ON THE MULTI-DIMENSIONAL CONTROLLER AND STOPPER GAMES

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Abstract. We consider a zero-sum stochastic differential controller-and-stopper game in which the state process is a controlled jump-diffusion evolving in a multi-dimensional Euclidean space. In this game, the controller affects both the drift and the volatility terms of the state process. Under appropriate conditions, we show that the lower value function of this game is a viscosity solution to an obstacle problem for a Hamilton-Jacobi-Bellman equation, by generalizing the weak dynamic programming principles in [3].

Key Words: Controller-stopper games, weak dynamic programming principle, viscosity solutions.

1. Introduction

While the game of control and stopping is closely related to some common problems in Mathematical Finance, such as pricing American-type contingent claims (see e.g. [9], [13] and [14]) and minimizing the probability of ruin (see [2]), it has not been studied to a great extent except under certain particular cases. Karatzas and Sudderth [12] study a zero-sum game in which the controller affects the coefficients of a linear diffusion along a given interval on $\mathbb{R}$, while the stopper decides the time to halt the diffusion. Under appropriate conditions, they not only prove that this game has a value but also describe fairly explicitly a saddle-point of optimal strategies. It is, however, difficult to extend their remarkable results to multi-dimensional cases following the same line of arguments because their techniques rely heavily on the optimal stopping results for one-dimensional diffusions. Karatzas and Zamfirescu [15] develop a martingale approach to deal with a multi-dimensional game of controll and stopping; but since their method makes use of Girsanov's theorem, which demands a nondegenerate condition on the volatility coefficient of the state process $X^{t,x,\alpha}_{\cdot}$, only the drift of $X^{t,x,\alpha}_{\cdot}$ is allowed to be controlled in this game.

In contrast, we investigate a much more general zero-sum controller-and-stopper game, at least under a Markovian framework. In our game, the state process $X^{t,x,\alpha}_{\cdot}$ is a controlled jump-diffusion evolving in a multi-dimensional Euclidean space. The controller intends to maximize his payoff, consisting of a running reward $\int_t^\tau f(s, X^{t,x,\alpha}_s, \alpha_s)ds$ and a terminal reward $g(X^{t,x,\alpha}_\tau)$, by selecting a control α that affects all the coefficients, the drift, the volatility and the jump terms, of $X^{t,x,\alpha}_{\cdot}$. The stopper intends to minimize his cost by choosing the duration of the game, in the form of a stopping time τ. We show that the lower value function V of this game, as defined in Section 2, is a viscosity solution to an obstacle problem for a Hamilton-Jacobi-Bellman equation.

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Our method differs largely from those in [12] and [15] mentioned above. We generalize the weak dynamic programming principles (WDPPs) introduced in [3] to the controller-and-stopper context. With the aid from the theory of Reflected Backward Stochastic Differential Equations (RBSDEs) (see e.g. [6], [4] and [5]), we modify the arguments in Theorems 3.1 and 4.1 of [3] and obtain a dynamic-programming-type result; see Proposition 3.1, which is the key to proving the supersolution property of V. On the other hand, the proof of the corresponding WDPP in our case requires some additional probabilistic techniques; see Theorem 4.1.

The structure of this paper is simple. We set up the framework of our study in Section 2, where assumptions and notations are introduced. In Sections 3 and 4, the supersolution property and the subsolution property of V are derived, respectively.

2. The Model

Consider the product space $\Omega := \Omega_W \times \Omega_N$, where $\Omega_W := C([0,T];\mathbb{R}^d)$ and Ω_N is the set of interger-valued measures on $[0,T]\times\mathbb{R}^n$. For any $\omega = (\omega^1,\omega^2)\in\Omega$, set $W(\omega)=\omega^1$ and $N(\omega)=\omega^2$. Now define $\mathbb{F}^W = \{\mathcal{F}_t^W\}_{t\in[0,T]}$ (resp. $\mathbb{F}^N = \{\mathcal{F}_t^N\}_{t\in[0,T]}$) as the smallest right-continuous filtration on Ω_W (resp. Ω_N) such that W (resp. N) is optional. Let $\mathbb{P}_W$ denote the Wiener measure on $(\Omega_W,\mathcal{F}_T^W)$, and $\mathbb{P}_N$ denote the measure on $(\Omega_N,\mathcal{F}_T^N)$ under which N is a Poisson random measure with intensity $\tilde{N}(dq,dt) = \lambda(dq)dt$, for some finite measure λ on $\mathbb{R}^n$. Now we define the probability measure $\mathbb{P} := \mathbb{P}_W\otimes\mathbb{P}_N$ on $(\Omega,\mathcal{F}_T^W\otimes\mathcal{F}_T^N)$ and let $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ be the natural right-continuous filtration generated by (W,N) which is augmented with $\mathbb{P}-$null sets. Note that under this construction, W and N are independent under $\mathbb{P}$. Let $X^{\tau,\xi,\alpha}$ denote a $\mathbb{R}^d$-valued process satisfying the following SDE:

$$dX_t^{\tau,\xi,\alpha} = b(t,X_t^{\tau,\xi,\alpha},\alpha_t)dt + \sigma(t,X_t^{\tau,\xi,\alpha},\alpha_t)dW_t + \int_{\mathbb{R}^n}\gamma(t,X_{t-}^{\tau,\xi,\alpha},\alpha_t,q)N(dq,dt), \quad t\in[\tau,T] \tag{2.1}$$

where α_t, the control, belongs to $\mathcal{A}$, a subset of all progressively measurable processes valued in $\mathbb{R}^m$; τ is a stopping time and $X_\tau^{\tau,\xi,\alpha}=\xi$ is such that $\mathbb{E}[\xi^2]<\infty$.

We consider a game of control and stopping in a finite time horizon T with running gain $f\geq 0$, terminal reward $g\geq 0$ and discount rate $c\geq 0$. The functions f, g, c are assumed to be measurable. We also assume that the discount rate is bounded above by some positive real number $\bar{c}$. Let $\mathcal{T}_{t,T}$ denote the set of all $\mathbb{F}$-stopping times with values in $[t,T]$. We introduce the lower value function concerning this game

$$V(t,x) := \sup_{\alpha\in\mathcal{A}_t}\inf_{\tau\in\mathcal{T}_{t,T}^t}\mathbb{E}\left[\int_t^\tau e^{-\int_t^s c(u,X_u^{t,x,\alpha})du} f(s,X_s^{t,x,\alpha},\alpha_s)ds + e^{-\int_t^\tau c(u,X_u^{t,x,\alpha})du} g(X_\tau^{t,x,\alpha})\right], \tag{2.2}$$

for any $(t,x)\in[0,T]\times\mathbb{R}^d$, where $\mathcal{A}_t$ is the set of all $\alpha\in\mathcal{A}$ that are independent of $\mathcal{F}_t$ and $\mathcal{T}_{t,T}^t$ is the set of all $\tau\in\mathcal{T}_{t,T}$ that are indepentdent of $\mathcal{F}_t$.

We assume that there exists a $K > 0$ such that for any $t \in [0,T]$, $x, y \in \mathbb{R}^d$, $q \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}^m$,

$$\begin{aligned}
&|b(t,x,\alpha) - b(t,y,\beta)| + |\sigma(t,x,\alpha) - \sigma(t,y,\beta)| \le K|x-y| + |\alpha - \beta|, \\
&|b(t,x,\alpha)| + |\sigma(t,x,\alpha)| \le K(1 + |x| + |\alpha|), \\
&|\gamma(t,x,\alpha,q) - \gamma(t,y,\beta,q)| \le K(|x-y| + |\alpha - \beta|), \\
&|\gamma(t,x,\alpha,q)| \le K(1 + |x| + |\alpha|).
\end{aligned} \tag{2.3}$$

The above conditions on the coefficients impliy that for any initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$ and any admissible control $\alpha \in \mathcal{A}$, equation (2.1) admits a unique strong solution $X^{\tau,\xi,\alpha}$ and satisfies the flow property; see Section 5 of [3]. In addition, we assume that g is continuous, and that f and g satisfy the polynomial growth condition

$$|f(t,x,\alpha)| + |g(x)| \le K(1 + |x|^p) \text{ for some } p \ge 1. \tag{2.4}$$

Remark 2.1. *Under assumption* (2.3) *the solution of* (2.1) *satisfies*

$$\mathbb{E}\left[\sup_{\tau \le r \le T} |X_r^{\tau,\xi,\alpha}|^p\right] < \infty, \tag{2.5}$$

for all $p \ge 1$, see Section 5 of [3]. *Therefore, the polynomial growth condition* (2.4) *on f and g implies that for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and control $\alpha \in \mathcal{A}$,*

$$\mathbb{E}\left[\int_t^T f(s, X_s^{t,x,\alpha}, \alpha_s) ds + \sup_{t \le r \le T} g(X_r^{t,x,\alpha})\right] < \infty, \tag{2.6}$$

as the following estimation demonstrates

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t \le r \le T}\left(\int_t^r e^{-\int_t^s c(u,X_u^{t,x,\alpha})du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^r c(u,X_u^{t,x,\alpha})du} g(X_r^{t,x,\alpha})\right)\right] \\
&\le \mathbb{E}\left[\sup_{t \le r \le T} \int_t^r f(s, X_s^{t,x,\alpha}, \alpha_s) ds\right] + \mathbb{E}\left[\sup_{t \le r \le T} g(X_r^{t,x,\alpha})\right] \\
&\le \int_t^T \mathbb{E}[f(s, X_s^{t,x,\alpha}, \alpha_s)] ds + \mathbb{E}\left[\sup_{t \le r \le T} g(X_r^{t,x,\alpha})\right] \\
&\le \int_t^T \mathbb{E}|K(1 + |X_s^{t,x,\alpha}|^p)| ds + \mathbb{E}\left[\sup_{t \le r \le T} K(1 + |X_r^{t,x,\alpha}|^p)\right] \\
&< \infty.
\end{aligned} \tag{2.7}$$

For $(t,x,p,A) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d$, define

$$H(t,x,p,A) := \inf_{\alpha \in \mathbb{R}^m} H^\alpha(t,x,p,A),$$

where

$$H^\alpha(t,x,p,A) := -b(t,x,\alpha) - \frac{1}{2} Tr[\sigma\sigma'(t,x,\alpha)A] - \int_{\mathbb{R}^n} [v(t, x + \gamma(t,x,\alpha,q)) - v(t,x)]\lambda(dq) - f(t,x,\alpha).$$

The Hamilton-Jacobi-Bellman equation associated with this game is the following nonlinear PDE

$$\max\left\{c(t,x)v - \frac{\partial v}{\partial t} + H(t,x,D_x v, D_x^2 v),\ v - g(x)\right\} = 0, \tag{2.8}$$

on $[0,T)\times\mathbb{R}^d$, with the terminal condition

$$v(T,x) = g(x),\ \forall\, x\in\mathbb{R}^d. \tag{2.9}$$

Also, consider the lower-semicontinuous envelope

$$H_*(z) := \liminf_{z'\to z} H(z'),$$

for any $z=(t,x,p,A)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{S}_d$.

2.1. **Notation.** First, observe that the value function can be written as

$$V(t,x) := \sup_{\alpha\in\mathcal{A}_t}\inf_{\tau\in\mathcal{T}^t_{t,T}} \mathbb{E}\left[\int_t^\tau Y_s^{t,x,1,\alpha} f(s,X_s^{t,x,\alpha},\alpha_s)ds + Y_\tau^{t,x,1,\alpha} g(X_\tau^{t,x,\alpha})\right],$$

where $dY_s^{t,x,y,\alpha} = -Y_s^{t,x,y,\alpha}c(s,X_s^{t,x,\alpha})ds,\ Y_t^{t,x,y,\alpha} = y > 0$. By increasing the state process to (X,Y,Z) with $dZ_s^{t,x,y,z,\alpha} = Y_s^{t,x,y,\alpha} f(s,X_s^{t,x,\alpha},\alpha_s)ds,\ Z_t^{t,x,y,z,\alpha} = z\in\mathbb{R}_+$, and considering the value function

$$\bar{V}(t,x,y,z) := \sup_{\alpha\in\mathcal{A}_t}\inf_{\tau\in\mathcal{T}^t_{t,T}} \mathbb{E}\left[F(X_\tau^{t,x,\alpha}, Y_\tau^{t,x,y,\alpha}, Z_\tau^{t,x,y,z,\alpha})\right],$$

where $F(x,y,z) := z + yg(x)$, we have

$$\bar{V}(t,x,y,z) = yV(t,x) + z.$$

It follows that $V(t,x) = \bar{V}(t,x,1,0)$ and $\bar{V}^*(t,x,y,z) = yV^*(t,x)+z$, where V^* is the upper semi-continuous envelope of V

$$V^*(t,x) := \limsup_{(t',x')\to(t,x)} V(t',x').$$

We denote the lower semi-continuous envelope by

$$V_*(t,x) := \liminf_{(t',x')\to(t,x)} V(t',x').$$

Remark 2.2. *When $|f(t,x,\alpha)-f(t,y,\alpha)| + |g(x)-g(y)| \le K|x-y|$, the value function V is continuous, i.e., $V^* = V_*$. However, in general this may not be true.*

Let $\mathcal{S} := \mathbb{R}^d\times\mathbb{R}_+^2$. Mimicking the relation between V and $\bar{V}$, we define for any real-valued φ with domain $[0,T]\times\mathbb{R}^d$ the function $\bar{\varphi}:[0,T]\times\mathcal{S}\mapsto\mathbb{R}$ by

$$\bar{\varphi}(t,x,y,z) := y\varphi(t,x) + z.$$

We also define Now set $\bar{x} := (x,y,z)$ and

$$\bar{X}_s^{t,\bar{x},\alpha} := \begin{pmatrix} X_s^{t,x,\alpha} \\ Y_s^{t,x,y,\alpha} \\ Z_s^{t,x,y,z,\alpha} \end{pmatrix}.$$

For any $(t, \bar{x}) \in [0, T) \times \mathcal{S}$, $(\alpha, \tau) \in \mathcal{A}_t \times \mathcal{T}_{t,T}$, introduce the function

$$J(t, \bar{x}; \alpha, \tau) := \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})].$$

Observe that $F(\bar{X}_\tau^{t,\bar{x},\alpha}) = z + yF(\bar{X}_\tau^{t,(x,1,0),\alpha})$; it follows that $J(t, \bar{x}; \alpha, \tau) = z + yJ(t, (x, 1, 0); \alpha, \tau)$.

Remark 2.3. *In the definition of $V(t, \cdot)$, we restrict to control processes and stopping times which are independent of $\mathcal{F}_t$ for some technical reasons, as can be seen in the proof of Lemma 3.1 below. It is, however, not restrictive under our model; namely, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, we demonstrate in Proposition 2.1 below that*

$$V(t, x) = \widetilde{V}(t, x),$$

where

$$\widetilde{V}(t, x) := \sup_{\alpha \in \mathcal{A}} \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau e^{-\int_t^s c(u, X_u^{t,x,\alpha})du} f(s, X_s^{t,x,\alpha}, \alpha_s)ds + e^{-\int_t^\tau c(u, X_u^{t,x,\alpha})du} g(X_\tau^{t,x,\alpha})\right]. \quad (2.10)$$

We first present a lemma that will be used in proving Proposition 2.1 and Theorem 4.1.

Lemma 2.1. *For any $(t, \bar{x}) \in [0, T] \times \mathcal{S}$, $\theta \in \mathcal{T}_{t,T}$ and $\alpha \in \mathcal{A}$, we have $\mathbb{P}$-a.s. that*

$$\operatorname*{essinf}_{\tau \in \mathcal{T}_{\theta,T}} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta] = \inf_{\tau \in \mathcal{T}_{\theta,T}} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta].$$

Proof. Let $\mathcal{T}_{\theta,T}^d$ be the stopping times in $\mathcal{T}_{\theta,T}$ that have values in the set of dyadic rationals. Then we have $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\left(\operatorname*{essinf}_{\tau \in \mathcal{T}_{\theta,T}} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta]\right)(\omega) \leq \left(\operatorname*{essinf}_{\tau \in \mathcal{T}_{\theta,T}^d} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta]\right)(\omega) = \inf_{\tau \in \mathcal{T}_{\theta,T}^d} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta](\omega). \quad (2.11)$$

Now we claim that

$$\inf_{\tau \in \mathcal{T}_{\theta(\omega),T}^d} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta](\omega) = \inf_{\tau \in \mathcal{T}_{\theta(\omega),T}} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta](\omega). \quad (2.12)$$

For any $\tau \in \mathcal{T}_{\theta,T}$, there exists a decreasing sequence of stopping times $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathcal{T}^d$ such that $\tau = \lim_{n \to \infty} \tau_n$. By the dominated convergence theorem, which we can apply thanks to (2.5), (2.4), and a calculation similar to (2.7), for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{n \to \infty} \mathbb{E}[F(\bar{X}_{\tau_n}^{t,\bar{x},\alpha}) | \mathcal{F}_\theta](\omega) = \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha}) | \mathcal{F}_\theta](\omega), \quad (2.13)$$

from which it follows that (2.12) is indeed true. The statement of the lemma is a consequence of (2.11) and (2.12). $\square$

Proposition 2.1. *For any $(t, x) \in [0, T] \times \mathbb{R}^d$, $V(t, x) = \widetilde{V}(t, x)$.*

Proof. Fix $\alpha \in \mathcal{A}_t$. It is obvious that

$$\inf_{\tau \in \mathcal{T}_{t,T}} J(t, (x, 1, 0); \alpha, \tau) \leq \inf_{\tau \in \mathcal{T}_{t,T}^t} J(t, (x, 1, 0); \alpha, \tau). \quad (2.14)$$

Take an arbitrary $\tau \in \mathcal{T}^t_{t,T}$. Observe that for any fix $(\omega_s)_{0\le s\le t}$, the map $\tau_{(\omega_s)_{0\le s\le t}} : (\omega_s - \omega_t)_{t\le s\le T} \mapsto \tau((\omega_s)_{0\le s\le t}, (\omega_s-\omega_t)_{t\le s\le T})$ is a stopping time independent of $\mathcal{F}_t$, thanks to the independence of increments of the Brownian motion and the compound Poisson process. It follows that

$$\begin{aligned} J(t,(x,1,0);\alpha,\tau) &= \mathbb{E}\left[\mathbb{E}[F(\bar{X}^{t,x,1,0,\alpha}_{\tau_{(\omega_s)_{0\le s\le t}}}) \mid \mathcal{F}_t]\right] = \int \mathbb{E}\left[F(\bar{X}^{t,x,1,0,\alpha}_{\tau_{(\omega_s)_{0\le s\le t}}})\right] d\mathbb{P}(\omega_s)_{0\le s\le t} \\ &\ge \inf_{\tau\in\mathcal{T}^t_{t,T}} J(t,(x,1,0);\alpha,\tau). \end{aligned} \tag{2.15}$$

This, together with (2.14), shows that

$$\inf_{\tau\in\mathcal{T}_{t,T}} J(t,(x,1,0);\alpha,\tau) = \inf_{\tau\in\mathcal{T}^t_{t,T}} J(t,(x,1,0);\alpha,\tau). \tag{2.16}$$

We can therefore conclude

$$\widetilde{V}(t,x) \ge \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}_{t,T}} J(t,(x,1,0);\alpha,\tau) = \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}^t_{t,T}} J(t,(x,1,0);\alpha,\tau) = V(t,x).$$

Now we want to show the opposite inequality. Fix $\bar{x}\in\mathcal{S}$. For any $\alpha\in\mathcal{A}$, thanks to Lemma D.1 in Appendix D of [11], we know that there exists a sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathcal{T}_{t,T}$ such that the sequence $\{\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})|\mathcal{F}_t]\}_{n\in\mathbb{N}}$ is nonincreasing and

$$\lim_{n\to\infty} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})|\mathcal{F}_t] = \operatorname*{essinf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})|\mathcal{F}_t].$$

(Although Lemma D.1 is for esssup, we can state the corresponding result for essinf). Note that by (2.5) and (2.4), $\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_1})|\mathcal{F}_t]$ is integrable. Therefore, by the dominated convergence theorem,

$$\begin{aligned} \inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})] &\le \lim_{n\to\infty} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})] = \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})|\mathcal{F}_t]] \\ &= \mathbb{E}[\lim_{n\to\infty} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})|\mathcal{F}_t]] = \mathbb{E}[\operatorname*{essinf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})|\mathcal{F}_t]] \\ &= \mathbb{E}[\inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})|\mathcal{F}_t]] = \int \inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[F(\bar{X}^{t,\bar{x},\alpha_{(\omega_s)_{0\le s\le t}}}_{\tau})\right] d\mathbb{P}(\omega_s)_{0\le s\le t} \\ &\le \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})], \end{aligned}$$

where the fourth equality comes from Lemma 2.1 and the last inequality is due to the fact that for any fix $(\omega_s)_{0\le s\le t}$, the map $\alpha_{(\omega_s)_{0\le s\le t}} : (\omega_s-\omega_t)_{t\le s\le T} \mapsto \alpha((\omega_s)_{0\le s\le t}, (\omega_s-\omega_t)_{t\le s\le T})$ is a control independent of $\mathcal{F}_t$, thanks again to the independence of increments of the Brownian motion and the compound Poisson process. Now by taking supremum over $\alpha\in\mathcal{A}$, we get

$$\sup_{\alpha\in\mathcal{A}} \inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})] \le \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})] \le \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}^t_{t,T}} \mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})].$$

Setting $\bar{x} = (x,1,0)$, we see that the above inequality yields $\widetilde{V}(t,x) \le V(t,x)$. □

3. Supersolution Property

In this section, with the aid of some results concerning RBSDEs, we are able to modify the arguments in [3] to show that the value function V_* is a viscosity supersolution to (2.8).

Lemma 3.1. *Fix $t \in [0,T]$. Then for any $\alpha \in \mathcal{A}_t$, the function*

$$W^\alpha(s,\bar{x}) := \inf_{\tau \in \mathcal{T}^s_{s,T}} J(s,\bar{x};\alpha,\tau)$$

is continuous on $[0,t] \times \mathcal{S}$.

Proof. Fix $t \in [0,T]$ and choose an arbitrary $\alpha \in \mathcal{A}_t$. For any $s \in [0,t]$ and $\bar{x} = (x,y,z) \in \mathcal{S}$, define the function $\tilde{f}^{(s,x)} : \Omega \times [s,T] \times \mathbb{R} \mapsto \mathbb{R}$ by

$$\tilde{f}^{(s,x)}(r,\mathfrak{y}) := f(r, X^{s,x,\alpha}_r, \alpha_r) - c(r, X^{s,x,\alpha}_r)\mathfrak{y}.$$

Moreover, set $\xi := g(X^{s,x,\alpha}_T)$ and $S_r := g(X^{s,x,\alpha}_r)$ for $r \in [s,T]$. Note that $c(r, X^{s,x,\alpha}_r)$ is a bounded process, and by (2.4) we have $f(\cdot, X^{s,x,\alpha}_{\cdot}, \alpha_{\cdot}) \in \mathbb{H}^2_{s,T}(\mathbb{R})$, $\xi \in \mathbb{L}^2$, and $\mathbb{E}[\sup_{r\in[s,T]} |S_r|^2] < \infty$. Now let $(\mathfrak{Y}^{s,x}_r, \mathfrak{Z}^{s,x}_r, \mathfrak{K}^{s,x}_r; s \le r \le T)$ be the unique solution to the RBSDE with RCLL obstacle (see e.g. [7], [8]) associated with the data $(\xi, \tilde{f}, S)$. Following the arguments in Proposition 3.5 of [5] with the help from equation (12) in [8], we have

$$\mathfrak{Y}^{s,x}_r = \operatorname*{essinf}_{\tau \in \mathcal{T}_{r,T}} \mathbb{E}\left[\int_r^\tau e^{-\int_r^l c(u,X^{s,x,\alpha}_u)du} f(l, X^{s,x,\alpha}_l, \alpha_l) dl + e^{-\int_r^\tau c(u,X^{s,x,\alpha}_u)du} g(X^{s,x,\alpha}_\tau) \middle| \mathcal{F}_r \right].$$

(Although the results in [5] and [8] are stated for RBSDEs that characterize a process $\mathfrak{Y}^{s,x}_{\cdot}$ bounded below by the obstacle $S_{\cdot}$, we can state analogous results in the case where $\mathfrak{Y}^{s,x}_{\cdot}$ is bounded above, not below, by $S_{\cdot}$). Now we claim that for all $s \in [0,t]$, $\mathfrak{Y}^{s,x}_s$ is deterministic and equals $W^\alpha(s,(x,1,0))$. Noting that $\alpha \in \mathcal{A}_s$ for all $s \in [0,t]$, we get

$$\mathfrak{Y}^{s,x}_s \le \operatorname*{essinf}_{\tau \in \mathcal{T}^s_{s,T}} \mathbb{E}[F(\bar{X}^{s,x,1,0,\alpha}_\tau) \mid \mathcal{F}_s] = \inf_{\tau \in \mathcal{T}^s_{s,T}} \mathbb{E}[F(\bar{X}^{s,x,1,0,\alpha}_\tau)] = W^\alpha(s,(x,1,0))..$$

Since the opposite inequality follows from calculations similar to (2.15), the claim is proved.

Note that $\tilde{f}^{(s,x)}$ and g satisfy (20), (21) and (22) in [4] and that the calculation in Proposition 3.6 of [4] still holds in our case with RCLL obstacle. We can therefore proceed as in Lemma 8.4 of [4] and conclude that $\mathfrak{Y}^{s,x}_s = W^\alpha(s,(x,1,0))$ is continuous on $[0,t] \times \mathbb{R}^d$. Finally, observing that

$$W^\alpha(s,\bar{x}) = z + yW^\alpha(s,(x,1,0)),$$

we conclude that $W^\alpha(s,\bar{x})$ is continuous on $[0,t] \times \mathcal{S}$. □

Now, we want to modify the arguments in [3] to get the following result, which is the key to proving the supersolution property of V.

Proposition 3.1. *Fix $(t,\bar{x}) \in [0,T] \times \mathcal{S}$ and $\varepsilon > 0$. Take arbitrary $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}^t_{t,T}$ and $\varphi \in USC([0,T] \times \mathbb{R}^d)$ with $\varphi \le V$. We have the following:*

(i) $\mathbb{E}[\bar{\varphi}^+(\theta, \bar{X}^{t,\bar{x},\alpha}_\theta)] < \infty$*;*

(ii) *If, moreover, $\mathbb{E}[\bar{\varphi}^-(\theta, \bar{X}^{t,\bar{x},\alpha}_\theta)] < \infty$, then there exists $\alpha^* \in \mathcal{A}_t$ with $\alpha^*_s = \alpha_s$ for $s \in [t,\theta]$ such that*

$$\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha^*}_\tau)] \ge \mathbb{E}[Y^{t,x,y,\alpha}_{\tau\wedge\theta} \varphi(\tau\wedge\theta, X^{t,x,\alpha}_{\tau\wedge\theta}) + Z^{t,x,y,z,\alpha}_{\tau\wedge\theta}] - 4\varepsilon, \tag{3.1}$$

for any $\tau \in \mathcal{T}^t_{t,T}$.

Proof. (i) First, observe that for any $\bar{x} = (x, y, z) \in \mathcal{S}$, $\bar{\varphi}(t, \bar{x}) = y\varphi(t, x) + z \leq yV(t, x) + z \leq yg(x) + z$, which implies $\bar{\varphi}^+(t, \bar{x}) \leq yg(x) + z$. It follows that

$$\begin{aligned}\bar{\varphi}^+(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) &\leq Y_\theta^{t,x,y,\alpha} g(X_\theta^{t,x,\alpha}) + Z_\theta^{t,x,y,z,\alpha} \\ &\leq Y_\theta^{t,x,y,\alpha} g(X_\theta^{t,x,\alpha}) + z + \int_t^\theta Y_s^{t,x,y,\alpha} f(s, X_s^{t,x,\alpha}, \alpha_s) ds,\end{aligned}$$

the right-hand-side is integrable as a result of (2.6).

(ii) For each $(s, \eta) \in [0, T] \times \mathcal{S}$, by the definition of $\bar{V}$, there exists $\alpha^{(s,\eta),\varepsilon} \in \mathcal{A}_s$ such that

$$\inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \eta; \alpha^{(s,\eta),\varepsilon}, \tau) \geq \bar{V}(s, \eta) - \varepsilon. \tag{3.2}$$

Note that $\varphi \in USC([0, T] \times \mathbb{R}^d)$ implies $\bar{\varphi} \in USC([0, T] \times \mathcal{S})$. Then by the upper semicontinuity of $\bar{\varphi}$ on $[0, T] \times \mathcal{S}$ and the lower semicontinuity of $W^{\alpha^{(s,\eta),\varepsilon}}$ on $[0, s] \times \mathcal{S}$ (from Lemma 3.1), there must exist an $r^{(s,\eta)} > 0$ such that

$$\bar{\varphi}(t', x') - \bar{\varphi}(s, \eta) \leq \varepsilon \text{ and } W^{\alpha^{(s,\eta),\varepsilon}}(s, \eta) - W^{\alpha^{(s,\eta),\varepsilon}}(t', x') \leq \varepsilon,$$

for any $(t', x') \in B(s, \eta; r^{(s,\eta)}) := \{(t', x') \in [0, T] \times \mathcal{S} \mid t' \in (s - r^{(s,\eta)}, s), |x' - \eta| < r^{(s,\eta)}\}$. It follows that if $(t', x') \in B(s, \eta; r^{(s,\eta)})$, we have

$$W^{\alpha^{(s,\eta),\varepsilon}}(t', x') \geq W^{\alpha^{(s,\eta),\varepsilon}}(s, \eta) - \varepsilon \geq \bar{V}(s, \eta) - 2\varepsilon \geq \bar{\varphi}(s, \eta) - 2\varepsilon \geq \bar{\varphi}(t', x') - 3\varepsilon,$$

where the second inequality is due to (3.2).

Note that $\{B(s, \eta; r) \mid (s, \eta) \in [0, T] \times \mathcal{S}, 0 \leq r \leq r^{(s,\eta)}\}$ forms an open covering of $[0, T) \times \mathcal{S}$. Then by the Lindelöf covering theorem (see e.g. Theorem 3.28 in [1]), there exists a countable subcovering $\{B(t_i, x_i; r_i)\}_{i \in \mathbb{N}}$ of $[0, T) \times \mathcal{S}$. Now set $A_0 := \{T\} \times \mathcal{S}$, $C_{-1} := \emptyset$ and define for all $i \in \mathbb{N} \cup \{0\}$

$$A_{i+1} := B(t_{i+1}, x_{i+1}; r_{i+1}) \setminus C_i, \text{ where } C_i := C_{i-1} \cup A_i.$$

Under this construction, we have

$$[0, T] \times \mathcal{S} \subseteq \cup_{i \in \mathbb{N} \cup \{0\}} A_i,\ A_i \cap A_j = \emptyset \text{ for } i \neq j, \text{ and } W^{\alpha^{i,\varepsilon}}(t', x') \geq \bar{\varphi}(t', x') - 3\varepsilon \text{ for } (t', x') \in A_i, \tag{3.3}$$

where $\alpha^{i,\varepsilon} := \alpha^{(t_i,x_i),\varepsilon}$.

For any $n \in \mathbb{N}$, set $A^n := \cup_{0 \leq i \leq n} A_i$ and define

$$\alpha_s^{\varepsilon,n} := 1_{[t,\theta]}(s)\alpha_s + 1_{(\theta,T]}(s) \left(\alpha_s 1_{(A^n)^c}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) + \sum_{i=0}^n 1_{A_i}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) \alpha_s^{i,\varepsilon} \right) \in \mathcal{A}_t.$$

Note that $\alpha_s^{\varepsilon,n} = \alpha_s$ for $s \in [t, \theta]$. Then for any $\tau \in \mathcal{T}_{t,T}^t$,

$$\begin{aligned}\mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha^{\varepsilon,n}}) 1_{\{\tau \geq \theta\}} | \mathcal{F}_\theta] 1_{A^n}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) &= 1_{\{\tau \geq \theta\}} \sum_{i=0}^n J(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}; \alpha^{i,\varepsilon}, \tau) 1_{A_i}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) \\ &\geq 1_{\{\tau \geq \theta\}} \sum_{i=0}^n W^{\alpha^{i,\varepsilon}}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) 1_{A_i}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) \\ &\geq 1_{\{\tau \geq \theta\}} [\bar{\varphi}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}) - 3\varepsilon] 1_{A^n}(\theta, \bar{X}_\theta^{t,\bar{x},\alpha}),\end{aligned} \tag{3.4}$$

where the last inequality follows from the last part of (3.3). Thus, we have

$$\begin{aligned}
\mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha^{\varepsilon,n}})] &= \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})1_{\{\tau<\theta\}}] + \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha^{\varepsilon,n}})1_{\{\tau\geq\theta\}}] \\
&= \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})1_{\{\tau<\theta\}}] + \mathbb{E}[\mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha^{\varepsilon,n}})1_{\{\tau\geq\theta\}}|\mathcal{F}_\theta]1_{A^n}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] \\
&+ \mathbb{E}[\mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha^{\varepsilon,n}})1_{\{\tau\geq\theta\}}|\mathcal{F}_\theta]1_{(A^n)^c}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] \\
&\geq \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})1_{\{\tau<\theta\}}] + \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^n}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] - 3\varepsilon \\
&\geq \mathbb{E}[1_{\{\tau<\theta\}}\bar{\varphi}(\tau,\bar{X}_\tau^{t,\bar{x},\alpha})] + \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^n}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] - 3\varepsilon, \qquad (3.5)
\end{aligned}$$

where the first inequality comes from (3.4), and the second inequality is due to the observation that

$$\begin{aligned}
F(\bar{X}_\tau^{t,\bar{x},\alpha}) &= Y_\tau^{t,x,y,\alpha}g(X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha} \geq Y_\tau^{t,x,y,\alpha}V(\tau,X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha} \\
&\geq Y_\tau^{t,x,y,\alpha}\varphi(\tau,X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha}.
\end{aligned}$$

Since $\mathbb{E}[\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] < \infty$, thanks to the first statement of this proposition, there exists $n^* \in \mathbb{N}$ such that

$$\mathbb{E}[\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] - \mathbb{E}[\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^{n^*}}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] < \varepsilon.$$

We observe the following holds for any $\tau \in \mathcal{T}_{t,T}^t$

$$\begin{aligned}
&\mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] - \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^{n^*}}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] \\
&\qquad \leq \mathbb{E}[\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] - \mathbb{E}[\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^{n^*}}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] < \varepsilon.
\end{aligned} \qquad (3.6)$$

Suppose $\mathbb{E}[\bar{\varphi}^-(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] < \infty$, then we can conclude from (3.6) that for any $\tau \in \mathcal{T}_{t,T}^t$

$$\begin{aligned}
\mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] &= \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] - \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}^-(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] \\
&\leq \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}^+(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^{n^*}}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] + \varepsilon - \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}^-(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^{n^*}}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] \\
&= \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})1_{A^{n^*}}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] + \varepsilon. \qquad (3.7)
\end{aligned}$$

Taking $\alpha^* = \alpha^{\varepsilon,n^*}$, we now conclude from (3.5) and (3.7) that

$$\begin{aligned}
\mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha^*})] &\geq \mathbb{E}[1_{\{\tau<\theta\}}\bar{\varphi}(\theta,\bar{X}_\tau^{t,\bar{x},\alpha})] + \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\bar{X}_\theta^{t,\bar{x},\alpha})] - 4\varepsilon \\
&= \mathbb{E}[\bar{\varphi}(\tau\wedge\theta,\bar{X}_{\tau\wedge\theta}^{t,\bar{x},\alpha})] - 4\varepsilon \\
&= \mathbb{E}[Y_{\tau\wedge\theta}^{t,x,y,\alpha}\varphi(\tau\wedge\theta,X_{\tau\wedge\theta}^{t,x,\alpha}) + Z_{\tau\wedge\theta}^{t,x,y,z,\alpha}] - 4\varepsilon.
\end{aligned}$$

□

We are ready to present the main result in this section.

Proposition 3.2. *The value function V_* defined in* (2.2) *is a viscosity supersolution of the HJB equation* (2.8).

Proof. Let $h \in C^{1,2}([0,T)\times\mathbb{R}^d)$ be such that

$$0 = (V_* - h)(t_0,x_0) < (V_* - h)(t,x), \text{ for any } (t,x) \in [0,T)\times\mathbb{R}^d, \ (t,x) \neq (t_0,x_0),$$

for some $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$. If $V(t_0,x_0)=g(x_0)$, then there is nothing to prove. We, therefore, assume that $V(t_0,x_0)<g(x_0)$. For such (t_0,x_0) it is enough to prove the following inequality:

$$0\le c(t_0,x_0)h(t_0,x_0)-\frac{\partial h}{\partial t}(t_0,x_0)+H(\cdot,D_xh,D_x^2h)(t_0,x_0). \tag{3.8}$$

Assume the contrary. Then there must exist $\zeta_0\in\mathbb{R}^m$ such that

$$0> c(t_0,x_0)h(t_0,x_0)-\frac{\partial h}{\partial t}(t_0,x_0)+H^{\zeta_0}(\cdot,D_xh,D_x^2h)(t_0,x_0). \tag{3.9}$$

Define the function $\tilde h$ by

$$\tilde h(t,x):=h(t,x)-|t-t_0|^2-|x-x_0|^4.$$

Note that $(\tilde h,\partial_t\tilde h,D_x\tilde h,D_x^2\tilde h)(x_0,t_0)=(h,\partial_t h,D_xh,D_x^2h)(x_0,t_0)$. We can choose $r>0$ with $t_0+r<T$ such that

$$0> c(t,x)\tilde h(t,x)-\frac{\partial \tilde h}{\partial t}(t,x)+H^{\zeta_0}(\cdot,D_x\tilde h,D_x^2\tilde h)(t_0,x_0),\ \text{for all } (t,x)\in B_r(t_0,x_0). \tag{3.10}$$

Define $\zeta\in\mathcal{A}$ by setting $\zeta_t=\zeta_0$ for all $t\ge 0$. Let (t_n,x_n) be a sequence in $B_r(t_0,x_0)$ such that $(t_n,x_n,V(t_n,x_n))\to(t_0,x_0,V_*(t_0,x_0))$, and introduce the stopping time

$$\theta_n:=\inf\left\{s\in[t_n,T]\ \middle|\ (s,X_s^{t_n,x_n,\zeta})\notin B_r(t_0,x_0)\right\}\in\mathcal{T}_{t_n,T}^{t_n}.$$

Note that we have $\theta_n\in\mathcal{T}_{t_n,T}^{t_n}$ because the control ζ is by definition independent of $\mathcal{F}_{t_n}$. Observe that $h\ge\tilde h+\eta$ on $([0,T]\times\mathbb{R}^d)\setminus B_{r/2}(t_0,x_0)$ for some $\eta>0$. Then by applying the product rule of stochastic calculus to $Y_s^{t_n,x_n,1,\zeta}\tilde h(s,X_s^{t_n,x_n,\zeta})$ and recalling (3.10) and $c\le\bar c$, we obtain that

$$\begin{aligned}\tilde h(t_n,x_n)=\mathbb{E}\Bigg[&Y_{\theta_n\wedge\tau}^{t_n,x_n,1,\zeta}\tilde h(\theta_n\wedge\tau,X_{\theta_n\wedge\tau}^{t_n,x_n,\zeta})\\&+\int_{t_n}^{\theta_n\wedge\tau}Y_s^{t_n,x_n,1,\zeta}\left(c\tilde h-\frac{\partial\tilde h}{\partial t}+H^{\zeta_0}(\cdot,D_x\tilde h,D_x^2\tilde h)+f\right)(s,X_s^{t_n,x_n,\zeta},\zeta_0)ds\Bigg]\\<\mathbb{E}\Bigg[&Y_{\theta_n\wedge\tau}^{t_n,x_n,1,\zeta}h(\theta_n\wedge\tau,X_{\theta_n\wedge\tau}^{t_n,x_n,\zeta})+\int_{t_n}^{\theta_n\wedge\tau}Y_s^{t_n,x_n,1,\zeta}f(s,X_s^{t_n,x_n,\zeta},\zeta_0)ds\Bigg]-e^{-\bar cT}\eta,\end{aligned} \tag{3.11}$$

for any $\tau\in\mathcal{T}_{t_n,T}$. Note that by construction, $(\tilde h-V)(t_n,x_n)\to 0$ as $n\to\infty$. This implies that we can find an $\hat n\in\mathbb{N}$ large enough such that

$$V(t_{\hat n},x_{\hat n})<\mathbb{E}\left[Y_{\theta_{\hat n}\wedge\tau}^{t_{\hat n},x_{\hat n},1,\zeta}h(\theta_{\hat n}\wedge\tau,X_{\theta_{\hat n}\wedge\tau}^{t_{\hat n},x_{\hat n},\zeta})+\int_{t_{\hat n}}^{\theta_{\hat n}\wedge\tau}Y_s^{t_{\hat n},x_{\hat n},1,\zeta}f(s,X_s^{t_{\hat n},x_{\hat n},\zeta},\zeta_0)ds\right]-\frac{e^{-\bar cT}\eta}{2}, \tag{3.12}$$

for any $\tau\in\mathcal{T}_{t_{\hat n},T}$. Let

$$\bar h(\theta_{\hat n},\bar X_{\theta_{\hat n}}^{t_{\hat n},x_{\hat n},1,0,\zeta}):=Y_{\theta_{\hat n}}^{t_{\hat n},x_{\hat n},1,\zeta}h(\theta_{\hat n},X_{\theta_{\hat n}}^{t_{\hat n},x_{\hat n},\zeta})+\int_{t_{\hat n}}^{\theta_{\hat n}}Y_s^{t_{\hat n},x_{\hat n},1,\zeta}f(s,X_s^{t_{\hat n},x_{\hat n},\zeta},\zeta_0)ds.$$

Note from (3.11) that $\mathbb{E}[\bar h(\theta_{\hat n},\bar X_{\theta_{\hat n}}^{t_{\hat n},x_{\hat n},1,0,\zeta})]$ is bounded from below. It follows from this fact that $\mathbb{E}[\bar h^-(\theta_{\hat n},\bar X_{\theta_{\hat n}}^{t_{\hat n},x_{\hat n},1,0,\zeta})]<\infty$ since we already have $\mathbb{E}[\bar h^+(\theta_{\hat n},\bar X_{\theta_{\hat n}}^{t_{\hat n},x_{\hat n},1,0,\zeta})]<\infty$ from Proposition 3.1(i).

We can therefore apply Proposition 3.1(ii) and conclude that there exists an $\alpha^* \in \mathcal{A}_{t_{\hat{n}}}$ such that

$$\mathbb{E}[F(\bar{X}_\tau^{t_{\hat{n}},x_{\hat{n}},1,0,\alpha^*})] \geq \mathbb{E}\left[Y_{\theta_{\hat{n}}\wedge\tau}^{t_{\hat{n}},x_{\hat{n}},1,\zeta} h(\theta_{\hat{n}}\wedge\tau, X_{\theta_{\hat{n}}\wedge\tau}^{t_{\hat{n}},x_{\hat{n}},\zeta}) + \int_{t_{\hat{n}}}^{\theta_{\hat{n}}\wedge\tau} Y_s^{t_{\hat{n}},x_{\hat{n}},1,\zeta} f(s, X_s^{t_{\hat{n}},x_{\hat{n}},\zeta}, \zeta_0)ds\right] - \frac{e^{-\bar{c}T}\eta}{4}, \tag{3.13}$$

for any $\tau \in \mathcal{T}_{t_{\hat{n}},T}^{t_{\hat{n}}}$. Next, observe that

$$\begin{aligned} V(t_{\hat{n}}, x_{\hat{n}}) &\geq \inf_{\tau\in\mathcal{T}_{t_{\hat{n}},T}^{t_{\hat{n}}}} \mathbb{E}\left[Y_\tau^{t_{\hat{n}},x_{\hat{n}},1,\alpha^*} g(\tau, X_\tau^{t_{\hat{n}},x_{\hat{n}},\alpha^*}) + \int_{t_{\hat{n}}}^{\tau} Y_s^{t_{\hat{n}},x_{\hat{n}},1,\alpha^*} f(s, X_s^{t_{\hat{n}},x_{\hat{n}},\alpha^*}, \alpha_s^*)ds\right] \\ &\geq \mathbb{E}\left[Y_{\hat{\tau}}^{t_{\hat{n}},x_{\hat{n}},1,\alpha^*} g(\hat{\tau}, X_{\hat{\tau}}^{t_{\hat{n}},x_{\hat{n}},\alpha^*}) + \int_{t_{\hat{n}}}^{\hat{\tau}} Y_s^{t_{\hat{n}},x_{\hat{n}},1,\alpha^*} f(s, X_s^{t_{\hat{n}},x_{\hat{n}},\alpha^*}, \alpha_s^*)ds\right] - \frac{e^{-\bar{c}T}\eta}{4}, \end{aligned} \tag{3.14}$$

for some $\hat{\tau} \in \mathcal{T}_{t_{\hat{n}},T}^{t_{\hat{n}}}$. Then we obtain from (3.13) and (3.14) that

$$V(t_{\hat{n}}, x_{\hat{n}}) \geq \mathbb{E}\left[Y_{\theta_{\hat{n}}\wedge\hat{\tau}}^{t_{\hat{n}},x_{\hat{n}},1,\zeta} h(\theta_{\hat{n}}\wedge\hat{\tau}, X_{\theta_{\hat{n}}\wedge\hat{\tau}}^{t_{\hat{n}},x_{\hat{n}},\zeta}) + \int_{t_{\hat{n}}}^{\theta_{\hat{n}}\wedge\hat{\tau}} Y_s^{t_{\hat{n}},x_{\hat{n}},1,\zeta} f(s, X_s^{t_{\hat{n}},x_{\hat{n}},\zeta}, \zeta_0)ds\right] - \frac{e^{-\bar{c}T}\eta}{2},$$

which contradicts (3.12). □

4. Subsolution Property

In this section, we will first derive a weak dynamic programming principle, which corresponds to the first statement in Theorem 3.1 in [3], for our value function V. Then we will show that the subsolution property of V^* follows from this weak dynamic programming principle.

Theorem 4.1. *For all $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\theta \in \mathcal{T}_{t,T}^t$, we have*

$$V(t,x) \leq \sup_{\alpha\in\mathcal{A}_t} \mathbb{E}[Y_\theta^{t,x,1,\alpha} V^*(\theta, X_\theta^{t,x,\alpha}) + Z_\theta^{t,x,1,0,\alpha}].$$

Proof. Fix $(t,x) \in [0,T) \times \mathbb{R}^d$. For any $\omega \in \Omega$ and $r \geq t$, set $\omega_\cdot^r := \omega_{\cdot\wedge r}$ and $\mathbf{T}_r(\omega)(\cdot) := \omega_{\cdot\vee r} - \omega_r$ so that $\omega_\cdot = \omega_\cdot^r + \mathbf{T}_r(\omega)(\cdot)$. Also, for any $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^t$, set $\tilde{\alpha}_\omega(\tilde{\omega}) := \alpha(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde{\omega})) \in \mathcal{A}_{\theta(\omega)}$. Then by the same calculation in Proposition 5.1 in [3], for any $\tau \in \mathcal{T}_{\theta,T}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$\mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})|\mathcal{F}_\theta](\omega) = J(\theta(\omega), \bar{X}_{\theta(\omega)}^{t,\bar{x},\alpha}; \tilde{\alpha}_\omega, \tau).$$

Then from Lemma 2.1, for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\begin{aligned} \left(\operatorname*{essinf}_{\tau\in\mathcal{T}_{\theta,T}} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})|\mathcal{F}_\theta]\right)(\omega) &= \inf_{\tau\in\mathcal{T}_{\theta,T}} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})|\mathcal{F}_\theta](\omega) \leq \inf_{\tau\in\mathcal{T}_{\theta,T}^\theta} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})|\mathcal{F}_\theta](\omega) \\ &= \inf_{\tau\in\mathcal{T}_{\theta,T}^\theta} J(\theta(\omega), \bar{X}_{\theta(\omega)}^{t,\bar{x},\alpha}; \tilde{\alpha}_\omega, \tau) \leq \bar{V}(\theta(\omega), \bar{X}_{\theta(\omega)}^{t,\bar{x},\alpha}) \\ &\leq \bar{V}^*(\theta(\omega), \bar{X}_{\theta(\omega)}^{t,\bar{x},\alpha}).. \end{aligned} \tag{4.1}$$

Applying Lemma D.1 in Appendix D of [11] (as what we did in the proof of Proposition 2.1), we know that there exists a sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathcal{T}_{\theta,T}$ such that the sequence $\{\mathbb{E}[F(\bar{X}_{\tau_n}^{t,\bar{x},\alpha})|\mathcal{F}_\theta]\}_{n\in\mathbb{N}}$ is nonincreasing and

$$\lim_{n\to\infty} \mathbb{E}[F(\bar{X}_{\tau_n}^{t,\bar{x},\alpha})|\mathcal{F}_\theta] = \operatorname*{essinf}_{\tau\in\mathcal{T}_{\theta,T}} \mathbb{E}[F(\bar{X}_\tau^{t,\bar{x},\alpha})|\mathcal{F}_\theta].$$

Note that by (2.5) and (2.4), $\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_1})|\mathcal{F}_\theta]$ is integrable. Thus, by the dominated convergence theorem and (4.1), we get

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})] &= \lim_{n\to\infty}\mathbb{E}[\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})|\mathcal{F}_\theta]] = \mathbb{E}[\lim_{n\to\infty}\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})|\mathcal{F}_\theta]] = \mathbb{E}[\operatorname*{essinf}_{\tau\in\mathcal{T}_{\theta,T}}\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})|\mathcal{F}_\theta]] \\ &\leq \mathbb{E}[\bar{V}^*(\theta,\bar{X}^{t,\bar{x},\alpha}_\theta)].\end{aligned}$$

We, therefore, conclude that

$$\inf_{\tau\in\mathcal{T}^t_{t,T}}\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})] = \inf_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})] \leq \inf_{\tau\in\mathcal{T}_{\theta,T}}\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau})] \leq \lim_{n\to\infty}\mathbb{E}[F(\bar{X}^{t,\bar{x},\alpha}_{\tau_n})] \leq \mathbb{E}[\bar{V}^*(\theta,\bar{X}^{t,\bar{x},\alpha}_\theta)],$$

where the first equality is due to (2.16) as a consequence of $\alpha\in\mathcal{A}_t$. Taking supremum over all $\alpha\in\mathcal{A}_t$, we get

$$\bar{V}(t,x,y,z) \leq \sup_{\alpha\in\mathcal{A}_t}\mathbb{E}[\bar{V}^*(\theta,\bar{X}^{t,\bar{x},\alpha}_\theta)] = \sup_{\alpha\in\mathcal{A}_t}\mathbb{E}[Y^{t,x,y,\alpha}_\theta V^*(\theta,X^{t,x,\alpha}_\theta) + Z^{t,x,y,z,\alpha}_\theta].$$

By taking $y=1, z=0$, we get the desired result. □

Proposition 4.1. *The value function V^* defined in* (2.2) *is a viscosity subsolution of the HJB equation*

$$\max\left\{c(t,x)v - \frac{\partial v}{\partial t} + H_*(t,x,D_x v, D_{xx}v), v - g(x)\right\} = 0.$$

Proof. Assume the contrary that there exist $h\in C^{1,2}([0,T)\times\mathbb{R}^d)$ and $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$ satisfying

$$0 = (V^*-h)(t_0,x_0) > (V^*-h)(t,x), \text{ for any } (t,x)\in[0,T)\times\mathbb{R}^d, \ (t,x)\neq(t_0,x_0),$$

such that

$$\max\left\{c(t_0,x_0)h(t_0,x_0) - \frac{\partial h}{\partial t}(t_0,x_0) + H_*(\cdot,D_x h,D_{xx}h)(t_0,x_0), h(t_0,x_0)-g(x_0)\right\} > 0. \quad (4.2)$$

Since $V^*(t_0,x_0) = h(t_0,x_0)$ and $V\leq g$ by definition, continuity of g implies that $h(t_0,x_0) = V^*(t_0,x_0)\leq g(x_0)$. Therefore, we can conclude from (4.2) that

$$c(t_0,x_0)h(t_0,x_0) - \frac{\partial h}{\partial t}(t_0,x_0) + H_*(\cdot,D_x h,D_{xx}h)(t_0,x_0) > 0$$

By the lower-semicontinuity of H_*, there exists $r>0$ with $t_0+r<T$ such that

$$c(t,x)h(t,x) - \frac{\partial h}{\partial t}(t,x) + H^\alpha(\cdot,D_x h,D_{xx}h)(t,x) > 0, \ \forall\alpha\in\mathbb{R}^m \text{ and } (t,x)\in B_r(t_0,x_0). \quad (4.3)$$

Now define $\eta>0$ by

$$-2\eta e^{\bar{c}T} := \max_{\partial B_r(t_0,x_0)}(V^*-h) < 0. \quad (4.4)$$

(One may need to modify h, so that (t_0,x_0) is a strict maximum of V^*-h, to obtain the strict inequality above.) Take $(\hat{t},\hat{x})\in B_r(t_0,x_0)$ such that $|(V-h)(\hat{t},\hat{x})|<\eta$. For any $\alpha\in\mathcal{A}_{\hat{t}}$, define the stopping time

$$\theta := \inf\left\{s\geq\hat{t} \mid (s,X^{\hat{t},\hat{x},\alpha}_s)\notin B_r(t_0,x_0)\right\}\in\mathcal{T}^{\hat{t}}_{\hat{t},\hat{x}}.$$

Note that we have $\theta \in \mathcal{T}^{\hat{t}}_{\hat{t},\hat{x}}$ because the control α is independent of $\mathcal{F}_{\hat{t}}$. Applying the product rule of stochastic calculus to $Y_s^{\hat{t},\hat{x},1,\alpha} h(s, X_s^{\hat{t},\hat{x},\alpha})$, we get

$$\begin{aligned} h(\hat{t},\hat{x}) &= \mathbb{E}\left[Y_\theta^{\hat{t},\hat{x},1,\alpha} h(\theta, X_\theta^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta} Y_s^{\hat{t},\hat{x},1,\alpha}\left(ch - \frac{\partial h}{\partial t} + H^\alpha(\cdot, D_x h, D_{xx} h) + f\right)(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s) ds\right] \\ &\geq \mathbb{E}\left[Y_\theta^{\hat{t},\hat{x},1,\alpha} V^*(\theta, X_\theta^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s) ds\right] + 2\eta, \end{aligned}$$

where the inequality follows from (4.4), (4.3) and $c \leq \bar{c}$. Finally, by our choice of $(\hat{t},\hat{x})$, we have $V(\hat{t},\hat{x}) + \eta > h(\hat{t},\hat{x})$. It follows that

$$V(\hat{t},\hat{x}) \geq \mathbb{E}\left[Y_\theta^{\hat{t},\hat{x},1,\alpha} V^*(\theta, X_\theta^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s) ds\right] + \eta.$$

Since $\alpha \in \mathcal{A}_{\hat{t}}$ is arbitrary, this inequality contradicts Theorem 4.1. □

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