Mathematical Structures Defined by Identities II

Constantin M. Petridi
cpetridi@hotmail.com

Abstract In our paper arXiv: math.RA/0110333 v1 Oct 2001 we showed that the number of algebras defined by a binary operation satisfying a formally irreducible identity between two n-iterates is $O\big(e^{-n/16}S_n^2\big)$ for $n \to \infty$, S_n being the nth-Catalan number. This was proved by using exclusively the series of tableaux A_n. By using also the series of tableaux B_n, we now sharpen this result to $O\Big(|\frac{n+2}{n}\,e^{-n/16} - \frac{2}{n}|S_n^2\Big)$.

The exposition follows, in abbreviated form, the outline of above arXiv paper, denoted by MS, to which we refer for explanation of concepts and symbols.

1. Since tableau A_n has n-lines and tableau B_n has 2 lines, tableau $A_n \oplus B_n$ has $n+2$ lines. The relations of MS 2.2 regarding the number of their (lines) common elements, have to be unchanged as follows, so that we again have for $k = 1, 2, \ldots, n+2$

$$|L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_k}| = \begin{cases} S_{n-1} & if \;\; i_1 = i_2 = \cdots = i_k \\ 0 & \text{if at least one } |i_1 - i_2|, \ldots, |i_{k-1} - i_k|, \\ & \text{all taken mod n, is equal to 1} \\ S_{n-k} & \text{otherwise.} \end{cases}$$

For example, for $n = 6$, $k = 2$, there are 8 lines $L_1, L_2, \ldots, L_8$ in tableau $A_6 \oplus B_6$. The 8×8 table $(|L_i \cap L_j|)$ looks as follows (only the entries on and above the diagonal are shown since $|L_i \cap L_j| = |L_j \cap L_i|$.)

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
L_1	S_6	0	S_5	S_5	S_5	S_5	S_5	0
L_2		S_6	0	S_5	S_5	S_5	S_5	S_5
L_3			S_6	0	S_5	S_5	S_5	S_5
L_4				S_6	0	S_5	S_5	S_5
L_5					S_6	0	S_5	S_5
L_6						S_6	0	S_5
L_7							S_6	0
L_8								S_6

=

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
L_1	42	0	14	14	14	14	14	0
L_2		42	0	14	14	14	14	14
L_3			42	0	14	14	14	14
L_4				42	0	14	14	14
L_5					42	0	14	14
L_6						42	0	14
L_7							42	0
L_8								42

2. The multiplicity $M(J_i^n)$ of an n-interate J_i^n is the number of times the iterate occurs in tableau $A_n \oplus B_n$.The number of n-iterates with same multiplicity k, is now denoted by $T_{n,k}^{A_n \oplus B_n}$. As stated at the end of MS 2.6 this number has been found to be, for $k \geq 1$

$$T_{n,k}^{A_n \oplus B_n} = T_{n,k} + 2T_{n-1,k-1} - 2T_{n-1,k}, \tag{1}$$

where $T_{n,k}$ are the corresponding numbers with regard to tableau A_n, i.e.

$$T_{n,k} = 2^{n-2k+1}\binom{n-1}{2k-2}S_{k-1}, \quad k = 1, 2, \ldots, \tag{2}$$

Calculating the terms of the left side of (1), we have from (2)

$$T_{n,k} = 2^{n-2k+1}\binom{n-1}{2k-2}S_{k-1} = 2^{n-2k+1}\frac{(n-1)\dots(n-2k+2)}{(2k-2)!}S_{k-1}$$

$$2T_{n-1,k-1} = 2^{n-2k+3}\binom{n-2}{2k-1}S_{k-2} = 2^{n-2k+3}\frac{(n-2)\dots(n-2k+3)}{(2k-4)!}S_{k-2}$$

$$-2T_{n-1,k} = -2^{n-2k+1}\binom{n-2}{2k-2}S_{k-1} = -2^{n-2k+1}\frac{(n-2)\dots(n-2k+1)}{(2k-2)!}S_{k-1}.$$

Adding and using the recursion $S_k = 2\frac{2k-1}{k+1}S_{k-1}$ for the Catalan numbers we get

$$T_{n,k}^{A_n\oplus B_n} = 2^{n-2k+1}\frac{(n+2)\dots(n-2k+3)}{(2k-4)!}\left\{\frac{(n-1)(n-2k+1)}{(2k-3)(2k-2)}S_{k-1}+\right.$$

$$\left.4S_{k-2} - \frac{(n-2k+2)(n-2k+1)}{(2k-3)(2k-2)}S_{k-1}\right\}$$

$$= 2^{n-2k+1}\binom{n-2}{2k-4}\left\{\frac{n-2k+2}{(2k-3)(2k-2)}(n-1-n+2k-1)S_{k-1} + 4S_{k-2}\right\}$$

$$= 2^{n-2k+1}\binom{n-2}{2k-4}\left\{\frac{n-2k+2}{2k-3}S_{k-1} + 4S_{k-2}\right\}$$

$$= 2^{n-2k+1}\binom{n-2}{2k-4}\left\{\frac{n-2k+2}{k} + 2\right\}2S_{k-2}$$

$$= 2^{n-2k+2}\binom{n-2}{2k-4}\frac{n+2}{k}S_{k-2}.$$

But from (2) we have that

$$T_{n-1,k-1} = 2^{n-2k+2}\binom{n-2}{2k-4}S_{k-2}$$

so that finally

$$T_{n,k}^{A_n\oplus B_n} = \frac{n+2}{k}T_{n-1,k-1}. \tag{3}$$

3. Formal reducibility of an identity $J_i^n = J_j^n$ and incidence matrix relative to tableau $A_n \oplus B_n$ are defined in the same way as for tableau A_n. The number of formally *reducible* identities $J_i^n = J_j^n$ of order n, which we denote by $I_n^{A_n\oplus B_n}$, to distinguish it from I_n relative to tableau A_n, is given by

$$I_n^{A_n\oplus B_n} = \sum_{1\le i,j\le S_n} \delta(J_i^n, J_j^n),$$

where

$$\delta(J_i^n, J_j^n) = \begin{cases} 1 & \text{if } J_i^n = J_j^n \text{ formally reducible} \\ 0 & \text{if } J_i^n = J_j^n \text{ formally irreducible} \end{cases}$$

As an example, we display the incidence matrices relative to tableaux A_3 and $A_3 \oplus B_3$, which clearly shows, as expected that $I_3^{A_3 \oplus B_3} > I_3$.

A_3

	J_1^3	J_2^3	J_3^3	J_4^3	J_5^3	$\sum_i 1$
J_1^3	1	1	0	0	0	2
J_2^3	1	1	0	0	1	3
J_3^3	0	0	1	1	0	2
J_4^3	0	0	1	1	0	2
J_5^3	0	1	0	0	1	2
						–
			I_3		=	11

$A_3 \oplus B_3$

	J_1^3	J_2^3	J_3^3	J_4^3	J_5^3	$\sum_i 1$
J_1^3	1	1	1	0	0	3
J_2^3	1	1	0	0	1	3
J_3^3	1	0	1	1	0	3
J_4^3	0	0	1	1	1	3
J_5^3	0	1	0	1	1	3
						–
				$I_3^{A_3 \oplus B_3}$	=	15

For $n = 4$ and $n = 5$ the corresponding findings are

$n = 4$	$I_4 = 88$	$I_4^{A_4 \oplus B_4} = 116$
$n = 5$	$I_5 = 834$	$I_5^{A_4 \oplus B_4} = 1050$

4. The arguments which led us to establish the fundamental Theorem of MS 2.4 can be applied verbatim, resulting in

$$\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) = \sum_{\nu=1}^{M(J_i^n)} (-1^{\nu-1}) \binom{M(J_i^n)}{\nu} S_{n-\nu}, \tag{4}$$

where $M(J_i^n$ is the multiplicity of J_i^n in tableau $A_n \oplus B_n$. (4) means that the number of formally reducible identities in the line L_i of tableau $I_n^{A_n \oplus B_n}$ does *not* depend on J_i^n but only on the multiplicity of $M(J_i^n)$, as was the case in MS 2.4, when solely the series of tableaux A_n was used. Since $\binom{M(J_i^n)}{\nu} = 0$, for $\nu > M(J_i^n)$ we can forget the upper limit $M(J_i^n)$ for the index ν and write instead

$$\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) = \sum_{\nu=0}^{\infty} (-1^{\nu-1}) \binom{M(J_i^n)}{\nu} S_{n-\nu}. \tag{5}$$

This convention will be used for all finite series of the form $\sum_{\nu=n}^{N} \binom{f(N)}{\nu} c_\nu$, $f(x)$ a positive arithmetic function which from now on will be written as $\sum_{\nu=n}^{\infty} \binom{f(N)}{\nu} c_\nu$, since $\binom{f(N)}{\nu} = 0$ for $\nu > f(N)$.

5. Following identity from MS 2.5 for various values of the indices n and k will be needed in the sequel

$$\sum_{\nu=0}^{[\frac{n+1}{2}]-k} \binom{k+\nu}{k} T_{n,k+\nu} = \binom{n-k+1}{k} S_{n-k},$$

which, because of said convention, will be written

$$\sum_{\nu=0}^{\infty} \binom{k+\nu}{k} T_{n,k+\nu} = \binom{n-k+1}{k} S_{n-k}. \tag{6}$$

6. On basis of above results we can now evaluate $I_n^{A_n \oplus B_n}$. By definition $I_n^{A_n \oplus B_n}$ is the sum of 1's in the incidence matrix relative to tableau $A_n \oplus B_n$. Counting them by lines and because of (4) we therefore have

$$I_n^{A_n \oplus B_n} = \sum_{1 \le i,j \le S_n} \delta(J_i^n, J_j^n) = \sum_{i=1}^{S_n} \Big(\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) \Big)$$

$$= \sum_{i=1}^{S_n} \Big(\sum_{\nu=1}^{M(J_i^n)} (-1)^{\nu-1} \binom{M(J_i^n)}{\nu} S_{n-\nu} \Big). \tag{7}$$

Since there are $T_{n,k}^{A_n \oplus B_n}$ n-iterates with multiplicity k, $k = 1, 2, \ldots$, the double sum can be rearranged by pooling together all terms with $M(J_i^n) = k$. As a consequence, using our convention, we have

$$I^{A_n \oplus B_n} = \sum_{k=1}^{\infty} T_{n,k}^{A_n \oplus B_n} \Big[\sum_{\nu=1}^{\infty} (-1)^{\nu-1} \binom{k}{\nu} S_{n-\nu} \Big]. \tag{8}$$

Substituting $T_{n,k}^{A_n \oplus B_n}$ by its value from (4) we obtain

$$I_n^{A_n \oplus B_n} = \sum_{k=1}^{\infty} \frac{n+2}{k} T_{n-1,k-1} \Big[\sum_{\nu=1}^{\infty} (-1)^{\nu-1} \binom{k}{\nu} S_{n-\nu} \Big],$$

and reversing the order of summation we get,

$$I_n^{A_n \oplus B_n} = (n+2) \Big\{ S_{n-1} \Big[\sum_{k=1}^{\infty} \frac{1}{k} \binom{k}{1} T_{n-1,k-1} \Big] - S_{n-2} \Big[\sum_{k=1}^{\infty} \frac{1}{k} \binom{k}{2} T_{n-1,k-1} \Big] + \ldots$$

$$+ (-1)^{n-\nu} S_{n-\nu} \Big[\sum_{k=1}^{\infty} \frac{1}{k} \binom{k}{\nu} T_{n-1,k-1} \Big] + \ldots \Big\}. \tag{9}$$

Observing that $\frac{1}{k}\binom{k}{\nu} = \frac{1}{\nu}\binom{k-1}{\nu-1}$ and setting $\mu = k-1$ as a new running index (9) becomes

$$I_n^{A_n \oplus B_n} = (n+2)\Big\{\frac{S_{n-1}}{1}\Big[\sum_{\mu=0}^{\infty}\binom{\mu}{0}T_{n-1,\mu}\Big] - \frac{S_{n-2}}{2}\Big[\sum_{\mu=0}^{\infty}\binom{\mu}{0}T_{n-1,\mu}\Big] + \dots$$

$$+(-1)^{\nu-1}\frac{S_{n-\nu}}{\nu}\Big[\sum_{\mu=0}^{\infty}\binom{\mu}{\nu-1}T_{n-1,\mu}\Big] + \dots\Big\}. \tag{10}$$

The expressions in brackets can be evaluated from (6) by replacing n by $n-1$, $\binom{k+\nu}{\nu}$ by $\binom{k+\nu}{k}$ and k successively by $0, 1, \dots$. This gives

$$\begin{array}{lll}
k=0 & \displaystyle\sum_{\nu=0}^{\infty}\binom{\nu}{0}T_{n-1,\nu} = \binom{n}{0}S_{n-1} \\
k=1 & \displaystyle\sum_{\nu=0}^{\infty}\binom{\nu+1}{1}T_{n-1,\nu+1} = \binom{n-1}{1}S_{n-2} \\
\dots & \dots \qquad\qquad \dots \\
k=k & \displaystyle\sum_{\nu=0}^{\infty}\binom{\nu+k}{k}T_{n-1,\nu+k} = \binom{n-k}{k}S_{n-k-1} \\
\dots & \dots \qquad\qquad \dots
\end{array}$$

Inserting these values in (10) we obtain

$$I_n^{A_n \oplus B_n} = (n+2)\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{1}{\nu+1}\binom{n-\nu}{\nu}S_{n-\nu-1}^2. \tag{11}$$

Actually this sum is finite since for $\nu > \frac{n}{2}$ all coefficients $\binom{n-\nu}{\nu}$ are zero. It can be asymptotically evaluated by the same heuristic method we used in MS 2.6 to evaluate I_n. Setting $\nu = k-1$ and after obvious transformations, (11) becomes

$$I_n^{A_n \oplus B_n} = (n+2)S_n^2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\binom{n-k+1}{k-1}\Big(\frac{S_{n-k}}{S_n}\Big)^2 \tag{12}$$

On the other hand for $n \to \infty$ and k finite

$$\frac{S_{n-k}}{S_n} \backsim \frac{1}{4^k}$$

$$\frac{1}{k}\binom{n-k+1}{k-1} \backsim \frac{n^{k-1}}{k!}$$

so that the general term of the series in (12) behaves for $n \to \infty$ like

$$(-1)^{k-1}\frac{n^{k-1}}{k!}\Big(\frac{1}{4^k}\Big)^2 = (-1)^{k-1}\frac{1}{k!}\frac{1}{4^{2k}}n^{k-1}.$$

We now sum over k to get

$$
\begin{aligned}
I_n^{A_n \oplus B_n} &\backsim (n+2)S_n^2 \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{4^2 k!}\frac{1}{4^{2k-2}}n^{k-1} \\
&\backsim \frac{(n+2)S_n^2}{4^2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k!}\Big(\frac{n}{4^2}\Big)^{k-1} \\
&\backsim \frac{(n+2)S_n^2}{n}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k!}\Big(\frac{n}{4^2}\Big)^{k} \\
&\backsim \frac{n+2}{n}\, S_n^2\big(1-e^{-\frac{n}{16}}\big) \qquad (13)
\end{aligned}
$$

As said in MS 2.6 above argument is not rigorous, but can be made so by estimating for $n \to \infty$ the differences $|\frac{S_{n-k}}{S_n} - \frac{1}{4^k}|$ and $|\frac{1}{k}\binom{n-k+1}{k} - \frac{n^{k-1}}{k!}|$. A stronger result can be obtained if we apply the Sterling formula to the binomial coefficients, taking into account that even $S_n = \frac{1}{n+1}\binom{2n}{n}$ contains binomial coefficients (see SP).

6. $I_n^{A_n \oplus B_n}$ was defined as the number of formally reducible identities $J_i^n = J_j^n$ of order n, so that the number of formally irreducible identities is $S_n^2 - I_n^{A_n \oplus B_n}$, since the total number of n-identities is S_n^2.

(13) means that the order of $I_n^{A_n \oplus B_n}$ for $n \to \infty$ is $O\big(\frac{n+2}{n}(1-e^{-\frac{n}{16}})S_n^2\big)$, which gives for $S_n^2 - I_n^{A_n \oplus B_n}$ the order $O\Big(|\frac{n+2}{n}\ e^{-n/16} - \frac{2}{n}|S_n^2\Big)$.

Summarizing and using the terminology of algebras (see BO), we have proved following.

Theorem. The number of algebras defined by a binary operation satisfying a formally irreducible identity between two n-iterates of the operation is, for $n \to \infty$, $O\Big(|\frac{n+2}{n}\ e^{-n/16} - \frac{2}{n}|S_n^2\Big)$.

Acknowledgment. I wish to thank Peter Krikelis, University of Athens, Dep. of Mathematics, for his help.

References.

1. MS Petridi, Constantin M., Krikelis, Peter B.: Mathematical Structures defined by identities, arXiv: math.RA/0110333 v1 Oct 2001
2. BO Bakhturin Yu.A., Ol'Shanskij A.Yu.: II Identities, Encyclopedia of Mathematical Sciences Vol. 18, Springer - Verlag Berlin Heideberg 1991.
3. SP Stanica, Pantelimon: Good lower and upper bounds on binomial coefficients, Journal of Inequalities in Pure and Applied Mathematics, Vol 2, Issue 3, Article 30, 2001