we have $\int_{M'} f \operatorname{vol}_{M'} = \int_{M'/\tilde{\mathcal{F}}} \left(\int_{\tilde{\mathcal{F}}} f \operatorname{vol}_{\tilde{\mathcal{F}}} \right) \operatorname{vol}_{M'/\tilde{\mathcal{F}}}$ for a function f on M' where $\operatorname{vol}_{M'/\tilde{\mathcal{F}}}$ is the volume form on M' determined by g. Hence we have

$$
\begin{aligned}
&\frac{1}{C}\int_{M'}\left(\int_{\tilde{\mathcal{F}}}\left(\int_{\gamma_x}\kappa_b\right)\mathrm{vol}_{\tilde{\mathcal{F}}}\right)\mathrm{vol}_{M'}(x)\\
&=\frac{1}{C}\int_{M'/\tilde{\mathcal{F}}}\left(\int_{\tilde{\mathcal{F}}}\left(\int_{\tilde{\mathcal{F}}}\left(\int_{\gamma_x}\kappa_b\right)\mathrm{vol}_{\tilde{\mathcal{F}}}\right)\mathrm{vol}_{\tilde{\mathcal{F}}}\right)\mathrm{vol}_{M'/\tilde{\mathcal{F}}}\\
&=\int_{M'/\tilde{\mathcal{F}}}\left(\int_{\tilde{\mathcal{F}}}\left(\int_{\gamma_x}\kappa_b\right)\mathrm{vol}_{\tilde{\mathcal{F}}}\right)\mathrm{vol}_{M'/\tilde{\mathcal{F}}}\\
&=\int_{M'}\left(\int_{\gamma_x}\kappa_b\right)\mathrm{vol}_{M'}(x).
\end{aligned}
\tag{28}
$$

The proof of Lemma 21 is completed. □

We fix a point x_0 on M. Let $\tilde{F}$ be the fiber of $\pi_{\tilde{\mathcal{F}}}$ containing x_0. We assume that the condition (d) in Proposition 20 is satisfied. We define transverse fields Y^1, $Y^2, \cdots, Y^{\mathrm{cod}(M,\mathcal{F})}$ on $(M,\mathcal{F})$ by

$$
Y^j=\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})} a_i^j X^i \tag{29}
$$

choosing a nondegenerate matrix $(a_i^j)_{1\leq j\leq \mathrm{cod}(M,\mathcal{F}),1\leq i\leq \mathrm{cod}(M,\mathcal{F})}$ so that $(Y^1)_{x_0}$, $(Y^2)_{x_0}, \cdots, (Y^{\mathrm{cod}(\tilde{F},\mathcal{F}|_{\tilde{F}})})_{x_0}$ are tangent to $\tilde{F}$. By the condition (d) in Proposition 20, vector fields $Y^1, Y^2, \cdots, Y^{\mathrm{cod}(\tilde{F},\mathcal{F}|_{\tilde{F}})}$ are basic with respect to $\tilde{\mathcal{F}}$. Hence $Y^1, Y^2, \cdots, Y^{\mathrm{cod}(\tilde{F},\mathcal{F}|_{\tilde{F}})}$ are tangent to $\tilde{F}$ at every point on $\tilde{F}$. We denote the vector subspace $\oplus_{j=1}^{\mathrm{cod}(\tilde{F},\mathcal{F}|_{\tilde{F}})}\mathbb{R}(Y^j|_{\tilde{F}})$ of the Lie algebra of transverse fields on $(\tilde{F},\mathcal{F}|_{\tilde{F}})$ by $\mathfrak{g}$.

Lemma 22. *Assume that the conditions (c) and (d) in Proposition 20 are satisfied.*

(i) $\mathfrak{g}$ *is a Lie subalgebra of the Lie algebra of transverse fields on* $(\tilde{F},\mathcal{F}|_{\tilde{F}})$.

(ii) $(\tilde{F},\mathcal{F}|_{\tilde{F}})$ *is a Lie foliation.*

Proof. We prove (i). We take basic 1-forms ζ_i by $\zeta_i(Y^j)=\delta_{ij}$ where δ_{ij} is the Kronecker's delta. By the condition (c), we have

$$
d\zeta_i=\sum_{1\leq j<k\leq \mathrm{cod}(M,\mathcal{F})} c_i^{jk}\zeta_j\wedge\zeta_k. \tag{30}
$$

where c_i^{jk} are functions on M whose restriction to $\tilde{F}$ is a constant. Clearly we have

$$
\zeta_i([Y^j,Y^k])=-2d\zeta_i(Y^j,Y^k)=-2c_i^{jk} \tag{31}
$$

For any transverse field Z on $(M,\mathcal{F})$, we have $Z=\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})}\zeta_i(Z)Y^i$. Hence we have

$$
[Y^j,Y^k]=\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})}\zeta_i([Y^j,Y^k])Y^i=-2\sum_{i=1}^{\mathrm{cod}(M,\mathcal{F})}c_i^{jk}Y^i. \tag{32}
$$

Consider the case of $1\leq j<k\leq \mathrm{cod}(\tilde{F},\mathcal{F}|_{\tilde{F}})$. Note that $[Y^j,Y^k]$ is also tangent to $\tilde{F}$ at every point on $\tilde{F}$, because Y^j and Y^k are tangent to $\tilde{F}$ at every point on $\tilde{F}$. Then it follows that c_i^{jk} must be 0 for $\mathrm{cod}(\tilde{F},\mathcal{F}|_{\tilde{F}})+1\leq i\leq \mathrm{cod}(M,\mathcal{F})$ from (32), because $Y^{\mathrm{cod}(\tilde{F},\mathcal{F}|_{\tilde{F}})+1}, \cdots, Y^{\mathrm{cod}(M,\mathcal{F})}$ are not tangent to $\tilde{F}$ at x_0. Hence $[Y^j,Y^k]$ is contained in $\mathfrak{g}$. (i) is proved.

We prove (ii). Here, $\mathfrak{g}$ is a Lie algebra by (i). We define a $\mathfrak{g}$-valued 1-form Ω on $\tilde{F}$ by

$$
\Omega_x((Y^j)_x)=Y^j \tag{33}
$$

for every point x on $\tilde{F}$ and every j. For the proof of (ii), it suffices to show that Ω satisfies the Maurer-Cartan equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$ by (III) and (IV) of Section 3. This is proved in a way similar to the argument of Theorem 4.24 of Moerdijk Mrčun [17] as follows: For $1 \leq j < k \leq \operatorname{cod}(\tilde{F}, \mathcal{F}|_{\tilde{F}})$, we have

$$\begin{aligned} & d\Omega(Y^j, Y^k) + \tfrac{1}{2}[\Omega, \Omega](Y^j, Y^k) \\ & = \tfrac{1}{2}\big(Y^j(\Omega(Y^k)) - Y^k(\Omega(Y^j)) - \Omega([Y^j, Y^k])\big) + \tfrac{1}{2}[\Omega(Y^j), \Omega(Y^k)] \\ & = \tfrac{1}{2}\big(Y^j(\Omega(Y^k)) - Y^k(\Omega(Y^j))\big) \\ & = 0. \end{aligned} \tag{34}$$

In the second equality, we used the equality $\Omega([Y^j, Y^k]) = [\Omega(Y^j), \Omega(Y^k)]$ which follows from the definition of Ω. The last equality follows from the fact that the $\mathfrak{g}$-valued functions $\Omega(Y^j)$ and $\Omega(Y^k)$ are constant on $\tilde{F}$. □

Let $\mathcal{F}_b$ be the foliation of M defined by the fibers of π_b.

Lemma 23. *Assume that the conditions (c) and (d) in Proposition 20 are satisfied. Assume that a basic closed* 1*-form* α *on* $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ *satisfies*

$$\int_{\tilde{F}} \Big(\int_{\gamma_x} \alpha \Big) \operatorname{vol}_{\tilde{F}}(x) = 0 \tag{35}$$

for every flow $\{\phi_t\}_{t \in [0,1]}$ *on* $\tilde{F}$ *generated by a vector field* X *such that* $\omega_i(X)$ *is constant on* $\tilde{F}$ *for each* i *where* γ_x *is the orbit of* x *of the flow* $\{\phi_t\}_{t \in [0,1]}$*. Then we have*

(i) $\alpha|_{L_b} = 0$ *for each leaf* L_b *of* $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ *and*
(ii) $[\alpha] = 0$ *in* $H^1(\tilde{F}; \mathbb{R})$.

Proof. We show (i) in the case where L_b is the leaf F_b of $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ which contains x_0. The proof of the general case is similar.

Here $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ is a Lie foliation by Lemma 22 (ii). We take a connected Lie group G so that $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ is a G-Lie foliation. We can assume the simply connectedness of G as noted in the second paragraph of Section 3.

Let $p^{\mathrm{univ}}_{\tilde{F}} \colon \tilde{F}^{\mathrm{univ}} \longrightarrow \tilde{F}$ be the universal covering of $\tilde{F}$. Fix a point x^{univ}_0 on the fiber of x_0. Let $\operatorname{dev} \colon \tilde{F} \longrightarrow G$ be the developing map of the Lie foliation $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ which maps x^{univ}_0 to the unit element e of G. Let hol be the holonomy homomorphism $\pi_1(\tilde{F}, x_0) \longrightarrow G$ of the Lie foliation $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$. By (I) of Section 3, every basic 1-form on $(\tilde{F}^{\mathrm{univ}}, (p^{\mathrm{univ}}_{\tilde{F}})^*(\mathcal{F}|_{\tilde{F}}))$ is the pullback of a 1-form on G by dev. Hence there exists a 1-form $\overline{\alpha}$ on G such that

$$(p^{\mathrm{univ}}_{\tilde{F}})^* \alpha = \operatorname{dev}^* \overline{\alpha}. \tag{36}$$

By (II) of Section 3 and the invariance of $(p^{\mathrm{univ}}_{\tilde{F}})^* \alpha$ under the $\pi_1(\tilde{F}, x_0)$-action on $\tilde{F}^{\mathrm{univ}}$, we have

$$R^*_{\mathrm{hol}(\gamma)} \overline{\alpha} = \overline{\alpha} \tag{37}$$

for γ in $\pi_1(\tilde{F}, x_0)$ where $R_{\mathrm{hol}(\gamma)} \colon G \longrightarrow G$ is the right multiplication map of an element $\mathrm{hol}(\gamma)$ of G. Let H be the closure of the image of hol in G. Note that H is a proper subgroup of G if the leaves of $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ are not dense by (II) in Section 3. It follows from (37) that

$$R^*_g \overline{\alpha} = \overline{\alpha} \tag{38}$$

for every g in H.

In the sequel, for a path γ on $\tilde{F}$, we denote a lift of γ to $\tilde{F}^{\mathrm{univ}}$ by γ^{univ}. By (38), we have

$$\int_{\gamma} \alpha = \int_{\gamma^{\mathrm{univ}}} (p_{\tilde{F}}^{\mathrm{univ}})^* \alpha = \int_{\gamma^{\mathrm{univ}}} \mathrm{dev}^* \overline{\alpha} = \int_{\mathrm{dev}_* \gamma^{\mathrm{univ}}} \overline{\alpha}. \tag{39}$$

Let γ be a closed path on $\tilde{F}$ whose endpoints are x_0. We denote the element of $\pi_1(\tilde{F}, x_0)$ represented by γ by the same symbol γ. Let $\overline{X}_\gamma$ be the left invariant vector field on G such that $\exp \overline{X}_\gamma = \mathrm{hol}(\gamma)$. Let X_γ^{univ} be a lift of $\overline{X}_\gamma$ to $\tilde{F}^{\mathrm{univ}}$ which is invariant by the action of $\mathrm{hol}(\pi_1(\tilde{F}, x_0))$. Let X_γ be the vector field on $\tilde{F}$ whose lift to $\tilde{F}^{\mathrm{univ}}$ is X_γ^{univ}. Let γ_x be the orbit of x on $\tilde{F}$ of the flow $\{\phi_t\}_{t \in [0,1]}$ generated by X_γ. It follows that $\mathrm{dev}_* \gamma_x^{\mathrm{univ}}$ is an orbit of the flow generated by $\overline{X}_\gamma$ from the definition of $\overline{X}_\gamma$ and γ_x. Hence we have

$$\mathrm{dev}_* \gamma_x^{\mathrm{univ}}(t) = \mathrm{dev}_* \gamma_x^{\mathrm{univ}}(0) \cdot \exp(t \overline{X}_\gamma). \tag{40}$$

We take the lifts γ^{univ} and $\gamma_{x_0}^{\mathrm{univ}}$ of γ and γ_{x_0} to $\tilde{F}^{\mathrm{univ}}$ so that $\gamma^{\mathrm{univ}}(0) = \gamma_{x_0}^{\mathrm{univ}}(0) = x_0^{\mathrm{univ}}$, respectively. Then, by (9) and (40), we have

$$\begin{aligned} \mathrm{dev}_* \gamma_{x_0}^{\mathrm{univ}}(0) = e = \mathrm{dev}_* \gamma^{\mathrm{univ}}(0), \\ \mathrm{dev}_* \gamma_{x_0}^{\mathrm{univ}}(1) = \mathrm{hol}(\gamma) = \mathrm{dev}_* \gamma^{\mathrm{univ}}(1). \end{aligned} \tag{41}$$

Since G is simply connected, (39), (41) and the Stokes theorem imply

$$\int_{\gamma} \alpha = \int_{\mathrm{dev}_* \gamma^{\mathrm{univ}}} \overline{\alpha} = \int_{\mathrm{dev}_* \gamma_{x_0}^{\mathrm{univ}}} \overline{\alpha} = \int_{\gamma_{x_0}} \alpha. \tag{42}$$

It follows from (40) and $\mathrm{dev}(\gamma_{x_0}^{\mathrm{univ}}(0)) = e$ that

$$\mathrm{dev}_* \gamma_x^{\mathrm{univ}} = (R_{\mathrm{dev}(\gamma_x^{\mathrm{univ}}(0)) \cdot \mathrm{dev}(\gamma_{x_0}^{\mathrm{univ}}(0))^{-1}})_* \mathrm{dev}_* \gamma_{x_0}^{\mathrm{univ}} = (R_{\mathrm{dev}(\gamma_x^{\mathrm{univ}}(0))})_* \mathrm{dev}_* \gamma_{x_0}^{\mathrm{univ}}. \tag{43}$$

If γ and x are contained in F_b, then we can take γ_x^{univ} so that $\mathrm{dev}(\gamma_x^{\mathrm{univ}}(0))$ is contained in H. By using (39), (43), (38) and (39) in this order, we have

$$\int_{\gamma_x} \alpha = \int_{\mathrm{dev}_* \gamma_x^{\mathrm{univ}}} \overline{\alpha} = \int_{(R_{\mathrm{dev}(\gamma_x^{\mathrm{univ}}(0))})_* \mathrm{dev}_* \gamma_{x_0}^{\mathrm{univ}}} \overline{\alpha} = \int_{\mathrm{dev}_* \gamma_{x_0}^{\mathrm{univ}}} \overline{\alpha} = \int_{\gamma_{x_0}} \alpha. \tag{44}$$

Since X_γ satisfies the condition of X in the statement of Lemma 23, we have

$$\int_{\tilde{F}} \Big(\int_{\gamma_x} \alpha \Big) \mathrm{vol}_{\tilde{F}}(x) = 0 \tag{45}$$

by the assumption. By (42), (44) and (45), we have

$$\int_{\gamma} \alpha = 0. \tag{46}$$

Then $\alpha|_{F_b}$ is exact. Hence there exists a basic function h on $(F_b, \mathcal{F}|_{F_b})$ such that $dh = \alpha|_{F_b}$. But since the leaves of $(F_b, \mathcal{F}|_{F_b})$ are dense, h is constant. Then we have $\alpha = dh = 0$. We complete the proof of (i).

We show

$$\int_{\gamma_x} \alpha = \int_{\gamma_{x_0}} \alpha \tag{47}$$

for every point x on $\tilde{F}$.

We show that $[\alpha]$ in $H^1(\tilde{F};\mathbb{R})$ is contained in the image of $(\pi_b|_{\tilde{F}})^* \colon H^1(\tilde{F}/\mathcal{F}_b;\mathbb{R}) \longrightarrow H^1(\tilde{F};\mathbb{R})$. Since α is basic with respect to $\mathcal{F}$, we have $\phi^*\alpha = \alpha$ for a diffeomorphism ϕ which maps each leaf of $\mathcal{F}$ to itself. Each leaf L of $\mathcal{F}$ is dense in the leaf L_b of $\mathcal{F}_b|_{\tilde{F}}$ which contains L. Hence the orbits of the group of diffeomorphisms which map each leaf of $\mathcal{F}$ to itself is dense in L_b. Hence we have $(L_Y\alpha)_x = 0$ for every point x on M and every Y in T_xM tangent to L_b. Since $\alpha|_{L_b}$ is zero by (i), α is basic with respect to $\mathcal{F}_b$. Thus $[\alpha]$ in $H^1(\tilde{F};\mathbb{R})$ is contained in $(\pi_b|_{\tilde{F}})^*(H^1(\tilde{F}/\mathcal{F}_b;\mathbb{R}))$ in $H^1(\tilde{F};\mathbb{R})$.

The path γ_x may not be closed in general. But we show that $(\pi_b)_*\gamma_x$ is closed where $\pi_b \colon \tilde{F} \longrightarrow \tilde{F}/\mathcal{F}_b$ is the projection to the leaf space. Let H be the Lie subgroup of G defined by the closure of $\mathrm{hol}(\pi_1(\tilde{F},x_0))$. The structural Lie algebra of the Lie foliation $(F_b, \mathcal{F}|_{F_b})$ is $\mathrm{Lie}(H)$, and hence $\dim H = \mathrm{cod}(F_b, \mathcal{F}|_{F_b})$. By the equivariance of the developing map in (9), the map $\mathrm{dev} \colon \tilde{F}^{\mathrm{univ}} \longrightarrow G$ induces a map $\mathrm{dev}_{G/H} \colon \tilde{F} \longrightarrow G/H$. Furthermore, $\mathrm{dev}_{G/H}$ induces a map $\varpi \colon \tilde{F}/\mathcal{F}_b \longrightarrow G/H$. Since ϖ is a submersion between two manifolds of the same dimension, ϖ is a covering map. Since ϖ is injective as easily confirmed, ϖ is a diffeomorphism.

Let X_b be a vector field on $\tilde{F}/\mathcal{F}_b$ induced from X_γ. Let $\overline{X}_{G/H}$ be a vector field on G/H induced from a vector field $\overline{X}_\gamma$ on G. By the definition of X_γ and $\overline{X}_\gamma$, we have

$$\varpi_*\overline{X}_{G/H} = X_b. \tag{48}$$

Recall that γ_x is the orbit of x of the flow $\{\phi_t\}_{0\leq t\leq 1}$ from time zero to time one generated by X_γ. Thus $(\pi_b)_*\gamma_x$ is the orbit of x of the flow from time zero to time one generated by X_b. By (48), ϖ maps an orbit of the flow from time zero to time one generated by X_b to an orbit of the flow from time zero to time one generated by the vector field $\overline{X}_{G/H}$ on G/H. Here the time one map of the flow generated by $\overline{X}_{G/H}$ is the identity, because this map is induced by the time one map of the flow on G generated by $\overline{X}_\gamma$, which is the right multiplication map of an element of $\mathrm{hol}(\pi_1(\tilde{F},x_0))$ by the definition of $\overline{X}_\gamma$. Hence $(\pi_b)_*\gamma_x$ is a closed path on $\tilde{F}/\mathcal{F}_b$ for each x.

The homology class determined by $(\pi_b)_*\gamma_x$ in $\tilde{F}/\mathcal{F}_b$ is independent of x. This is because γ_x and γ_y are bounded by a 1-parameter family of closed paths on $\tilde{F}/\mathcal{F}_b$ of the form $\{\gamma_{l(s)}\}_{0\leq s\leq 1}$ where l is a path on $\tilde{F}$ such that $l(0) = x$ and $l(1) = y$ for every two points x and y in $\tilde{F}/\mathcal{F}_b$.

Thus, by the argument in the previous three paragraphs, $[\alpha]$ is contained in $(\pi_b|_{\tilde{F}})^*(H^1(\tilde{F}/\mathcal{F}_b;\mathbb{R}))$, and $(\pi_b)_*\gamma_x$ determines the same homology class in $\tilde{F}/\mathcal{F}_b$ for every x. Hence (47) is proved.

By (42), (45) and (47), we have

$$\int_\gamma \alpha = 0. \tag{49}$$

Hence (ii) is proved. □

5.4. **Two lemmas on a fiber bundle over S^1 with fiberwise Lie foliations.** We prove two lemmas to use in the next section. Note that $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ is transversely orientable by the assumption of the orientability of both of $\tilde{\mathcal{F}}$ and the basic fibration π_b of $(M, \mathcal{F})$.

Lemma 24. *Assume that the conditions (c) and (d) in Proposition 20 are satisfied. Let* $\gamma\colon S^1 \longrightarrow M$ *be a smooth embedding in* M*. Assume that* γ *is transverse to the fibers of* $\pi_{\tilde{\mathcal{F}}}$*, and that* $\pi_{\tilde{\mathcal{F}}} \circ \gamma$ *is an embedding. We put* $M' = \pi_{\tilde{\mathcal{F}}}^{-1}(\pi_{\tilde{\mathcal{F}}}(S^1))$*. Then there exists a flat connection* ∇ *on* $\pi_{\tilde{\mathcal{F}}}|_{M'}$ *which satisfies the following four conditions:*

- **(A):** *The holonomy map* $f\colon \tilde{F} \longrightarrow \tilde{F}$ *of* ∇ *preserves the foliation* $\mathcal{F}_b|_{\tilde{F}}$*.*
- **(B):** *The holonomy map* f *of* ∇ *preserves a transverse volume form* $\mu_{\tilde{F}/\mathcal{F}_b}$ *of* $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$*.*
- **(C):** *We denote the inverse map* $\pi_{\tilde{\mathcal{F}}} \circ \gamma(S^1) \longrightarrow S^1$ *of* $\pi_{\tilde{\mathcal{F}}} \circ \gamma$ *by* φ*. We define a section* γ_1 *of* $\pi_{\tilde{\mathcal{F}}}|_{M'}$ *by* $\gamma_1 = \gamma \circ \varphi$*. Then* γ_1 *is a section parallel to* ∇*.*
- **(D):** *There exists a vector field* Z_∇ *such that the orbits of the flow generated by* Z_∇ *is parallel to* ∇ *and the restriction of* $\omega_i(Z_\nabla)$ *to each fiber of* $\pi_{\tilde{\mathcal{F}}}$ *is constant.*

Proof. Let $\hat{X}^j$ be a section of $TM|_{M'}$ on M' such that $\hat{X}^j$ is projected to X^j by the canonical projection $C^\infty(TM|_{M'}) \longrightarrow C^\infty((TM/T\mathcal{F})|_{M'})$ for $1 \leq j \leq \mathrm{cod}(M,\mathcal{F})$. There exists a vector field Y tangent to $\mathcal{F}$ defined on $\gamma_1(S^1)$ and functions h_1, h_2, $\cdots$, $h_{\mathrm{cod}(M,\mathcal{F})}$ on S^1 such that

$$(D\gamma_1)_t\left(\frac{\partial}{\partial t}\right) = Y_{\gamma_1(t)} + \sum_{j=1}^{\mathrm{cod}(M,\mathcal{F})} h_j(t)(\hat{X}^j)_{\gamma_1(t)} \tag{50}$$

where $(D\gamma_1)_t$ is the differential map of γ_1 at a point t. Let Y' be a vector field on M' which is tangent to $\mathcal{F}$ and whose restriction to $\gamma_1(S^1)$ is equal to Y. We define a vector field Z_∇ on M' by

$$Z_\nabla = Y' + \sum_{j=1}^{\mathrm{cod}(M,\mathcal{F})} ((\pi_{\tilde{\mathcal{F}}}|_{M'})^* h_j)\hat{X}^j. \tag{51}$$

The restriction of Z_∇ to $\gamma_1(S^1)$ is equal to the tangent vectors of γ_1 by (50). Z_∇ is basic with respect to $\mathcal{F}$ and transverse to the fibers of $\pi_{\tilde{\mathcal{F}}}$. We define a connection ∇ on $\pi_{\tilde{\mathcal{F}}}$ by the line field tangent to Z_∇ at each point on M'. It is trivial that ∇ is flat, because every connection on a fiber bundle over S^1 is flat.

We show that ∇ satisfies the conditions (A), (B), (C) and (D). Here, $\hat{X}^j$ is basic with respect to $\tilde{\mathcal{F}}|_{M'}$ by the condition (d) in Proposition 20. Hence Z_∇ is also basic with respect to $\tilde{\mathcal{F}}|_{M'}$ by the definition. On a foliated manifold, the flow generated by a basic vector field maps leaves of the foliation to the leaves by Proposition 2.2 of Molino [18]. Then the flow generated by Z_∇ also maps the fibers of $\pi_{\tilde{\mathcal{F}}}|_{M'}$ to the fibers of $\pi_{\tilde{\mathcal{F}}}|_{M'}$. The time one map of the flow generated by Z_∇ maps $\tilde{F}$ to $\tilde{F}$ itself. Since the orbits of the flow generated by Z_∇ are parallel to ∇ by the definition, the time one map of the flow generated by Z_∇ is the holonomy of ∇. This proves that ∇ satisfies the condition (D). Since the restriction of Z_∇ to $\gamma_1(S^1)$ is equal to the tangent vectors of γ_1, the condition (C) is satisfied. Since Z_∇ is basic with respect to $\mathcal{F}$, the flow generated by Z_∇ maps the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}$. Since the leaves of $\mathcal{F}_b$ are the closures of the leaves of $\mathcal{F}$, the flow generated by Z_∇ maps the leaves of $\mathcal{F}_b$ to the leaves of $\mathcal{F}_b$. Hence f satisfies the condition (A). By the conditions (c), (d) in Proposition 20 and Lemma 22 (ii), $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ is a Lie foliation. Let G be a connected Lie group such that $(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ is a G-Lie foliation. We can assume the simply connectedness of G as noted in the second paragraph of Section 3. Let H be the Lie subgroup of G such that $\mathrm{Lie}(H)$ is the structural Lie algebra of the Lie foliation $(F_b, \mathcal{F}|_{F_b})$. We denote the universal cover of $\tilde{F}$ by

$$p_{\tilde{F}}^{\mathrm{univ}} : \tilde{F}^{\mathrm{univ}} \longrightarrow \tilde{F}. \tag{52}$$

Note that $\dim G = \mathrm{cod}(\tilde{F}, \mathcal{F}|_{\tilde{F}})$ and the codimension of H in G is equal to $\mathrm{cod}(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$. We regard $(\mathrm{Lie}(G)/\,\mathrm{Lie}(H))^*$ as a subset of $\mathrm{Lie}(G)^*$ consisting of the elements whose restriction to $\mathrm{Lie}(H)$ is 0. Fix a basis $\{\overline{\beta}_1, \overline{\beta}_2, \cdots, \overline{\beta}_{\mathrm{cod}(\tilde{F}, \mathcal{F}|_{\tilde{F}})}\}$ of $\mathrm{Lie}(G)^*$ so that $\{\overline{\beta}_1, \overline{\beta}_2, \cdots, \overline{\beta}_{\mathrm{cod}(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})}\}$ is a basis of $(\mathrm{Lie}(G)/\,\mathrm{Lie}(H))^*$. Note that $\mathrm{dev}_G^* \overline{\beta}_j$ is $\pi_1 \tilde{F}$-invariant. Let β_j be the 1-form on $\tilde{F}$ induced by the $\pi_1 \tilde{F}$-invariant 1-form $\mathrm{dev}_G^* \overline{\beta}_j$ on $\tilde{F}^{\mathrm{univ}}$ for $1 \leq j \leq \mathrm{cod}(\tilde{F}, \mathcal{F}|_{\tilde{F}})$. Then the restriction of β_j on each leaf of $\tilde{\mathcal{F}}$ is zero for $1 \leq j \leq \mathrm{cod}(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$. Note that the Maurer-Cartan form of $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ is given by the equation (33). Hence, by (V) of Section 3, we can write

$$\beta_j = \sum_{i=1}^{\mathrm{cod}(\tilde{F}, \mathcal{F}|_{\tilde{F}})} b_j^i (\omega_i|_{\tilde{F}}) \tag{53}$$

for each j for some constants b_j^i, $1 \leq i \leq \mathrm{cod}(\tilde{F}, \mathcal{F}|_{\tilde{F}})$.

We define a transverse volume form $\mu_{\tilde{F}/\mathcal{F}_b}$ on $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ by

$$\mu_{\tilde{F}/\mathcal{F}_b} = \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_{\mathrm{cod}(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})}. \tag{54}$$

Since $\mu_{\tilde{F}/\mathcal{F}_b}$ is closed, we have

$$L_{Z_\nabla} \mu_{\tilde{F}/\mathcal{F}_b} = d\iota_{Z_\nabla} \mu_{\tilde{F}/\mathcal{F}_b} = d\iota_{Z_\nabla} (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_{\mathrm{cod}(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})}). \tag{55}$$

Note that $\omega_i(Z_\nabla)$ is constant on $\tilde{F}$ by the definition of Z_∇. Then $\beta_j(Z_\nabla)$ is also constant on $\tilde{F}$. We can write $d\beta_i$ as a sum of $\beta_j \wedge \beta_k$ on M as

$$d\beta_i = \sum_{1 \leq j < k \leq \mathrm{cod}(M, \mathcal{F})} c_i^{jk} \beta_j \wedge \beta_k. \tag{56}$$

Then the restriction of c_i^{jk} to $\tilde{F}$ is a constant by the condition (c) in Proposition 20. Then there exists a constant C_0 such that

$$d\iota_{Z_\nabla} (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_{\mathrm{cod}(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})})_x = C_0 \big(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_{\mathrm{cod}(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})}\big)_x \tag{57}$$

for every point x on $\tilde{F}$. By (55) and (57), we have

$$(L_{Z_\nabla} \mu_{\tilde{F}/\mathcal{F}_b})_x = C_0 (\mu_{\tilde{F}/\mathcal{F}_b})_x \tag{58}$$

for every point x on $\tilde{F}$. In the same way, there exists a constant C_0^t such that

$$(L_{Z_\nabla} \mu_{\tilde{F}/\mathcal{F}_b})_x = C_0^t (\mu_{\tilde{F}/\mathcal{F}_b})_x \tag{59}$$

for every point x on $\pi_{\tilde{\mathcal{F}}}^{-1}(t)$ for each t in $\pi_{\tilde{\mathcal{F}}} \circ \gamma(S^1)$. By (59), f satisfies

$$f^* \mu_{\tilde{F}/\mathcal{F}_b} = C_1 \mu_{\tilde{F}/\mathcal{F}_b} \tag{60}$$

for a constant C_1. Let $\tilde{F}/\mathcal{F}_b$ be the leaf space of $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$. $\tilde{F}/\mathcal{F}_b$ is a closed manifold. This $\tilde{F}/\mathcal{F}_b$ is orientable by the transverse orientability of $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$. Let

$\overline{f}$ be the map induced by f on $\tilde{F}/\mathcal{F}_b$. Since $\mu_{\tilde{F}/\mathcal{F}_b}$ is basic, $\mu_{\tilde{F}/\mathcal{F}_b}$ is a pull back of a volume form $\overline{\mu}_{\tilde{F}/\mathcal{F}_b}$ on $\tilde{F}/\mathcal{F}_b$. We have

$$\overline{f}^*\overline{\mu}_{\tilde{F}/\mathcal{F}_b} = C_1\overline{\mu}_{\tilde{F}/\mathcal{F}_b} \tag{61}$$

Hence C_1 is equal to 1, because C_1 is equal to the mapping degree of a diffeomorphism $\overline{f}\colon \tilde{F}/\mathcal{F}_b \longrightarrow \tilde{F}/\mathcal{F}_b$. It follows from (60) that ∇ satisfies the condition (B). □

A section of $C^\infty(\wedge^k T^*\mathcal{F})$ is called a leafwise k-form on $(M,\mathcal{F})$. If $k = \dim \mathcal{F}$, a leafwise k-form is called a leafwise volume form on $(M,\mathcal{F})$. The wedge product induces a natural operation $C^\infty(\wedge^k T^*\mathcal{F}) \otimes C^\infty(\wedge^{\mathrm{cod}(M,\mathcal{F})}(TM/T\mathcal{F})^*) \longrightarrow C^\infty(\wedge^{\mathrm{cod}(M,\mathcal{F})+k}T^*M)$.

Lemma 25. *Let $\gamma\colon S^1 \longrightarrow M$ be a smooth embedding in M. Assume that γ is transverse to the fibers of $\pi_{\tilde{\mathcal{F}}}$, and that $\pi_{\tilde{\mathcal{F}}} \circ \gamma$ is an embedding. We put $M' = \pi_{\tilde{\mathcal{F}}}^{-1}(\pi_{\tilde{\mathcal{F}}}(\gamma(S^1)))$. Fix a fiber $\tilde{F}$ of $\pi_{\tilde{\mathcal{F}}}|_{M'}$ and a point x_0 on $\tilde{F}$. Let ∇ be a flat connection on $\pi_{\tilde{\mathcal{F}}}|_{M'}$. Let $f\colon \tilde{F} \longrightarrow \tilde{F}$ be the holonomy of the flat connection ∇ with respect to the path $\pi_{\tilde{\mathcal{F}}} \circ \gamma$. For x on $\tilde{F}$, let γ_x^∇ be the lift of the path $\pi_{\tilde{\mathcal{F}}} \circ \gamma$ to M' such that $\gamma_x(0) = x$ and γ_x^∇ is parallel to ∇. Let α be a closed 1-form on M such that $[\alpha|_{\tilde{F}}] = 0$ in $H^1(\tilde{F};\mathbb{R})$ and $\alpha|_{L_b} = 0$ on each leaf L_b of $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$.*

We assume that ∇ satisfies the conditions (A), (B) and (C) in the statement of Lemma 24. Let $\mathrm{vol}_{\tilde{F}}$ be a volume form on $\tilde{F}$. Then we have

$$\int_{\tilde{F}}\left(\int_{\gamma_x^\nabla}\alpha\right)\mathrm{vol}_{\tilde{F}} = \left(\int_{\tilde{F}}\mathrm{vol}_{\tilde{F}}\right)\left(\int_{\gamma_{x_0}^\nabla}\alpha\right). \tag{62}$$

Proof. First, we shall show that there exists an isotopy $\{\phi_s\}_{s\in[0,1]}$ on $\tilde{F}$ such that

(i) $\phi_0 = \mathrm{id}_{\tilde{F}}$,
(ii) $f \circ \phi_1$ preserves $\mathrm{vol}_{\tilde{F}}$,
(iii) ϕ_s maps each leaf of $\tilde{\mathcal{F}}$ to itself and
(iv) ϕ_s fixes x_0

by the assumption and a leafwise version of Moser's argument in below. The leafwise version of Moser's argument was used by Ghys in [8] and by Hector, Macias and Saralegui in [12]. Let $\eta_{\mathcal{F}_b}$ be the leafwise volume form on $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ such that

$$\eta_{\mathcal{F}_b} \wedge \mu_{\tilde{F}/\mathcal{F}_b} = \mathrm{vol}_{\tilde{F}}\,. \tag{63}$$

Since each leaf L_b of $\mathcal{F}_b|_{\tilde{F}}$ is compact and oriented by the assumption, f maps the fundamental class of L_b to the fundamental class of $f(L_b)$. Then we have a leafwise $(\dim \mathcal{F}_b - 1)$-form σ on $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ such that $d\sigma = f^*\eta_{\mathcal{F}_b} - \eta_{\mathcal{F}_b}$. By adding a closed $(\dim \mathcal{F}_b - 1)$-form supported on an open neighborhood of x_0 to σ, we can modify σ so that $\sigma_{x_0} = 0$ and $d\sigma = f^*\eta_{\mathcal{F}_b} - \eta_{\mathcal{F}_b}$ are satisfied. Since $\eta_{\mathcal{F}_b}$ is a leafwise volume form on $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$, there exists a vector field Y on $\tilde{F}$ tangent to leaves of $\tilde{\mathcal{F}}$ such that $-\iota_Y\eta_{\mathcal{F}_b} = \sigma$. We put $\eta_s = \eta_{\mathcal{F}_b} + s d\sigma$. We have

$$\begin{aligned} \frac{d(\phi_s^*\eta_s)}{ds} &= \phi_s^*(L_Y\eta_s + \tfrac{d\eta_s}{ds}) \\ &= \phi_s^* d(\iota_Y\eta_{\mathcal{F}_b} + \sigma) \\ &= 0. \end{aligned} \tag{64}$$

Hence we have

$$\phi^{1*} f^* \eta_{\mathcal{F}_b} = \phi^{1*}\eta_1 = \eta_0 = \eta_{\mathcal{F}_b}. \tag{65}$$

Thus $f \circ \phi_1$ preserves $\eta_{\mathcal{F}_b}$. Here, ϕ_s maps each leaf of $\tilde{\mathcal{F}}$ to itself, because Y is tangent to leaves of $\tilde{\mathcal{F}}$. Then clearly ϕ_1 preserves the transverse volume form $\mu_{\tilde{F}/\mathcal{F}_b}$. Since f preserves $\mu_{\tilde{F}/\mathcal{F}_b}$ by the assumption, by (63) and (65), we have

$$(f \circ \phi_1)^* \operatorname{vol}_{\tilde{F}} = (f \circ \phi_1)^* \eta_{\mathcal{F}_b} \wedge (f \circ \phi_1)^* \mu_{\tilde{F}/\mathcal{F}_b} = \eta_{\mathcal{F}_b} \wedge \mu_{\tilde{F}/\mathcal{F}_b} = \operatorname{vol}_{\tilde{F}}. \tag{66}$$

Hence $f \circ \phi_1$ preserves $\operatorname{vol}_{\tilde{F}}$. Since $\sigma_{x_0} = 0$, we have $Y_{x_0} = 0$. This implies that ϕ_s fixes x_0.

Using $\{\phi_s\}_{s\in[0,1]}$, we can construct a smooth family $\{\nabla_s\}_{s\in[0,1]}$ of flat connections on $\pi_{\tilde{\mathcal{F}}}|_{M'}$ such that $\nabla_0 = \nabla$ and the holonomy of ∇_s with respect to $\pi_{\tilde{\mathcal{F}}} \circ \gamma$ is $f \circ \phi_s$. Since each ϕ_s fixes x_0, we can take $\{\nabla_s\}_{s\in[0,1]}$ so that

$$\gamma_{x_0}^{\nabla} = \gamma_{x_0}^{\nabla_s} \tag{67}$$

for $0 \le s \le 1$. For x in $\tilde{F}$, let $\gamma_x^{\nabla_1}$ be the lift of the path $\pi_{\tilde{\mathcal{F}}} \circ \gamma$ to M such that $\gamma_x(0) = x$ and γ_x is parallel to ∇_1. We take a function h on $\tilde{F}$ so that $dh = \alpha|_{\tilde{F}}$. For each point x on $\tilde{F}$, we have

$$\int_{\gamma_x^{\nabla_0}} \alpha + \big(h(\phi_1 \circ f(x)) - h(f(x))\big) - \int_{\gamma_x^{\nabla_1}} \alpha = 0 \tag{68}$$

by the Stokes' theorem. By the assumption on α, the restriction of h to each leaf of $\mathcal{F}_b|_{\tilde{F}}$ is constant. Since $\phi_1 \circ f$ maps each leaf of $\tilde{\mathcal{F}}$ to itself, we have $h(\phi_1 \circ f(x)) - h(f(x)) = 0$. Hence, by (68), we have

$$\int_{\gamma_x^{\nabla_0}} \alpha - \int_{\gamma_x^{\nabla_1}} \alpha = 0 \tag{69}$$

for each point x on $\tilde{F}$. Then we have

$$\int_{\tilde{F}} \left(\int_{\gamma_x^{\nabla_0}} \alpha\right) \operatorname{vol}_{\tilde{F}} - \int_{\tilde{F}} \left(\int_{\gamma_x^{\nabla_1}} \alpha\right) \operatorname{vol}_{\tilde{F}} = \int_{\tilde{F}} \left(\int_{\gamma_x^{\nabla_0}} \alpha - \int_{\gamma_x^{\nabla_1}} \alpha\right) \operatorname{vol}_{\tilde{F}} = 0. \tag{70}$$

For each point x on $\tilde{F}$, we have

$$\big(h(x) - h(x_0)\big) + \int_{\gamma_x^{\nabla_1}} \alpha + \big(h(f \circ \phi_1(x_0)) - h(f(x))\big) - \int_{\gamma_{x_0}^{\nabla_1}} \alpha = 0 \tag{71}$$

by the Stokes' theorem. Then we have

$$\begin{aligned}
& \int_{\tilde{F}} \left(\int_{\gamma_x^{\nabla_1}} \alpha\right) \operatorname{vol}_{\tilde{F}} - \left(\int_{\tilde{F}} \operatorname{vol}_{\tilde{F}}\right) \left(\int_{\gamma_{x_0}^{\nabla}} \alpha\right) \\
= & \int_{\tilde{F}} \left(\int_{\gamma_x^{\nabla_1}} \alpha\right) \operatorname{vol}_{\tilde{F}} - \left(\int_{\tilde{F}} \operatorname{vol}_{\tilde{F}}\right) \left(\int_{\gamma_{x_0}^{\nabla_1}} \alpha\right) \\
= & \int_{\tilde{F}} \left(\int_{\gamma_x^{\nabla_1}} \alpha\right) \operatorname{vol}_{\tilde{F}} - \int_{\tilde{F}} \left(\int_{\gamma_{x_0}^{\nabla_1}} \alpha\right) \operatorname{vol}_{\tilde{F}} \\
= & \int_{\tilde{F}} \left(\int_{\gamma_x^{\nabla_1}} \alpha - \int_{\gamma_{x_0}^{\nabla_1}} \alpha\right) \operatorname{vol}_{\tilde{F}} \\
= & \int_{\tilde{F}} \Big(- \big(h(x) - h(x_0)\big) - \big(h(f \circ \phi_1(x_0)) - h(f \circ \phi_1(x))\big)\Big) \operatorname{vol}_{\tilde{F}} \\
= & -\int_{\tilde{F}} (h(x) - h(x_0)) \operatorname{vol}_{\tilde{F}} + \int_{\tilde{F}} (h(x) - h(x_0)) \phi_1^* f^* \operatorname{vol}_{\tilde{F}} \\
= & \; 0.
\end{aligned} \tag{72}$$

Here we used (67) in the first equality, (71) in the fourth equality and $\phi_1^* f^* \operatorname{vol}_{\tilde{F}} = \operatorname{vol}_{\tilde{F}}$ in the last equality. The equation (62) follows from (70) and (72). □

5.5. $\xi(\mathcal{F}) = [\tilde{\kappa}_b]$ **on** M**.** We prove a lemma which will be used to complete the proof of Proposition 20 (ii). Note that α will be considered to be $\xi(\mathcal{F}) - [\tilde{\kappa}_b]$ in the application in Section 5.6.

Lemma 26. *Assume that the conditions (a), (b), (c) and (d) in Proposition 20 are satisfied. Assume that* γ_0 *in* $\pi_1(M, x_0)$ *satisfies the following conditions:* γ_0 *is represented by a smooth path* $l'_0 \colon [0,1] \longrightarrow M$ *which factors a smooth embedding* $l_0 \colon S^1 \longrightarrow M$ *and is transverse to the fibers of* $\pi_{\tilde{\mathcal{F}}}$*, and* $\pi_{\tilde{\mathcal{F}}} \circ l_0$ *is a smooth embedding. Let* α *be a closed basic* 1*-form on* $(M, \mathcal{F})$ *which satisfies the following conditions:*

(i) $\alpha|_{L_b} = 0$ *for each leaf* L_b *of* $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$.
(ii) $\alpha|_{\tilde{F}}$ *is exact.*
(iii) *For any submanifold* M' *of* M *which is a union of fibers of* $\pi_{\tilde{\mathcal{F}}}$*,*

$$\int_{M'} \left(\int_{\gamma_x} \alpha \right) \mathrm{vol}_{M'}(x) = 0 \tag{73}$$

is satisfied for every flow $\{\phi_t\}_{t \in [0,1]}$ *generated by vector field* X *such that* $\omega_i(X)$ *is constant on each fiber of* $\pi_{\tilde{\mathcal{F}}}$ *for each* i *where* γ_x *is the orbit of* x *of the flow* $\{\phi_t\}_{t \in [0,1]}$.

Then we have

$$\int_{\gamma_0} \alpha = 0. \tag{74}$$

Proof. We put

$$K = \pi_{\tilde{\mathcal{F}}} \circ l_0(S^1), M' = \pi_{\tilde{\mathcal{F}}}^{-1}(K). \tag{75}$$

M' is a submanifold of M which is a union of fibers of $\pi_{\tilde{\mathcal{F}}}$ by the assumption on γ_0.

By the condition (a), we have

$$\int_{M'} \left(\int_{\gamma_x} \alpha \right) \mathrm{vol}_{M'}(x) = \int_K \left(\int_{\pi_{\tilde{\mathcal{F}}}} \left(\int_{\gamma_x} \alpha \right) \mathrm{vol}_{\tilde{\mathcal{F}}} \right) \mathrm{vol}_K . \tag{76}$$

By the conditions (c), (d) and Lemma 24, there exists a flat connection ∇ on the fiber bundle $M' \longrightarrow K$ which satisfies the conditions (A), (B), (C) and (D) in the statement of Lemma 24. By condition (D) and the assumption (iii), we have

$$\int_{M'} \left(\int_{\gamma_x} \alpha \right) \mathrm{vol}_{M'}(x) = 0. \tag{77}$$

Since the assumptions of Lemma 25 are satisfied by the conditions (A), (B) and (C), we have

$$\int_{\pi_{\tilde{\mathcal{F}}}^{-1}(t)} \left(\int_{\gamma_x} \alpha \right) \mathrm{vol}_{\pi_{\tilde{\mathcal{F}}}^{-1}(t)} = \left(\int_{\pi_{\tilde{\mathcal{F}}}^{-1}(t)} \mathrm{vol}_{\pi_{\tilde{\mathcal{F}}}^{-1}(t)} \right) \left(\int_{\gamma_{x_0}} \alpha \right) \tag{78}$$

for each t in K by Lemma 25. By condition (b) and Lemma 19, the volume of fibers of $\pi_{\tilde{\mathcal{F}}}$ is constant. By (76), (77) and (78), we have

$$\int_{\gamma_{x_0}} \alpha = 0. \tag{79}$$

Hence Lemma 26 is proved. □

5.6. **Proof of Proposition 20 (ii).** By the homotopy exact sequence of the fiber bundle $\pi_{\tilde{\mathcal{F}}}$, we have an exact sequence
(80)

$$\pi_2(M/\tilde{\mathcal{F}}, \pi_{\tilde{\mathcal{F}}}(x_0)) \longrightarrow \pi_1(\tilde{F}, x_0) \xrightarrow{\iota} \pi_1(M, x_0) \xrightarrow{(\pi_{\tilde{\mathcal{F}}})_*} \pi_1(M/\tilde{\mathcal{F}}, \pi_{\tilde{\mathcal{F}}}(x_0)) \longrightarrow 0.$$

By Lemmas 21 and 23, we have $\kappa_b|_{L_b} - \tilde{\kappa}_b|_{L_b}$ for every leaf L_b of $(\tilde{F}, \mathcal{F}_b|_{\tilde{F}})$ and $[\kappa_b|_{\tilde{F}}] - [\tilde{\kappa}_b|_{\tilde{F}}] = 0$. This implies that $[\kappa_b] - [\tilde{\kappa}_b]$ vanishes on the image of ι. Then Lemma 26 implies $\int_\gamma(\kappa_b - \tilde{\kappa}_b) = 0$ for γ in $\pi_1(M, x_0)$ such that

- γ is transverse to the fibers of $\pi_{\tilde{\mathcal{F}}}$ and
- $\pi_{\tilde{\mathcal{F}}} \circ \gamma$ is an embedding.

Note that $\pi_1(M/\tilde{\mathcal{F}}, \pi_{\tilde{\mathcal{F}}}(x_0))$ is generated by the loops of the forms $(\pi_{\tilde{\mathcal{F}}})_*\gamma$ where γ runs all of the closed paths satisfying these two conditions. Hence we have $[\kappa_b] = [\tilde{\kappa}_b]$ in $H^1(M; \mathbb{R})$.

6. Continuity of the Álvarez classes

We show Theorem 1 for smooth families of orientable transversely parallelizable foliations using Proposition 20.

6.1. **A family version of the Molino theory.** Let U be a connected open set in $\mathbb{R}^L$ which contains 0. Let M be a closed manifold, and $\{\mathcal{F}^t\}_{t \in U}$ be a smooth family of orientable transversely parallelizable foliations of M over U given by a smooth foliation $\mathcal{F}^{\mathrm{amb}}$ of $M \times U$. We define a distribution D on $M \times U$ by

$$D_{(x,t)} = \{v \in T_{(x,t)}(M \times U) \,|\, vf(x,t) = 0, \forall f \in C_b^\infty(M \times U, \mathcal{F}^{\mathrm{amb}})\}. \tag{81}$$

where $C_b^\infty(M \times U, \mathcal{F}^{\mathrm{amb}})$ is the space of basic functions on $(M \times U, \mathcal{F}^{\mathrm{amb}})$. By the standard argument of the Molino theory on D, we shall obtain the following properties of D similar to the properties of the basic foliation of transversely parallelizable foliations:

Lemma 27. (i) *$(M \times U, \mathcal{F}^{\mathrm{amb}})$ is fiberwise transitive, that is, for each two points (x,t) and (y,t) in $M \times U$ with the same second coordinates, there exists a diffeomorphism f of $M \times U$ which preserves $\mathcal{F}^{\mathrm{amb}}$ and satisfies $f(x) = y$.*

(ii) *The dimension of $D_{(x,t)}$ is independent of x.*

(iii) *$D|_{M \times \{t\}}$ is integrable, and we have a foliation $\mathcal{D}^t$ of $M \times \{t\}$ defined by $D|_{M \times \{t\}}$.*

(iv) *The leaf space $(M \times \{t\})/\mathcal{D}^t$ is a closed manifold and the canonical projection $M \times \{t\} \longrightarrow (M \times \{t\})/\mathcal{D}^t$ is a smooth fiber bundle with compact fibers for each t.*

Proof. Fix t_0 on U. Let $\hat{X}^j_{\mathrm{amb}}$ be a vector field on $M \times U$ which is projected to X^j_{amb} by the canonical projection $C^\infty(TM) \longrightarrow C^\infty(TM/T\mathcal{F})$. By the compactness of M, for a relative compact open neighborhood U' of t_0 in U, each $\hat{X}^i_{\mathrm{amb}}$ generates a flow $\{\phi^s_i\}_{s \in \mathbb{R}}$ on $M \times U'$. By the proof of Theorem 4.8 of Moerdijk and Mrčun [17], for each two points (x, t_0) and (y, t_0) in $M \times \{t_0\}$, there exists a diffeomorphism f of $M \times \{t_0\}$ which is a composition of $\phi_1^{s_1}|_{M \times \{t_0\}}$, $\phi_2^{s_2}|_{M \times \{t_0\}}$, $\cdots$, $\phi_{\mathrm{cod}(M,\mathcal{F})}^{s_{\mathrm{cod}(M,\mathcal{F})}}|_{M \times \{t_0\}}$ for some s_i and diffeomorphisms of $M \times \{t_0\}$ preserving each leaf of $\mathcal{F}^{t_0}$ whose supports are contained in a foliated chart of $\mathcal{F}^{t_0}$. Since X^i_{amb} is basic with respect to $(M \times U', \mathcal{F}^{\mathrm{amb}})$, $\phi_i^{s_i}$ preserves $\mathcal{F}^{\mathrm{amb}}$. We can extend diffeomorphisms of $M \times \{t_0\}$

which preserve each leaf of $\mathcal{F}^{t_0}$ and whose supports are contained in a foliated chart of $\mathcal{F}^{t_0}$ to diffeomorphisms of $M \times U'$ preserving each leaf of $\mathcal{F}^{\mathrm{amb}}$. Then f extends to a diffeomorphism of $M \times U'$ preserving $\mathcal{F}^{\mathrm{amb}}$ as a composite of $\phi_1^{s_1}$, $\phi_2^{s_2}$, $\cdots$, $\phi_{\mathrm{cod}(M,\mathcal{F})}^{s_{\mathrm{cod}(M,\mathcal{F})}}$ and diffeomorphisms of $M \times U'$ preserving each leaf of $\mathcal{F}^{\mathrm{amb}}$. This proves (i).

(ii), (iii) and (iv) follow from (i) and the proof of Theorem 4.3 of Moerdijk and Mrčun [17]. We write down the proof for the sake of completeness. Since $\mathcal{D}|_{M\times U'}$ is preserved by a diffeomorphism of $M \times U'$ preserving $\mathcal{F}^{\mathrm{amb}}$ by the definition, (ii) directly follows from (i). Let (x, t_0) be a point on $M \times \{t_0\}$. By (ii), $\mathcal{D}|_{M\times\{t_0\}}$ is a vector bundle on $M \times \{t_0\}$. Let Z_1 and Z_2 be two local sections of $\mathcal{D}|_{M\times\{t_0\}}$ defined near (x, t_0). For every f in $C_b^\infty(M \times U, \mathcal{F}^{\mathrm{amb}})$, we have $[Z_1, Z_2]f = Z_1Z_2f - Z_2Z_1f = 0$. Then $[Z_1, Z_2]$ is a section of $\mathcal{D}|_{M\times\{t\}}$. Hence $\mathcal{D}|_{M\times\{t\}}$ is integrable. (iii) is proved. Let L be a leaf of the foliation defined by $\mathcal{D}|_{M\times\{t_0\}}$. Let x be a point in L. We put $k(t_0) = \dim M - \dim D^{t_0}$. By the definition of $\mathcal{D}$, there exists basic functions f_1, f_2, $\cdots$, $f_{k(t_0)}$ on $(M \times U, \mathcal{F}^{\mathrm{amb}})$ such that $df_1 \wedge df_2 \wedge \cdots \wedge df_{k(t_0)}$ is nonzero at x. Since each f_i is basic, $df_1 \wedge df_2 \wedge \cdots \wedge df_{k(t_0)}$ is nowhere vanishing on an open saturated neighborhood U' of L in $(M \times U, \mathcal{F}^{\mathrm{amb}})$. Then the map ϕ defined by

$$\begin{array}{cccc} \phi\colon & U' & \longrightarrow & \mathbb{R}^{k(t_0)} \\ & z & \longmapsto & (f_1(z), f_2(z), \cdots, f_{k(t_0)}(z)) \end{array} \tag{82}$$

is a submersion such that one of the fibers of ϕ is equal to L. Shrinking U', we can assume that the fibers of ϕ is connected. Since each fiber of ϕ is saturated by D^t, each leaf of D^t near L coincides with a fiber of ϕ. Then ϕ gives a local trivialization of a fiber bundle. Hence (iv) is proved. □

Since D is a closed subset of $T(M \times U)$ by the definition of D, the dimension of $D_{(x,t)}$ is upper semicontinuous with respect to t. If the dimension of $D_{(x,t)}$ is constant with respect to t, then the leaves of $\mathcal{D}$ are fibers of a smooth submersion whose restriction to $M \times \{t\}$ is equal to the canonical map $M \times \{t\} \longrightarrow (M \times \{t\})/\mathcal{D}^t$ for each t. In this case, the continuity of the Álvarez class follows without Proposition 20 (see Example 7.1). When the dimension of D jumps, we have only a family of smooth proper submersions defined by D which changes discontinuously with respect to t.

6.2. **Verification of the conditions in Proposition 20.** Let U be a connected open set in $\mathbb{R}^l$ which contains 0. Let M be a closed manifold, and $\{\mathcal{F}^t\}_{t\in U}$ be a smooth family of orientable transversely parallelizable foliations of M over U given by a smooth foliation $\mathcal{F}^{\mathrm{amb}}$ of $M \times U$. We define a distribution D on $M \times U$ by (81). By Lemma 27 (iv), $D|_{M\times\{t\}}$ defines a foliation $\mathcal{D}^t$ of $M \times \{t\}$ whose leaves are fibers of a submersion. We denote the projection $M \times \{0\} \longrightarrow (M \times \{0\})/\mathcal{D}^0$ by $\pi_{\tilde{\mathcal{F}}}^0$.

To apply Proposition 20 to our situation, we prepare two lemmas.

Lemma 28. *There exist an open neighborhood U' of 0 in U and a smooth proper submersion $\pi_{\tilde{\mathcal{F}}}^{\mathrm{amb}}\colon M \times U' \longrightarrow (M \times \{0\})/\mathcal{D}^0$ such that*

(i) $\pi_{\tilde{\mathcal{F}}}^{\mathrm{amb}}|_{M\times\{0\}} = \pi_{\tilde{\mathcal{F}}}^0$ *and*
(ii) *each fiber of $\pi_{\tilde{\mathcal{F}}}|_{M\times\{t\}}$ is saturated by the leaves of $\mathcal{F}^t$ for each t in U'.*

Proof. We put $k = \dim M - \dim D^0$. For each point $(x, 0)$ on $M \times \{0\}$, there exists a k-tuple of leafwise constant functions f_{x1}, f_{x2}, $\cdots$, f_{xk} globally defined on $(M \times U, \mathcal{F}^{\mathrm{amb}})$ such that $(df_{x1} \wedge df_{x2} \wedge \cdots \wedge df_{xk})_{(x,0)}$ is nonzero by the definition of D. Then $df_{x1} \wedge df_{x2} \wedge \cdots \wedge df_{xk}$ is nowhere vanishing on an open saturated neighborhood V_x of $(x, 0)$ in $(M \times U, \mathcal{F}^{\mathrm{amb}})$. We define a map ϕ_x by

$$\begin{array}{rccc} \phi_x\colon & V_x & \longrightarrow & \mathbb{R}^k \\ & z & \longmapsto & (f_{x1}(z), f_{x2}(z), \cdots, f_{xk}(z)). \end{array} \tag{83}$$

This ϕ_x is a submersion, because $df_{x1} \wedge df_{x2} \wedge \cdots \wedge df_{xk}$ has no zero on V_x. We can assume that the fibers of ϕ_x is connected after shrinking V_x. We put $V'_x = \phi_x^{-1}\big(\phi_x(V_x \cap (M \times \{0\}))\big)$. This V'_x is also an open neighborhood of x in $M \times U$. Since $\phi_x(V'_x) = \phi_x(V_x)$, for each point z on V'_x, there exists a leaf L_z of D^0 such that $\phi_x(z) = \phi_x(L_z)$. Since the fibers of ϕ_x are connected, L_z is unique. We define a map by

$$\begin{array}{rccc} \psi_x\colon & V'_x & \longrightarrow & V'_x/\mathcal{D}^0 \\ & z & \longmapsto & L_z. \end{array} \tag{84}$$

ψ_x is a smooth submersion which maps each leaf of $\mathcal{F}^t$ to a point. Note that $\psi_x|_{M\times\{0\}}$ is the restriction of the projection $\pi^0_{\tilde{\mathcal{F}}}$ to $M \times \{0\}$ by the definition. It follows that $\psi_x|_{M\times\{t\}}$ is a submersion, because $\psi_x|_{M\times\{t\}}$ is of the same rank with $\psi_x|_{M\times\{0\}}$. By the compactness of M, there exists finite points $\{x_j\}_{j=1}^n$ such that $\cup_{j=1}^n V'_{x_j}$ contains $M \times \{0\}$. There exists an open neighborhood U_1 of 0 in U such that $M \times U_1$ is contained in $\cup_{j=1}^n V'_{x_j}$. Let $\{\rho_j\}_{j=1}^n$ be a partition of unity on $(M \times \{0\})/\mathcal{D}^0$ with respect to a covering $\{\pi\big(V'_{x_j} \cap (M \times \{0\})\big)\}_{j=1}^n$. We fix a smooth embedding $\iota\colon M \times \{0\} \longrightarrow \mathbb{R}^m$ to the m-dimensional Euclidean space. We define a map Ψ_1 by

$$\begin{array}{rccc} \Psi_1\colon & M \times U_1 & \longrightarrow & \mathbb{R}^m \\ & z & \longmapsto & \sum_{j=1}^n \rho_j(\psi_{x_{sj}}(z))\iota(\psi_{x_{sj}}(z)). \end{array} \tag{85}$$

Note that each leaf of $\mathcal{F}^t$ is mapped to a point by Ψ_1 by the definition. $\Psi|_{M\times\{0\}}$ is equal to $j \circ \pi^0_{\mathcal{D}}$ by the definition. Since $\psi_x|_{M\times\{t\}}$ is a submersion to $(M \times \{0\})/\tilde{\mathcal{F}}$ and ι is an embedding, there exists an open neighborhood U_2 of 0 in U_1 such that $\Psi_1|_{M\times\{t\}}$ is a map of constant rank for every t in U_2. Hence $\Psi_1(M \times \{t\})$ is a smooth submanifold of $\mathbb{R}^m$, and $\Psi_1|_{M\times\{t\}}$ is a smooth submersion on the image for t in U_2. There exists an open neighborhood U' of 0 in U_2 such that $\Psi_1(M \times \{t\})$ is the image of a section of the normal bundle in a tubular neighborhood W of $\iota(M \times \{0\})$ for every t in U'. We denote the projection $W \longrightarrow \iota(M \times \{0\})$ of the tubular neighborhood by p_W. We put $\pi^{\mathrm{amb}}_{\tilde{\mathcal{F}}} = p_W \circ j \circ \Psi_1|_{M\times U'}$. Then $\pi^{\mathrm{amb}}_{\tilde{\mathcal{F}}}$ is a submersion and an extension of $\pi^0_{\tilde{\mathcal{F}}}$ which satisfies the given conditions. □

We put $\pi^t_{\tilde{\mathcal{F}}} = \pi^{\mathrm{amb}}_{\tilde{\mathcal{F}}}|_{M\times\{t\}}$ for t in U'. We write $\tilde{\mathcal{F}}^t$ for a foliation of $M \times \{t\}$ defined by the fibers of $\pi^t_{\tilde{\mathcal{F}}}$.

Lemma 29. *There exists a smooth family $\{g^t\}_{t\in U'}$ of Riemannian metrics on M such that*

(i) *g^t is bundle-like with respect to both of $(M, \mathcal{F}^t)$ and $(M, \tilde{\mathcal{F}}^t)$ and*
(ii) *the leaves of $\tilde{\mathcal{F}}^t$ are minimal submanifolds of $(M \times \{t\}, g^t)$ for each t in U'.*

In particular, the conditions (a) and (b) in Proposition 20 are satisfied by $M \times \{0\}$, $\mathcal{F}^0$, $\tilde{\mathcal{F}}^0$ *and* g^0.

Proof. It is well known that a Riemannian foliation $\mathcal{G}$ on a closed manifold N defined by a proper submersion is minimizable. For example, see Corollary 2 of Haefliger [11]. Then there exists a Riemannian metric g_1^{amb} on $M \times U'$ which is bundle-like with respect to the foliation defined by the fibers of $\pi_{\tilde{\mathcal{F}}}$ and each leaf of $\tilde{\mathcal{F}}$ is a minimal submanifold of $(M \times U', g_1^{\mathrm{amb}})$. We put $g_1^t = g_1^{\mathrm{amb}}|_{M\times\{t\}}$. Then the leaves of $\tilde{\mathcal{F}}^t$ are minimal submanifolds of $(M \times \{t\}, g_1^t)$. Let $\tilde{\chi}^t$ be the characteristic form of $(M \times \{t\}, \tilde{\mathcal{F}}^t, g_1^t)$. We can take a family of metrics $\{g_2^t\}_{t\in U}$ on a family of vector bundles $T(M \times \{t\})/T\mathcal{F}^t \cong (T\mathcal{F}^t)^{\perp}$ on M such that g_2^t is transverse with respect to both of $\mathcal{F}^t$ and $\tilde{\mathcal{F}}^t$. We can extend the family of metrics $\{g_2^t\}_{t\in U}$ on $\{(T\mathcal{F}^t)^{\perp}\}_{t\in U}$ to a family of Riemannian metrics $\{g^t\}_{t\in U}$ on M so that the characteristic form of $(M, \tilde{\mathcal{F}}^t, g^t)$ is equal to $\tilde{\chi}^t$. Then g^t is bundle-like with respect to both of $\mathcal{F}^t$ and $\tilde{\mathcal{F}}^t$. By the Rummler's formula (see the second formula in the proof of Proposition 1 in Rummler [23] or Lemma 10.5.6 of Candel and Conlon [4]), the mean curvature form of a Riemannian manifold with a foliation is determined only by the characteristic form and the orthogonal complement of the tangent bundle of the foliation. Since the characteristic form of $(M \times \{t\}, \tilde{\mathcal{F}}^t, g^t)$ is equal to χ^t, the leaves of $\tilde{\mathcal{F}}^t$ are minimal submanifolds of $(M \times \{t\}, g^t)$. □

We will confirm that the conditions (c) and (d) in Proposition 20 are satisfied in the present situation.

Lemma 30. *The conditions (c) and (d) in Proposition 20 are satisfied by* $(M, \mathcal{F}^0, g^0)$ *and* $\pi_{\tilde{\mathcal{F}}}^0$.

Proof. Let $\{\omega_i\}_{i=1}^{\mathrm{cod}(M,\mathcal{F})}$ on $(M \times U, \mathcal{F}^{\mathrm{amb}})$ be the set of basic 1-forms on $(M \times U, \mathcal{F}^{\mathrm{amb}})$ such that $\omega_i(X_{\mathrm{amb}}^j) = \delta_{ij}$ where δ_{ij} is the Kronecker's delta. $d\omega_i$ is written as

$$d\omega_i = \sum_{1\leq j<k\leq \mathrm{cod}(M,\mathcal{F})} c_i^{jk} \omega_j \wedge \omega_k \tag{86}$$

for some functions c_i^{jk} on $M \times U$. We have

$$d\omega_i(X_{\mathrm{amb}}^j, X_{\mathrm{amb}}^k) = c_i^{jk}. \tag{87}$$

Since $d\omega_i$ and each ω_j are basic forms and X_{amb}^j is a transverse field on $(M \times U, \mathcal{F}^{\mathrm{amb}})$, c_i^{jk} is a basic function on $(M \times U, \mathcal{F}^{\mathrm{amb}})$. Hence the restriction of c_i^{jk} to each fiber of $\tilde{\mathcal{F}}^0$ is a constant by the definition of $\mathcal{D}$. This proves that the condition (c) is satisfied.

Let β be a 1-form on $(M \times \{0\})/\tilde{\mathcal{F}}^0$. Then $\pi_{\tilde{\mathcal{F}}}^*\beta$ is a basic 1-forms on $(M \times U, \mathcal{F}^{\mathrm{amb}})$. It follows that $\pi_{\tilde{\mathcal{F}}}^*\beta(X_{\mathrm{amb}}^i)$ is a global basic function on $(M \times U, \mathcal{F}^{\mathrm{amb}})$. Hence the restriction of $\pi_{\tilde{\mathcal{F}}}^*\beta(X_{\mathrm{amb}}^i)$ to each fiber of $\tilde{\mathcal{F}}^0$ is a constant by the definition of $\mathcal{D}$. Hence the image of $X_{\mathrm{amb}}^i|_{M\times\{0\}}$ by the canonical projection $C^\infty(T(M \times \{0\})/T\mathcal{F}^0) \longrightarrow C^\infty(T(M \times \{0\})/T\tilde{\mathcal{F}}^0)$ is a transverse field on $(M \times \{0\}, \tilde{\mathcal{F}}^0)$. This proves that the condition (d) is satisfied. □

6.3. **Proof of the continuity Theorem 1.** Let M be a closed manifold and $\{\mathcal{F}^t\}_{t\in U}$ be a smooth family of transversely parallelizable foliations of M over U. We consider the distribution D defined by the equation (81). By Lemma 27, a proper submersion $\pi^0_{\tilde{\mathcal{F}}} \colon M\times\{0\} \longrightarrow (M\times\{0\})/\mathcal{D}^0$ is defined by the restriction of D to $M\times\{0\}$. By Lemma 28, we can take an open neighborhood U' of 0 in U and a proper submersion $\pi^{\mathrm{amb}}_{\tilde{\mathcal{F}}} \colon M\times U' \longrightarrow (M\times\{0\})/\mathcal{D}^0$ such that $\pi_{\tilde{\mathcal{F}}}|_{M\times\{0\}} = \pi^0_{\tilde{\mathcal{F}}}$ and each fiber of $\pi_{\tilde{\mathcal{F}}}|_{M\times\{t\}}$ are saturated by the leaves of $\mathcal{F}^t$ for each t in U'. We denote the foliation of $M\times\{t\}$ defined by the fibers of $\pi_{\tilde{\mathcal{F}}}|_{M\times\{t\}}$ by $\tilde{\mathcal{F}}^t$. By Lemma 29, we take a smooth family of metrics $\{g^t\}_{t\in U'}$ on M such that the fibers of $\pi^t_{\tilde{\mathcal{F}}}$ are minimal submanifolds of $(M\times\{t\}, g^t)$, and g^t is bundle-like with respect to both of $(M,\mathcal{F}^t)$ and $(M,\tilde{\mathcal{F}}^t)$ for each t in U'. We denote the mean curvature form and the Álvarez form of $(M,\mathcal{F}^t,g^t)$ by κ^t and κ^t_b, respectively. We define $\tilde{\kappa}^t_b = \rho_{\tilde{\mathcal{F}}}(\kappa^t)$.

We prove Theorem 1 by using Proposition 20 and Corollary 4.23 of Domínguez in [5].

Proof of Theorem 1. By Lemma 17, it suffices to show the case where M, $\tilde{\mathcal{F}}^0$ and the basic fibration of $(M,\mathcal{F}^0)$ are orientable. By Lemmas 29 and 30, the conditions (a), (b), (c) and (d) in Proposition 20 are satisfied. Hence, by Proposition 20, $\tilde{\kappa}^0_b$ is closed and $[\tilde{\kappa}^0_b] = [\kappa^0_b]$. By Corollary 4.23 of Domínguez [5], we can modify the component $g^t|_{T\mathcal{F}^t\otimes T\mathcal{F}^t}$ along the leaves of $\{g^t\}_{t\in T'}$ so that $\kappa^0 = \tilde{\kappa}^0_b$. Note that $\tilde{\kappa}^t_b$ may not be closed for nonzero parameter t in T'.

For a smooth loop γ in M, we have the following evaluation:

$$\begin{aligned}
&\left|\int_\gamma \left(\tilde{\kappa}^0_b - \kappa^t_b\right)\right| \\
\leq &\left|\int_\gamma \left(\tilde{\kappa}^0_b - \tilde{\kappa}^t_b\right)\right| + \left|\int_\gamma \left(\tilde{\kappa}^t_b - \kappa^t_b\right)\right| \\
= &\left|\int_\gamma \left(\tilde{\kappa}^0_b - \tilde{\kappa}^t_b\right)\right| + \left|\int_\gamma \rho_{\mathcal{F}}\left(\tilde{\kappa}^t_b - \kappa^t\right)\right| \\
\leq &\left|\int_\gamma \left(\tilde{\kappa}^0_b - \tilde{\kappa}^t_b\right)\right| + \left(\sup_{s\in S^1} \left|\left|\tfrac{d\gamma}{ds}(s)\right|\right|\right)\left(\sup_{x\in M\times\{t\}} \left|\left|\tilde{\kappa}^t_b(x) - \kappa^t(x)\right|\right|\right)
\end{aligned} \tag{88}$$

where $||\cdot||$ is a norm induced by g^t. Since $\tilde{\kappa}^t_b$ converges to $\tilde{\kappa}^0_b = \kappa^0$, the first and the second term converges to 0 as t tends to 0. Then we have $\lim_{t\to 0}\int_\gamma \kappa^t_b = \int_\gamma \tilde{\kappa}^0_b$ and the proof is completed. □

By Proposition 5.3 of Álvarez López [1], every closed 1-form cohomologous to the Álvarez class of $(M,\mathcal{F}^0)$ is realized as the Álvarez form of $(M,\mathcal{F}^0,g)$ for some bundle-like metric g. Proposition 5.3 of of Álvarez López is simpler to prove than the Corollary 4.23 of Domínguez [5] used in the proof of Theorem 1 above. But we do not know if we can replace Corollary 4.23 of [5] by Proposition 5.3 of [1] in the proof of Theorem 1.

In fact, by Proposition 5.3 of [1], we can modify the component $g^t|_{T\mathcal{F}^t\otimes T\mathcal{F}^t}$ along the leaves of $\{g^t\}_{t\in T'}$ so that $\kappa^0_b = \tilde{\kappa}^0_b$. But we do not know if $\left|\int_\gamma\left(\tilde{\kappa}^t_b - \kappa^t_b\right)\right|$ converges to 0 as t goes to 0 here. Note that $\left|\int_\gamma\left(\tilde{\kappa}^t_b - \kappa^t\right)\right|$ may not converge to 0 as t goes to 0 in this situation. This is because κ^t_b is defined by integrating the mean curvature form on each leaf closure of $\mathcal{F}^t$ and the dimension of the closures of leaves of $\mathcal{F}^t$ can change on any small open neighborhood of 0.

7. Examples of Riemannian foliations

7.1. **A special case where the families of Molino's commuting sheaves are smooth.** Let $\{\mathcal{F}^t\}_{t\in T}$ be a family of Riemannian foliation on a closed manifold M. If the dimension of the closures of generic leaves of $\mathcal{F}^t$ is constant with respect to t, then the family of Molino's commuting sheaves of $\{\mathcal{F}^t\}$ is smooth (see pages 125–130 and Section 5.3 of Molino [18] for the definition of the Molino's commuting sheaf of a Riemannian foliation). Since the Álvarez class of $(M,\mathcal{F}^t)$ is computed from the holonomy homomorphism of the Molino's commuting sheaf of $(M,\mathcal{F}^t)$ by Theorem 1.1 of Álvarez López [2], the Álvarez classes of this family are continuous with respect to t. Our main continuity Theorem 1 is essential in the case where the dimension of the closures of leaves change. If the dimension of the closures of leaves change, we cannot prove the continuity of the Álvarez class as above or directly by an application of deformation theory to Molino's commuting sheaves. In fact, the family of the Molino's commuting sheaves must be discontinuous in this case, because the rank of the Molino's commuting sheaf is equal to the dimension of the closures of generic leaves of $\mathcal{F}^t$.

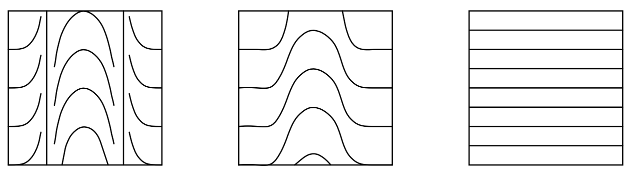
7.2. **Families of homogeneous Lie foliations.** Let $p\colon L \longrightarrow G$ be a surjective homomorphism between Lie groups. Let Γ be a uniform lattice of L. A foliation $\mathcal{F}$ on a homogeneous space $\Gamma\backslash L$ is induced by the fibers of p. This $\mathcal{F}$ has a structure of a G-Lie foliation. Such $\mathcal{F}$ is called a homogeneous G-Lie foliation. By deforming L, G, p and Γ, we may produce families of homogeneous Lie foliations. The Álvarez class is computed in terms of Lie theory by the interpretation of the Álvarez class as a first secondary characteristic class of Molino's commuting sheaf by Álvarez López (Theorem 1.1 of [2]). But the author does not know an example of a family of Riemannian foliations whose Álvarez classes change nontrivially obtained in this way. In many cases, the Álvarez class does not change as we will see in the following. If G is nilpotent, then the $\mathcal{F}$ is of polynomial growth. Then the Álvarez class does not change under deformation of $\mathcal{F}$ by Corollary 3. If L is solvable, then Γ is polycyclic (see Proposition 3.7 of Raghunathan [21]). Then the Álvarez class does not change under deformation of $\mathcal{F}$ by Corollary 3. If G is semisimple, the structural Lie algebra of the Lie foliation defined on the closure of leaves of $\mathcal{F}$ is semisimple. Then $\mathcal{F}$ is minimizable by Theorem 2 of Nozawa [19].

7.3. **Meigniez's examples: Families of solvable Lie foliations.** Meigniez constructed plenty of families of solvable Lie foliations which are not homogeneous by a surgery construction on homogeneous Lie foliations in [15] (see also [16], in particular, pages 119–122 for an explicit example). These families contains many examples of families of Lie foliations whose Álvarez classes change nontrivially.

7.4. **Basic cohomology of Riemannian foliations is not invariant under deformation.** We present an example of a family of Riemannian foliations whose basic cohomology changes. Let $M = S^1 \times S^3$. Let σ be the free S^1-action on S^3 whose orbits are fibers of the Hopf fibration. Let ρ be the T^2-action on M which is the product of the principal S^1-action on the first S^1-component and σ. For each element v of $\mathrm{Lie}(T^2) - \{0\}$, let $\mathcal{F}_v$ be the Riemannian flow on M whose leaves are the orbits of an $\mathbb{R}$-subaction of ρ whose infinitesimal action is given by v. Then we have a smooth family $\{\mathcal{F}_v\}_{v\in\mathrm{Lie}(T^2)-\{0\}}$ of Riemannian flows on M. Let v_1 and v_2 be the infinitesimal generators of the principal S^1-action on the first S^1-component and σ, respectively. Since $M/\mathcal{F}_{v_1} = S^3$ and $M/\mathcal{F}_{v_2} = M/\sigma = S^1 \times S^2$, clearly the dimension of $H^1_b(M/\mathcal{F}_{v_1})$ and $H^1_b(M/\mathcal{F}_{v_2})$ are different.

8. Examples of non-Riemannian foliations

8.1. **Turburization.** Let $\mathcal{F}^1$ be a product foliation $T^2 = \sqcup_{\theta_1 \in S^1} \{\theta_1\} \times S^1$ on T^2. Let $\mathcal{F}^0$ be a turbulization of $\mathcal{F}^1$ along a closed curve $\{\frac{1}{2}\} \times S^1$. This $\mathcal{F}^0$ is not minimizable by a theorem of Sullivan (see [25]), because $\mathcal{F}^0$ has a tangent homology defined by the Reeb component. Note that $\mathcal{F}^0$ is a limit of 1-dimensional foliations on T^2 which are diffeomorphic to $\mathcal{F}^1$. Thus we have a family of 1-dimensional foliations on T^2 parametrized by $[0,1]$ such that only $\mathcal{F}^0$ is not minimizable.

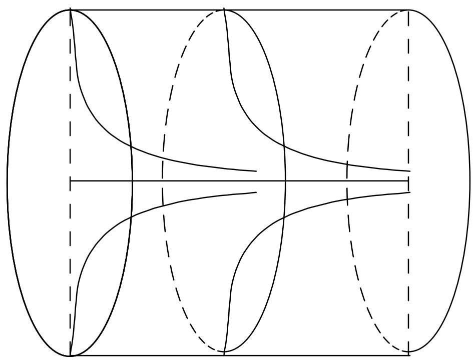


$\mathcal{F}^0$ $\mathcal{F}^1$

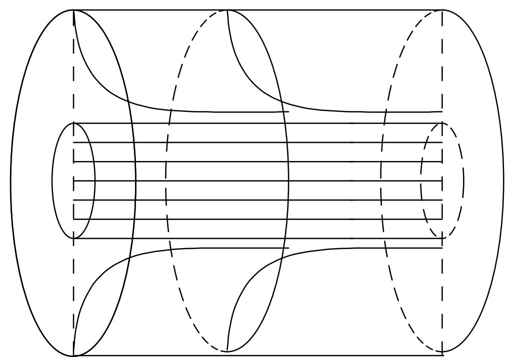
8.2. **Deformation of an example of Candel and Conlon.** We present an example of a family $\{\mathcal{F}^t\}_{t\in[0,1]}$ of 1-dimensional foliations on S^3 such that $\mathcal{F}^0$ is not minimizable and $\mathcal{F}^t$ is minimizable if t is nonzero. In a similar way, we will construct a family $\{\mathcal{H}^t\}_{t\in[0,1]}$ of 1-dimensional foliations on S^3 such that $\mathcal{H}_1$ is minimizable and $\mathcal{H}_t$ is minimizable if t is not equal to 1. Here $\mathcal{F}^0$ and $\mathcal{H}^0$ are the example constructed by Candel and Conlon in Example 10.5.19 of [4].

We restate the construction of the example $\mathcal{F}^0$ of Candel and Conlon here. We consider the 2-dimensional product foliation $S^1 \times D^2 = \sqcup_{t\in S^1}\{t\} \times D^2$ on the solid torus. Turburizing this product foliation around the axis $S^1 \times \{0\}$, we obtain a singular foliation $\mathcal{S}$ on $S^1 \times D^2$ whose leaves are trumpet-like surfaces and the axis $S^1 \times \{0\}$. We foliate $S^1 \times D^2$ by a 1-dimensional foliation $\mathcal{G}^0$ so that each leaf of $\mathcal{S}$ is saturated by leaves of $\mathcal{G}^0$ and the leaves of $\mathcal{G}^0$ is transverse to the boundary of the solid torus. We obtain a foliation $\mathcal{F}^0$ on S^3 by pasting two copies of $(S^1 \times D^2, \mathcal{G}^0)$. This $(S^3, \mathcal{F}^0)$ is nonminimizable as Candel and Conlon showed by a theorem of Sullivan in [4].

We construct $\mathcal{F}^t$ for nonzero t in $[0,1]$. Let L_1 and L_2 be two closed leaves of $\mathcal{F}^0$ which are axes of solid tori. For t in $[0,1]$, let $\mathcal{F}^t$ be the smooth foliation obtained from $\mathcal{F}^0$ by replacing both of L_1 and L_2 to solid tori K^t_1 and K^t_2 of radius t with the product foliation $K^t_i = S^1 \times D^2 = \sqcup_{x\in D^2} S^1 \times \{x\}$ for $i = 1$ and 2. Thus we have a smooth family $\{\mathcal{F}^t\}_{t\in[0,1]}$ of 1-dimensional foliations on S^3.



$(S^1 \times T^2, \mathcal{G}^0)$



After inserting a sorid torus.

We can decompose S^3 into $\mathcal{F}^t$-saturated subsets K_1^t, K_2^t and $T^2 \times [0,1]$. Let (θ_1, θ_2, s) be the coordinates on $T^2 \times [0,1]$ such that

- θ_1 parametrizes a meridian of K_1^t and a longitude of K_2^t, and
- θ_2 parametrizes a meridian of K_2^t and a longitude of K_1^t.

By the construction, we can construct $\mathcal{F}^t$ so that the leaves of $\mathcal{F}^t$ are transverse to a 1-form $d\theta_1 + d\theta_2$ on $T^2 \times [0,1]$.

By the Rummler-Sullivan criterion, we show

Proposition 31. *$\mathcal{F}^t$ is minimizable for nonzero t in $[0,1]$.*

Proof. By the Rummler-Sullivan criterion (see Sullivan [24]), $\mathcal{F}^t$ is minimizable if and only if there exist a 1-form χ on S^3 such that $\chi|_{T\mathcal{F}^t}$ has no zero and $d\chi|_{T\mathcal{F}^t} = 0$.

We take the decomposition of S^3 into $\mathcal{F}^t$-saturated subsets

$$S^3 = K_1^t \sqcup K_2^t \sqcup (T^2 \times [0,1]) \tag{89}$$

as above. We take a coordinate (θ_1, θ_2, s) on $T^2 \times [0,1]$ as noted in the paragraph previous to Proposition 31. We assume that $\mathcal{F}^t$ is transverse to $d\theta_1 + d\theta_2$ on $T^2 \times [0,1]$, while $\mathcal{F}^t|_{K_i^t}$ is the product foliation on a solid torus for $i = 1$ and 2. Let A_i be the axis of K_i^t. We can extend θ_1 from $T^2 \times [0,1]$ to $S^3 - A_1$ so that

- θ_1 is the composite of a diffeomorphism $S^3 - A_1 \cong S^1 \times D^2$ and the first projection $S^1 \times D^2 \longrightarrow S^1$ and
- $d\theta_1$ is transverse to $\mathcal{F}^t$ on $S^3 - K_2^t$.

We extend θ_2 to $S^3 - A_2$ in a similar way.

Let r_i be the radius coordinate on the D^2-component of K_i^t. Let ϕ_i be a non-negative smooth function on S^3 such that

- $\phi_i = 0$ on $S^3 - K_i^t$,
- ϕ_i is a function of r_i on K_i^t and
- ϕ_i is 1 on an open neighborhood of A_i.

We define a 1-form χ on S^3 by

$$\chi = (1 - \phi_1) d\theta_1 + (1 - \phi_2) d\theta_2. \tag{90}$$

Note that $(1 - \phi_i) d\theta_i$ is well-defined on S^3, though $d\theta_i$ is not defined on the axis A_i of K_i^t.

We will confirm that χ satisfies the conditions in the Rummler-Sullivan's criterion for $\mathcal{F}^t$ on the each component of the decomposition (89). Since the restriction of χ to $T^2 \times [0,1]$ is equal to $d\theta_1 + d\theta_2$, $\chi|_{T^2\times[0,1]}$ is transverse to $\mathcal{F}^t|_{T^2\times[0,1]}$ and $d\chi = 0$. On K_1^t, we have

$$\chi|_{K_1^t} = (1 - \phi_1)d\theta_1 + d\theta_2 \tag{91}$$

Since $d\theta_2$ is transverse to $\mathcal{F}^t|_{K_1^t}$ and $d\theta_1|_{T\mathcal{F}^t}$ is zero, $\chi|_{K_1^t}$ is transverse to $\mathcal{F}^t|_{K_1^t}$. We have $d\chi|_{K_1^t} = d\phi_1 \wedge d\theta_1$, and hence $(d\chi|_{K_1^t})|_{T\mathcal{F}^t} = 0$. We can prove that $\chi|_{K_2^t}$ also satisfies the two conditions in the Rummler-Sullivan's characterization in the same way. Hence $\mathcal{F}^t$ is minimizable. □

Let X^s be a nowhere vanishing vector field tangent to $\mathcal{F}^s$ for $s = 0$ and 1. We put $X^t = tX^1 + (1-t)X^0$. Then X^t is also nowhere vanishing and defines a foliation $\mathcal{H}^t$. In this family $\{\mathcal{H}^t\}_{t\in[0,1]}$ of foliations, $\mathcal{H}^t$ is not minimizable for $0 \leq t < 1$ and $\mathcal{H}^1$ is minimizable. Hence

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Unité de Mathématiques Pures et Appliqueés, École Normale Supérieure de Lyon, 46 allée d'Italie 69364 Lyon Cedex 07, France
E-mail address: nozawahiraku@06.alumni.u-tokyo.ac.jp