ON WEYL CALCULUS IN INFINITELY MANY VARIABLES

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Abstract. We outline an abstract approach to the pseudo-differential Weyl calculus for operators in function spaces in infinitely many variables. Our earlier approach to the Weyl calculus for Lie group representations is extended to the case of representations associated with infinite-dimensional coadjoint orbits. We illustrate the approach by the case of infinite-dimensional Heisenberg groups. The classical Weyl-Hörmander calculus is recovered for the Schrödinger representations of the finite-dimensional Heisenberg groups.

1. Introduction

The pseudo-differential Weyl calculus which takes into account a magnetic field on $\mathbb{R}^n$ was recently developed in a series of papers including [MP04], [IMP07], [MP10], and [IMP10]. We have shown ([BB09a], [BB09b], [BB10a], [BB10d]) that a representation theoretic approach to that calculus can lead to a number of improvements such as an extension to the situation of magnetic fields on any nilpotent Lie group instead of the abelian group $(\mathbb{R}^n,+)$ and, more importantly, establishing the relationship to the Weyl quantization discussed for instance in [Ca07]. The latter point was settled by recovering the magnetic calculus as the Weyl quantization for a *finite*-dimensional coadjoint orbit of a Lie group which is in general infinite-dimensional.

In the present paper we wish to point out that this representation theoretic approach can also be applied in the case of certain *infinite*-dimensional coadjoint orbits. As a by-product of this method, we provide a generalized version for the pseudo-differential calculus developed in [AD96] and [AD98] for the differential operators of infinite-dimensional analysis (see e.g., [Kuo75], [Be86], [DF91], or [Bg98]).

2. An abstract framework for the Weyl calculus

In this section we develop a version of the localized Weyl calculus of [BB09a] and [BB10d], which is general enough for dealing with Weyl quantizations of some infinite-dimensional coadjoint orbits.

Setting 2.1. Let M be a locally convex Lie group with Lie algebra $\mathbf{L}(M)=\mathfrak{m}$ and smooth exponential map $\exp_M\colon \mathfrak{m}\to M$ (see [Ne06]), and $\pi\colon M\to\mathcal{B}(\mathcal{Y})$ a continuous unitary representation on the complex Hilbert space $\mathcal{Y}$. We shall think of the dual space $\mathfrak{m}^*$ as a locally convex space with respect to the weak*-topology. Let $\mathcal{UC}_b(\mathfrak{m}^*)$ be the commutative unital C^*-algebra of uniformly continuous bounded functions on the locally convex space $\mathfrak{m}^*$ and for every $\mu\in\mathcal{UC}_b(\mathfrak{m}^*)^*$ define the function

$$\widehat{\mu}\colon \mathfrak{m}\to\mathbb{C},\quad \widehat{\mu}(X)=\langle\mu,\mathrm{e}^{\mathrm{i}\langle\cdot,X\rangle}\rangle,$$

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where either of the duality pairings $\mathfrak{m}^* \times \mathfrak{m} \to \mathbb{R}$ and $\mathcal{UC}_b(\mathfrak{m}^*)^* \times \mathcal{UC}_b(\mathfrak{m}^*) \to \mathbb{C}$ is denoted by $\langle \cdot, \cdot \rangle$. Assume the setting defined by the following data:

- a locally convex real vector space Ξ and a Borel measurable map $\theta\colon \Xi \to \mathfrak{m}$,
- a locally convex space $\Gamma \hookrightarrow \mathcal{UC}_b(\mathfrak{m}^*)^*$ with continuous inclusion map, where $\mathcal{UC}_b(\mathfrak{m}^*)^*$ is endowed with the weak*-topology,
- a locally convex space $\mathcal{Y}_{\Xi,\infty} \hookrightarrow \mathcal{Y}$ with continuous inclusion map,

subject to the following conditions:

(1) The linear mapping
$$\mathcal{F}_\Xi\colon \Gamma \to \mathcal{UC}_b(\Xi), \quad \mu \mapsto \widehat{\mu} \circ \theta$$
is well defined and injective. Let us denote $\mathcal{Q}_\Xi := \mathcal{F}_\Xi(\Gamma) \hookrightarrow \mathcal{UC}_b(\Xi)$ and endow it with the topology which makes the *Fourier transform*
$$\mathcal{F}_\Xi\colon \Gamma \to \mathcal{Q}_\Xi$$
into a linear toplogical isomorphism. Note that there also exists the linear toplogical isomorphism $(\mathcal{F}_\Xi^*)^{-1}\colon \Gamma^* \to \mathcal{Q}_\Xi^*$.

(2) We have the well-defined continuous sesquilinear functional
$$\mathcal{Y}_{\Xi,\infty} \times \mathcal{Y}_{\Xi,\infty} \to \mathcal{Q}_\Xi, \quad (\phi, \psi) \mapsto (\pi(\exp_M(\theta(\cdot)))\phi \mid \psi).$$

Definition 2.2. In this framework, the *quasi-localized Weyl calculus for π along θ* is the linear map $\mathrm{Op}\colon \Gamma^* \to \mathcal{L}(\mathcal{Y}_{\Xi,\infty}, \overline{\mathcal{Y}}^*_{\Xi,\infty})$ defined by
$$(\mathrm{Op}(a)\phi \mid \psi) = \langle \underbrace{(\mathcal{F}_\Xi^*)^{-1}(a)}_{\in \mathcal{Q}_\Xi^*}, \underbrace{(\pi(\exp_M(\theta(\cdot)))\phi \mid \psi)}_{\in \mathcal{Q}_\Xi} \rangle \tag{2.1}$$
for $a \in \Gamma^*$ and $\phi, \psi \in \mathcal{Y}_{\Xi,\infty}$, where $\overline{\mathcal{Y}}^*_{\Xi,\infty}$ denotes the space of antilinear continuous functionals on $\mathcal{Y}_{\Xi,\infty}$. $\square$

Remark 2.3. In the setting of Definition 2.2, let us assume that the linear functional $(\mathcal{F}_\Xi^*)^{-1}(a) \in \mathcal{Q}_\Xi^*$ is defined by a complex Borel measure on Ξ denoted in the same way. For arbitrary $\phi, \psi \in \mathcal{Y}_{\Xi,\infty}$, the function $(\pi(\exp_M(\theta(\cdot)))\phi \mid \psi)$ is uniformly bounded on Ξ, hence it is integrable with respect to the measure $(\mathcal{F}_\Xi^*)^{-1}(a)$ and equation (2.1) takes the form
$$(\mathrm{Op}(a)\phi \mid \psi) = \int_\Xi (\pi(\exp_M(\theta(\cdot)))\phi \mid \psi) \mathrm{d}(\mathcal{F}_\Xi^*)^{-1}(a) \tag{2.2}$$
which is very similar to the definition of the Weyl-Pedersen calculus for irreducible representations of finite-dimensional nilpotent Lie groups with the locally convex space Ξ in the role of a predual of the coadjoint orbit under consideration (see for instance [BB09b] and [BB10g]).

Moreover, for arbitrary $\phi, \psi \in \mathcal{Y}$ we have $\|(\pi(\exp_M(\theta(\cdot)))\phi \mid \psi)\|_\infty \le \|\phi\| \cdot \|\psi\|$, hence by (2.2) we get $|(\mathrm{Op}^\theta(a)\phi \mid \psi)| \le \|(\mathcal{F}_\Xi^*)^{-1}(a)\| \cdot \|\phi\| \cdot \|\psi\|$. Thus $\mathrm{Op}(a) \in \mathcal{B}(\mathcal{Y})$ and $\|\mathrm{Op}(a)\| \le \|(\mathcal{F}_\Xi^*)^{-1}(a)\|$. Here $\|(\mathcal{F}_\Xi^*)^{-1}(a)\|$ denotes the norm of the measure $(\mathcal{F}_\Xi^*)^{-1}(a)$ viewed as an element of the dual Banach space $\mathcal{UC}_b(\Xi)^*$. $\square$

Remark 2.4. Assume the setting of Definition 2.2 again. We note that, due to the continuous inclusion map $\Gamma \hookrightarrow \mathcal{UC}_b(\mathfrak{m}^*)^*$, every function $f \in \mathcal{UC}_b(\mathfrak{m}^*)$ gives rise to a functional $a_f \in \Gamma^*$, $a_f(\gamma) = \langle \gamma, f \rangle$ for every $\gamma \in \Gamma$.

Furthermore, let us assume that the function $f \in \mathcal{UC}_b(\mathfrak{m}^*)$ is the Fourier transform of a Radon measure $\mu \in \mathcal{M}_{\mathrm{t}}(\Xi)$, in the sense that $f(\cdot) = \int_\Xi \mathrm{e}^{\mathrm{i}\langle \cdot, \theta(X) \rangle} \mathrm{d}\mu(X)$.

Then it is straightforward to check that $\mathcal{F}^*_\Xi(\mu) = a_f$, hence one can use Remark 2.3 to see that $\mathrm{Op}(a_f) = \int\limits_\Xi \pi(\exp_M(\theta(\cdot)))\mathrm{d}\mu$ and $\mathrm{Op}(a_f) \in \mathcal{B}(\mathcal{Y})$. □

Preduals for coadjoint orbits. The following notion recovers the magnetic preduals of [BB09a] as very special cases.

Definition 2.5. Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra and pick any coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ of the corresponding Lie group $G = (\mathfrak{g}, *)$ defined by the Baker-Campbell-Hausdorff multiplication $*$. A *predual* for $\mathcal{O}$ is any pair (Ξ, θ), where Ξ is a locally convex real vector space and $\theta\colon \Xi \to \mathfrak{g}$ is a continuous linear map such that $\theta^*|_{\mathcal{O}}\colon \mathcal{O} \to \Xi^*$ is injective. If Ξ is a closed linear subspace of $\mathfrak{g}$ and θ is the inclusion map $\Xi \hookrightarrow \mathfrak{g}$, then we say simply that Ξ is a predual for the coadjoint orbit $\mathcal{O}$. □

The following statement provides a useful criterion for proving that condition (1) in Definition 2.2 is satisfied in the situation of Fréchet preduals (see for instance Sect. 7 in Ch. II of [Sch66]). Here $\mathcal{M}_{\mathrm{t}}(\cdot)$ stands for the space of complex Radon measures on some topological space.

Proposition 2.6. *Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra with the Lie group $G = (\mathfrak{g}, *)$. If (Ξ, θ) is a predual for the coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ such that Ξ is barreled, then the linear mapping $\mathcal{F}_\Xi\colon \mathcal{M}_{\mathrm{t}}(\mathcal{O}) \to \mathcal{UC}_b(\Xi)$, $\mu \mapsto \widehat{\mu} \circ \theta$ is well defined and injective.*

Proof. See [BB10f]. □

Flat coadjoint orbits. We now introduce some terminology that claims its origins in the results of [MW73] on representations of *finite-dimensional* nilpotent Lie groups.

Definition 2.7. Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra with the corresponding Lie group $G = (\mathfrak{g}, *)$, and denote by $\mathfrak{z}$ the center of $\mathfrak{g}$. We shall say that a coadjoint orbit $\mathcal{O}$ $(\hookrightarrow \mathfrak{g}^*)$ is *flat* if the coadjoint isotropy algebra at some point $\xi_0 \in \mathcal{O}$ satisfies the condition $\mathfrak{g}_{\xi_0} = \mathfrak{z}$. □

Proposition 2.8. *Assume that $\mathfrak{g}$ is a nilpotent locally convex Lie algebra with the Lie group $G = (\mathfrak{g}, *)$, and denote by $\mathfrak{z}$ the center of $\mathfrak{g}$. If $\dim \mathfrak{z} = 1$, the coadjoint orbit $\mathcal{O}$ is flat, and $\xi_0 \in \mathcal{O}$ satisfies the condition $\xi_0|_{\mathfrak{z}} \not\equiv 0$, then $\mathrm{Ker}\,\xi_0$ is a predual for $\mathcal{O}$ and the mapping $\mathrm{Ker}\,\xi_0 \simeq \mathcal{O}$, $\quad X \mapsto (\mathrm{Ad}^*_G X)\xi_0$ is a diffeomorphism.*

Proof. See [BB10f]. □

Weyl calculus on flat coadjoint orbits.

Setting 2.9. Until the end of this section we assume the following setting:

- $\mathfrak{g}$ is a nilpotent locally convex Lie algebra with the Lie group $G = (\mathfrak{g}, *)$;
- the topological vector space underlying $\mathfrak{g}$ is barreled (for instance, $\mathfrak{g}$ is a Fréchet-Lie algebra);
- the center $\mathfrak{z}$ of $\mathfrak{g}$ satisfies the condition $\dim \mathfrak{z} = 1$;
- the coadjoint orbit $\mathcal{O}$ $(\hookrightarrow \mathfrak{g}^*)$ is flat, and $\xi_0 \in \mathcal{O}$ satisfies $\xi_0|_{\mathfrak{z}} \not\equiv 0$;
- $\pi\colon G \to \mathcal{B}(\mathcal{Y})$ is an irreducible unitary representation such that for every $X \in \mathfrak{z}$ we have $\pi(X) = \mathrm{e}^{\mathrm{i}\langle \xi_0, X\rangle}\mathrm{id}_{\mathcal{Y}}$.

In addition we shall denote $\Xi := \operatorname{Ker}\xi_0$. This is a closed hyperplane in $\mathfrak{g}$, hence it is in turn a barreled space (see Sect. 7 in Ch. II of [Sch66]).

We have the linear isomorphism $\mathfrak{g}/\mathfrak{z} \simeq \Xi$, so Ξ is also a nilpotent Lie algebra. Let $*_e$ be the corresponding Baker-Campbell-Hausdorff multiplication. We also need the mapping $s\colon \Xi\times\Xi\to\mathfrak{z}$, $s(X,Y) = X*Y - X*_e Y$. □

The following notion is inspired by [Ma07, Def. 2].

Definition 2.10. For arbitrary Radon measures $\mu_1,\mu_2\in\mathcal{M}_{\rm t}(\Xi)$ we define their *twisted convolution product* $\mu_1 *_{\xi_0} \mu_2 \in \mathcal{M}_{\rm t}(\Xi)$ as the push-forward of the measure

$$\mathrm{e}^{\mathrm{i}\langle\xi_0, s(X_1,X_2)\rangle}\mathrm{d}(\mu_1\otimes\mu_2)(X_1,X_2)\in\mathcal{M}_{\rm t}(\Xi\times\Xi)$$

under the multiplication map $\Xi\times\Xi\to\Xi$, $(X_1,X_2)\mapsto X_1 *_e X_2$. □

Theorem 2.11. *Let us assume that we have a locally convex space Γ such that there exists the continuous inclusion map $\Gamma\hookrightarrow\mathcal{M}_{\rm t}(\mathcal{O})$. Then the following assertions hold:*

1. *The linear mapping $\Gamma\to\mathcal{UC}_b(\Xi)$, $\mu\mapsto\widehat{\mu}|_\Xi$ is well defined and injective, and gives rise to the topological linear isomorphism $\mathcal{F}_\Xi\colon\Gamma\to\mathcal{Q}_\Xi$ $(\hookrightarrow\mathcal{UC}_b(\Xi))$.*
2. *If $a_1,a_2\in\Gamma^*$ and $(\mathcal{F}_\Xi^*)^{-1}(a_j)\in\mathcal{Q}_\Xi^*\cap\mathcal{M}_{\rm t}(\Xi)$ for $j=1,2$, then*

$$\mathrm{Op}(a_1)\mathrm{Op}(a_2) = \pi((\mathcal{F}_\Xi^*)^{-1}(a_1) *_{\xi_0} (\mathcal{F}_\Xi^*)^{-1}(a_2)).$$

Proof. See [BB10f]. □

3. The special case of infinite-dimensional Heisenberg groups

Infinite-dimensional Heisenberg groups.

Definition 3.1. Let $\mathcal{V}$ be a real Hilbert space endowed with a symmetric injective operator $A\in\mathcal{B}(\mathcal{V})$.

The *Heisenberg algebra* associated with the pair $(\mathcal{V},A)$ is $\mathfrak{h}(\mathcal{V},A)=\mathcal{V}\dotplus\mathcal{V}\dotplus\mathbb{R}$ with the Lie bracket $[(x_1,y_1,t_1),(x_2,y_2,t_2)] = (0,0,(Ax_1\mid y_2)-(Ax_2\mid y_1))$. This is a nilpotent Lie algebra. The corresponding Lie group $\mathbb{H}(\mathcal{V},A)=(\mathfrak{h}(\mathcal{V},A),*)$ is the *Heisenberg group* associated with $(\mathcal{V},A)$, with the multiplication given by

$$(x_1,y_1,t_1)*(x_2,y_2,t_2) = (x_1+x_2, y_1+y_2, t_1+t_2+((Ax_1\mid y_2)-(Ax_2\mid y_1))/2)$$

whenever $(x_1,y_1,t_1),(x_2,y_2,t_2)\in\mathbb{H}(\mathcal{V},A)$. □

Example 3.2. If $\mathfrak{h}(\mathcal{V},A)=\mathcal{V}\dotplus\mathcal{V}\dotplus\mathbb{R}$ is a Heisenberg algebra and $\xi_0\in\mathfrak{h}(\mathcal{V},A)^*$ as in Definition 3.1, then it is easy to see that $\Xi:=\mathcal{V}\times\mathcal{V}\times\{0\}$ is a predual for the coadjoint orbit of ξ_0. □

Remark 3.3. Let **LieGr** denote the category of infinite-dimensional Lie groups modeled on locally convex spaces (see [Ne06]) and denote by **QuadrHilb** the category whose objects are the pairs $(\mathcal{V},A)$ where $\mathcal{V}$ is a real Hilbert space and $A\in\mathcal{B}(\mathcal{V})$ is a symmetric, nonnegative, injective operator. The morphisms between two objects $(\mathcal{V}_1,A_1)$ and $(\mathcal{V}_2,A_2)$ in **QuadrHilb**, are defined as the continuous linear operators $T\colon\mathcal{V}_1\to\mathcal{V}_2$ satisfying the condition $T^*A_2T=A_1$. (Equivalently, T becomes an isometry if $\mathcal{V}_j$ is endowed with the continuous inner product $(x,y)\mapsto(Ax\mid y)_{\mathcal{V}_j}$ for $j=1,2$.)

Then we have a natural functor $\mathbb{H}\colon$ **QuadrHilb** $\to$ **LieGr** such that the image of any object $(\mathcal{V},A)$ in **QuadrHilb** is the corresponding Heisenberg group

$\mathbb{H}(\mathcal{V}, A)$ constructed in Definition 3.1. For every morphism $T\colon (\mathcal{V}_1, A_1) \to (\mathcal{V}_2, A_2)$ in **QuadrHilb** as above, we have the corresponding morphism

$$\mathbb{H}(T)\colon \mathbb{H}(\mathcal{V}_1, A_1) \to \mathbb{H}(\mathcal{V}_2, A_2), \quad (x, y, t) \mapsto (Tx, Ty, t)$$

in **LieGr**. □

Gaussian measures and Schrödinger representations. Let $\mathcal{V}_-$ be a real Hilbert space with the scalar product denoted by $(\cdot \mid \cdot)_-$. For every vector $a \in \mathcal{V}_-$ and every symmetric, nonnegative, injective, *trace-class* operator K on $\mathcal{V}_-$ there exists a unique probability Borel measure γ on $\mathcal{V}_-$ such that

$$(\forall x \in \mathcal{V}_-) \quad \int_{\mathcal{V}_-} \mathrm{e}^{\mathrm{i}(x|y)_-} \mathrm{d}\gamma(y) = \mathrm{e}^{\mathrm{i}(a|x)_- - \frac{1}{2}(Kx|x)_-}$$

(see for instance [Bg98, Th. 2.3.1]). We also have

$$a = \int_{\mathcal{V}_-} y \mathrm{d}\gamma(y) \quad \text{and} \quad Kx = \int_{\mathcal{V}_-} (x \mid y)_- \cdot (y - a) \mathrm{d}\gamma(y) \text{ for all } x \in \mathcal{V}_-,$$

where the integrals are weakly convergent, and γ is called the *Gaussian measure with the mean a and the variance K*.

Let us assume that the Gaussian measure γ is centered, that is, $a = 0$. Denote $\mathcal{V}_+ := \operatorname{Ran} K$ and $\mathcal{V}_0 := \operatorname{Ran} K^{1/2}$ endowed with the scalar products $(Kx \mid Ky)_+ := (x \mid y)_-$ and $(K^{1/2}x \mid K^{1/2}y)_0 := (x \mid y)_-$, respectively, for all $x, y \in \mathcal{V}_-$, which turn the linear bijections $K\colon \mathcal{V}_- \to \mathcal{V}_+$ and $K^{1/2}\colon \mathcal{V}_- \to \mathcal{V}_0$ into isometries. We thus get the real Hilbert spaces

$$\mathcal{V}_+ \hookrightarrow \mathcal{V}_0 \hookrightarrow \mathcal{V}_-$$

where the inclusion maps are Hilbert-Schmidt operators, since $K^{1/2} \in \mathcal{B}(\mathcal{V}_-)$ is so. Also, the scalar product of $\mathcal{V}_0$ extends to a duality pairing $(\cdot \mid \cdot)_0\colon \mathcal{V}_- \times \mathcal{V}_+ \to \mathbb{R}$.

We also recall that for every $x \in \mathcal{V}_+$ the translated measure $\mathrm{d}\gamma(-x + \cdot)$ is absolutely continuous with respect to $\mathrm{d}\gamma(\cdot)$ and we have the Cameron-Martin formula

$$\mathrm{d}\gamma(-x + \cdot) = \rho_x(\cdot)\mathrm{d}\gamma(\cdot) \quad \text{with } \rho_x(\cdot) = \mathrm{e}^{(\cdot|x)_0 - \frac{1}{2}(x|x)_0}.$$

(This actually holds true for every $x \in \mathcal{V}_0$; see for instance [Bg98, Cor. 2.4.3].)

Definition 3.4. Let $(\mathcal{V}_+, A)$ be an object in the category **QuadrHilb** such that $A\colon \mathcal{V}_+ \to \mathcal{V}_+$ is a nonnegative, symmetric, injective, trace-class operator. Denote the scalar product of $\mathcal{V}_+$ by $(x, y) \mapsto (x \mid y)_+$ and let $\mathcal{V}_0$ and $\mathcal{V}_-$ be the completions of $\mathcal{V}_+$ with respect to the scalar products $(x, y) \mapsto (x \mid y)_0 := (A^{1/2} \mid A^{1/2}y)$ and $(x, y) \mapsto (x \mid y)_- := (Ax \mid Ay)$, respectively. Then the operator A has a unique extension to a nonnegative, symmetric, injective, trace-class operator $K \in \mathcal{B}(\mathcal{V}_-)$ such that the above setting is recovered (see for instance [Be86, Ch. 1, §1]), hence we get the centered Gaussian measure γ on $\mathcal{V}_-$ with the variance K.

On the other hand, we can construct the Heisenberg group $\mathbb{H}(\mathcal{V}_+, A)$. The *Schrödinger representation* $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma))$ is defined by

$$\pi(x, y, t)\phi = \rho_x(\cdot)^{1/2} \mathrm{e}^{\mathrm{i}(t + (\cdot|y)_0 + \frac{1}{2}(x|y)_0)} \phi(-x + \cdot)$$

whenever $(x, y, t) \in \mathbb{H}(\mathcal{V}_+, A)$ and $\phi \in L^2(\mathcal{V}_-, \gamma)$. This is a continuous unitary irreducible representation of the Heisenberg group $\mathbb{H}(\mathcal{V}_+, A)$; see [Höb06, Th. 5.2.9 and 5.2.10]. □

Remark 3.5. We note that more general Schrödinger representations of infinite-dimensional Heisenberg groups are described in [Ne00, Prop. II.4.6] by using cocycles and reproducing kernel Hilbert spaces. $\square$

Remark 3.6. One way to see that the representation $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma))$ of Definition 3.4 is irreducible is the following For every integer $n \geq 1$ let $\mathcal{V}_{n,+}$ denote the spectral space for A corresponding to the interval $[1/n, \infty)$. That is, $\mathcal{V}_{n,+}$ is spanned by the eigenvectors of A corresponding to eigenvalues $\geq 1/n$. Since A is a compact operator, it follows that $\dim \mathcal{V}_{n,+} < \infty$. We have

$$\mathcal{V}_{1,+} \subseteq \mathcal{V}_{2,+} \subseteq \cdots \subseteq \bigcup_{n\geq 1} \mathcal{V}_{n,+} \subseteq \mathcal{V}_+$$

and $\bigcup\limits_{n\geq 1} \mathcal{V}_{n,+}$ is a dense subspace of $\mathcal{V}_+$. Let us denote by A_n the restriction of A to $\mathcal{V}_{n,+}$. Then $\mathbb{H}(\mathcal{V}_{n,+}, A_n)$ is a finite-dimensional Heisenberg group, hence it is well known that its Schrödinger representation $\pi_n\colon \mathbb{H}(\mathcal{V}_{n,+}, A_n) \to \mathcal{B}(L^2(\mathcal{V}_{n,-}, \gamma_n))$ is irreducible, where γ_n is the Gaussian measure on the finite-dimensional space $\mathcal{V}_{n,-}$ obtained out of the pair $(\mathcal{V}_{+,n}, A_n)$ by the construction outlined at the very beginning of Definition 3.4. Note that

$$\mathbb{H}(\mathcal{V}_{1,+}, A_1) \subseteq \mathbb{H}(\mathcal{V}_{2,+}, A_2) \subseteq \cdots \subseteq \bigcup_{n\geq 1} \mathbb{H}(\mathcal{V}_{n,+}, A_n) =: \mathbb{H}(\mathcal{V}_{\infty,+}, A_\infty) \subseteq \mathbb{H}(\mathcal{V}_+, A)$$

and $\mathbb{H}(\mathcal{V}_{\infty,+}, A_\infty)$ is a dense subgroup of $\mathbb{H}(\mathcal{V}_+, A)$, hence the Schrödinger representation $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma))$ is irreducible if and only if so is its restriction $\pi|_{\mathbb{H}(\mathcal{V}_{\infty,+}, A_\infty)}$.

On the other hand, if we denote by $\mathbf{1}_n$ the function identically equal to 1 on the orthogonal complement $\mathcal{V}_{n+1,+} \ominus \mathcal{V}_{n,+}$, then it is straightforward to check that the operator

$$L^2(\mathcal{V}_{n,-}, \gamma_n) \to L^2(\mathcal{V}_{n+1,-}, \gamma_{n+1}), \quad f \mapsto f \otimes \mathbf{1}_n$$

is unitary and intertwines the representations π_n and π_{n+1}. We can thus make the sequence of representations $\{\pi_n\}_{n\geq 1}$ into an inductive system of irreducible unitary representations and then their inductive limit $\pi|_{\mathbb{H}(\mathcal{V}_{\infty,+}, A_\infty)} = \operatorname*{ind}\limits_{n\to\infty} \pi_n$ is irreducible (see for instance [KS77]). As noted above, this implies that the Schrödinger representation $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma))$ of Definition 3.4 is irreducible. $\square$

The infinite-dimensional pseudo-differential calculus of [AD96] and [AD98] can be recovered as a quasi-localized Weyl calculus for the Schrödinger representations introduced in Definition 3.4 above. Compare for instance [AD98, Prop. 3.7].

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