

COHEN-MACAULAYNESS OF LARGE POWERS OF STANLEY-REISNER IDEALS

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ABSTRACT. We prove that for $m \geq 3$, the symbolic power $I_{\Delta}^{(m)}$ of a Stanley-Reisner ideal is Cohen-Macaulay if and only if the simplicial complex Δ is a matroid. Similarly, the ordinary power I_{Δ}^m is Cohen-Macaulay for some $m \geq 3$ if and only if I_{Δ} is a complete intersection. These results solve several open questions on the Cohen-Macaulayness of ordinary and symbolic powers of Stanley-Reisner ideals. Moreover, they have interesting consequences on the Cohen-Macaulayness of symbolic powers of facet ideals and cover ideals.

1. INTRODUCTION

In this article we study the Cohen-Macaulayness of ordinary and symbolic powers of squarefree ideals. More concretely, we consider such an ideal as the Stanley-Reisner ideal I_{Δ} of a simplicial complex Δ and we give conditions on Δ such that the ordinary or symbolic powers I_{Δ}^m or $I_{\Delta}^{(m)}$ is Cohen-Macaulay for a fixed number $m \geq 3$.

A classical theorem [3] implies that all ordinary powers I_{Δ}^m are Cohen-Macaulay if and only if I_{Δ} is a complete intersection. On the other hand, there are examples such that the second power I_{Δ}^2 is Cohen-Macaulay but the third power I_{Δ}^3 is not Cohen-Macaulay. It does not deny any logical possibility that there may exist a Stanley-Reisner ideal such that its 100-th power is Cohen-Macaulay but its 101-th power is not. Hence it is of great interest to find combinatorial conditions on Δ which guarantees the Cohen-Macaulayness of a given ordinary power of I_{Δ} . Since I_{Δ}^m is Cohen-Macaulay if and only if $I_{\Delta}^{(m)}$ is Cohen-Macaulay and $I_{\Delta}^{(m)} = I_{\Delta}^m$, one may raise the same problem on the Cohen-Macaulayness of a given symbolic power of I_{Δ} .

Studies in this direction began in [4, 15, 17] for the cases $\dim \Delta = 1$ (Δ is a graph) or Δ is a flag complex (I_{Δ} is generated by quadratic monomials), where one could give precise combinatorial conditions for the Cohen-Macaulayness of each power I_{Δ}^m or $I_{\Delta}^{(m)}$ in terms of Δ . For the general case, one was only able to give conditions for the Cohen-Macaulayness of $I_{\Delta}^{(2)}$ and I_{Δ}^2 [16, 18]. To characterize the

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Cohen-Macaulayness of $I_{\Delta}^{(m)}$ and I_{Δ}^m seems to be a difficult problem. In fact, there were open questions which would have an answer if one could solve this problem.

First of all, results on the preservation of Cohen-Macaulayness among large powers of I_{Δ} [16] led to the following

Question 1. Is $I_{\Delta}^{(m)}$ Cohen-Macaulay if $I_{\Delta}^{(m+1)}$ is Cohen-Macaulay?

Question 2. Does there exist a number t depending on $\dim \Delta$ such that if $I_{\Delta}^{(t)}$ is Cohen-Macaulay, then $I_{\Delta}^{(m)}$ is Cohen-Macaulay for every $m \geq 1$?

If $\dim \Delta = 2$, there is a complete classification of complexes possessing a Cohen-Macaulay ordinary power in [25]. Together with results for $\dim \Delta = 1$ [15] and for flag complexes [17] this suggests the following

Question 3. Is I_{Δ}^m Cohen-Macaulay for some $m \geq 3$ if and only if I_{Δ} is a complete intersection?

On the other hand, works on the Buchsbaumness or on Serre condition (S_2) of ordinary and symbolic powers of Stanley-Reisner ideals in [13, 14, 17, 24] showed that these properties are strongly related to the Cohen-Macaulayness and that one may raise the following

Question 4. Is $I_{\Delta}^{(m)}$ (or I_{Δ}^m) Cohen-Macaulay if $I_{\Delta}^{(m)}$ (or I_{Δ}^m) is (quasi-)Buchsbaum or satisfies (S_2) for large m ?

This paper will give a positive answer to all these questions. More precisely, we prove the following characterizations of the Cohen-Macaulayness of I_{Δ}^m and $I_{\Delta}^{(m)}$ for $m \geq 3$.

Theorem 1.1. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then the following conditions are equivalent:*

- (i) $I_{\Delta}^{(m)}$ is Cohen-Macaulay for every $m \geq 1$.
- (ii) $I_{\Delta}^{(m)}$ is Cohen-Macaulay for some $m \geq 3$.
- (iii) $I_{\Delta}^{(m)}$ satisfies (S_2) for some $m \geq 3$.
- (iv) $I_{\Delta}^{(m)}$ is (quasi-)Buchsbaum for some $m \geq 3$.
- (v) Δ is a matroid.

Theorem 1.2. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then the following conditions are equivalent:*

- (i) I_{Δ}^m is Cohen-Macaulay for every $m \geq 1$.
- (ii) I_{Δ}^m is Cohen-Macaulay for some $m \geq 3$.
- (iii) I_{Δ}^m satisfies (S_2) for some $m \geq 3$.
- (iv) I_{Δ}^m is (quasi-)Buchsbaum for some $m \geq 3$.
- (v) Δ is a complete intersection.

These theorems are remarkable in the sense that the Cohen-Macaulayness of $I_{\Delta}^{(m)}$ or I_{Δ}^m is equivalent to much weaker properties and that it can be characterized in purely combinatorial terms *which are the same for all $m \geq 3$* . Both theorems except condition (iii) also hold for $\dim \Delta = 1$ [15]. If $\dim \Delta = 1$, the Buchsbaumness

behaves a little bit differently. Minh-Nakamura [13] showed that for a graph Δ , $I_\Delta^{(3)}$ is Buchsbaum if and only if $I_\Delta^{(2)}$ is Cohen-Macaulay and that for $m \geq 4$, $I_\Delta^{(m)}$ is Cohen-Macaulay if $I_\Delta^{(m)}$ is Buchsbaum. Similar results also hold for the Buchsbaum property of I_Δ^m [14].

The equivalence (i) \Leftrightarrow (v) of the first theorem was discovered in [16, 26]. This result establishes an unexpected link between a purely algebraic property with a vast area of combinatorics. The equivalence (i) \Leftrightarrow (vi) of the second theorem was proved by Cowsik-Nori [3] under much more general setting. Our new contributions are (iii) \Rightarrow (v) and (iv) \Rightarrow (v) in both theorems. The proofs involve both algebraic and combinatorial arguments.

The idea for the proof of (iii) \Rightarrow (v) comes from the fact that matroids and complete intersection complexes can be characterized by properties of their links. We call Δ locally a matroid or a complete intersection if the links of Δ at the vertices are matroids or complete intersections, respectively. It turns out that a complex is a matroid if and only if it is connected and locally a matroid (Theorem 2.7). A similar result on complete intersection was already proved in [24]. The connectedness of Δ can be studied by using Takayama's formula for the local cohomology modules of monomial ideals [23]. Since the links of Δ correspond to localizations of I_Δ , these results allow us to reduce our investigation to the one-dimensional case for which everything is known by [15].

The proof of (iv) \Rightarrow (v) follows from our investigation on the generalized Cohen-Macaulayness of $I_\Delta^{(m)}$ or I_Δ^m . It is known that (quasi-)Buchsbaum rings are generalized Cohen-Macaulay. In general, it is difficult to classify generalized Cohen-Macaulay ideals because this class of ideals is too large. However, we can prove that $I_\Delta^{(m)}$ is generalized Cohen-Macaulay for some $m \geq 3$ or for every $m \geq 1$ if and only if Δ is a union of disjoint matroids of the same dimension (Theorem 3.7). Similarly, I_Δ^m is generalized Cohen-Macaulay for some $m \geq 3$ or for every $m \geq 1$ if and only if Δ is a union of disjoint complete intersections of the same dimension (Theorem 4.5). A weaker version of this result was proved earlier by Goto-Takayama [8]. For flag complexes it was proved in [17].

As applications we study the Cohen-Macaulayness of symbolic powers of the facet ideal $I(\Delta)$ and the cover ideal $J(\Delta)$, which are generated by the squarefree monomials of the facets of Δ or of their covers, respectively.

To study ideals generated by squarefree monomials of the same degree r means to study facet ideals of pure complexes of dimension $r - 1$. By [17] we know that for a pure complex Δ with $\dim \Delta = 1$, $I(\Delta)^{(m)}$ is Cohen-Macaulay for some $m \geq 3$ or for every $m \geq 1$ if and only if Δ is a union of disjoint 1-uniform matroids, where a matroid is called r -uniform if it is generated by the r -dimensional faces of a simplex. This gives a structure theorem for squarefree monomial ideals generated in degree 2 whose symbolic powers are Cohen-Macaulay. It is natural to ask whether there are similar results for squarefree monomial ideals generated in degree ≥ 3 . We prove that for a pure complex Δ with $\dim \Delta = 2$, $I(\Delta)^{(m)}$ is Cohen-Macaulay for some

$m \geq 3$ or for every $m \geq 1$ if and only if Δ is a union of disjoint 2-uniform matroids (Theorem 5.2), and we show that a similar result doesn't hold for $\dim \Delta \geq 3$.

A cover ideal $J(\Delta)$ can be understood as the intersection of prime ideals generated by the variables of the facets of Δ . It turns out that for $m \geq 3$, $J(\Delta)^{(m)}$ is Cohen-Macaulay if and only if Δ is a matroid (Theorem 5.6) so that the Cohen-Macaulayness of $J(\Delta)^{(m)}$ is equivalent to that of $I_\Delta^{(m)}$. This result is somewhat surprising because the Cohen-Macaulayness of I_Δ usually has nothing to do with that of $J(\Delta)$. As a consequence we can say exactly when all symbolic powers of a squarefree monomial ideal of codimension 2 is Cohen-Macaulay.

Summing up we can say that our results provide a framework for the study of the Cohen-Macaulayness and other ring-theoretic properties of large ordinary and symbolic powers of squarefree monomial ideals.

The paper is organized as follows. In Section 2 we prepare basic facts and properties of simplicial complexes, especially of localizations and matroids. Section 3 and Section 4 deal with the Cohen-Macaulayness of symbolic and ordinary powers of Stanley-Reisner ideals. In Section 5 we study the Cohen-Macaulayness of symbolic powers of facet ideals and cover ideals. For unexplained terminology we refer to [1] and [21].

2. LOCALIZATIONS AND MATROIDS

Throughout this article let K be an arbitrary field and $[n] = \{1, \dots, n\}$. Let Δ be a (simplicial) complex on the vertex set $V(\Delta) = [n]$. The *Stanley-Reisner ideal* I_Δ of Δ (over K) is defined as the squarefree monomial ideal

$$I_\Delta = (x_{i_1}x_{i_2}\cdots x_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n, \{i_1, \dots, i_p\} \notin \Delta)$$

in the polynomial ring $S = K[x_1, x_2, \dots, x_n]$. Note that the sets $\{i_1, \dots, i_p\} \notin \Delta$ are called *nonfaces* of Δ . It is obvious that this association gives an one-to-one correspondence between simplicial complexes on the vertex set $[n]$ and squarefree monomial ideals in S which do not contain any variable.

We will consider a graph as a complex, and we will always assume that a graph has no multiple edges, no loops and no isolated vertices.

For every subset $F \subseteq [n]$, we set $P_F = (x_i \mid i \in F)$. Let $\mathcal{F}(\Delta)$ denote the set of the facets of Δ . We have the following prime decomposition of I_Δ :

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\overline{F}},$$

where \overline{F} denotes the complement of F .

To describe the localization of I_Δ at a set of variables we need the following notations. For every face $G \in \Delta$, we define

$$\begin{aligned} \text{st}_\Delta G &:= \{F \in \Delta : F \cup G \in \Delta\}, \\ \text{lk}_\Delta G &:= \{F \setminus G \in \Delta : G \subset F \in \Delta\}, \end{aligned}$$

and call these subcomplexes of Δ the *star* of G or the *link* of G , respectively.

By the definition of the Stanley-Reisner ideal, $I_{\text{st}_\Delta G}$ and $I_{\text{lk}_\Delta G}$ have the same (minimal) monomial generators though they lie in different polynomial subrings of S if $G \neq \emptyset$.

Lemma 2.1. *Let $Y = \{x_i \mid i \notin V(\text{st}_\Delta G)\}$. Then*

$$I_\Delta S[x_i^{-1} \mid i \in G] = (I_{\text{lk}_\Delta G}, Y)S[x_i^{-1} \mid i \in G].$$

Proof. It is easily seen that

$$\begin{aligned} I_\Delta S[x_i^{-1} \mid i \in G] &= \bigcap_{F \in \mathcal{F}(\text{st}_\Delta G)} P_{\bar{F}} S[x_i^{-1} \mid i \in G] \\ &= (I_{\text{st}_\Delta G}, Y)S[x_i^{-1} \mid i \in G] \\ &= (I_{\text{lk}_\Delta G}, Y)S[x_i^{-1} \mid i \in G]. \end{aligned}$$

□

Remark 2.2. Let $R = K[x_i \mid i \in V(\text{lk}_\Delta G)]$ and $T = K[x_i \mid x_i \notin G]$. Then $T = R[Y]$ is a polynomial ring over R and $S[x_i^{-1} \mid i \in G] = T[x_i^{\pm 1} \mid x_i \in G]$ is a Laurent polynomial ring over T . Since $I_{\text{lk}_\Delta G}$ is an ideal in R , the variables of Y forms a regular sequence on $I_{\text{lk}_\Delta G}T$. Therefore, properties of $I_\Delta S[x_i^{-1} \mid i \in G]$ and $I_{\text{lk}_\Delta G}$ are strongly related to each other. So we may think of $I_{\text{lk}_\Delta G}$ as the combinatorial localization of Δ at G .

For simplicity we say that a simplicial complex Δ is a *complete intersection* if I_Δ is a complete intersection. Combinatorially, this means that the minimal nonfaces of Δ are disjoint. We say that Δ is *locally a complete intersection* if $\text{lk}_\Delta\{i\}$ is a complete intersection for $i = 1, \dots, n$.

There are the following relationship between complete intersections and locally complete intersections, which will play an essential role in our investigation on ordinary powers.

Lemma 2.3. [24, Theorem 1.5] *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then Δ is a complete intersection if and only if Δ is connected and locally a complete intersection.*

Lemma 2.4. [24, Theorem 1.15] *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then Δ is locally a complete intersection if and only if Δ is a union of disjoint complete intersections.*

Now we want to prove similar results for matroids. Recall that a *matroid* is a collection of subsets of a finite set, called independent sets, with the following properties:

- (i) The empty set is independent.
- (ii) Every subset of an independent set is independent.
- (iii) If F and G are two independent sets and F has more elements than G , then there exists an element in F which is not in G that when added to G still gives an independent set.

We may consider a matroid as a simplicial complex. It is easy to see that induced subcomplexes, stars and links of faces of matroids are again matroids. Moreover, every matroid is a pure complex, that is, all facets have the same dimension.

We shall need the following criterion for a simplicial complex to be a matroid.

Lemma 2.5. [20, Theorem 39.1] *A simplicial complex Δ is a matroid if and only if for any pair of faces F, G of Δ with $|F \setminus G| = 1$ and $|G \setminus F| = 2$, there is a vertex $i \in G \setminus F$ with $F \cup \{i\} \in \Delta$.*

For a graph this characterization can be reformulated as follows.

Corollary 2.6. *A graph Γ without isolated vertices is a matroid if and only if every pair of disjoint edges is contained in a 4-cycle.*

Proof. Assume that Γ is a matroid. By Theorem 2.5, every vertex i is connected to every edge $\{u, v\}$ by an edge (of Γ). Let $\{i, j\}$ and $\{u, v\}$ be two disjoint edges. We may assume that $\{i, u\}$ is an edge. If $\{j, v\}$ is an edge, then i, u, v, j are the ordered vertices of a 4-cycle of G . If $\{j, v\}$ is not an edge, then $\{j, u\}$ must be an edge. Since either i or j is connected with v by an edge, we conclude that $\{i, j\}$ and $\{u, v\}$ are always contained in a 4-cycle.

Conversely, assume that every pair of disjoint edges is contained in a 4-cycle. Let i be an arbitrary vertex and $\{u, v\}$ an edge not containing i . If i isn't connected with $\{u, v\}$ by an edge, there is an edge $\{i, j\}$ such that $j \neq u, v$. But then $\{i, j\}$ and $\{u, v\}$ are contained in a 4-cycle so that i is connected with $\{u, v\}$ by an edge, which is a contradiction. \square

We say that Δ is *locally a matroid* if $\text{lk}_\Delta\{i\}$ is a matroid for every vertex i of Δ . This notion will play an essential role in our investigation on symbolic powers.

Theorem 2.7. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then Δ is a matroid if and only if Δ is connected and locally a matroid.*

Proof. The necessity can be easily seen from the definition of matroids. To prove the sufficiency assume that Δ is connected and locally a matroid.

We show first that Δ is pure. Let $\dim \Delta = d$. Assume for the contrary that there is a facet F with $\dim F < d$. Since $\text{lk}_\Delta\{v\}$ is a matroid for every vertex v and since every matroid is pure, no vertex v of F is contained in a facet G with $\dim G = d$. Let Γ be the graph of the one-dimensional faces of Δ . Let r be the minimal length of a path of Γ which connects a vertex of F with a vertex of a facet G with $\dim G = d$. Then $r \geq 1$. Let v_0, \dots, v_r be the ordered vertices of such a path. Let H be a facet containing the edge $\{v_{r-1}, v_r\}$. Since $G \setminus \{v_r\}$ and $H \setminus \{v_r\}$ are facets of the matroid $\text{lk}_\Delta\{x_r\}$, they have the same dimension. Therefore, $\dim H = \dim G = d$. Since x_{r-1} is a vertex of H , we obtain a contradiction to the minimality of r .

Next we show that two arbitrary vertices u, v are connected by a path of Γ of length at most 2. For that we only need to prove that if u, v, w, t are the ordered vertices of a path of length 3, then u and t are connected by a path of length 2. We may assume that $\{u, w\}, \{v, t\} \notin \Delta$. Since Δ is pure and $\dim \Delta \geq 2$, there is another vertex s such that $\{v, w, s\} \in \Delta$. Therefore, $\{w, s\} \in \text{lk}_\Delta\{v\}$. Since $\text{lk}_\Delta\{v\}$

is a matroid, so is $\text{st}_\Delta\{v\}$. Therefore, the graph of the one-dimensional faces of $\text{st}_\Delta\{v\}$ is also a matroid. By Corollary 2.6, every pair of disjoint edges of $\text{st}_\Delta\{v\}$ is contained in a 4-cycle. Since $\{u, w\} \notin \Delta$, we must have $\{u, s\} \in \Delta$. Similarly, we also have $\{s, t\} \in \Delta$. Hence $\{u, s\}, \{s, t\}$ form a path of length 2.

By Theorem 2.5, to show that Δ is a matroid we only need to show that if F and G are two faces of Δ with $|F \setminus G| = 1$ and $|G \setminus F| = 2$, then there is a vertex $i \in G \setminus F$ with $F \cup \{i\} \in \Delta$. Since Δ is locally a matroid, we may assume that F and G are not faces of the link of any vertex. From this it follows that $F \cap G = \emptyset$. Hence $|F| = 1$ and $|G| = 2$.

Let $F = \{u\}$ and $G = \{v, w\}$. Assume for the contrary that $\{u, v\}, \{u, w\} \notin \Delta$. Since u, v are connected by a path of length 2, there is a vertex t such that $\{u, t\}, \{v, t\} \in \Delta$. Since $\{u\} \in \text{lk}_\Delta\{t\}$, $\{v, w\} \notin \text{lk}_\Delta\{t\}$ by our assumption on F and G . This implies $\{t, w\} \notin \text{lk}_\Delta\{v\}$. Let s be a vertex of a facet of Δ containing $\{t, v\}$. Since $\{w\}, \{s, t\} \in \text{lk}_\Delta\{v\}$ and since $\text{lk}_\Delta\{v\}$ is a matroid, we must have $\{s, w\} \in \text{lk}_\Delta\{v\}$ by Theorem 2.5. This implies $\{v, w\} \in \text{lk}_\Delta\{s\}$. Hence $\{u\} \notin \text{lk}_\Delta\{s\}$ by the assumption on F and G . On the other hand, since $\{u\}, \{s, v\} \in \text{lk}_\Delta\{t\}$ and since $\text{lk}_\Delta\{t\}$ is a matroid, we must have $\{s, u\} \in \text{lk}_\Delta\{t\}$, which implies $\{u\} \in \text{lk}_\Delta\{s\}$, a contradiction. \square

Corollary 2.8. *Let Δ be a pure simplicial complex with $\dim \Delta \geq 2$. Then Δ is locally a matroid if and only if Δ is a union of disjoint matroids.*

Proof. Assume that Δ is locally a matroid. Let Γ be a connected component of Δ . Since Δ is pure, $\dim \Gamma = \dim \Delta \geq 2$. For any vertex v of Γ we have $\text{lk}_\Gamma v = \text{lk}_\Delta v$. Hence Γ is also locally a matroid. By Theorem 2.7, this implies that Γ is a matroid.

Conversely, let Δ be a union of disjoint matroids. For any vertex v of Δ we have $\text{lk}_\Delta v = \text{lk}_\Gamma v$, where Γ is the connected component containing v . Since Γ is a matroid, $\text{lk}_\Gamma v$ is a matroid. Hence Δ is locally a matroid. \square

Theorem 2.7 and Corollary 2.8 don't hold if $\dim \Delta = 1$. In fact, any graph is locally a matroid but there are plenty connected graphs which aren't matroids.

3. COHEN-MACAULAYNESS OF LARGE SYMBOLIC POWERS

Let Δ be a simplicial complex on the vertex set $V(\Delta) = [n]$. For every number $m \geq 1$, the m -th symbolic power of I_Δ is the ideal

$$I_\Delta^{(m)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^m.$$

By [16, Theorem 3.5] or [26, Theorem], $I_\Delta^{(m)}$ is Cohen-Macaulay for every $m \geq 1$ if and only if Δ is a matroid. We will show in this section that Δ is a matroid if $I_\Delta^{(m)}$ satisfies some weaker property than the Cohen-Macaulayness. The idea comes from Theorem 2.7 which says that a complex is a matroid if and only if it is locally a matroid and connected. The property of being locally a matroid can be passed to the one-dimensional case where everything is known [15]. Therefore, it remains to study the connectedness of Δ . For that we need the following result of Takayama on local cohomology modules of monomial ideals.

Let I be a monomial ideal of $S = K[x_1, \dots, x_n]$. Since S/I has a natural \mathbb{N}^n -graded structure, its local cohomology modules $H_{\mathfrak{m}}^i(S/I)$ with respect to the ideal $\mathfrak{m} = (x_1, \dots, x_n)$ have an \mathbb{Z}^n -graded structure. For every $\mathbf{a} \in \mathbb{Z}^n$ let $[H_{\mathfrak{m}}^i(S/I)]_{\mathbf{a}}$ denote the \mathbf{a} -component of $H_{\mathfrak{m}}^i(S/I)$.

Lemma 3.1. [23, Theorem 2.2] *There is a simplicial complex $\Delta_{\mathbf{a}}$ such that*

$$[H_{\mathfrak{m}}^i(S/I)]_{\mathbf{a}} \cong \tilde{H}^{i-1}(\Delta_{\mathbf{a}}, K),$$

where $\tilde{H}^{i-1}(\Delta_{\mathbf{a}}, K)$ is the $(i-1)$ th reduced cohomology group of $\Delta_{\mathbf{a}}$ with coefficients in K .

By [16, Lemma 1.2] the complex $\Delta_{\mathbf{a}}$ can be described as follows. For $\mathbf{a} = (a_1, \dots, a_n)$ let $G_{\mathbf{a}} = \{i \mid a_i < 0\}$ and $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Then $\Delta_{\mathbf{a}}$ is the complex of all sets of the form $F \setminus G_{\mathbf{a}}$, where F is a subset of $[n]$ containing $G_{\mathbf{a}}$ such that $x^{\mathbf{a}} \notin IS[x_i^{-1} \mid i \in F]$.

Example 3.2. For $\mathbf{a} = \mathbf{0}$ we have $G_{\mathbf{0}} = \emptyset$ and $x_{\mathbf{0}} = 1$. Hence $\Delta_{\mathbf{0}}$ is the complex of all $F \subseteq [n]$ such that $1 \notin IS[x_i^{-1} \mid i \in F]$ or, equivalently, $1 \notin \sqrt{I}S[x_i^{-1} \mid i \in F]$. Since \sqrt{I} is a squarefree monomial ideal, we can find a complex Δ such that

$$\sqrt{I} = I_{\Delta} = \bigcap_{G \in \mathcal{F}(\Delta)} P_{\overline{G}}.$$

Thus, $F \in \Delta_{\mathbf{0}}$ if and only if $F \cap \overline{G} = \emptyset$ for all $G \in \mathcal{F}(\Delta)$. Hence $\Delta_{\mathbf{0}} = \Delta$.

Lemma 3.3. *Let Δ be a simplicial complex with $\dim \Delta \geq 1$ and I a monomial ideal with $\sqrt{I} = I_{\Delta}$. Then Δ is connected if $\text{depth } S/I \geq 2$.*

Proof. If $\text{depth } S/I \geq 2$, we have $H_{\mathfrak{m}}^1(S/I) = 0$. By Lemma 3.1, this implies $\tilde{H}^0(\Delta_{\mathbf{0}}, K) = 0$. By the above example we know that $\Delta_{\mathbf{0}} = \Delta$. Hence Δ is connected. \square

We say that an ideal I in S (or S/I) satisfies *Serre condition* (S_2) if $\text{depth}(S/I)_P \geq \min\{2, \text{ht } P\}$ for every prime ideal P of S .

The condition (S_2) as well as the Cohen-Macaulayness of $I_{\Delta}^{(m)}$ can be passed to the links of Δ . To see that we need the following observation.

Lemma 3.4. *Let I be a squarefree monomial ideal in a polynomial ring R . Let $T := R[y]$ be a polynomial ring over R . Then*

- (i) $(I, y)^{(m)}$ satisfies (S_2) if and only if $I^{(k)}$ satisfies (S_2) for every k with $1 \leq k \leq m$.
- (ii) $(I, y)^{(m)}$ is Cohen-Macaulay if and only if $I^{(k)}$ is Cohen-Macaulay for every k with $1 \leq k \leq m$.

Proof. Let $I = \bigcap_j P_j$ be the minimal prime decomposition of I . Since $(I, y)T = \bigcap_j (P_j, y)$ is the minimal prime decomposition of $(I, y)T$,

$$(I, y)^{(m)} = \bigcap_j (P_j, y)^m = \bigcap_j \sum_{k=0}^{\ell} P_j^k y^{m-k} = \sum_{k=0}^m (\bigcap_j P_j^k) y^{\ell-k} = \sum_{k=0}^m I^{(k)} y^{m-k}.$$

Therefore,

$$T/(I, y)^{(m)} \cong R/I \oplus R/I^{(2)} \oplus \cdots \oplus R/I^{(m)}$$

as R -modules. The assertions (i) and (ii) follow from this isomorphism. \square

Corollary 3.5. *Let G be a face of Δ . Then*

(i) $I_{\Delta}^{(m)}S[x_i^{-1} \mid i \in G]$ satisfies (S_2) if and only if $I_{\text{lk}_{\Delta}G}^{(k)}$ satisfies (S_2) for every k with $1 \leq k \leq m$.

(ii) $I_{\Delta}^{(m)}S[x_i^{-1} \mid i \in G]$ is Cohen-Macaulay if and only if $I_{\text{lk}_{\Delta}G}^{(k)}$ is Cohen-Macaulay for every k with $1 \leq k \leq m$.

Proof. Let $Y = \{x_i \mid i \notin V(\text{st}_{\Delta}G)\}$ and $T = K[x_i \mid x_i \notin G]$. By Lemma 2.1 and Remark 2.2, $I_{\Delta}^{(m)}S[x_i^{-1} \mid i \in G]$ satisfies (S_2) if and only if $(I_{\text{lk}_{\Delta}G}, Y)^{(m)}T$ satisfies (S_2) . By Lemma 3.4, $(I_{\text{lk}_{\Delta}G}, Y)^{(m)}T$ satisfies (S_2) if and only if $I_{\text{lk}_{\Delta}G}^{(k)}$ satisfies (S_2) for every k with $1 \leq k \leq m$. This proves (i). The proof of (ii) is similar. \square

Now we are able to prove the following characterization for the Cohen-Macaulayness of a symbolic power $I_{\Delta}^{(m)}$, $m \geq 3$.

Theorem 3.6. *Let Δ be a simplicial complex with $\dim \Delta \geq 1$. Then the following conditions are equivalent:*

- (i) $I_{\Delta}^{(m)}$ is Cohen-Macaulay for every $m \geq 1$.
- (ii) $I_{\Delta}^{(m)}$ is Cohen-Macaulay for some $m \geq 3$.
- (iii) $I_{\Delta}^{(m)}$ satisfies (S_2) for some $m \geq 3$.
- (iv) Δ is a matroid.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (iv). If $\dim \Delta = 1$, (S_2) means that $I_{\Delta}^{(m)}$ is Cohen-Macaulay. In this case, the assertion follows from [15, Theorem 2.4]. Let $\dim \Delta \geq 2$. By Theorem 2.7 we only need to show that Δ is connected and locally a matroid. Since (S_2) implies $\text{depth } S/I_{\Delta}^{(m)} \geq 2$, Δ is connected by Lemma 3.3. By Corollary 3.5, $I_{\text{lk}_{\Delta}\{i\}}^{(m)}$ satisfies (S_2) for all $i = 1, \dots, n$. Using induction on $\dim \Delta$ we may assume that $\text{lk}_{\Delta}\{i\}$ is a matroid for all $i = 1, \dots, n$. Hence Δ is locally a matroid.

(iv) \Rightarrow (i) follows from [16, Theorem 3.5] or [26, Theorem 2.1]. \square

The implication (ii) \Rightarrow (i) gives a positive answer to the question of [16] whether there exists a number t depending on $\dim \Delta$ such that if $I_{\Delta}^{(t)}$ is Cohen-Macaulay, then $I_{\Delta}^{(m)}$ is Cohen-Macaulay for every $m \geq 1$. As the Cohen-Macaulayness of $I_{\Delta}^{(2)}$ implies that of I_{Δ} [9, Theorem 3.7] we also obtain a positive answer to the question of [16] whether the Cohen-Macaulayness of $I_{\Delta}^{(m+1)}$ implies that of $I_{\Delta}^{(m)}$ for every $m \geq 1$.

Combinatorial criteria for the Cohen-Macaulayness of $I_{\Delta}^{(2)}$ can be found in [15] for $\dim \Delta = 1$ and in [16] for arbitrary dimension. Using these criteria one can easily find simplicial complexes Δ such that $I_{\Delta}^{(2)}$ is Cohen-Macaulay but $I_{\Delta}^{(m)}$ is not Cohen-Macaulay for every $m \geq 3$. Such an example is the 5-cycle, which is not a matroid.

Next we consider the generalized Cohen-Macaulayness of large symbolic powers of Stanley-Reisner ideals.

Recall that a homogeneous ideal I in S (or S/I) is called *generalized Cohen-Macaulay* if $\dim_K H_m^i(S/I) < \infty$ for $i = 0, 1, \dots, \dim S/I - 1$. It is well-known that I is generalized Cohen-Macaulay if and only if $IS[x_i^{-1}]$ is Cohen-Macaulay for $i = 1, \dots, n$ and S/I is equidimensional, that is, $\dim S/P = \dim S/I$ for every minimal prime P of I (see [5]). The class of generalized Cohen-Macaulay ideals is rather large. For instance, $I_\Delta^{(m)}$ is generalized Cohen-Macaulay for every $m \geq 1$ if Δ is pure and $\dim \Delta = 1$. For $\dim \Delta \geq 2$, the situation is completely different.

Theorem 3.7. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then the following conditions are equivalent:*

- (i) $I_\Delta^{(m)}$ is generalized Cohen-Macaulay for every $m \geq 1$.
- (ii) $I_\Delta^{(m)}$ is generalized Cohen-Macaulay for some $m \geq 3$.
- (iii) Δ is a union of disjoint matroids of the same dimension.

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii). Since $I_\Delta^{(m)}S[x_i^{-1}]$ is Cohen-Macaulay for $i = 1, \dots, n$, $I_{\text{lk}_\Delta\{i\}}^{(m)}$ is Cohen-Macaulay by Corollary 3.5. By Theorem 3.6, $\text{lk}_\Delta\{i\}$ is a matroid for $i = 1, \dots, n$. Hence Δ is locally a matroid. By Corollary 2.8, Δ is a union of disjoint matroids. On the other hand, since S/I_Δ is equidimensional, Δ is pure. Therefore, the connected components of Δ have the same dimension.

(iii) \Rightarrow (i). By Corollary 2.8, Δ is locally a matroid. Since $\text{lk}_\Delta\{i\}$ is a matroid for $i = 1, \dots, n$, $I_{\text{lk}_\Delta\{i\}}^{(m)}$ is Cohen-Macaulay for every $m \geq 1$ by Theorem 3.6. By Corollary 3.5, this implies the Cohen-Macaulayness of $I_\Delta^{(m)}S[x_i^{-1}]$ for every $m \geq 1$, $i = 1, \dots, n$. Since Δ is pure, S/I_Δ is equidimensional. Therefore, $I_\Delta^{(m)}$ is generalized Cohen-Macaulay for every $m \geq 1$. \square

A homogeneous ideal I in S (or S/I) is called *Buchsbaum* or *quasi-Buchsbaum* if the natural map $\text{Ext}_S^i(K, S/I) \rightarrow H_m^i(S/I)$ is surjective or $\mathfrak{m}H_m^i(S/I) = 0$ for $i = 0, \dots, \dim S/I - 1$ (see e.g. [22] and [7]). We have the following implications:

$$\begin{aligned} \text{Cohen-Macaulayness} &\Rightarrow \text{Buchsbaumness} \Rightarrow \text{quasi-Buchsbaumness} \\ &\Rightarrow \text{generalized Cohen-Macaulayness.} \end{aligned}$$

We will use Theorem 3.7 to study the Buchsbaumness and quasi-Buchsbaumness of large symbolic powers of Stanley-Reisner ideals. For that we need the following observation.

Lemma 3.8. *Let Δ be a simplicial complex with $\dim \Delta \geq 1$. Then Δ is connected if $I_\Delta^{(m)}$ is quasi-Buchsbaum for some $m \geq 2$.*

Proof. Set $I = I_\Delta^{(m)}$ and $\mathbf{e} = (1, 0, \dots, 0)$. We have $\Delta_0 = \Delta$ by Example 3.2 and

$$\Delta_{\mathbf{e}} = \{F \subset [n] \mid x_1 \notin IS[x_i \mid i \in F]\}.$$

For $F \in \mathcal{F}(\Delta)$ we have $IS[x_i \mid i \in F] = P_F^m S[x_i \mid i \in F]$. Since $x_1 \notin P_F^m S[x_i \mid i \in F]$ ($m \geq 2$) we get $F \in \Delta_{\mathbf{e}}$. For $F \notin \Delta$ we have $IS[x_i \mid i \in F] = S[x_i \mid i \in F]$. Hence $\Delta_{\mathbf{e}} = \Delta$. By [13, Lemma 2.3] there is a commutative diagram

$$\begin{array}{ccc}
 H_m^1(S/I)_0 & \xrightarrow{x_1} & H_m^1(S/I)_e \\
 \downarrow & & \downarrow \\
 \tilde{H}^0(\Delta_0, K) & \longrightarrow & \tilde{H}^0(\Delta_e, K),
 \end{array}$$

where the vertical maps are the isomorphisms of Lemma 3.1 and the lower horizontal map is induced from the natural embedding $\Delta_e \hookrightarrow \Delta_0$. Since $\Delta_e = \Delta_0$, this map is an identity. Therefore,

$$\tilde{H}_0(\Delta, K) \cong H_m^1(S/I)_e = x_1 H_m^1(S/I)_0 = 0$$

because I is quasi-Buchsbaum. The vanishing of $\tilde{H}_0(\Delta, K)$ just means that Δ is connected. \square

The above lemma doesn't hold for $m = 1$. It is well known that if I_Δ is generalized Cohen-Macaulay, then I_Δ is Buchsbaum [19, Theorem 3.2], [21, Theorem 8.1]. Hence I_Δ is Buchsbaum if $\dim \Delta = 1$, even when Δ is unconnected.

The following result shows that the Cohen-Macaulayness of $I_\Delta^{(m)}$, $m \geq 3$, is equivalent to the Buchsbaumness and quasi-Buchsbaumness.

Theorem 3.9. *Let Δ be a pure simplicial complex with $\dim \Delta \geq 2$. Then the following conditions are equivalent:*

- (i) $I_\Delta^{(m)}$ is Cohen-Macaulay for every $m \geq 1$.
- (ii) $I_\Delta^{(m)}$ is Buchsbaum for some $m \geq 3$.
- (iii) $I_\Delta^{(m)}$ is quasi-Buchsbaum for some $m \geq 3$.
- (iv) Δ is a matroid.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (iv). The quasi-Buchsbaumness implies that $I_\Delta^{(m)}$ is generalized Cohen-Macaulay. By Theorem 3.7, Δ is a union of disjoint matroids. On the other hand, Δ is connected by Lemma 3.8. Hence Δ is a matroid.

(iv) \Rightarrow (i) follows from [16, Theorem 3.5] or [26, Theorem 2.1]. \square

The situation is a bit different if $\dim \Delta = 1$. Minh-Nakamura [13, Theorem 3.7] showed that for a graph Δ , $I_\Delta^{(3)}$ is Buchsbaum if and only if $I_\Delta^{(2)}$ is Cohen-Macaulay and that for $m \geq 4$, $I_\Delta^{(m)}$ is Cohen-Macaulay if $I_\Delta^{(m)}$ is Buchsbaum. For instance, if Δ is a 5-cycle, $I_\Delta^{(3)}$ is Buchsbaum but not Cohen-Macaulay.

4. COHEN-MACAULAYNESS OF LARGE ORDINARY POWERS

In this section we study the Cohen-Macaulayness of ordinary powers of Stanley-Reisner ideals.

It is well known that I_Δ^m is Cohen-Macaulay for every $m \geq 1$ if and only if Δ is a complete intersection [3]. If $\dim \Delta \leq 2$, we know that Δ is a complete intersection if I_Δ^m is Cohen-Macaulay for some $m \geq 3$ [15], [25]. It was asked whether this result holds in general [25, Question 3]. We shall give a positive answer to this question

by showing that Δ is a complete intersection if S/I_Δ^m satisfies Serre condition (S_2) for some $m \geq 3$.

To study the relationship between the ordinary powers of Δ and of its links we need the following observation.

Lemma 4.1. *Let I be a monomial ideal in a polynomial ring R . Let $T := R[y]$ be a polynomial ring over R . Then*

- (i) $(I, y)^m$ satisfies (S_2) if and only if I^k satisfies (S_2) for every k with $1 \leq k \leq m$.
- (ii) $(I, y)^m$ is Cohen-Macaulay if and only if I^k is Cohen-Macaulay for every k with $1 \leq k \leq m$.

Proof. Since $(I, y)^m = \sum_{k=0}^m I^k y^{m-k}$, we have

$$T/(I, y)^m \cong R/I^m \oplus R/I^{m-1} \oplus \cdots \oplus R/I$$

as R -modules. The assertion follows from this isomorphism. \square

Corollary 4.2. *Let G be a face of Δ . Then*

- (i) $I_\Delta^m S[x_i^{-1} \mid i \in G]$ satisfies (S_2) if and only if $I_{\text{lk}_\Delta G}^k$ satisfies (S_2) for every k with $1 \leq k \leq m$.
- (ii) $I_\Delta^m S[x_i^{-1} \mid i \in G]$ is Cohen-Macaulay if and only if $I_{\text{lk}_\Delta G}^k$ is Cohen-Macaulay for every k with $1 \leq k \leq m$.

Proof. The assertions follow from Lemma 4.1 similarly as in the proof of Corollary 3.5. \square

Now we are able to prove the following characterizations of the Cohen-Macaulayness of I_Δ^m , $m \geq 3$.

Theorem 4.3. *Let Δ be a simplicial complex with $\dim \Delta \geq 1$. Then the following conditions are equivalent:*

- (i) I_Δ^m is Cohen-Macaulay for every $m \geq 1$.
- (ii) I_Δ^m is Cohen-Macaulay for some $m \geq 3$.
- (iii) I_Δ^m satisfies (S_2) for some $m \geq 3$.
- (iv) Δ is a complete intersection.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) and (iv) \Rightarrow (i) are clear.

(iii) \Rightarrow (iv). If $\dim \Delta = 1$, (S_2) means that I_Δ^m is Cohen-Macaulay. In this case, the assertion follows from [15, Corollary 3.5]. Let $\dim \Delta \geq 2$. By Lemma 2.3 we only need to show that Δ is connected and locally a complete intersection. Since (S_2) implies $\text{depth } S/I_\Delta^m \geq 2$, Δ is connected by Lemma 3.3. By Corollary 4.2, $I_{\text{lk}_\Delta \{i\}}^m$ satisfies (S_2) for all $i = 1, \dots, n$. Using induction on $\dim \Delta$ we may assume that $\text{lk}_\Delta \{i\}$ is a complete intersection for all $i = 1, \dots, n$. Hence Δ is locally a complete intersection. \square

There is a complete description of all complexes Δ such that I_Δ^2 is Cohen-Macaulay in [15, Corollary 3.4] for $\dim \Delta = 1$ and in [25, Theorem 3.7] for $\dim \Delta = 2$. Using these results one can find examples such that I_Δ^2 is Cohen-Macaulay but I_Δ^m is not Cohen-Macaulay for every $m \geq 3$. The 5-cycle is such an example.

Next we consider the generalized Cohen-Macaulayness of I_Δ^m . Goto-Takayama [8, Theorem 2.5] showed that I_Δ^m is generalized Cohen-Macaulay for every integer $m \geq 1$ if and only if Δ is pure and locally a complete intersection. This result can be improved in the case $\dim \Delta = 1$ as follows.

Theorem 4.4. *Let Δ be a graph. Then the following conditions are equivalent:*

- (i) I_Δ^m is generalized Cohen-Macaulay for every $m \geq 1$.
- (ii) I_Δ^m is generalized Cohen-Macaulay for some $m \geq 3$.
- (iii) Δ is a union of disjoint paths and cycles.

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii). The generalized Cohen-Macaulayness of I_Δ^m implies that $I_\Delta^m S[x_i^{-1}]$ is Cohen-Macaulay for $i = 1, \dots, n$. Therefore, $I_{\text{lk}_\Delta\{i\}}^m$ is Cohen-Macaulay by Corollary 4.2. Since $\dim \Delta = 1$, $\text{lk}_\Delta\{i\}$ is a collection of vertices. Hence $I_{\text{lk}_\Delta\{i\}}$ is the edge ideal of a complete graph. By [17, Theorem 3.8], the Cohen-Macaulayness of $I_{\text{lk}_\Delta\{i\}}^m$ for some $m \geq 3$ implies that $I_{\text{lk}_\Delta\{i\}}$ is a complete intersection. Thus, $\text{lk}_\Delta\{i\}$ consists of either one point or two points. From this it follows that every connected component of Δ must be a path or a cycle.

(iii) \Rightarrow (i). Condition (iii) implies that $\text{lk}_\Delta\{i\}$ consists of either one point or two points for $i = 1, \dots, n$. Hence Δ is locally a complete intersection. Therefore, I_Δ^m is generalized Cohen-Macaulay for every $m \geq 1$ by [8, Theorem 2.5]. \square

For $\dim \Delta \geq 2$ we have the following characterization.

Theorem 4.5. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then the following conditions are equivalent:*

- (i) I_Δ^m is generalized Cohen-Macaulay for every $m \geq 1$.
- (ii) I_Δ^m is generalized Cohen-Macaulay for some $m \geq 3$.
- (iii) Δ is a union of disjoint complete intersections of the same dimension.

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii). The generalized Cohen-Macaulayness of I_Δ^m implies that $I_\Delta^m S[x_i^{-1}]$ is Cohen-Macaulay for $i = 1, \dots, n$ and I_Δ is equidimensional. The last property means that Δ is pure. By Corollary 4.2, the Cohen-Macaulayness of $I_\Delta^m S[x_i^{-1}]$ implies that of $I_{\text{lk}_\Delta\{i\}}^m$. Hence $\text{lk}_\Delta\{i\}$ is a complete intersection by Theorem 4.3. Thus, Δ is locally a complete intersection. By Lemma 2.4, Δ is a union of disjoint complete intersections. Since Δ is pure, these complete intersections have the same dimension.

(iii) \Rightarrow (i). By Lemma 2.4, Δ is locally a complete intersection. Hence $I_{\text{lk}_\Delta\{x_i\}}^m$ is Cohen-Macaulay for every $m \geq 1$, $i = 1, \dots, n$. By Corollary 4.2, this implies the Cohen-Macaulayness of $I_\Delta^m S[x_i^{-1}]$ for every $m \geq 1$. Since Δ is pure, S/I_Δ is equidimensional. Therefore, I_Δ^m is generalized Cohen-Macaulay for every $m \geq 1$. \square

The above two theorems are similar but they can't be put together because a path of length ≥ 4 or a cycle of length ≥ 5 is not a complete intersection. Compared with the mentioned result of Goto-Takayama, these theorems are stronger in the sense that it gives a combinatorial characterization of the generalized Cohen-Macaulayness of each power I_Δ^m , $m \geq 3$.

Now we consider the Buchsbaumness and quasi-Buchsbaumness of ordinary powers of Stanley-Reisner ideals. By [24, Theorem 2.1] we know that Δ is a complete intersection if I_Δ^m is Buchsbaum for all (large) $m \geq 1$. It was asked [24, Question 2.10] whether Δ is a complete intersection if I_Δ^m is quasi-Buchsbaum for every $m \geq 1$. In the following we give a positive answer to this question.

The case $\dim \Delta = 1$ was already studied by Minh-Nakamura in [14], where they describe all graphs Δ with a Buchsbaum ideal I_Δ^m . In particular, they showed that I_Δ^m is Buchsbaum for some $m \geq 4$ if and only if Δ is a complete intersection. We extend this result for the quasi-Buchsbaumness as follows.

Theorem 4.6. *Let Δ be a graph. Then the following conditions are equivalent:*

- (i) I_Δ^m is Cohen-Macaulay for every $m \geq 1$.
- (ii) I_Δ^m is Buchsbaum for some $m \geq 4$.
- (iii) I_Δ^m is quasi-Buchsbaum for some $m \geq 4$.
- (iv) Δ is a complete intersection.

Proof. We only need to prove (iii) \Rightarrow (iv). Consider the exact sequence

$$0 \rightarrow I_\Delta^{(m)}/I_\Delta^m \rightarrow S/I_\Delta^m \rightarrow S/I_\Delta^{(m)} \rightarrow 0.$$

Since $H_m^0(S/I_\Delta^m) = I_\Delta^{(m)}/I_\Delta^m$, $H_m^0(S/I_\Delta^{(m)}) = 0$ and $H_m^i(S/I_\Delta^{(m)}) = H_m^i(S/I_\Delta^m)$ for $i \geq 1$. Hence the quasi-Buchsbaumness of I_Δ^m implies that $\mathfrak{m}H_m^1(S/I_\Delta^{(m)}) = 0$. Since $\dim S/I_\Delta^{(m)} = 2$, $I_\Delta^{(m)}$ is Buchsbaum by [22, Proposition 2.12]. By [13, Theorem 3.7], this implies the Cohen-Macaulayness of $I_\Delta^{(m)}$. Hence every pair of disjoint edges of Δ is contained in a 4-cycle by [15, Theorem 2.4]. On the other hand, Δ is locally a complete intersection by Goto-Takayama [8, Theorem 2.5]. By [24, Proposition 1.11], Δ is either an r -cycle, $r \geq 3$, or a path of length $r \geq 2$. Thus, Δ must be an r -cycle, $r \leq 4$, or a path of length 2, which are complete intersections. \square

We can not lower m to 3 in (ii) and (iii) of the above theorem. In fact, if Δ is a 5-cycle, then I_Δ^3 is Buchsbaum but not Cohen-Macaulay by [14, Theorem 4.11].

For $\dim \Delta \geq 2$ we need the following observation.

Lemma 4.7. *Let Δ be a simplicial complex with $\dim \Delta \geq 1$. Then Δ is connected if I_Δ^m is quasi-Buchsbaum for some $m \geq 2$.*

Proof. This can be shown similarly as for Lemma 3.8. \square

Theorem 4.8. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Then the following conditions are equivalent:*

- (i) I_Δ^m is Cohen-Macaulay for every $m \geq 1$.
- (ii) I_Δ^m is Buchsbaum for some $m \geq 3$.
- (iii) I_Δ^m is quasi-Buchsbaum for some $m \geq 3$.
- (iv) Δ is a complete intersection.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (iv). Since S/I_Δ^m is generalized Cohen-Macaulay, Δ is a union of disjoint complete intersections by Theorem 4.5. By Lemma 4.7, the quasi-Buchsbaumness of I_Δ^m implies that Δ is connected. Therefore, Δ is a complete intersection.

(iv) \Rightarrow (i) is well known. \square

Below we give an example where $\dim \Delta = 2$ and I_Δ^2 is Buchsbaum but not Cohen-Macaulay.

Example 4.9. Let Δ be the simplicial complex generated by the sets

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{3, 4, 5\}$$

and $S = K[x_1, x_2, x_3, x_4, x_5]$. It is easy to check that

$$\begin{aligned} I_\Delta^2 &= (x_1x_2x_3x_4, x_1x_5, x_2x_5)^2, \\ I_\Delta^{(2)} &= (x_1x_2x_3x_4, x_1x_5, x_2x_5)^2 + (x_1x_2x_3x_4x_5). \end{aligned}$$

From this it follows that $\mathfrak{m}I_\Delta^{(2)} \subseteq I_\Delta^2$. On the other hand, $I_\Delta^{(2)}$ is Cohen-Macaulay by [16, Example 4.7]. This implies $H_{\mathfrak{m}}^0(S/I_\Delta^2) = I_\Delta^{(2)}/I_\Delta^2$, hence $\mathfrak{m}H_{\mathfrak{m}}^0(S/I_\Delta^2) = 0$. Therefore, I_Δ^2 is Buchsbaum [22, Proposition 2.12] and not Cohen-Macaulay.

5. APPLICATIONS

In this section we investigate the Cohen-Macaulayness of symbolic powers of the cover ideal and the facet ideal of a simplicial complex.

Let Δ be a simplicial complex on the vertex set $V = [n]$ and $S = k[x_1, \dots, x_n]$. The *facet ideal* $I(\Delta)$ is defined as the ideal generated by all squarefree monomials $x_{i_1} \cdots x_{i_r}$, $\{i_1, \dots, i_r\} \in \mathcal{F}(D)$ (see [6]). For instance, squarefree monomial ideals generated in degree r are just facet ideals of pure complexes of dimension $r - 1$.

Let Δ^* denote the simplicial complex with $I_{\Delta^*} = I(\Delta)$. By Theorem 3.6, to study the Cohen-Macaulayness of large symbolic powers of $I(\Delta)$ we have to study when Δ^* is a matroid. Note that facets of Δ are minimal nonfaces of Δ^* and that Δ and Δ^* have the same vertex set.

If $\dim \Delta = 1$, we may consider Δ as a graph and $I(\Delta)$ as the Stanley-Reisner ideal of a flag complex. This case was already dealt with in [17].

Theorem 5.1. [17, Theorem 3.6] *Let Γ be a graph. Then the following conditions are equivalent:*

- (i) $I(\Gamma)^{(m)}$ is Cohen-Macaulay for every $m \geq 1$.
- (ii) $I(\Gamma)^{(m)}$ is Cohen-Macaulay for some $m \geq 3$.
- (iii) Γ is a union of disjoint complete graphs.

The original proof of this theorem didn't involve matroids. However, using a recent result in matroid theory we can give another proof as follows.

Proof. It suffices to show that Γ^* is a matroid if and only if Γ is a union of disjoint complete graphs. Note first that Γ^* is the clique complex of the graph $\bar{\Gamma}$ of the nonedges of Γ . By [12, Theorem 3.3], the clique complex of a graph is a matroid if and only if there is a partition of the vertices into independent sets (which contain no adjacent vertices) such that every nonedge of the graph is contained in an independent set. But an independent set of $\bar{\Gamma}$ is just a complete graph in Γ . \square

Now we are going to prove a similar characterization for the case that Δ is a pure complex with $\dim \Delta = 2$.

In the following we call a simplicial complex *r-uniform* if it is generated by the r -dimensional faces of a simplex. Complete graphs are just 1-uniform matroids.

Theorem 5.2. *Let Δ be a pure complex with $\dim \Delta = 2$. Then the following conditions are equivalent:*

- (i) $I(\Delta)^{(m)}$ is Cohen-Macaulay for every $m \geq 1$.
- (ii) $I(\Delta)^{(m)}$ is Cohen-Macaulay for some $m \geq 3$.
- (iii) Δ is a union of disjoint 2-uniform matroids.

As observed above, it suffices to show that Δ^* is a matroid if and only if Δ is a union of disjoint 2-uniform matroids. The proof is based on the following observation.

Lemma 5.3. *Let Δ be a matroid. Let F, G be two maximal proper subsets of a minimal nonface of Δ . Then $\text{lk}_\Delta F = \text{lk}_\Delta G$.*

Proof. By symmetry, it suffices to show that every non-empty face H of $\text{lk}_\Delta F$ is also a face of $\text{lk}_\Delta G$. Note that $F \cup H \in \Delta$ and $F \cap H = \emptyset$. Since $|F \cup H| = |F| + |H| = |G| + |H| > |G|$, we can extend G by elements of $F \cup H$ to a face L of Δ such that $|L| = |F \cup H|$. Since $|F \cap G| = |F| - 1$, there is only a vertex of F not contained in G . Since $F \cup G$ is a minimal nonface of Δ , L does not contain this vertex. Therefore, $L \subseteq G \cup H$. Since $|L| = |G| + |H|$, we must have $L = G \cup H$ and $G \cap H = \emptyset$. This shows $H \in \text{lk}_\Delta G$. \square

Using the above lemma we prove the following structure theorem for matroids whose minimal nonfaces have dimension 2.

Recall that the *join* $\Delta_1 * \Delta_2$ of two simplicial complexes Δ_1 and Δ_2 on different vertex sets is the simplicial complex whose faces are the unions of two faces of Δ_1 and Δ_2 .

Proposition 5.4. *Let Δ be a simplicial complex whose minimal nonfaces have dimension 2. Then Δ is a matroid if and only if Δ is the join of 1-uniform matroids with possibly a simplex.*

Proof. It is obvious that the join of two matroids is a matroid. Therefore, Δ is a matroid if Δ is the join of 1-uniform matroids with eventually a simplex.

Conversely, assume that Δ is a matroid. Let $d = \dim \Delta$. Then $d \geq 1$. If $d = 1$, Δ is a 1-uniform matroid because every set of two vertices of Δ is a face of Δ . If $d \geq 2$, we choose an edge F which is contained in a minimal nonface H of Δ .

Let $\Delta_1 = \text{lk}_\Delta F$. Then $\dim \Delta_1 = d - 2 \geq 0$. Let W be the vertex set of Δ_1 . We will show that Δ_1 is the induced subcomplex Δ_W of Δ on W . Assume for the contrary that there is a nonempty face G of Δ_W such that $F \cup G$ is a nonface of Δ . Since $F \cup \{v\}$ is a face of Δ for every vertex $v \in W$, $|G| \geq 2$. Choose G as small as possible. Then $F \cup G'$ is a face of Δ for any subset $G' \subset G$ with $|G'| = |G| - 1$. Since $|F \cup G| \geq 4$, $F \cup G$ is not a minimal nonface of Δ . Therefore, there exists a vertex $v \in F$ such that $\{v\} \cup G$ is a nonface of Δ . By the choice of G , this nonface

must be minimal. Hence $|\{v\} \cup G| = 2$, which implies $|G| = 2$. Let u be a vertex of G . Then $F \cup \{u\}$ is a face of Δ . Let $F = \{v, x\}$. By the definition of matroids, $\{x\} \cup G$ must be a face of Δ . By the choice of F , there is a minimal nonface H of Δ containing F . Let $H = \{v, x, y\}$. By Lemma 2.2, $\text{lk}_\Delta \{x, y\} = \Delta_1$. Hence $\{y, u\}$ is a face of Δ . By the definition of matroids, $\{y\} \cup G$ must be a face of Δ . Since $G \notin \Delta_1$, $\{x, y\} \cup G$ is a nonface of Δ . Since every proper subset of $\{x, y\} \cup G$ is a face of Δ , this nonface is minimal, which is a contradiction to the assumption that every nonface has dimension 2. So we have shown that Δ_1 is an induced subcomplex of Δ . From this it follows that Δ_1 is a matroid whose minimal nonfaces have dimension 2.

Let Γ_1 be the induced subcomplex of Δ on the vertices not contained in Δ_1 . It is clear that Γ_1 is a matroid. Since every face of Δ containing F properly must contain a vertex of Δ_1 , F is a facet of Γ_1 . Therefore, all facets of Γ_1 have dimension 1. Since every set of 2 vertices is a face of Δ , Γ_1 is a 1-uniform matroid. We will show that $\text{lk}_\Delta G = \Delta_1$ for every facet G of Γ_1 . By the definition of matroids, we can find a sequence of facets $F = F_1, F_2, \dots, F_r = G$ of Γ_1 such that $|F_i \cap F_{i+1}| = 1$, $i = 1, \dots, r - 1$. From this it follows that $|F_i \cup F_{i+1}| = 3$. Since the vertices of F_2 are not contained in Δ_1 , $F_1 \cup F_2$ is a nonface of Δ . Since $|F_1 \cup F_2| = 3$, $F_1 \cup F_2$ is a minimal nonface of Δ . By Lemma 5.3, we have $\text{lk}_\Delta F_1 = \text{lk}_\Delta F_2 = \Delta_1$. Similarly, we can show that $\text{lk}_\Delta F_2 = \dots = \text{lk}_\Delta F_r = \Delta_1$. So we obtain $\text{lk}_\Delta G = \Delta_1$. From this it follows that every facet of Δ is a union of a facet of Γ_1 and a facet of Δ_1 . Therefore, $\Delta = \Delta_1 * \Gamma_1$.

If Δ_1 has no nonface, then Δ_1 is a simplex. If Δ_1 has a nonface and if Δ_1 is not a 1-uniform matroid, we use the above argument to show that there are a 1-uniform matroid Γ_2 and a matroid Δ_2 such that $\Delta_1 = \Gamma_2 * \Delta_2$. Proceeding like that we will see that Δ is the join of 1-uniform matroids with possibly a simplex. \square

Now we are able to prove Theorem 5.2.

Proof. Let Δ^* denote the simplicial complex with $I_{\Delta^*} = I(\Delta)$. Then the minimal nonfaces of Δ^* have dimension 2. Moreover, every vertex of Δ^* appears in at least a minimal nonface. Hence the facets of Δ^* don't have common vertices. From this it follows that Δ^* can not be the join of a complex with a simplex. By Proposition 5.4, Δ^* is a matroid if and only if Δ^* is the join of 1-uniform matroids. In this case, the faces of Δ^* are unions of faces of the 1-uniform matroids. Hence the minimal nonfaces of Δ^* are the minimal nonfaces of the 1-uniform matroids. The complex generated by the minimal nonfaces of a 1-uniform matroid is just the 2-uniform matroid on the same vertex set. Therefore, Δ is the union of these 2-uniform matroids. Similarly, we can also show that Δ^* is the join of 1-uniform matroids if Δ is a union of disjoint 2-uniform matroid. Thus, Δ^* is a matroid if and only if Δ is a union of disjoint 2-uniform matroids. By Theorem 3.6, this implies the assertion. \square

Replacing condition (iii) of Theorem 5.2 by the condition that Δ is a union of disjoint r -uniform matroids we may expect that the theorem could be extended to

arbitrary r -dimensional pure complexes. But the following example shows that this is not the case.

Example 5.5. Let Δ be the complex generated by $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$, $\{3, 4, 5, 6\}$. Then

$$I(\Delta) = (x_1, x_3) \cap (x_1, x_4) \cap (x_1, x_5) \cap (x_1, x_6) \cap (x_2, x_3) \cap (x_2, x_4) \\ \cap (x_2, x_5) \cap (x_2, x_6) \cap (x_3, x_5) \cap (x_3, x_6) \cap (x_3, x_5) \cap (x_3, x_6).$$

From this it follows that Δ^* is generated by the 4-subsets of $\{1, \dots, 6\}$ different than $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$, $\{3, 4, 5, 6\}$. It is easy to check that Δ^* is a matroid. Hence $I(\Delta)^{(m)}$ is Cohen-Macaulay for every $m \geq 1$ by Theorem 3.6. However, Δ can not be a union of disjoint 3-uniform matroids.

The *cover ideal* $J(\Delta)$ of the simplicial complex Δ is defined by

$$J(\Delta) = \bigcap_{F \in \mathcal{F}(\Delta)} P_F,$$

where $P_F = (x_i \mid i \in F)$. For instance, unmixed squarefree monomial ideals of codimension 2 are just cover ideals of graphs.

The name cover ideal comes from the fact that $J(\Delta)$ is generated by squarefree monomials $x_{i_1} \cdots x_{i_r}$ with $\{i_1, \dots, i_r\} \cap F \neq \emptyset$ for every facet F of Δ , which correspond to covers of the facets of Δ . Note that the equality between ordinary and symbolic powers of $J(\Delta)$ were already studied in [9], [10].

Using a well known result in matroid theory we are able to derive from Theorem 3.6 the following combinatorial characterization of the Cohen-Macaulayness of large symbolic powers of $J(\Delta)$.

Theorem 5.6. *Let Δ be a simplicial complex. The following conditions are equivalent:*

- (i) $J(\Delta)^{(m)}$ is Cohen-Macaulay for every $m \geq 1$.
- (ii) $J(\Delta)^{(m)}$ is Cohen-Macaulay for some $m \geq 3$.
- (iii) Δ is a matroid.

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii) and (iii) \Rightarrow (i). Let Δ^c be the simplicial complex generated by the complements of the facets of Δ in $[n]$. Then $I_{\Delta^c} = J(\Delta)$. It is well known that Δ^c is a matroid if and only if so is Δ [20, Theorem 39.2]. Therefore, the assertion follows from Theorem 3.6. \square

By Lemma 2.5 we can easily test a matroid. For instance, if $\dim \Delta = 1$, Δ can be considered as a graph and we obtain the following result. This result is also proved in [2] by a different method.

Corollary 5.7. *Let Γ be a graph without isolated vertices. Then the following conditions are equivalent:*

- (i) $J(\Gamma)^{(m)}$ is Cohen-Macaulay for every $m \geq 1$.
- (ii) $J(\Gamma)^{(m)}$ is Cohen-Macaulay for some $m \geq 3$.
- (iii) Any pair of disjoint edges of Γ is contained in a 4-cycle.

Proof. The assertion follows from the above theorem and Corollary 2.6. \square

By Theorem 3.6 and Theorem 5.6, the Cohen-Macaulayness of $I_{\Delta}^{(m)}$ is equivalent to that of $J(\Delta)^{(m)}$ for $m \geq 3$. Therefore, we may ask whether this holds also for $m = 1, 2$. The following example shows that the answer is no.

Example 5.8. Let Δ be the graph of a 5-cycle. Then $I_{\Delta}^{(2)}$ is Cohen-Macaulay [15, Theorem 2.3]. We have

$$J(\Delta) = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1).$$

It can be easily checked that $J(\Delta)$ is not Cohen-Macaulay so that $J(\Delta)^{(m)}$ is neither for any $m \geq 2$ by [11, Theorem 2.6].

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