UHLENBECK-DONALDSON COMPACTIFICATION FOR FRAMED SHEAVES ON PROJECTIVE SURFACES

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ABSTRACT. We construct a compactification $M^{\mu ss}$ of the Uhlenbeck-Donaldson type for the moduli space of slope stable framed bundles. This is a kind of a moduli space of slope semistable framed sheaves. We show that there exists a projective morphism $\gamma\colon M^s \to M^{\mu ss}$, where M^s is the moduli space of S-equivalence classes of Gieseker-semistable framed sheaves. The space $M^{\mu ss}$ has a natural set-theoretic stratification which allows one, via a Hitchin-Kobayashi correspondence, to compare it with the moduli spaces of framed ideal instantons.

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1. Introduction

Let X be a smooth complex projective surface, and let $M^\mu(c)$, with $c \in H^\bullet(X, \mathbb{Q})$, be the moduli space of slope stable locally free coherent $\mathcal{O}_X$-modules, having Chern character c. One can obtain a compactification of $M^\mu(c)$ by taking closure in the Gieseker-Maruyama moduli space space $M^{ss}(c)$, formed by S-equivalence classes of Gieseker-semistable coherent $\mathcal{O}_X$-modules. On the other hand, by the so-called Hitchin-Kobayashi correspondence [9], $M^\mu(c)$ may be regarded as a moduli space of bundles carrying a Hermitian-Yang-Mills metric; as such, it admits a differential-geometric compactification, called the Uhlenbeck-Donaldson compactification $N(c)$, which is obtained by adding to $M^\mu(c)$ points corresponding to "ideal" (degenerated) Hermitian-Yang-Mills bundles. In a 1993 paper [8], Jun Li showed that $N(c)$ may be given a structure of scheme over $\mathbb{C}$, and constructed a morphism $M^{ss}(c) \to N(c)$, which on $M^\mu(c)$ restricts to an isomorphism. With that scheme structure, $N(c)$ may be regarded as a sort of moduli space of slope semistable sheaves, under an identification which is somehow stronger than S-equivalence [7].

In this paper we consider pairs formed by a bundle on a smooth polarized projective surface, together with a framing. A notion of stability exists for such objects, and one can construct corresponding moduli spaces [5, 6]. The main result of this paper is the construction of an Uhlenbeck-Donaldson compactification for the slope-stable part of this moduli space. This is accomplished by following rather closely the construction of the Uhlenbeck-Donaldson compactification of the moduli space of (unframed) vector bundles, as done, e.g., in [7]. A first key ingredient is, as always, a boundedness result for the family $\mathcal{S}^{\mu ss}(c)$ of semistable framed sheaves on X (Proposition 3.1) with Chern character c. After introducing an appropriate Quot scheme, this family is realized as a locally closed subset $R^{\mu ss}(c)$ in the Quot scheme, and a suitable semiample line bundle on $R^{\mu ss}(c)$ is picked out. The moduli scheme $M^{ss}(c)$ cannot be defined as a geometric quotient, hence it is defined in an *ad hoc* way, cf. Definition 4.4. The Jordan-Hölder filtration allows one to introduce a set-theoretic stratification in the space $\mathcal{S}^{\mu ss}(c)$.

Let X be a smooth projective surface, D a divisor on X satisfying some numerical conditions, and $\mathcal{F}$ a rank r vector bundle on D, which is semistable or satisfies a slightly more general stability condition. The following property was proved in [1]: given a torsion-free rank r sheaf $\mathcal{E}$ on X and an isomorphism $\phi\colon \mathcal{E}|D \to \mathcal{F}$, one can choose a polarization H in X and a stability condition for framed sheaves in such a way that the pair $(\mathcal{E}, \phi)$ is stable in

Huybrechts-Lehn's sense. Moreover, the choice of the polarization and that of the stability condition only depend on the pair $(D, \mathcal{F})$ and on the Chern character of $\mathcal{E}$. This means that the moduli space of such pairs embeds into a moduli space of stable pairs, and therefore we can restrict the Uhlenbeck-Donaldson compactification to it. Via the Hitchin-Kobayashi correspondence, this allows one to look at $M^{\mu ss}(c)$ as a quasi-projective scheme structure on the moduli space of ideal instantons. For framed moduli spaces, a quasi-projective Uhlenbeck-Donaldson type compactification has been previously known only in the case of $\mathbb{P}^2$. It was constructed by Nakajima in [12] by completely different techniques, using ADHM construction and hyperkähler quotients.

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2. A Quot scheme for framed sheaves

Let X be a smooth d-dimensional projective variety over an algebraically closed field $\Bbbk$ of characteristic zero, H an ample class on it, $\mathcal{F}$ a coherent sheaf on X, $c \in K(X)_{\mathrm{num}}$ a numerical K-theory class, P_c the corresponding Hilbert polynomial. We shall consider pairs $(\mathcal{E}, [\phi])$, where $\mathcal{E}$ is a coherent sheaf on X with Hilbert polynomial $P_{\mathcal{E}} = P_c$, and $[\phi] \in \mathbb{P}(\mathrm{Hom}(\mathcal{E}, \mathcal{F}))$ is the proportionality class of nonzero sheaf morphism $\phi\colon \mathcal{E} \to \mathcal{F}$. We call each such pair $(\mathcal{E}, [\phi])$ a *framed sheaf*. Later on, to simplify notation, we shall write a framed sheaf as $(\mathcal{E}, \phi)$. A homomorphism between two framed sheaves $(\mathcal{E}_1, [\phi_1])$, $(\mathcal{E}_2, [\phi_2])$ is a sheaf homomorphism $f : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\phi_2 f = \lambda \phi_1$ for some $\lambda \in k$. An isomorphism is an invertible homomorphism. At some stage, when we consider (semi)stability of framed sheaves, also the choice of a polynomial δ will come into play.

Let V be a vector space of dimension $P_c(m)$ for some $m \gg 0$, let $\mathcal{H} = V \otimes \mathcal{O}_X(-m)$, and let $\mathrm{Quot}(\mathcal{H}, P_c)$ be the Quot scheme parametrizing the coherent quotients of $\mathcal{H}$ with Hilbert polynomial P_c. On $\mathrm{Quot}(\mathcal{H}, P_c) \times X$ there is a universal quotient $\tilde{\mathcal{Q}}$, and a morphism

$$\mathcal{O}_{\mathrm{Quot}(\mathcal{H}, P_c)} \boxtimes \mathcal{H} \xrightarrow{\tilde{g}} \tilde{\mathcal{Q}}$$

Let $\mathbb{P} = \mathbb{P}[\mathrm{Hom}(V, H^0(X, \mathcal{F}(m)))]^*$; a point $[a] \in \mathbb{P}$ induces a morphism $a\colon \mathcal{H} \to \mathcal{F}$, defined up to a constant factor.

Let $\mathbb{Q}\mathrm{uot}(\mathcal{H}, P_c, \mathcal{F})$ be the closed subscheme of $\mathrm{Quot}(\mathcal{H}, P_c) \times \mathbb{P}$ formed by the pairs $([g], [a])$ such that there is a morphism $\phi\colon \mathcal{G} \to \mathcal{F}$ for which the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{g} & \mathcal{G} \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle \phi} \\ & & \mathcal{F} \end{array}$$

commutes. Obviously, such ϕ is uniquely determined by a. We denote by $\mathcal{Q}$ the restriction to $\mathbb{Q}\mathrm{uot}(\mathcal{H}, P_c, \mathcal{F}) \times X$ of the pullback of $\tilde{\mathcal{Q}}$ to $\mathrm{Quot}(\mathcal{H}, P_c) \times \mathbb{P} \times X$. There is a morphism $\Phi\colon \mathcal{Q} \to p^*\mathcal{F}$, defined locally over the base, where $p\colon \mathbb{Q}\mathrm{uot}(\mathcal{H}, P_c, \mathcal{F}) \times X \to X$ is the natural projection. In some sense, $(\mathcal{Q}, \Phi)$ is a locally defined universal pair (see Proposition 2.3).

Definition 2.1. *A family of framed sheaves on X parametrized by a scheme S is a sheaf $\mathcal{G}$ on $S \times X$, flat over S, with a collection Ψ of sections Ψ_α of $\mathrm{pr}_{1*}\mathcal{H}om(\mathcal{G}, \mathrm{pr}_2^*\mathcal{F})$ defined on the elements U_α of some open covering (U_α) of S such that $\Psi_\alpha|_{U_\alpha \cap U_\beta} = (\mathrm{pr}_1^* f_{\alpha\beta})\Psi_\beta|_{U_\alpha \cap U_\beta}$ for some $f_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_S^*)$. Two collections Ψ, Ψ' are equivalent if their union is again a collection with the same property.*

Two families $(\mathcal{E}, \Phi)$, $(\mathcal{G}, \Psi)$ of framed sheaves over the same base scheme S are called locally isomorphic over S if there is a collection of isomorphisms $h_\alpha : (\mathcal{E}, \Phi) \xrightarrow{\sim} (\mathcal{G}, \Psi)$ defined over the elements U_α of some open covering (U_α) of S such that $h_\alpha|_{U_\alpha \cap U_\beta} = (\mathrm{pr}_1^* f_{\alpha\beta}) h_\beta|_{U_\alpha \cap U_\beta}$ for some $f_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_S^*)$.

Definition 2.2. *Let $(\mathcal{E}, [\phi])$ be a framed sheaf on X. A pair $(\mathcal{G}, [\psi])$ is a quotient of $(\mathcal{E}, [\phi])$ if $\mathcal{G}$ is a quotient of $\mathcal{E}$, and the diagram*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{G} \\ & {\scriptstyle \phi}\searrow & \downarrow{\scriptstyle \psi} \\ & & \mathcal{F} \end{array}$$

commutes modulo a scalar factor.

If $(\mathcal{E}, [\phi])$ is a framed sheaf on X, a family of framed quotients of $(\mathcal{E}, [\phi])$ is a family of framed sheaves $(\mathcal{G}, \Psi)$ with a sheaf epimorphism $g\colon \mathrm{pr}_2^*\mathcal{E} \to \mathcal{G}$ such that the diagram

$$\begin{array}{ccc} \mathrm{pr}_2^*\mathcal{E} & \xrightarrow{g} & \mathcal{G} \\ & {\scriptstyle \mathrm{pr}_2^*\phi}\searrow & \downarrow{\scriptstyle \Psi_\alpha} \\ & & \mathrm{pr}_2^*\mathcal{F} \end{array}$$

commutes over U_α *for each* α *up to a factor of the form* $\mathrm{pr}_1^* f_\alpha$ *for some* $f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_S^*)$.

The universality property of the Quot scheme implies the following result.

Proposition 2.3. *Let* $(\mathcal{G}, \Psi)$ *be a family of framed quotients of* $\mathcal{H}$*, parametrized by a scheme* S*. Assume that the Hilbert polynomial of* $\mathcal{G}_s =: \mathcal{G} \otimes \Bbbk(s)$ *is* P_c *for any* $s \in S$*. Then there is a morphism* $f\colon S \to \mathbb{Q}\mathrm{uot}(\mathcal{H}, P_c, \mathcal{F})$ *(unique up to a unique isomorphism) such that* $(\mathcal{G}, \Psi)$ *is locally isomorphic to* $(f \times \mathrm{id})^*(\mathcal{Q}, \Phi)$ *over* S.

The action of $\mathrm{SL}(V)$ on V induces well-defined actions on $\mathrm{Quot}(\mathcal{H}, P_c)$ and $\mathbb{P}$ which are compatible, so that one has an action of $\mathrm{SL}(V)$ on $Y := \mathbb{Q}\mathrm{uot}(\mathcal{H}, P_c, \mathcal{F})$. The moduli space of semistable framed sheaves is constructed as the GIT quotient of Y by this action of $\mathrm{SL}(V)$.

3. A family of μ-semistable framed sheaves on a surface

From now on we assume that
(i) X is a surface (i.e. $d = 2$),
(ii) $\mathcal{F}$ is a $\mathcal{O}_D$-module, where $D \subset X$ is a fixed big and nef curve,
(iii) $\deg P_c(m) = 2$.

Besides, we fix a polynomial

$$\delta(m) = \delta_1 m + \delta_0 \in \mathbb{Q}[m] \text{ with } \delta_1 > 0.$$

For an arbitrary framed sheaf $(E, \alpha : E \to \mathcal{F})$ of rank $\mathrm{rk}\, E > 0$, denote

$$\deg(E, \alpha) := \deg E - \varepsilon(\alpha)\delta_1 \ , \quad \mu(E, \alpha) := \deg(E, \alpha)/\,\mathrm{rk}\, E,$$

where $\deg E := c_1(E) \cdot H$ and where we set $\varepsilon(\alpha) := 1$ if $\alpha \neq 0$, respectively, $\varepsilon(\alpha) := 0$ otherwise. Recall that a framed sheaf $(E, \alpha : E \to \mathcal{F}) \in Y$ is called *μ-(semi)stable with respect to* δ_1 in the sense of Huybrechts-Lehn [5, Def. 1.8] if $\ker \alpha$ is torsion-free, and for all framed subsheaves (E', α') of (E, α), where $0 \leq \mathrm{rk}(E') \leq \mathrm{rk}\, E$ and $\alpha' : E' \hookrightarrow E \xrightarrow{\alpha} \mathcal{F}$ is the induced framing, one has $\mathrm{rk}\, E' \cdot \deg(E, \alpha) - \mathrm{rk}\, E \cdot \deg(E', \alpha') \underset{(\geq)}{>} 0$. (If $\mathrm{rk}\, E' > 0$, then the latter inequality can be written as $\mu(E', \alpha') \underset{(\leq)}{<} \mu(E, \alpha)$.)

We shall need a boundedness result. Denote by $\mathcal{S}^{\mu ss}(c, \delta)$ the family of all framed sheaves (E, α) of class c on X that are *μ-semistable with respect to* δ_1 (shortly: *μ-semistable*).

Proposition 3.1. *The family $\mathcal{S}^{\mu ss}(c,\delta)$ is bounded.*

Proof. The sheaves E from the pairs $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)$ may have torsion. We use the following trick of Huybrechts–Lehn (Remark 1.9 and Lemma 2.5 from [5]) in order to replace them by torsion free ones. Let $\hat{\mathcal{F}}$ be any locally free sheaf with a surjection $\phi : \hat{\mathcal{F}} \to \mathcal{F}$ and $\hat{E} = E \times_{\mathcal{F}} \hat{\mathcal{F}}$. Then $\hat{E}$ is torsion free, and there is an exact triple $0 \to \mathcal{K} \to \hat{E} \to E \to 0$, where $\mathcal{K} = \ker\phi$. Thus if we fix $\hat{\mathcal{F}}$ and ϕ, then $P_{\hat{E}} = P_c + P_{\mathcal{K}}$ does not depend on (E,α).

Let now $\hat{F}$ be any nonzero subsheaf of $\hat{E}$. Then $\operatorname{rk}\hat{F} > 0$, as $\hat{E}$ is torsion free. We have an exact triple $0 \to \mathcal{K}_F \to \hat{F} \to F \to 0$, where $F = \phi(\hat{F})$ and $\mathcal{K}_F = \ker(\phi|_{\hat{F}})$. By the μ-semistability of (E,α), we have $\deg(F) \le \operatorname{rk} F \cdot (\mu(E) + \delta_1)$. Hence

$$\mu(\hat{F}) = \frac{\deg F + \deg \mathcal{K}_F}{\operatorname{rk}\hat{F}} \le \frac{\operatorname{rk} F \cdot (\mu_c + \delta_1) + \operatorname{rk}\mathcal{K}_F \cdot \mu_{\max}(\mathcal{K})}{\operatorname{rk}\hat{F}} \ ,$$

where $\mu_{\max}$ stands for the slope of the maximal destabilizing subsheaf.

This shows that $\mu_{\max}(\hat{E})$ is uniformly bounded as (E,α) runs over $\mathcal{S}^{\mu ss}(c,\delta)$. Hence by a theorem of Le Potier-Simpson [7, Thm. 3.3.1], there exist constants C_0, C_1, C_2 and an $(\hat{E},\phi)$-regular sequence of two hyperplane sections $H_1, H_2 \in |\mathcal{O}_X(H)|$ such that $h^0(\hat{E}) \le C_0$, $h^0(\hat{E}|_{H_1}) \le C_1$, $h^0(\hat{E}|_{H_1\cap H_2}) \le C_2$. Now apply Kleiman's boundedness criterion [7, Thm. 1.7.8] to obtain the boundedness of the family of the sheaves $\hat{E}$ associated to the pairs (E,α) from $\mathcal{S}^{\mu ss}(c,\delta)$. The boundedness of the family of the pairs (E,α) themselves then follows by the same argument as in the proof of Lemma 2.5 in [5].

□

By Proposition 3.1 and semicontinuity we can fix a sufficiently large number m such that for each pair (E,α) in $\mathcal{S}^{\mu ss}(c,\delta)$ the sheaf E is m-regular. We define now $R^{\mu ss}(c,\delta)$ as the locally closed subscheme of the scheme

$$Y := \mathbb{Q}\mathrm{uot}(\mathcal{H}, P_c, \mathcal{F}),$$

with $\mathcal{H} = V \otimes \mathcal{O}_X(-m)$ and $\dim V = P_c(m)$, formed by the pairs $([g : \mathcal{H} \to E], [a : \mathcal{H} \to \mathcal{F}])$ such that $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)$, is μ-semistable with respect to δ_1, where the framing α is defined by the relation $a = \alpha \circ g$, and g induces an isomorphism $V \to H^0(E(m))$.

3.1. **Choosing a semiample sheaf $\mathcal{L}(n_1,n_2)$ on $R^{\mu ss}(c,\delta)$.** For any framed sheaf (E,α) on X we set

$$P_{(E,\alpha)}(l) := P_E(l) - \varepsilon(\alpha)\delta(l)$$

Take a sheaf $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)$. We then have a surjective quotient morphism $V \otimes \mathcal{O}_X(-m) \to E$. Since the family of subsheaves E' of E generated by all subspaces V' of V is bounded, the set $\mathcal{N}_{(E,\alpha)}$ of their Hilbert polynomials $P_{E'}$ is finite. Hence, since the scheme $\mathcal{S}^{\mu ss}(c,\delta)$ is noetherian, the set

$$\mathcal{N}_X(c,\delta) := \bigcup_{(E,\alpha)\in\mathcal{S}^{\mu ss}(c,\delta)} \mathcal{N}_{(E,\alpha)}$$

is finite.

Now for each polynomial $B \in \mathcal{N}_X(c,\delta)$, where $B = P_{E'}$, for E' a subsheaf of some framed sheaf $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)$, defined by a subspace V' of V, together with the induced framing α', we denote

$$G_B(l) := \dim V \left(1 + \varepsilon(\alpha')\frac{\delta(m)}{P_{(E',\alpha')}(m)}\right) P_{(E',\alpha')}(l) - \dim V' \left(1 + \frac{\delta(m)}{P_{(E,\alpha)}(m)}\right) P_{(E,\alpha)}(l).$$

Since the set $\{G_B | B \in \mathcal{N}_X(c,\delta)\}$ is finite, there exists a rational number ℓ_0 such that for any $\ell' \geq \ell_0$ the implication

$$G_B(\ell') > 0 \ \Rightarrow\ G_B(l) \text{ is positive for } l \gg 0 \tag{1}$$

is true for all $B \in \mathcal{N}_X(c,\delta)$.

Fix an integer $k > 0$ big enough so that

$$H^1(X, E(m-k)) = 0, \qquad (E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta).$$

Consider the linear series $|kH|$ and its dense open subset $|kH|^* = \{C \in |kH| \mid C \text{ is a smooth curve}\}$. For any $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)$ and any curve $C \in |kH|^*$ we have a Hilbert polynomial

$$P_{c|_C}(l) := P_{E|C}(l) = P_E(l) - P_E(l-k) = P_c(l) - P_c(l-k).$$

Take $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)$ and consider the rational functions

$$A_X(l) := P_{(E,\alpha)}(l)\frac{\delta(m)}{P_{(E,\alpha)}(m)} - \delta(l) \in \mathbb{Q}(l),$$

$$A_C(l) := P_{(E|_C,\alpha|_C)}(l)\frac{\delta_C}{P_{(E|_C,\alpha|_C)}(m)} - \delta_C \in \mathbb{Q}(l), \tag{2}$$

where, as before, $\delta(l) := \delta_1 l + \delta_0$ and where we set $\delta_C := a\delta_1$. Next, let

$$P_c(l) = p_2 l^2 + p_1 l + p_0, \quad p_i \in \mathbb{Q}.$$

The equality

$$A_X(l) = A_C(\tilde{l}), \tag{3}$$

considered as an equation on $\tilde{l}$, in view of (6) and (2) yields

$$\tilde{l} = L(l) := \frac{1}{2p_2}\left(\left(1 + \frac{A_X(l)}{a\delta_1}\right)\left(p_2(2m+k) + p_1 - \delta_1\right) - p_1 + \delta_1\right) - \frac{1}{2}k \tag{4}$$

For an arbitrary curve $C \in |kH|^*$ set

$$\mathcal{H}_C := V_C \otimes \mathcal{O}_C(-m), \qquad \dim V_C := P_c(m) - P_c(m-k),$$

$$P_{c|_C}(l) := P_c(l) - P_c(l-k) = k(p_2(2l+k) + p_1)$$

and consider the Quot scheme $Y_C := \mathrm{Quot}(\mathcal{H}_C, P_{c|_C})$. For any point $(E,\alpha) \in Y$ and any $C \subset |kH|^*$ consider the framed sheaf $(E|_C, \alpha|_C)$. The family of subsheaves E'_C of $E|_C$ generated by all subspaces W' of W is bounded, so that the set $\mathcal{N}_{(E|_C,\alpha|_C)}$ of polynomials $P_{E'_C}$ is finite. Hence, since the scheme $\mathcal{S}^{\mu ss}(c,\delta)$ is noetherian, the set $\mathcal{N}_C(c|_C, \delta_C) := \bigcup_{(E,\alpha)\in\mathcal{S}^{\mu ss}(c,\delta)} \mathcal{N}_{(E|_C,\alpha|_C)}$ is finite. Respectively, the set

$$\mathcal{N}(c|_C, \delta_C) := \bigcup_{C\in|kH|^*} \mathcal{N}_C(c|_C, \delta_C)$$

is also finite.

Now for each polynomial $B \in \mathcal{N}(c|_C, \delta_C)$, where $B = P_{E'_C}$, E'_C a subsheaf of a sheaf $(E|_C, \alpha|_C)$ for some framed sheaf $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)$, defined by a subspace W' of W, together with the induced framing α'_C, we denote

$$\widetilde{G}_B(l) := \dim W \cdot \left(P_{(E'|_C,\alpha'|_C)}(l) + \varepsilon(\alpha'|_C)\frac{\delta(m)}{P_{(E|_C,\alpha|_C)}(m)}\right) - \dim W' \cdot \left(1 + \frac{\delta_C(m)}{P_{(E|_C,\alpha|_C)}(m)}\right) P_{(E|_C,\alpha|_C)}(l).$$

Since the set $\{\widetilde{G}_B | B \in \mathcal{N}(c|_C, \delta_C)\}$ is finite, there exists a rational number ℓ_{0C} such that for any $\ell' \geq \ell_{0C}$ the implication

$$\widetilde{G}_B(\ell') > 0 \ \Rightarrow\ \widetilde{G}_B(l) \text{ is positive for } l \gg 0 \tag{5}$$

is true for all $B \in \mathcal{N}(c|_C, \delta_C)$.

Now choose a number $\ell_X \geq \ell_0$ such that $L(\ell_X) \geq \ell_{0C}$, where ℓ_{0C} was defined before formula (5) and $L(l)$ was defined earlier in (4). Set

$$\ell_C := L(\ell_X)$$

By (3) we have

$$A_X(\ell_X) = A_C(\ell_C), \qquad \ell_X \geq \ell_0, \quad \ell_C \geq \ell_{0C}. \tag{6}$$

Let

$$\mathcal{L}(n_1, n_2) = \left[\mathrm{pr}_1^* \lambda_{\tilde{Q}}(u_1)^{\otimes n_1} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}}(n_2)\right]|_{R^{\mu ss}(c,\delta)}$$

where we set

$$\frac{n_1}{n_2} := A_X(\ell_X) = \delta(m)\frac{P_c(\ell_X)}{P_c(m)} - \delta(\ell_X) \tag{7}$$

and where $\lambda_{\tilde{Q}}(u_1)$ is the determinant line bundle on $\mathrm{Quot}(\mathcal{H}, P_c)$ according to Huybrechts-Lehn's notation [7].

Now one has the following analogue of theorems of Mehta and Ramanathan [10, 11].

Theorem 3.2. *Let $(E, \alpha) \in \mathcal{S}^{\mu ss}(c, \delta)$ be a μ-semistable framed sheaf of positive rank. Then for all sufficiently big k, and for a generic curve $C \in |kH|$, the framed sheaf $(E|_C, \alpha|_C)$ is μ-semistable on C with respect to δ_C.*

Proof. See [15]. □

Proposition 3.3. *For $\nu \gg 0$ the line bundle $\mathcal{L}(n_1, n_2)^\nu$ on $R^{\mu ss}$ is generated by its $SL(V)$-invariant sections.*

Proof. Let S be a scheme parametrizing a flat family $(\mathbf{E}, \alpha_{\mathbf{E}})$ of μ-semistable framed sheaves $(E, \alpha : E \to \mathcal{F})$ on X with Chern character $c = (r, \mathcal{A}, c_2)$. Let $C \in |kH|$ be a general curve and $k \gg 0$. Then C is smooth and transversal to D, and the restriction of $(\mathbf{E}, \alpha_{\mathbf{E}})$ to $S \times C$ yields a family $(\mathcal{E}, \alpha_{\mathcal{E}})$ of framed sheaves $(E_C, \alpha_C : E_C \to i^*\mathcal{F})$ on C, where $i : C \to X$ is the inclusion. We may assume that the general element in this family is μ-semistable (by "general" we mean that the property holds true for all closed points in a nonempty open subset). Let $M_C := M^{ss}(c|_C, \delta_C)$ be the moduli space of framed sheaves on C with Mukai vector $c|_C = i^*c$ that are semistable with respect to δ_C. Note that, since C is a curve, semistability coincides with μ-semistability. By Theorem 3.2 a rational map $S \dashrightarrow M_C$ is defined.

For any $w \in K(X)$, let $w|_C = i^*w \in K(C)$ be its restriction. The class $c|_C$ is uniquely determined by its rank and by $\mathcal{A}|_C$. Let m' be a large positive integer, $P' := P_{c|_C}$, let V_C be a vector space of dimension $P'(m')$, let $\mathcal{H}' := V_C \otimes \mathcal{O}_C(-m')$ and let $Q_C \subset \mathrm{Quot}_C(\mathcal{H}', P')$ be the closed subset of quotients with determinant $\mathcal{A}|_C$, together with the universal quotient $\mathcal{O}_{Q_C} \boxtimes \mathcal{H}' \to \mathcal{E}'$. Furthermore, let $\mathbb{P}_C = \mathbb{P}\big(\mathrm{Hom}(V_C, H^0(C, i^*\mathcal{F}(m')))^*\big)$, so that a point $[a] \in \mathbb{P}_C$ corresponds to a morphism $a : \mathcal{H}' \to i^*\mathcal{F}$. Consider the closed subscheme $Y_C = \mathrm{Quot}(\mathcal{H}', P', i^*\mathcal{F})$ of $Q_C \times \mathbb{P}_C$ with projections $Q_C \xleftarrow{p_1} Y_C \xrightarrow{p_2} \mathbb{P}_C$, defined similarly to the scheme Y above. Clearly, the group $\mathrm{SL}(V_C)$ acts on Y_C. Denote $\deg C = C \cdot H$, and consider the line bundle

$$\mathcal{L}_0'(n_1, n_2k) := p_1^*\lambda_{\mathcal{E}'}(u_0(c|_C))^{n_1 \deg C} \otimes p_2^*\mathcal{O}_{\mathbb{P}_C}(n_2k)$$

on Y_C. If m' is sufficiently large the following results hold (see [5]).

Lemma 3.4. *Given a point $([g : \mathcal{H}' \to E_C], [a : \mathcal{H}' \to i^*\mathcal{F}]) \in Y_C$, the following assertions are equivalent:*

1. $(E_C, [a])$ *is a semistable pair and* $V_C \to H^0(E_C(m'))$ *is an isomorphism.*
2. $([g], [a])$ *is a semistable point in* Y_C *for the action of* $\mathrm{SL}(V_C)$ *with respect to the canonical linearization of* $\mathcal{L}_0'(n_1, n_2k)$.
3. *There is an integer* ν *and an* $\mathrm{SL}(V_C)$*-invariant section* σ *of* $\mathcal{L}_0'(n_1, n_2k)^\nu$ *such that* $\sigma([g], [a]) \neq 0$.

Jordan-Hölder filtrations for semistable framed sheaves were introduced in [5], Proposition 1.13, and the ensuing notion of S-equivalence was given there in Definition 1.14. In Section 4.2 we shall also use the notion of a μ-Jordan-Hölder filtration of a framed sheaf (E, α). It is constructed in the same way, but the associated graded object is not necessarily unique: two graded objects may differ by subsheaves supported in codimension ≥ 2. To avoid this difficulty, we shall only consider *saturated* μ-Jordan-Hölder filtrations, in which every term is a maximal proper μ-semistable framed subsheaf of the next term. We call (E, α) μ-polystable if E has a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$ such that: (i) it is split, that is the natural map from E to the associated graded object $\bigoplus_{i=1}^{n} E_i/E_{i-1}$ is an isomorphism, and (ii) the filtration $\ldots \subset (E_i, \alpha|_{E_i}) \subset (E_{i+1}, \alpha|_{E_{i+1}}) \subset \ldots$ is a saturated μ-Jordan-Hölder filtration of $(\mathcal{E}, \alpha)$.

Lemma 3.5. *Two points $([g_j : \mathcal{H}' \to E_{jC}], [a_j : \mathcal{H}' \to i^*\mathcal{F}])$, $j = 1, 2$ are separated by an $\mathrm{SL}(V_C)$-invariant section in some tensor power of $\mathcal{L}_0'(n_1, n_2k)$ if and only if either both are* *semistable points but the corresponding framed sheaves* (E_{1C}, α_{1C}) *and* (E_{2C}, α_{2C}) *are not* S*-equivalent, or one of them is semistable but the other is not.*

Consider now the exact sequence

$$0 \to \mathbf{E} \otimes \big(\mathcal{O}_S \boxtimes \mathcal{O}_X(-k)\big) \to \mathbf{E} \to \mathcal{E} \to 0. \tag{8}$$

Assume that m' is big enough so that, not only the results in Lemmas 3.4 and 3.5 hold, but one also has:

$\mathcal{E}_s$ *is* m'*-regular for all* $s \in S$.

Then $p_*(\mathcal{E}(m'))$ is a locally free $\mathcal{O}_S$-module of rank $P'(m')$, where $\mathcal{E}(m') = \mathcal{E} \otimes \mathcal{O}_S \boxtimes \mathcal{O}_C(m')$ and $p : S \times C \to S$ is the projection. Let $\widetilde{S} := \mathbb{P}(\mathrm{Isom}(V_C \otimes \mathcal{O}_S, p_*(\mathcal{E}(m')))^\vee)$, $\quad \pi : \widetilde{S} \to S$ the associated projective frame bundle and $\pi_C : \widetilde{S} \times C \to S \times C$ the induced projection. On $\tilde{S} \times C$ there is a universal quotient $\mathbf{g} : \mathcal{O}_{\widetilde{S}} \boxtimes \mathcal{H}' \twoheadrightarrow \pi_C^* \mathcal{E} \otimes \mathcal{O}_{\pi_C}(1)$ and a system $\Psi_{\mathcal{E}}$ of locally defined framings $\pi_C^* \mathcal{E} \otimes \mathcal{O}_{\pi_C}(1) \xrightarrow{\pi_C^* \alpha_{\mathcal{E}} \otimes \mathrm{id}} \pi_C^*(\mathcal{O}_S \boxtimes i^* \mathcal{F}) \otimes \mathcal{O}_{\pi_C}(1)$ which induce by Proposition 2.3 a $\mathrm{SL}(P'(m'))$-invariant morphism

$$\mathbf{f}_{\mathcal{E}} : \widetilde{S} \to Y_C.$$

By analogy with [7, Prop. 8.2.3] and using the relations (6) and (7) we obtain the isomorphism of line bundles

$$\mathbf{f}_{\mathcal{E}}^* \mathcal{L}_0'(n_1, n_2 k) \cong \pi^* \mathcal{L}(n_1, n_2)^{\otimes k}. \tag{9}$$

Now set $S = R^{\mu ss}(c, \delta)$. The group $\mathrm{SL}(V)$ acts on S, hence also on $\widetilde{S}$. Thus we have an action of $\mathrm{SL}(V) \times \mathrm{SL}(V_C)$ on $\widetilde{S}$ and by construction the morphism $\mathbf{f}_{\mathcal{E}}$ is $\mathrm{SL}(V) \times \mathrm{SL}(V_C)$-invariant. Take an arbitrary $\mathrm{SL}(V_C)$-invariant section σ of $\mathcal{L}_0'(n_1, n_2 k)^{\otimes \nu}$. Then $\mathbf{f}_{\mathcal{E}}^* \sigma$ is a $\mathrm{SL}(V) \times \mathrm{SL}(V_C)$-invariant section. Therefore, since π is a principal $\mathrm{PSL}(V_C)$-bundle, this section descends to a $\mathrm{SL}(V)$-invariant section of the line bundle $\mathcal{L}(n_1, n_2)^{\otimes \nu k}$. We thus obtain a monomorphism

$$s_{\mathcal{E}} : H^0(Y_C, \mathcal{L}_0'(n_1, n_2 k)^{\otimes \nu})^{\mathrm{SL}(V_C)} \to H^0(S, \mathcal{L}(n_1, n_2)^{\otimes \nu k})^{\mathrm{SL}(V)}. \tag{10}$$

By analogy with [7, Lemma 8.2.4], and using [7, Prop. 3.1-3.3], we obtain the following lemma.

Lemma 3.6. *1. If* $s \in R^{\mu ss}(c, \delta)$ *is a point such that* $(i^* E_s, i^* \alpha_s : i^* E_s \to i^* \mathcal{F})$ *is semistable with respect to* δ_C*, there is a* $\mathrm{SL}(V)$*-invariant section* $\bar{\sigma} \in H^0(R^{\mu ss}(c, \delta), \mathcal{L}(n_1, n_2)^{\otimes \nu k})^{\mathrm{SL}(V)}$ *such that* $\bar{\sigma}(s) \neq 0$.

2. If s_1 and s_2 are the two points in $R^{\mu ss}(c,\delta)$ such that $i^*E_{s_1}$ and $i^*E_{s_2}$ are both semistable but not S-equivalent, or one of them is semistable and the other is not, then for some ν there are $\mathrm{SL}(V)$-invariant sections of $\mathcal{L}(n_1,n_2)^{\otimes \nu k}$ that separate s_1 and s_2.

Proposition 3.3 now follows from the first assertion of Lemma 3.6. □

4. The Uhlenbeck-Donaldson compactification for framed sheaves

4.1. **Construction of $M^{\mu ss}(c,\delta)$.** By Proposition 3.3, the sheaf $\mathcal{L}(n_1,n_2)^\nu$ is generated by its invariant sections. Thus we can find a finite-dimensional subspace $W \subset W_\nu := H^0(R^{\mu ss}, \mathcal{L}(n_1,n_2)^\nu)^{\mathrm{SL}(V)}$ that generates $\mathcal{L}(n_1,n_2)^\nu$. Let $\phi_W : R^{\mu ss}(c,\delta) \to \mathbb{P}(W)$ be the induced $\mathrm{SL}(P_c(m))$-invariant morphism.

Proposition 4.1. *$M_W := \phi_W(R^{\mu ss}(c,\delta))$ is a projective scheme.*

The proof of this Proposition goes as in [7, Prop. 8.2.5], by using the following Lemma, which generalizes a classical result by Langton.

Lemma 4.2. *Let $(R,\mathfrak{m})$ be a discrete valuation ring with residue field k and quotient field K and let X be a smooth projective variety over k. Let $\mathcal{E}$ be an R-flat family of framed sheaves on X such that $\mathcal{E}_K = K \otimes_R \mathcal{E}$ is a μ-semistable framed sheaf. Then there is a subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $\mathcal{F}_K = \mathcal{E}_K$ and $\mathcal{F}_k$ is μ-semistable.*

By using Proposition 4.1, and proceeding as in [7], Proposition 8.2.6, we can prove the following result.

Proposition 4.3. *There is an integer $N > 0$ such that $\bigoplus_{l\geq 0} W_{lN}$ is a finitely generated graded ring.*

We can eventually define the Uhlenbeck-Donaldson compactification.

Definition 4.4. *Let N be a positive integer as in the above proposition. Then $M^{\mu ss} = M^{\mu ss}(c,\delta)$ is defined by*

$$M^{\mu ss} = \mathrm{Proj}\left(\bigoplus_{k\geq 0} H^0(R^{\mu ss}(c,\delta), \mathcal{L}(n_1,n_2)^{kN})^{SL(P(m))} \right) .$$

It is equiped with a natural morphism $\pi : R^{\mu ss}(c,\delta) \to M^{\mu ss}$ and is called the moduli space of μ-semistable framed sheaves.

Now we will explain, in which sense $M^{\mu ss}$ is the moduli space of μ-semistable framed sheaves. In fact, though $M^{\mu ss}$ is not in general a categorial quotient of $R^{\mu ss}$, still $M^{\mu ss}$ has the following universal property. Let $\mathcal{M}^{\mu ss}$ denote the functor which associates to S the set of isomorphism classes of S-flat families of μ-semistable framed sheaves of class c on X. Then one easily constructs a natural transformation of functors $\mathcal{M}^{\mu ss} \to \mathrm{Hom}(-, M^{\mu ss})$ with the property that for any S-flat family $\mathbf{F} = (\mathbf{E}, \alpha_{\mathbf{E}})$ of μ-semistable framed sheaves from $\mathcal{S}^{\mu ss}(c, \delta)$ and classifying morphism $\Phi_{\mathbf{F}} : S \to M^{\mu ss}$ the pullback of $\mathcal{O}_{M^{\mu ss}}(1)$ via $\Phi_{\mathbf{F}}$ is isomorphic to $(\lambda_{\mathbf{E}}(u_1)^{\otimes n_1} \otimes \mathrm{pr}^* \mathcal{O}_{\mathbb{P}}(n_2))^N$:

$$\Phi_{\mathbf{F}}^* \mathcal{O}_{M^{\mu ss}}(1) \cong (\lambda_{\mathbf{E}}(u_1)^{\otimes n_1} \otimes \mathrm{pr}^* \mathcal{O}_{\mathbb{P}}(n_2))^N, \tag{11}$$

where $\mathrm{pr} : S \to \mathbb{P}$ is a natural morphism defined by the framing $\alpha_{\mathbf{E}}$. The triple $(M^{\mu ss}, \mathcal{O}_{M^{\mu ss}}(1), N)$ is characterized by this property in a unique way, up to a unique isomorphism and up to replacing $(\mathcal{O}_{M^{\mu ss}}(1), N)$ by some multiple $(\mathcal{O}_{M^{\mu ss}}(d), dN)$. In particular, the construction of $M^{\mu ss}$ does not depend on the choice of the integer m.

Let $M = M(c, \mathcal{F})$ denote the moduli space of semistable framed sheaves $(E, \alpha : E \to \mathcal{F})$ on X with $\mathrm{ch}(E) = c$. It co-represents the moduli functor $\mathcal{M} = \mathcal{M}(c, \mathcal{F})$ which associates to a scheme T the set of all classes of T-flat families of framed sheaves $(E, \alpha : E \to \mathcal{F})$ with $ch(E) = c$ modulo isomorphisms, defined locally over the base (see [7]). As every semistable framed sheaf is μ-semistable, this implies:

Theorem 4.5. *The morphism of functors $\mathcal{M} \to \mathcal{M}^{\mu ss}$ induces a morphism of modui spaces $\gamma : M \to M^{\mu ss}$ such that $\gamma^* \mathcal{O}(1) \cong (\lambda_{\mathbf{E}}(u_1)^{\otimes n_1} \otimes \mathrm{pr}^* \mathcal{O}_{\mathbb{P}}(n_2))^N$.*

Let now $M^{\mu\text{-stable}}$, $M^{\mu\text{-poly}}$ be the open subsets of M corresponding to μ-stable, resp. μ-polystable pairs (E, α) with E locally free. We are assuming that $M^{\mu\text{-stable}}$ is nonempty. We shall see (Theorem 4.6) that the restriction $M^{\mu\text{-poly}} \xrightarrow{\gamma} M^{\mu ss}$ is injective. Actually, when restricted to $M^{\mu\text{-stable}}$, this map is an embedding, so that by taking the closure of $\gamma(M^{\mu\text{-stable}})$ in $M^{\mu ss}$, we obtain a compactification of $M^{\mu\text{-stable}}$. By analogy with the nonframed case, we will call it the *Uhlenbeck-Donaldson compactification* of $M^{\mu\text{-stable}}$.

With the reference to the notation introduced in the beginning of Section 3, we set

$$\mathcal{S}^{\mu ss}(c, \delta)^* := \{(E, \alpha) \in \mathcal{S}^{\mu ss}(c, \delta) \mid E \text{ is locally free at all points of } D \\ \text{and } \alpha \text{ induces an isomorphism } E|_D \simeq \mathcal{F}\},$$

$$R^{\mu ss}(c, \delta)^* := \{([g : \mathcal{H} \to E], [\alpha \circ g]) \in R^{\mu ss}(c, \delta) \mid (E, \alpha) \in \mathcal{S}^{\mu ss}(c, \delta)^*\},$$

$$M^{\mu ss}(c,\delta)^* := \pi(R^{\mu ss}(c,\delta)^*)\ , \qquad M^* := \gamma^{-1}(M^{\mu ss}(c,\delta)^*).$$

Note that these are open subsets of $\mathcal{S}^{\mu ss}(c,\delta)$, $R^{\mu ss}(c,\delta)$, $M^{\mu ss}(c,\delta)$ and M, respectively, and that $M^{\mu\text{-poly}} \subset M^*$.

We proceed now to a more detailed study of the morphism $\gamma : M^* \to M^{\mu ss}(c,\delta)^*$.

4.2. **Description of the morphism $\gamma : M^* \to M^{\mu ss}(c,\delta)^*$.** Let $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)^*$. Consider the *graded framed sheaf* $gr^\mu(E,\alpha) = (gr^\mu E, gr^\mu\alpha)$ associated to a saturated μ-Jordan-Hölder filtration of (E,α). It is μ-polystable as a framed sheaf. Remark that, applying the definition of μ-semistability to $E(-D) = \ker\alpha \subset E$, we conclude that $\delta_1 \leq r\deg D$. Moreover, in the case of equality, $(E(-D),0) \subset (E,\alpha)$ is the upper level of the Jordan–Hölder filtration with torsion quotient. Under our hypotheses, this is the only possible torsion in the graded object associated to the Jordan–Hölder filtration. To eliminate it, we impose, from now on, the additional hypothesis: $\delta_1 < r\deg D$.

By taking the double dual we get a μ-polystable locally free framed sheaf $(gr^\mu E)^{\vee\vee}$. The function $l_E : X \to \mathbb{N}\cup\{0\} : x \mapsto \operatorname{length}\big((gr^\mu E)^{\vee\vee}/gr^\mu E\big)_x$ can be considered as an element in the symmetric product $S^l(X\setminus D)$ with $l = c_2(E) - c_2((gr^\mu E)^{\vee\vee})$. Both $(gr^\mu E)^{\vee\vee}$ and l_E are well-defined invariants of (E,α), i.e., they do not depend on the choice of the saturated μ-Jordan-Hölder filtration of (E,α).

Theorem 4.6. *Assume that $\delta_1 < r\deg D$. The framed sheaves (E_1,α_1), $(E_2,\alpha_2) \in \mathcal{S}^{\mu ss}(c,\delta)^*$ define the same closed point in $M^{\mu ss}(c,\delta)^*$ if and only if*

$$(gr^\mu E_1)^{\vee\vee} = (gr^\mu E_2)^{\vee\vee} \quad \textit{and} \quad l_{E_1} = l_{E_2}.$$

Proof. The proof goes along the same lines as that of [7, Theorem 8.2.11]. We start with the "if" part. Take any framed sheaf $(E,\alpha) \in \mathcal{S}^{\mu ss}(c,\delta)^*$ and consider the graded framed sheaf $gr^\mu(E,\alpha)$ obtained from some saturated μ-Jordan-Hölder filtration of (E,α). Then one can naturally construct a flat family $(\mathbf{E},\mathbf{A})$ of framed sheaves over $\mathbb{A}^1$ such that

i) $(E_t,\alpha_t) \cong (E,\alpha)$ for all $0 \neq t \in \mathbb{A}^1$, and

ii) $(E_0,\alpha_0) \cong gr^\mu(E,\alpha)$.

The classifying morphism $\Phi_\mu : \mathbb{A}^1 \to M^{\mu ss}(c,\delta)^*$ factors into the composition $\Phi_\mu : \mathbb{A}^1 \stackrel{\Phi}{\to} M^* \stackrel{\gamma}{\to} M^{\mu ss}(c,\delta)^*$, where Φ is the classifying morphism. By i) $\Phi(\mathbb{A}^1)$ is a point, hence also $[(E,\alpha)] := \Phi_\mu(E,\alpha) = \Phi_\mu(\mathbb{A}^1)$ is a point, and by ii) we have $[(E,\alpha)] = [gr^\mu(E,\alpha)]$. It follows that it is enough to consider μ-polystable framed sheaves from $\mathcal{S}^{\mu ss}(c,\delta)^*$.

Thus, let (E,α) be a μ-polystable framed sheaf from $\mathcal{S}^{\mu ss}(c,\delta)^*$. Then $\mathcal{E} := E^{\vee\vee}$ is μ-polystable and locally free, and there is an exact sequence

$$0 \to E \xrightarrow{can} \mathcal{E} \xrightarrow{\epsilon} T \to 0$$

where T is a torsion sheaf with $l(T) = l_E$. Furthermore, E is locally free along the framing curve D by the definition of $\mathcal{S}^{\mu ss}(c,\delta)^*$, hence there exists a morphism $\alpha_D : \mathcal{E}|_D \to \mathcal{F}$ such that the framing $\alpha : E \to \mathcal{F}$ decomposes as

$$\alpha\colon E \xrightarrow{\otimes \mathcal{O}_D} E|_D \cong \mathcal{E}|_D \xrightarrow{\alpha_D} \mathcal{F}. \tag{12}$$

Take another μ-polystable framed sheaf (E',α') from $\mathcal{S}^{\mu ss}(c,\delta)^*$ such that $(E')^{\vee\vee} = \mathcal{E}$ and $l_E = l_{E'}$. The framing $\alpha' : E' \to \mathcal{F}$ decomposes in a similar way as above:

$$\alpha'\colon E' \xrightarrow{\otimes \mathcal{O}_D} E'|_D \cong \mathcal{E}|_D \xrightarrow{\alpha_D} \mathcal{F}. \tag{13}$$

Consider the morphism $\psi : \mathrm{Quot}(\mathcal{E}, l) \to S^l X : [\mathcal{E} \overset{\epsilon}{\twoheadrightarrow} T] \mapsto l_{E_\epsilon}$, where $E_\epsilon := \ker \epsilon$, and set

$$Y := \psi^{-1}(l_E).$$

There is a universal exact triple

$$0 \to \mathbb{E} \to \mathcal{O}_Y \boxtimes \mathcal{E} \to \mathbb{T} \to 0$$

of families on X parametrized by Y, where $\mathbb{T} = \mathcal{O}_Y \boxtimes \Bbbk(l_E)$. Let $p_1 : Y \times X \to Y$ be the projection onto the first factor and set $W := \mathrm{Isom}(V \otimes \mathcal{O}_Y, p_{1*}(\mathcal{E} \otimes \mathcal{O}_Y \boxtimes \mathcal{O}_X(m))) \overset{p_W}{\to} Y$ and $\mathbb{E}_W := (p_W \times id_X)^*\mathbb{E}$. For any $w \in W$ we have a tautological epimorphism $g_w : \mathcal{H} \to E_w := \mathbb{E}_W|\{w\} \times X$ By the universal property of $R^{\mu ss}(c,\delta)^*$ there is a well defined morphism

$$\begin{aligned} \Phi_W\colon W &\to R^{\mu ss}(c,\delta)^* \\ w &\mapsto (g_w, z)\ , \quad \text{where} \\ z &= [\mathcal{H} \overset{g_w}{\to} E_w \xrightarrow{\otimes \mathcal{O}_D} E_w|_D \cong \mathcal{E}|_D \xrightarrow{\alpha_D} \mathcal{F}]\ . \end{aligned}$$

Here z does not depend on $w \in W$, so that

$$\mathrm{im}(\Phi_W) \subset \mathrm{pr}^{-1}(z),$$

where $\mathrm{pr}: R^{\mu ss}(c,\delta)^* \to \mathbb{P}$ is the projection. We thus have a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{\Phi_W} & \mathrm{pr}^{-1}(z) & \hookrightarrow & R^{\mu ss}(c,\delta)^* \\ \downarrow{\scriptstyle p_W} & & & & \downarrow{\scriptstyle \pi} \\ Y & & \xrightarrow{\Phi_Y} & & M^{\mu ss}(c,\delta)^*, \end{array}$$

where

$$\begin{aligned} &(14) \qquad \Phi_Y : Y \;\to\; M^{\mu ss}(c,\delta)^* \\ &(15) \qquad y \;\mapsto\; [(E_y = \mathbb{E}|\{y\}\times X, \alpha_y : E_y \xrightarrow{\otimes \mathcal{O}_D} E_y|_D \cong \mathcal{E}|_D \xrightarrow{\alpha_D} \mathcal{F})] \end{aligned}$$

is the classifying morphism. From this diagram and formula (11) it follows that

$$(\Phi_Y \circ p_W)^* \mathcal{O}_{M^{\mu ss}(c,\delta)^*}(1) \cong (\lambda_{\mathbf{E}}(u_1)^{\otimes n_1} \otimes \mathrm{pr}^* \mathcal{O}_{\mathbb{P}}(n_2))^N, \tag{16}$$

One shows that the right hand side of (16) is trivial. In fact, since $\psi(Y) = l_E$ is a point, it follows from the computations in [7, Example 8.2.1] that $\lambda_{\mathbf{E}}(u_1) = \mathcal{O}_Y$, hence $\lambda_{\mathbf{E}_W}(u_1) = \mathcal{O}_W$. On the other hand, the above diagram shows that $\Phi_W^* \,\mathrm{pr}^* \mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_W$. Whence (16) yields

$$(\Phi_Y \circ p_W)^* \mathcal{O}_{M^{\mu ss}(c,\delta)^*}(1) \cong \mathcal{O}_W. \tag{17}$$

Note that Y is irreducible and projective (see, e.g., [3]) and $p_w : W \to Y$ is a projective bundle; hence W is also an irreducible projective scheme. It follows now from (17) that $y = \Phi_Y(Y)$ is a point. In particular, (14) shows that $y = [(E, \alpha_E)] = [(E, \alpha_{E'})]$ which proves the "if" part of the theorem.

The proof of the "only if" part goes as in [7, Theorem 8.2.11]. In particular, this requires a version of the restriction theorem 3.2 for *stable* framed sheaves [15]. □

From this theorem we obtain a set-theoretic stratification of the Uhlenbeck-Donaldson compactification.

Corollary 4.7. *Let $c = (r, \mathcal{Q}, c_2)$ and $M^{\mu\text{-poly}}(r, \mathcal{Q}, c_2, \delta)^* \subset M^{\mu ss}(c,\delta)^*$ denote the subset corresponding to μ-polystable locally free sheaves. Assume, as before, that $\delta_1 < r \deg D$. One has the following set-theoretic stratification:*

$$M^{\mu ss}(c,\delta)^* = \coprod_{l \geq 0} M^{\mu\text{-poly}}(r, \mathcal{Q}, c_2 - l, \delta)^* \times S^l(X \setminus D).$$

5. Concluding remarks

Let X be a smooth projective surface, and let D be a big and nef irreducible divisor in X. Let E_D be a locally free sheaf on D such that there exists a real number A_0, $0 \leq A_0 < \frac{1}{r}D^2$ with the following property: for any locally free subsheaf $F \subset E_D$ of constant positive rank, one has $\frac{1}{\operatorname{rk} F} \deg c_1(\mathcal{F}) \leq \frac{1}{\operatorname{rk} E_D} \deg c_1(E_D) + A_0$. Considering E_D as a sheaf on X, we say that a framed sheaf $(E, \alpha\colon E \to E_D)$ is (D, E_D)-framed if (E, α) satisfies the condition of the definition of $\mathcal{S}^{\mu ss}(c, \delta)^*$, that is E is locally free along D and $\alpha_{|D}$ is an isomorphism between $E_{|D}$ and E_D. It was shown in [1] that for any $c \in H^*(X, \mathbb{Q})$ there exists an ample divisor H on X and a real number $\delta > 0$ such that all the (D, E_D)-framed sheaves $\mathcal{E}$ on X with Chern character $\operatorname{ch}(\mathcal{E}) = c$ are (H, δ)-stable. As a consequence, one has a moduli space for (D, E_D)-framed sheaves on X, which embeds as an open subset into the moduli space of stable pairs. These moduli spaces have been quite extensively studied in connection with instanton counting and Nekrasov partition functions (see [14, 2, 4] among others).

Let us in particular consider the open subset formed by locally free (D, E_D)-framed sheaves on X. By restricting the previous construction to this open subset we construct a Uhlenbeck-Donaldson partial compactification for it (we call this "partial" because the moduli space of slope semistable framed bundles is not projective in general in this case). This generalizes the construction done by Nakajima, using ADHM data, when X is the complex projective plane. An extension to a general projective surface was hinted at in [13] but was not carried out.

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