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On Asymptotic Power Utility-Based Pricing and Hedging

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Abstract

Kramkov and Sîrbu [24, 25] have shown that first-order approximations of power utility-based prices and hedging strategies can be computed by solving a mean-variance hedging problem under a specific equivalent martingale measure and relative to a suitable numeraire. In order to avoid the introduction of an additional state variable necessitated by the change of numeraire, we propose an alternative representation in terms of the original numeraire. More specifically, we characterize the relevant quantities using semimartingale characteristics similarly as in Černý and Kallsen [4] for mean-variance hedging.

Key words: Utility-based pricing and hedging, incomplete markets, mean-variance hedging, numeraire, semimartingale characteristics

1 Introduction

In incomplete markets, derivative prices cannot generally be based on perfect replication. A number of alternatives have been suggested in the literature, relying e.g. on superreplication, mean-variance hedging, calibration of parametric families, utility-based concepts, or ad-hoc approaches. This paper focuses on utility indifference prices as studied by [12] and many others. They make sense for over-the-counter trades of a fixed quantity of contingent claims. Suppose that a client approaches a potential seller in order to buy q European-style contingent claims maturing at T. The seller is supposed to be a utility maximizer with given preference structure. She will enter into the contract only if her maximal expected utility is increased by the trade. The utility indifference price is the lowest acceptable premium

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for the seller. If the trade is made, the seller's optimal position in the underlyings changes due to the presence of the option. This adjustment in the optimal portfolio process is called utility-based hedging strategy for the claim. Both the utility indifference price and the corresponding utility-based hedging strategy are typically hard to compute even if relatively simple incomplete market models are considered. A reasonable way out for practical purposes is to consider approximations for small q, i.e. the limiting structure for small numbers of contingent claims. Extending earlier work on the limiting price, [24, 25] show that first order approximations of the utility indifference price and the utility-based hedging strategy can be expressed in terms of a Galtchouk-Kunita-Watanabe (GKW) decomposition of the claim after changing both the numeraire and the underlying probability measure. Similar results for exponential utility are obtained by [26, 2, 20], cf. also [18]. In this case no numeraire change occurs. We focus on power-type utility functions in this paper, which constitute the most popular and tractable ones on the positive real line. In particular, we work in the setup of [24, 25].

In order to compute the first-order approximation to utility-based pricing and hedging, one may proceed in two steps. Firstly, the pure investment problem in absence of the claim is solved. Its solution provides the numeraire and measure changes for the subsequent GKW decomposition. In a second step the GKW decomposition itself is computed, which corresponds to solving a mean-variance hedging problem for an option on a stock which follows a martingale. Both steps are in principle feasible e.g. for exponential Lévy processes or, more generally, many affine stochastic volatility models which have been studied in the empirical literature. However, the change of numeraire in this approach leads to an extra state variable, which e.g. in the integral transform approach of [13, 3] leads to high-dimensional integrals.

The present paper aims at reformulating the asymptotic results in an alternative form which does not involve a numeraire change. This reduces the computational complexity in concrete models. The representation of the asymptotic results in [24, 25] is inspired by a similar representation of the solution to mean-variance hedging problems by [11]. In [11] a measure and numeraire change reduce the mean-variance hedging problem for arbitrary price processes to the simpler martingale case of [9]. The solution to the latter is easily expressed in terms of a GKW decomposition.

An alternative characterization of the mean-variance hedging problem is derived in [4]. Its representation of the solution is closer in spirit to [31], and it does not require a numeraire change. Roughly speaking, the present paper replaces the [11]-type representation of the solution to the implied mean-variance hedging problem by the alternative formulation of [4]. Consequently, our representation of the first-order approximation to utility-based hedging and pricing in Theorem 4.4 resembles the solution to a mean-variance hedging problem as well. However, this is true only on a formal level. Rigorously, the involved numeraire change may not be compatible with the set of admissible trading strategies. Nevertheless, the formulas from [4] still characterize the objects of interest.

The remainder of the paper is organized as follows. After briefly reviewing the general theory of power utility-based pricing and hedging in Section 2, we introduce the asymptotic results of Kramkov and Sîrbu [24, 25]. Subsequently, we develop our alternative representation in Section 4. Finally, the appendix summarizes some notions and results concerning semimartingale calculus for the convenience of the reader.

Unexplained notation is generally used as in the monograph of Jacod and Shiryaev [16]. In particular, for a semimartingale X, we denote by $L(X)$ the predictable X-integrable processes and by $\varphi \bullet X$ the stochastic integral of $\varphi \in L(X)$ with respect to X. We write $\mathscr{E}(X)$ for the stochastic exponential of a semimartingale X and denote by $\mathscr{L}(Z) := \frac{1}{Z_-} \bullet Z$ the *stochastic logarithm* of a semimartingale Z satisfying $Z, Z_- \neq 0$. For semimartingales X and Y, $\langle X, Y \rangle$ represents the predictable compensator of $[X, Y]$, provided that the latter is a special semimartingale (cf. [15, page 37]). Finally, we write c^{-1} for the *Moore-Penrose pseudoinverse* of a matrix or matrix-valued process c (cf. [1]) and denote by E_d the identity matrix on $\mathbb{R}^d$.

2 Utility-based pricing and hedging

Our mathematical framework for a frictionless market model is as follows. Fix a terminal time $T > 0$ and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, P)$ in the sense of [16, I.1.2]. For ease of exposition, we assume that $\mathscr{F}_T = \mathscr{F}$ and $\mathscr{F}_0 = \{\varnothing, \Omega\}$, i.e. all $\mathscr{F}_0$-measurable random variables are almost surely constant.

We consider a securities market which consists of $d+1$ assets, a bond and d stocks. As is common in Mathematical Finance, we work in *discounted* terms. That means we suppose that the bond has constant value 1 and denote by $S = (S^1, \dots, S^d)$ the *discounted price process* of the d stocks in terms of multiples of the bond. The process S is assumed to be an $\mathbb{R}^d$-valued semimartingale. Self-financing *trading strategies* are described by $\mathbb{R}^d$-valued predictable stochastic processes $\varphi = (\varphi^1, \dots, \varphi^d)$, where φ^i_t denotes the number of shares of security i held at time t. In this financial model, we consider an investor whose preferences are modelled by a *power utility function* $u(x) = x^{1-p}/(1-p)$ for some $p \in \mathbb{R}_+ \backslash \{0, 1\}$. Given an initial endowment $v > 0$, the investor solves the *pure investment problem*

$$U(v) := \sup_{\varphi \in \Theta(v)} E(u(v + \varphi \bullet S_T)), \tag{2.1}$$

where the set $\Theta(v)$ of *admissible strategies* for initial endowment v is given by

$$\Theta(v) := \{\varphi \in L(S) : v + \varphi \bullet S \geq 0\}.$$

To ensure that the optimization problem (2.1) is well-posed, we make the following two standard assumptions.

Assumption 2.1 There exists an *equivalent local martingale measure*, i.e. a probability measure $Q \sim P$ such that S is a local Q-martingale.

Assumption 2.2 The maximal expected utility in the pure investment problem (2.1) is finite, i.e. $U(v) < \infty$.

In view of [23, Theorem 2.2], Assumptions 2.1 and 2.2 imply that the supremum in (2.1) is attained for some strategy $\widehat{\varphi} \in \Theta(v)$ with strictly positive wealth process $v + \widehat{\varphi} \bullet S$. By Assumption 2.1 and [16, I.2.27], $v + \widehat{\varphi} \bullet S_-$ is strictly positive as well and we can write

$$v + \widehat{\varphi} \bullet S = v\mathscr{E}(-\widetilde{a} \bullet S)$$

for the *optimal number of shares per unit of wealth*

$$-\widetilde{a} := \widehat{\varphi}/(v + \widehat{\varphi} \bullet S_-),$$

which is independent of the initial endowment v for power utility. Finally, [23, Theorem 2.2] also establishes the existence of a *dual minimizer*, i.e. a strictly positive supermartingale $\widehat{Y}$ with $\widehat{Y}_T = \mathscr{E}(-\widetilde{a} \bullet S)_T^{-p}$ such that $(v+\varphi \bullet S)\widehat{Y}$ is a supermartingale for all $\varphi \in \Theta(v)$ and $(v + \widehat{\varphi} \bullet S)\widehat{Y}$ is a true martingale. Alternatively, one can represent this object in terms of the *opportunity process* $L := L_0\mathscr{E}(K) := \mathscr{E}(-\widetilde{a} \bullet S)^p\widehat{Y}$ of the power utility maximization problem (cf. [4, 27, 28] for motivation and more details).

Remark 2.3 The optimal strategy $\widehat{\varphi}$ as well as the joint characteristics of the assets and the opportunity process L satisfy a semimartingale *Bellman equation* (cf. [29, Theorem 3.2]). In concrete models, this sometimes allows to determine $\widehat{\varphi}$ and L by making an appropriate ansatz, cf. e.g. [27, Chapter 4].

In addition to the traded securities, we now also consider a nontraded European contingent claim with maturity T and payment function H, which is a $\mathscr{F}_T$-measurable random variable. Following [24, 25], we assume that H can be superhedged by some admissible strategy.

Assumption 2.4 $|H| \leq w + \varphi \bullet S_T$ for some $w \in (0, \infty)$ and $\varphi \in \Theta(w)$.

If the investor sells q units of H at time 0, her terminal wealth should be sufficiently large to cover the payment $-qH$ due at time T. This leads to the following definition (cf. [14, 8] for more details).

Definition 2.5 A trading strategy $\varphi \in \Theta(v)$ is called *maximal*, if the terminal value $v + \varphi \bullet S_T$ of its wealth process cannot be dominated by that of any other strategy in $\Theta(v)$. An arbitrary strategy φ is called *acceptable*, if its wealth process can be written as

$$v + \varphi \bullet S = v' + \varphi' \bullet S - (v'' + \varphi'' \bullet S),$$

where $v' + \varphi' \bullet S \geq 0$, and $v'' + \varphi'' \bullet S \geq 0$ for $v', v'' \in \mathbb{R}_+$ and $\varphi', \varphi'' \in L(S)$ and, in addition, φ'' is maximal. For $v \in (0, \infty)$ and $q \in \mathbb{R}$ we denote by

$$\Theta^q(v) := \{\varphi \in L(S) : \varphi \text{ is acceptable}, v + \varphi \bullet S_T - qH \geq 0\},$$

the set of acceptable strategies whose terminal value dominates qH.

Remark 2.6 Given Assumption 2.1, we have $\Theta(v) = \Theta^0(v)$ by [8, Theorem 5.7] combined with [17, Lemma 3.1 and Proposition 3.1] .

Let an initial endowment of $v \in (0, \infty)$ be given. If the investor sells a number of q units of H for a price of $x \in \mathbb{R}$ each, her initial position consists of $v + qx$ in cash as well as $-q$ units of the contingent claim H. Hence $\Theta^q(v + qx)$ represents the natural set of admissible trading strategies for utility functions defined on $\mathbb{R}_+$. The maximal expected utility the investor can achieve by dynamic trading in the market is then given by

$$U^q(v + qx) := \sup_{\varphi \in \Theta^q(v+qx)} E(u(v + qx + \varphi \bullet S_T - qH)).$$

Definition 2.7 Fix $q \in \mathbb{R}$. A number $\pi^q \in \mathbb{R}$ is called *utility indifference price* of H if

$$U^q(v + q\pi^q) = U(v). \tag{2.2}$$

Existence of indifference prices does not hold in general for power utility. However, a unique indifference price π^q always exists if the number q of contingent claims sold is sufficiently small or conversely, if the initial endowment v is sufficiently large.

Lemma 2.8 *Suppose Assumptions 2.1, 2.2 and 2.4 hold. Then a unique indifference price exists for sufficiently small q. More specifically,* (2.2) *has a unique solution π^q if $q < \frac{v}{2w}$, respectively if $q < \frac{v}{w}$ and $H \geq 0$, where w denotes the initial endowment of the superhedging strategy for H from Assumption 2.4.*

PROOF. First notice that $g_v^q : x \mapsto U^q(v+qx)$ is concave and strictly increasing on its effective domain. By [23, Theorem 2.1], $g_v^q(x) \leq U(v+qx+qw) < \infty$ for all $x \in \mathbb{R}$. For $H \geq 0$ and $q < \frac{v}{w}$ we have $g_v^q(x) > -\infty$ for $x > w - \frac{v}{q}$. In particular, g_v^q is continuous and strictly increasing on $(w - \frac{v}{q}, \infty)$ and in particular on $[0, w]$ by [30, Theorem 10.1]. By $H \geq 0$ we have $g_v^q(0) \leq U(v)$. Moreover, Assumption 2.4 implies $g_v^q(w) \geq U(v)$. Hence there exists a unique solution $\pi^q \in [0, w]$ to $g_v^q(x) = U(v)$. Similarly, for general H and $q < \frac{v}{2w}$, the function g_v^q is finite, continuous and strictly increasing on an open set containing $[-w, w]$. Moreover, $g_v^q(-w) \leq U(v)$ and $g_v^q(w) \geq U(v)$. Hence there exists a unique $\pi^q \in (-w, w)$ such that $g_v^q(\pi^q) = U(v)$. This proves the assertion. □

We now turn to optimal trading strategies for random endowment. Their existence has been established by [7] resp. [14] in the bounded resp. general case.

Theorem 2.9 *Fix $q \in \mathbb{R}$ satisfying the conditions of Lemma 2.8 and suppose Assumptions 2.1, 2.2 and 2.4 are satisfied. Then there exists $\varphi^q \in \Theta^q(v + q\pi^q)$ such that*

$$E(u(v + q\pi^q + \varphi^q \bullet S_T - qH)) = U^q(v + q\pi^q).$$

Moreover, the corresponding optimal value process $v + q\pi^q + \varphi^q \bullet S$ is unique.

PROOF. This follows from [14, Theorem 2 and Corollary 1], because the proof of Lemma 2.8 shows that $(v + q\pi^q, q)$ belongs to the interior of $\{(x,r) \in \mathbb{R}^2 : \Theta^r(x) \neq \varnothing\}$. □

Without contingent claims, the investor will trade according to the strategy $\widehat{\varphi}$, whereas she will invest into φ^q if she sells q units of H for π^q each. Hence the difference between both strategies represents the action the investors needs to take in order to compensate for the risk of selling q units of H. This motivates the following

Definition 2.10 The trading strategy $\varphi^q - \widehat{\varphi}$ is called *utility-based hedging strategy*.

3 The asymptotic results of Kramkov and Sîrbu

We now give a brief exposition of some of the deep results of [24, 25] concerning the existence and characterization of first-order approximations of utility-based prices and hedging strategies in the following sense.

Definition 3.1 Real numbers π^0 and π' are called *marginal utility-based price* resp. *risk premium* per option sold if

$$\pi^q = \pi^0 + q\pi' + o(q^2)$$

for $q \to 0$, where π^q is well-defined for sufficiently small q by Lemma 2.8. A trading strategy $\varphi' \in L(S)$ is called *marginal utility-based hedging strategy*, if there exists $v' \in \mathbb{R}$ such that

$$\lim_{q \to 0} \frac{(v + q\pi^q + \varphi^q \bullet S_T) - (v + \widehat{\varphi} \bullet S_T) - q(v' + \varphi' \bullet S_T)}{q} = 0$$

in P-probability and $(v' + \varphi' \bullet S)\widehat{Y}$ is a martingale for the dual minimizer $\widehat{Y}$ of the pure investment problem.

Remark 3.2 [24, Theorems A.1, 8 and 4] show that for power utility functions, a trading strategy φ' is a marginal utility-based hedging strategy in the sense of Definition 3.1 if and only if it is a marginal hedging strategy in the sense of [25, Definition 2].

The asymptotic results of [24, 25] are derived subject to two technical assumptions.

Assumption 3.3 The following process is σ*-bounded*:

$$S^{\$} := \left(\frac{1}{\mathscr{E}(-\widetilde{a} \bullet S)}, \frac{S}{\mathscr{E}(-\widetilde{a} \bullet S)} \right).$$

The reader is referred to [24] for more details on σ-bounded processes as well as for sufficient conditions that ensure the validity of this assumption.

Since $\mathscr{E}(-\widetilde{a} \bullet S)\widehat{Y}$ is a martingale with terminal value $\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}$, we can define an equivalent probability measure $Q^\$ \sim P$ via

$$\frac{dQ^\$}{dP} := \frac{\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}}{C_0}, \quad C_0 := E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{1-p}).$$

Let $\mathscr{H}_0^2(Q^\$)$ be the space of square-integrable $Q^\$$-martingales starting at 0 and set

$$\mathscr{M}_\$^2 := \left\{ M \in \mathscr{H}_0^2(Q^\$) : M = \varphi \bullet S^\$ \text{ for some } \varphi \in L(S^\$) \right\}.$$

Assumption 3.4 There exists a constant $w^\$ \geq 0$ and a process $M^\$ \in \mathscr{M}_\$^2$, such that

$$|H^\$| \leq w^\$ + M_T^\$$$

for

$$H^\$:= \frac{H}{\mathscr{E}(-\widetilde{a} \bullet S)_T}.$$

Remark 3.5 By [24, Remark 1], Assumption 3.4 implies that Assumption 2.4 holds. In particular, indifference prices and utility-based hedging strategies exist for sufficiently small q if Assumptions 2.1, 2.2 and 3.4 are satisfied.

In the proof of [25, Lemma 1] it is shown that the process

$$V_t^\$:= E_{Q^\$}\left(H^\$ | \mathscr{F}_t\right), \quad t \in [0,T]$$

is a square-integrable $Q^\$$-martingale. Hence it admits a decomposition

$$V^\$ = E_{Q^\$}\left(H^\$\right) + \xi \bullet S^\$ + N^\$ = \frac{1}{C_0} E\left(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-p} H\right) + \xi \bullet S^\$ + N^\$, \tag{3.1}$$

where $\xi \bullet S^\$ \in \mathscr{M}_\$^2$ and $N^\$$ is an element of the orthogonal complement of $\mathscr{M}_\$^2$ in $\mathscr{H}_0^2(Q^\$)$. Note that this decomposition coincides with the classical *Galtchouk-Kunita-Watanabe* decomposition, if $S^\$$ itself is a square-integrable martingale. The following theorem is a reformulation of the results of [24, 25] applied to power utility.

Theorem 3.6 *Suppose Assumptions 2.1, 2.2, 3.3 and 3.4 hold. Then the marginal utility-based price π^0 and the risk premium π' exist and are given by*

$$\pi^0 = \frac{1}{C_0} E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-p} H), \quad \pi' = \frac{p}{2v} E_{Q^\$}((N_T^\$)^2).$$

A marginal-utility-based hedging strategy ϕ' is given in feedback form as

$$\phi' = (\widetilde{a}, E_d + \widetilde{a} S_-^\top)\xi - \left(\pi^0 + \phi' \bullet S_-\right)\widetilde{a},$$

with ξ from (3.1).

PROOF. The first two assertions follow immediately from [24, Theorems A.1, 8 and 4] adapted to the present notation. For the third, [25, Theorem 2] and [24, Theorems A.1, 8 and 4] yield

$$\lim_{q\to 0}\frac{(v+q\pi^q+\varphi^q\bullet S_T)-(v+\widehat{\varphi}\bullet S_T)-q\mathscr{E}(-\widetilde{a}\bullet S)_T(\pi^0+\xi\bullet S^\$_T)}{q}=0. \tag{3.2}$$

because the process $X'_T(x)$ from [25, Equation (23)] coincides with $\mathscr{E}(-\widetilde{a}\bullet S)$ for power utility. Set

$$\xi^0:=\pi^0+\xi\bullet S^\$-\xi^\top S^\$=\pi^0+\xi\bullet S^\$_- -\xi^\top S^\$_-.$$

Then we have $(\xi^0,\xi^2,\dots,\xi^{d+1})\in L((\mathscr{E}(-\widetilde{a}\bullet S),S))$ and

$$\pi^0+(\xi^0,\xi^2,\dots,\xi^{d+1})\bullet(\mathscr{E}(-\widetilde{a}\bullet S),S)=\mathscr{E}(-\widetilde{a}\bullet S)(\pi^0+\xi\bullet S^\$) \tag{3.3}$$

by [10, Proposition 2.1]. The predictable sets $D_n:=\{|\widetilde{a}|\leq n,|S_-|\leq n,|(\xi^0,\xi)|\leq n\}$ increase to $\Omega\times\mathbb{R}_+$, the predictable process $(\widetilde{a},E_d+\widetilde{a}S_-^\top)\xi 1_{D_n}$ is bounded and we have

$$\begin{aligned}
&((\widetilde{a},E_d+\widetilde{a}S_-^\top)\xi 1_{D_n})\bullet S\\
&=((\mathscr{E}(-\widetilde{a}\bullet S)_-\xi^\top S^\$_-\widetilde{a}+(\xi^2,\dots,\xi^{d+1}))1_{D_n})\bullet S\\
&=((\xi^0,\xi^2,\dots,\xi^{d+1})1_{D_n})\bullet(\mathscr{E}(-\widetilde{a}\bullet S),S)+(\mathscr{E}(-\widetilde{a}\bullet S)_-(\pi^0+\xi\bullet S^\$_-)1_{D_n})\bullet(\widetilde{a}\bullet S)\\
&=1_{D_n}\bullet((\xi^0,\xi^2,\dots,\xi^{d+1})\bullet(\mathscr{E}(-\widetilde{a}\bullet S),S)+(\mathscr{E}(-\widetilde{a}\bullet S)_-(\pi^0+\xi\bullet S^\$_-))\bullet(\widetilde{a}\bullet S)).
\end{aligned}$$

By [17, Lemma 2.2] and (3.3) this implies $(\widetilde{a},E_d+\widetilde{a}S_-^\top)\xi\in L(S)$ as well as

$$\pi^0+((\widetilde{a},E_d+\widetilde{a}S_-^\top)\xi)\bullet S=\mathscr{E}(-\widetilde{a}\bullet S)(\pi^0+\xi\bullet S^\$)+(\mathscr{E}(-\widetilde{a}\bullet S)_-(\pi^0+\xi\bullet S^\$_-))\bullet(\widetilde{a}\bullet S).$$

Hence $\mathscr{E}(-\widetilde{a}\bullet S)(\pi^0+\xi\bullet S^\$)$ solves the stochastic differential equation

$$G=\pi^0+((\widetilde{a},E_d+\widetilde{a}S_-^\top)\xi)\bullet S-G_-\bullet(\widetilde{a}\bullet S). \tag{3.4}$$

By [15, (6.8)] this solution is unique. Since we have shown $(\widetilde{a},E_d+\widetilde{a}S_-^\top)\xi\in L(S)$ above, it follows as in the proof of [4, Lemma 4.9] that ϕ' is well-defined. $\pi^0+\phi'\bullet S$ also solves (3.4), hence we obtain

$$\mathscr{E}(-\widetilde{a}\bullet S)(\pi^0+\xi\bullet S^\$)=\pi^0+\phi'\bullet S,$$

which combined with (3.2) yields the third assertion. □

Remark 3.7 If the dual minimizer $\widehat{Y}$ is a martingale and hence – up to the constant C_0 – the density process of the q-*optimal martingale measure* Q_0 with respect to P, the generalized Bayes formula yields $V^\$_t=E_{Q_0}(H|\mathscr{F}_t)/\mathscr{E}(-\widetilde{a}\bullet S)_t$. In particular, the marginal utility-based price of the claim H is given by its expectation $\pi^0=E_{Q_0}(H)$ under Q_0 in this case.

The computation of the optimal strategy $\widehat{\varphi}$ and the corresponding dual minimizer $\widehat{Y}$ in the pure investment problem 2.1 has been extensively studied in the literature. In particular, these objects have been determined explicitly in a variety of Markovian models using stochastic control theory resp. martingale methods. Given $\mathscr{E}(-\widetilde{a} \bullet S)$, the computation of π^0 can then be dealt with using integral transform methods or variants of the Feynman-Kac formula. Consequently, we suppose in the remainder of this section that $\widehat{\varphi}$ and π^0 are known and focus on how to obtain π' and φ'.

As reviewed above, [24, 25] show that φ' and π' can be obtained by calculating the generalized Galtchouk-Kunita-Watanabe decomposition (3.1). Since $S^\$$ is generally only a $Q^\$$-supermartingale rather than a martingale, this is typically very difficult. If however, $S^\$$ happens to be a square-integrable $Q^\$$-martingale, (3.1) coincides with the classical Galtchouk-Kunita-Watanabe decomposition. By [9], this shows that ξ represents the mean-variance optimal hedging strategy for the claim H hedged with $S^\$$ under the measure $Q^\$$ and $E_{Q^\$}((N_T^\$)^2)$ is given by the corresponding minimal expected squared hedging error in this case. Moreover, ξ and $E_{Q^\$}((N_T^\$)^2)$ can then be characterized in terms of semimartingale characteristics.

Assumption 3.8 $S^\$$ is a square-integrable $Q^\$$-martingale.

Remark 3.9 In applications, Assumption 3.8 is typically equivalent to Assumption 3.3, cf. [27, Chapter 6] for more details.

Lemma 3.10 *Suppose Assumptions 2.1, 2.2, 3.3, 3.4 and 3.8 hold. Denote by $\tilde{c}^{(S^\$,V^\$)\$}$ the modified second $Q^\$$-characteristic of $(S^\$, V^\$)$ with respect to some $A \in \mathscr{A}^+_{\mathrm{loc}}$ (cf. Appendix A). Then*

$$\xi = (\tilde{c}^{S^\$\$})^{-1}\tilde{c}^{S^\$,V^\$\$}, \tag{3.5}$$
$$E_{Q^\$}((N_T^\$)^2) = E_{Q^\$}\left((\tilde{c}^{V^\$\$} - (\tilde{c}^{S^\$,V^\$\$})^\top(\tilde{c}^{S^\$\$})^{-1}\tilde{c}^{S^\$,V^\$\$}) \bullet A_T\right).$$

PROOF. Since $S^\$$ is a square integrable $Q^\$$-martingale by Assumption 3.8, the claim follows from [4, Theorems 4.10 and 4.12] applied to the martingale case. □

The key to using Lemma 3.10 in concrete models is the computation of the joint characteristics of $S^\$$ and $V^\$$. In principle, this problem can be tackled using PDE methods as in [6] or by applying the Laplace transform approach of [13] as in [19]. However, the change of numeraire in this direct approach introduces an additional state variable, which makes the ensuing numerics considerably more involved. This can be avoided by the alternative approach put forward in the next section.

4 An alternative representation

Subject to Assumption 3.8 we can define a probability measure $P^€ \sim P$ via

$$\frac{dP^€}{dP} := \frac{\mathscr{E}(-\widetilde{a} \bullet S)_T^{-1-p}}{C_1}, \quad C_1 := E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-1-p}).$$

Remark 4.1 If we write the density process of $P^{€}$ w.r.t. P as $L^{€}\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p}/C_1$ for a semimartingale $L^{€}>0$ with $L^{€}_T=1$, the local P-characteristics of $K^{€}:=\mathscr{L}(L^{€})$ and S satisfy

$$\int_{\{|x|>1\}}(1+x_2)(1-\widetilde{a}^\top x_1)^{-1-p}F^{(S,K^{€})}(dx)<\infty, \tag{4.1}$$

and solve

$$\begin{aligned}0&=b^{K^{€}}+(1+p)\widetilde{a}^\top b^S+(1+p)\widetilde{a}^\top c^{S,K^{€}}+\frac{(p+1)(p+2)}{2}\widetilde{a}^\top c^S\widetilde{a}\\&\quad+\int\left((1+x_2)(1-\widetilde{a}^\top x_1)^{-1-p}-1-h_2(x_2)-(1+p)\widetilde{a}^\top h_1(x_1)\right)F^{(S,K^{€})}(dx),\end{aligned} \tag{4.2}$$

relative to some truncation function (h_1,h_2) on $\mathbb{R}^d\times\mathbb{R}$ by [17, Lemma 3.1] and Propositions A.2, A.3 . Conversely, if a strictly positive semimartingale $L^{€}=L^{€}_0\mathscr{E}(K^{€})$ satisfies $L^{€}_T=1$ and (4.1), (4.2), then $L^{€}\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p}/C_1$ is a σ-martingale and the density process of $P^{€}$ if it is a true martingale. In concrete models, this often allows to determine $L^{€}$ by making an appropriate parametric ansatz for $K^{€}$ (cf. [27, Chapter 6]).

The measures $P^{€}$ and $Q^{\$}$ are linked as follows.

Lemma 4.2 *Suppose Assumptions 2.1, 2.2 and 3.8 hold. Then the process*

$$L^{\$}_t:=E_{P^{€}}\left(\frac{\mathscr{E}(-\widetilde{a}\bullet S)^2_T}{\mathscr{E}(-\widetilde{a}\bullet S)^2_t}\middle|\mathscr{F}_t\right),\quad 0\le t\le T,$$

satisfies $L^{\$}_T=1$ and the density process of $Q^{\$}$ with respect to $P^{€}$ is given by

$$E_{P^{€}}\left(\frac{dQ^{\$}}{dP^{€}}\middle|\mathscr{F}_t\right)=\frac{C_1}{C_0}L^{\$}_t\mathscr{E}(-\widetilde{a}\bullet S)^2_t=\frac{L^{\$}_t\mathscr{E}(-\widetilde{a}\bullet S)^2_t}{L^{\$}_0}.$$

In particular, $L^{\$},L^{\$}_->0$ and the stochastic logarithm $K^{\$}:=\mathscr{L}(L^{\$})$ is well-defined.

PROOF. The first part of the assertion is trivial, whereas the second follows from $\frac{dQ^{\$}}{dP^{€}}=\frac{C_1}{C_0}\mathscr{E}(-\widetilde{a}\bullet S)^2_T$. Since $\mathscr{E}(-\widetilde{a}\bullet S),\mathscr{E}(-\widetilde{a}\bullet S)_->0$, [16, I.2.27] yields $L^{\$},L^{\$}_->0$ and hence the third part of the assertion by [16, II.8.3]. □

Remark 4.3 $L^{\$}$ is linked to the opportunity process L of the pure investment problem and the process $L^{€}$ from Remark 4.1 via

$$L^{\$}_0\mathscr{E}(K^{\$})=L^{\$}=\frac{L}{L^{€}}=\frac{L_0\mathscr{E}(K)}{L^{€}_0\mathscr{E}(K^{€})},$$

by the generalized Bayes' formula, $L_T=L^{€}_T=1$ and because the processes $L\mathscr{E}(-\widetilde{a}\bullet S)^{1-p}$ and $L^{€}\mathscr{E}(-\widetilde{a}\bullet S)^{-1-p}$ are martingales. In view of Yor's formula, this implies

$$K^{\$}=K-K^{€}-\langle K^c-(K^{€})^c,(K^{€})^c\rangle-\sum_{s\le\cdot}\frac{(\Delta K_s-\Delta K^{€}_s)\Delta K^{€}_s}{1+\Delta K^{€}_s}, \tag{4.3}$$

which allows to derive $K^{\$}$ and its characteristics from K (which is determined by the pure investment problem) and $K^{€}$ (compare Remark 4.1 above).

Set

$$V_t := \mathscr{E}(-\widetilde{a} \bullet S)_t V_t^\$ = \frac{E(\mathscr{E}(-\widetilde{a} \bullet S)_T^{-p} H | \mathscr{F}_t)}{L_t \mathscr{E}(-\widetilde{a} \bullet S)_t^{-p}}, \quad 0 \leq t \leq T,$$

denote by

$$\left(\begin{pmatrix} b^{S€} \\ b^{V€} \\ b^{K^\$ €} \end{pmatrix}, \begin{pmatrix} c^{S€} & c^{S,V€} & c^{S,K^\$ €} \\ c^{V,S€} & c^{V€} & c^{V,K^\$ €} \\ c^{K^\$,S€} & c^{K^\$,V€} & c^{K^\$ €} \end{pmatrix}, F^{(S,V,K^\$)€}, A \right)$$

$P^€$-differential characteristics of the semimartingale $(S, V, K^\$)$ and define

$$\begin{aligned}
\tilde{c}^{S\star} &:= \frac{1}{1+\Delta A^{K^\$}} \left(c^{S€} + \int (1+x_3) x_1 x_1^\top F^{(S,V,K^\$)€}(dx) \right), \\
\tilde{c}^{S,V\star} &:= \frac{1}{1+\Delta A^{K^\$}} \left(c^{S,V€} + \int (1+x_3) x_1 x_2 F^{(S,V,K^\$)€}(dx) \right), \\
\tilde{c}^{V\star} &:= \frac{1}{1+\Delta A^{K^\$}} \left(c^{V€} + \int (1+x_3) x_2^2 F^{(S,V,K^\$)€}(dx) \right),
\end{aligned}$$

where $K^\$ = K_0^\$ + A^{K^\$} + M^{K^\$}$ denotes an arbitrary $P^€$-semimartingale decomposition of $K^\$$. We then have the following representation of the marginal utility-based hedging strategy φ' and the risk premium π' in terms of semimartingale characteristics, which is the main result of this paper.

Theorem 4.4 *Suppose Assumptions 2.1, 2.2, 3.3, 3.4 and 3.8 hold. Then* $\tilde{c}^{S\star}, \tilde{c}^{S,V\star}, \tilde{c}^{V\star}$ *are well-defined,* φ' *given in feedback form as*

$$\varphi' = (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} - \left(\pi^0 + \varphi' \bullet S_- - V_-\right) \widetilde{a}$$

is a marginal utility-based hedging strategy and the corresponding risk premium is

$$\pi' = \frac{pC_1}{2vC_0} E_{P^€}\left(\left(\left(\tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^\top (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star} \right) L^\$ \right) \bullet A_T \right).$$

PROOF. An application of Propositions A.2 and A.3 yields the $P^€$-differential characteristics of $(S, V, \mathscr{E}(-\widetilde{a} \bullet S), \mathscr{L}(\frac{C_1}{C_0} L^\$ \mathscr{E}(-\widetilde{a} \bullet S)^2))$. Since $\frac{C_1}{C_0} L^\$ \mathscr{E}(-\widetilde{a} \bullet S)^2$ is the density process of $Q^\$$ with respect to $P^€$, the $Q^\$$-characteristics of $(S, V, \mathscr{E}(-\widetilde{a} \bullet S))$ can now be obtained with Proposition A.4. Another application of Proposition A.3 then allows to compute the $Q^\$$-characteristics of $(S^\$, V^\$)$.

Since $S^\$ \in \mathscr{H}^2(Q^\$)$ by Assumption 3.8 and $V^\$ \in \mathscr{H}^2(Q^\$)$ by the proof of [25, Lemma 1], the modified second characteristics $\tilde{c}^{V^\$\$}$, $\tilde{c}^{S^\$,V^\$\$}$ and $\tilde{c}^{S^\$\$}$ exist and are given by

$$\tilde{c}^{S^\$\$} = \frac{1+\Delta A^{K^\$}}{\mathscr{E}(-\widetilde{a} \bullet S)_-^2} \begin{pmatrix} \widetilde{a}^\top \tilde{c}^{S\star} \widetilde{a} & \widetilde{a}^\top \tilde{c}^{S\star} R^\top \\ R \tilde{c}^{S\star} \widetilde{a} & R \tilde{c}^{S\star} R^\top \end{pmatrix}, \tag{4.4}$$

$$\tilde{c}^{S^\$,V^\$\$} = \frac{1+\Delta A^{K^\$}}{\mathscr{E}(-\widetilde{a} \bullet S)_-^2} \begin{pmatrix} \widetilde{a}^\top \\ R \end{pmatrix} \left(\tilde{c}^{S,V\star} + \tilde{c}^{S\star} \widetilde{a} V_- \right), \tag{4.5}$$

$$\tilde{c}^{V^\$\$} = \frac{1+\Delta A^{K^\$}}{\mathscr{E}(-\widetilde{a} \bullet S)_-^2} \left(\tilde{c}^{V\star} + 2V_- \widetilde{a}^\top \tilde{c}^{S,V\star} + V_-^2 \widetilde{a}^\top \tilde{c}^{S\star} \widetilde{a} \right), \tag{4.6}$$

for $R := E_d + S_- \widetilde{a}^\top$. In particular it follows that $\tilde{c}^{V\star}$, $\tilde{c}^{S,V\star}$ and $\tilde{c}^{S\star}$ are well defined. By the definition of ξ in Equation (3.5) and [1, Theorem 9.1.6] we have

$$\tilde{c}^{S^{\$}\$}\xi = \tilde{c}^{S^{\$},V^{\$}\$}.$$

In view of Equations (4.5) and (4.4), this yields

$$\begin{pmatrix} \widetilde{a}^\top \tilde{c}^{S\star}\widetilde{a} & \widetilde{a}^\top \tilde{c}^{S\star} R^\top \\ R\tilde{c}^{S\star}\widetilde{a} & R\tilde{c}^{S\star}R^\top \end{pmatrix} \xi = \begin{pmatrix} \widetilde{a}^\top \\ R \end{pmatrix} \left(\tilde{c}^{S,V\star} + \tilde{c}^{S\star}\widetilde{a}V_- \right),$$

or equivalently, decomposed into the first and last d components,

$$\widetilde{a}^\top \tilde{c}^{S\star}(\widetilde{a}, R^\top)\xi = \widetilde{a}^\top (\tilde{c}^{S,V\star} + \tilde{c}^{S\star}\widetilde{a}V_-) \tag{4.7}$$

and

$$R\tilde{c}^{S\star}(\widetilde{a}, R^\top)\xi = R(\tilde{c}^{S,V\star} + \tilde{c}^{S\star}\widetilde{a}V_-). \tag{4.8}$$

By multiplying both sides of (4.7) with S_- from the left and subtracting the result from (4.8), this leads to

$$\tilde{c}^{S\star}(\widetilde{a}, R^\top)\xi = \tilde{c}^{S,V\star} + \tilde{c}^{S\star}\widetilde{a}V_-, \tag{4.9}$$

since $R - S_-\widetilde{a}^\top = E_d$. By Theorem 3.6,

$$\phi' = (\widetilde{a}, R^\top)\xi - (\pi^0 + \phi' \bullet S_-)\widetilde{a}$$

defines a marginal utility-based hedging strategy. Let

$$\psi' := \phi' - ((\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} - (\pi^0 + \phi' \bullet S_- - V_-)\widetilde{a}) = (\widetilde{a}, R^\top)\xi - (\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} - V_-\widetilde{a}.$$

Then it follows from the definition of ψ' and (4.9) that

$$\tilde{c}^{S\star}\psi' = \tilde{c}^{S,V\star} + \tilde{c}^{S\star}V_-\widetilde{a} - \tilde{c}^{S,V\star} - \tilde{c}^{S\star}V_-\widetilde{a} = 0,$$

because $\tilde{c}^{S\star}(\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} = \tilde{c}^{S,V\star}$ by [1, Theorem 9.1.6]. In particular, $(\psi')^\top \tilde{c}^{S\star}\psi' = 0$. Since $L^{\$}/L^{\$}_0 = \mathscr{E}(K^{\$}) > 0$ and hence $\Delta K^{\$} > -1$ by [16, I.4.61], this implies

$$(\psi')^\top \tilde{c}^{S}\psi' = 0. \tag{4.10}$$

For $n \in \mathbb{N}$, define the predictable sets $D_n := \{|\psi'| \leq n\}$. By Proposition A.2 and (4.10), we have $\tilde{c}^{\psi' 1_{D_n} \bullet S} = 0$ and hence $c^{\psi' 1_{D_n} \bullet S} = 0$ and $F^{\psi' 1_{D_n} \bullet S} = 0$. Together with Proposition A.4, this implies that the local characteristics of $\psi' 1_{D_n} \bullet S$ under the equivalent local martingale measure Q from Assumption 2.1 vanish by [17, Lemma 3.1]. Hence $\psi' 1_{D_n} \bullet S = 0$ and it follows from [17, Lemma 2.2] that $\psi' \in L(S)$ with $\psi' \bullet S = 0$. Taking into account the definition of ψ', this shows

$$\phi' \bullet S = ((\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} - (\pi^0 - V_-)\widetilde{a}) \bullet S - (\phi' \bullet S_-) \bullet (\widetilde{a} \bullet S),$$

i.e. $\phi' \bullet S$ solves the feedback equation

$$G = ((\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} - (\pi^0 - V_-)\widetilde{a}) \bullet S - G_- \bullet (\widetilde{a} \bullet S). \tag{4.11}$$

Since $\psi' \in L(S)$ and $L(S)$ is a vector space, it follows that $(\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} \in L(S)$, too. As in the proof of [4, Lemma 4.9], this in turn yields that φ' is well-defined and in $L(S)$. Apparently, $\varphi' \bullet S$ also solves (4.11) and, since the solution is unique by [15, (6.8)], we obtain $\varphi' \bullet S = \phi' \bullet S$. Therefore φ' is a marginal utility-based hedging strategy.

We now turn to the risk premium π'. First notice that by [1, Theorem 9.1.6],

$$C^{\$} := \tilde{c}^{V^{\$}\$} - (\tilde{c}^{S^{\$},V^{\$}\$})^{\top}\xi = \tilde{c}^{V^{\$}\$} - (\tilde{c}^{S^{\$},V^{\$}\$})^{\top}(\tilde{c}^{S^{\$}\$})^{-1}\tilde{c}^{S^{\$},V^{\$}\$} \geq 0,$$
$$C^{€} := \tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^{\top}(\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star} \geq 0.$$

Hence $C^{\$} \bullet A$ is an increasing predictable process and by Lemmas 3.10 and A.5,

$$\begin{aligned} E_{Q^{\$}}((N_T^{\$})^2) &= E_{Q^{\$}}(C^{\$} \bullet A_T) \\ &= \frac{C_1}{C_0} E_{P^{€}}\left(L_-^{\$}\mathscr{E}(-\widetilde{a} \bullet S)_-^2 C^{\$} \bullet A_T\right) \\ &= \frac{C_1}{C_0} E_{P^{€}}\left(L_-^{\$}\mathscr{E}(-\widetilde{a} \bullet S)_-^2 \bullet (\langle V^{\$}, V^{\$}\rangle_T^{Q^{\$}} - \langle V^{\$}, \xi \bullet S^{\$}\rangle_T^{Q^{\$}})\right). \end{aligned}$$

Since we have shown $\phi' \bullet S = \varphi' \bullet S$ above, [10, Proposition 2.1] and the proof of Theorem 3.6 yield $\xi \bullet S^{\$} = (\varphi'^0, \varphi') \bullet S^{\$}$ for $\varphi'^0 := \pi^0 + \varphi' \bullet S - \varphi' S$. Hence

$$\begin{aligned} E_{Q^{\$}}((N_T^{\$})^2) &= \frac{C_1}{C_0} E_{P^{€}}\left(L_-^{\$}\mathscr{E}(-\widetilde{a} \bullet S)_-^2 \bullet \left(\langle V^{\$}, V^{\$}\rangle_T^{Q^{\$}} - \langle V^{\$}, (\varphi'^0, \varphi') \bullet S^{\$}\rangle_T^{Q^{\$}}\right)\right) \\ &= \frac{C_1}{C_0} E_{P^{€}}\left(L_-^{\$}\mathscr{E}(-\widetilde{a} \bullet S)_-^2 \left(\tilde{c}^{V^{\$}\$} - (\tilde{c}^{S^{\$},V^{\$}\$})^{\top}(\varphi'^0, \varphi')\right) \bullet A_T\right). \end{aligned}$$

After inserting $\tilde{c}^{V^{\$}\$}$, $\tilde{c}^{S^{\$},V^{\$}\$}$ from (4.6) resp. (4.5) and the definition of (φ'^0, φ'), this leads to

$$E_{Q^{\$}}((N_T^{\$})^2) = \frac{C_1}{C_0} E_{P^{€}}\left(\left(1 + \Delta A^{K^{\$}}\right) L_-^{\$} C^{€} \bullet A_T\right). \tag{4.12}$$

Now notice that the definition of the stochastic exponential and [16, I.4.36] imply

$$L^{\$} = \left(1 + \Delta A^{K^{\$}} + \Delta M^{K^{\$}}\right) L_-^{\$}.$$

By [16, I.4.49] the process $\Delta M^{K^{\$}} \bullet (L_-^{\$} C^{€} \bullet A)$ is a local martingale. If $(T_n)_{n\in\mathbb{N}}$ denotes a localizing sequence, this yields

$$\begin{aligned} E_{P^{€}}(L^{\$} C^{€} \bullet A_{T\wedge T_n}) &= E_{P^{€}}\left(\left(1 + \Delta A^{K^{\$}} + \Delta M^{K^{\$}}\right) L_-^{\$} C^{€} \bullet A_{T\wedge T_n}\right) \\ &= E_{P^{€}}\left(\left(1 + \Delta A^{K^{\$}}\right) L_-^{\$} C^{€} \bullet A_{T\wedge T_n}\right), \end{aligned}$$

and hence

$$E_{P^{€}}(L^{\$} C^{€} \bullet A_T) = E_{P^{€}}\left(\left(1 + \Delta A^{K^{\$}}\right) L_-^{\$} C^{€} \bullet A_T\right)$$

by monotone convergence. Combining this with (4.12), we obtain

$$E_{Q^\$}((N_T^\$)^2) = \frac{C_1}{C_0} E_{P^€}\big(\big(\tilde{c}^{V\star} - (\tilde{c}^{S,V\star})^\top (\tilde{c}^{S\star})^{-1} \tilde{c}^{S,V\star}\big)\, L^\$ \bullet A_T\big).$$

In view of Theorem 3.6 this completes the proof. $\square$

Remarks.

1. The arguments used to show $\varphi' \bullet S = \phi' \bullet S$ in the proof of Theorem 4.4 also yield that one obtains a marginal utility-based hedging strategy, if the *pure hedge coefficient* $(\tilde{c}^{S\star})^{-1}\tilde{c}^{S,V\star}$ is replaced by any other solution ζ of $\tilde{c}^{S\star}\zeta = \tilde{c}^{S,V\star}$.

2. An inspection of the proof of Theorem 4.4 shows that the formulas for φ' and π' are independent of the specific semimartingale decomposition of $K^\$$ that is used. In particular, the not necessarily predictable term $1 + \Delta A^{K^\$€}$ disappears in the formula for φ' by [1, Theorem 3.9]. If the semimartingale $K^\$$ is $P^€$-special, one can choose the *canonical* decomposition [16, II.2.38]. By [16, II.2.29], this yields

   $$\Delta A^{K^\$} = \Delta A \int x F^{K^\$€}(dx).$$

   If additionally $K^\$$ has no fixed times of discontinuity, [16, II.2.9] shows that A can be chosen to be continuous such that $\Delta A^{K^\$} = 0$.

3. For *continuous* S, our feedback representation of φ' coincides with [25, Theorem 3], because the modified second characteristic is invariant with respect to equivalent changes of measure for continuous processes.

In view of [4, Theorems 4.10 and 4.12], Theorem 4.4 states that the first-order approximations for φ^q and π^q can *essentially* be computed by solving the mean-variance hedging problem for the claim H under the (non-martingale) measure $P^€$ relative to the original numeraire. However, this assertion only holds true *literally* if the dual minimizer $\widehat{Y}$ is a martingale and if the optimal strategy $\widehat{\varphi}$ in the pure investment problem is *admissible* in the sense of [4, Corollary 2.5], i.e. if $\widehat{\varphi} \bullet S_T \in L^2(P^€)$ and $(\widehat{\varphi} \bullet S)Z^Q$ is a $P^€$-martingale for any absolutely continuous signed σ-martingale measure Q with density process Z^Q and $\frac{dQ}{dP^€} \in L^2(P^€)$.

More precisely, in this case the strategy $-\widetilde{a}1_{]\!]\tau,T]\!]}\mathscr{E}(-\widetilde{a}1_{]\!]\tau,T]\!]} \bullet S)_-$ is *efficient* on the stochastic interval $]\!]\tau, T]\!]$ in the sense of [4, Section 3.1] and $\widetilde{a}$ is the corresponding *adjustment process* in the sense of [4, Definition 3.8]. By [4, Corollary 3.4] this in turn implies that $L^\$$ is the *opportunity process* in the sense of [4, Definition 3.3]. Hence it follows along the lines of [4, Lemma 3.15] that the *opportunity neutral measure* $P^\star$ with density process

$$Z^{P\star} := \frac{L^\$}{L_0^\$ \mathscr{E}(A^{K^\$})}$$

exists. By [4, Lemma 3.17 and Theorem 4.10], $\tilde{c}^{S\star}, \tilde{c}^{V\star}, \tilde{c}^{S,V\star}$ indeed coincide with the corresponding modified second characteristics of (S,V,K) under $P^\star$. Hence [4, Theorems 4.10 and 4.12] yield that subject to the probability measure $P^€$, φ' represents the variance-optimal hedging strategy for H whereas the minimal expected squared hedging error of H is given by the $2C_0v/(pC_1)$-fold of π'. Moreover, $V_t = E_{Q_0}(H|\mathscr{F}_t)$ and in particular the marginal utility-based price $\pi^0 = E_{Q_0}(H)$ are given as conditional expectations under the *variance-optimal martingale measure* Q_0 with respect to $P^€$, which coincides with the q-optimal martingale measure with respect to P.

However, let us emphasize that it is easier to apply Theorem 4.4 than to solve the corresponding mean-variance hedging problem, because admissibility of a given candidate strategy is typically hard to verify even in concrete models (cf. e.g. [5]).

Whereas the representations in Theorem 4.4 allow for a nice economic interpretation, it is often more convenient for applications to formulate them in terms of P- rather than $P^€$-characteristics. This is done in the following

Lemma 4.5 *Suppose the prerequisites of Theorem 4.4 hold. Then we have*

$$\begin{aligned}
\tilde{c}^{S\star} &= \frac{1}{1+\Delta A^{K^\$}}\left(c^S + \int \left((1+x_3)(1-\widetilde{a}^\top x_1)^{-1-p} x_1 x_1^\top\right) F^{(S,V,K)}(dx)\right),\\
\tilde{c}^{S,V\star} &= \frac{1}{1+\Delta A^{K^\$}}\left(c^{S,V} + \int \left((1+x_3)(1-\widetilde{a}^\top x_1)^{-1-p} x_1 x_2\right) F^{(S,V,K)}(dx)\right),\\
\tilde{c}^{V\star} &= \frac{1}{1+\Delta A^{K^\$}}\left(c^V + \int \left((1+x_3)(1-\widetilde{a}^\top x_1)^{-1-p} x_2^2\right) F^{(S,V,K)}(dx)\right).
\end{aligned}$$

Moreover, the semimartingale $K^\$$ is $P^€$-special if and only if

$$\int_0^T \int_{\{|x|>1\}} \left(|x_3 - x_4|(1-\widetilde{a}^\top x_1)^{-1-p}\right) F_t^{(S,V,K,K^€)}(dx) dA_t < \infty,$$

and in this case we have

$$\Delta A^{K^\$} = \Delta A \int \left((x_3 - x_4)(1-\widetilde{a}^\top x_1)^{-1-p}\right) F^{(S,V,K,K^€)}(dx). \tag{4.13}$$

PROOF. By Remark 4.2, the density process of $P^€$ with respect to P can be written as $\mathscr{E}(K^€)\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p}$. Now define $N^€ := \mathscr{L}(\mathscr{E}(K^€)\mathscr{E}(-\widetilde{a} \bullet S)^{-1-p})$. Then a straightforward computation using [16, I.4.36] shows $\Delta N^€ = (1+\Delta K^€_-)(1-\widetilde{a}^\top \Delta S_-)^{-1-p} - 1$. Moreover, Equation (4.3) implies $\Delta K^\$ = (\Delta K - \Delta K^€)/(1+\Delta K^€)$. Hence

$$F^{(S,V,K^\$,N^€)}(G) = \int 1_G\left(x_1, x_2, \frac{x_3 - x_4}{1+x_4}, (1+x_4)(1-\widetilde{a}^\top x_1)^{1-p} - 1\right) F^{(S,V,K,K^€)}(dx),$$

for all $G \in \mathscr{B}^{d+3}$ with $0 \notin G$, and Proposition A.4 shows $c^{S,V€} = c^{S,V}$ as well as

$$F^{(S,V,K^\$)€}(G) = \int \left(1_G\left(x_1, x_2, \frac{x_3 - x_4}{1+x_4}\right)(1+x_4)(1-\widetilde{a}^\top x_1)^{-1-p}\right) F^{(S,V,K,K^€)}(dx), \tag{4.14}$$

for all $G \in \mathscr{B}^{d+2}$ with $0 \notin G$. The first three assertions now follow by insertion. The last assertion is a direct consequence of (4.14) and [16, II.2.29]. $\square$

In view of Theorem 4.4 and Lemma 4.5 one can therefore proceed as follows in order to determine first-order approximations of the indifference price and the utility-based hedging strategy of a claim H:

1. Solve the pure investment problem, i.e. determine the optimal strategy $\widehat{\varphi}$ and the corresponding dual minimizer $\widehat{Y}$. This also yields the optimal number $-\widetilde{a} = \widehat{\varphi}/(v + \widehat{\varphi} \bullet S)_-$ of shares per unit of wealth and the opportunity process $L = L_0\mathscr{E}(K) = \widehat{Y}\mathscr{E}(-\widetilde{a} \bullet S)^p$.

2. Compute $V_t = E(\widehat{Y}_T H|\mathscr{F}_t)/\widehat{Y}_t$, which yields the marginal utility-based price $\pi^0 = V_0$, and determine the joint local characteristics of S, V and K.

3. Solve the linear system $\tilde{c}^{S\star}\psi' = \tilde{c}^{S,V\star}$ pointwise for any (ω, t) to obtain a marginal utility-based hedging strategy φ' in feedback form as $\varphi' = \psi' + (\pi^0 + \varphi' \bullet S_- - V_-)\widetilde{a}$.

4. Compute $K^{€}$ from the drift condition (4.2) and use it to calculate the risk premium π' as

$$\pi' = \frac{p}{2vC_0} \int_0^T E_P\left(\left(\left(\tilde{c}_t^{V\star} - (\tilde{c}_t^{S,V\star})^\top (\tilde{c}_t^{S\star})^{-1} \tilde{c}_t^{S,V\star}\right) \mathscr{E}(K)_t \mathscr{E}(-\widetilde{a} \bullet S)_t^{-1-p}\right)\right) dA_t$$

via (4.13).

The above program is carried out for Lévy processes and affine stochastic volatility models in [27], leading to formulas of the same complexity as for mean-variance hedging in [19, 22].

A Appendix

In this appendix we summarize some basic notions regarding semimartingale characteristics (cf. [16] for more details). In addition, we state and prove an auxiliary result which is used in the proof of Theorem 4.4.

To any $\mathbb{R}^d$-valued semimartingale X there is associated a triplet (B, C, ν) of *characteristics*, where B resp. C denote $\mathbb{R}^d$- resp. $\mathbb{R}^{d\times d}$-valued predictable processes and ν a random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ (cf. [16, II.2.6]). The first characteristic B depends on a *truncation function* $h : \mathbb{R}^d \to \mathbb{R}^d$ such as $h(x) = x1_{\{|x|\leq 1\}}$. Instead of the characteristics themselves, we typically use the following notion.

Definition A.1 Let X be an $\mathbb{R}^d$-valued semimartingale with characteristics (B, C, ν) relative to some truncation function h on $\mathbb{R}^d$. In view of [16, II.2.9], there exist a predictable process $A \in \mathscr{A}^+_{\rm loc}$, an $\mathbb{R}^d$-valued predictable process b, an $\mathbb{R}^{d\times d}$-valued predictable process c and a transition kernel F from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$ such that

$$B_t = b \bullet A_t, \quad C_t = c \bullet A_t, \quad \nu([0,t] \times G) = F(G) \bullet A_t \quad \text{for } t \in [0,T],\ G \in \mathscr{B}^d,$$

where we implicitly assume that (b, c, F) is a good version in the sense that the values of c are non-negative symmetric matrices, $F_s(\{0\}) = 0$ and $\int(1 \wedge |x|^2)F_s(dx) < \infty$. We call (b, c, F, A) *local characteristics* of X.

If (b, c, F, A) denote local characteristics of some semimartingale X, we write

$$\tilde{c} := c + \int xx^\top F(dx),$$

and call $\tilde{c}$ the *modified second characteristic* of X provided that the integral exists. This notion is motivated by the fact that $\langle X, X\rangle = \tilde{c} \bullet A$ by [16, I.4.52] if the corresponding integral is finite. We write (b^X, c^X, F^X, A) and $\tilde{c}^X$ for the differential characteristics and the modified second characteristic of a semimartingale X. Likewise, the joint local characteristics of two semimartingales X, Y are denoted by

$$(b^{(X,Y)}, c^{(X,Y)}, F^{(X,Y)}, A) = \left(\begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \begin{pmatrix} c^X & c^{X,Y} \\ c^{Y,X} & c^Y \end{pmatrix}, F^{(X,Y)}, A \right)$$

and

$$\tilde{c}^{(X,Y)} = \begin{pmatrix} \tilde{c}^X & \tilde{c}^{X,Y} \\ \tilde{c}^{Y,X} & \tilde{c}^Y \end{pmatrix},$$

if the modified second characteristic of (X, Y) exists. The characteristics of a semimartingale X under some other measure $Q^\$$ are denoted by $(b^{X\$}, c^{X\$}, F^{X\$}, A)$. The following rules are used repeatedly in the proofs of this paper.

Proposition A.2 (Stochastic integration) *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b^X, c^X, F^X, A) and H an $\mathbb{R}^{n\times d}$-valued predictable process with $H^{j\cdot} \in L(X)$ for $j = 1, \dots, n$. Then local characteristics of the $\mathbb{R}^n$-valued integral process $H \bullet X := (H^{j\cdot} \bullet X)_{j=1,\dots,n}$ are given by $(b^{H\bullet X}, c^{H\bullet X}, F^{H\bullet X}, A)$, where*

$$\begin{aligned} b_t^{H\bullet X} &= H_t b_t^X + \int (\widetilde{h}(H_t x) - H_t h(x)) F_t^X(dx), \\ c_t^{H\bullet X} &= H_t c_t^X H_t^\top, \\ F_t^{H\bullet X}(G) &= \int 1_G(H_t x) F_t^X(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G. \end{aligned}$$

Here $\widetilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the truncation function which is used on $\mathbb{R}^n$.

PROOF. [21, Lemma 3]. □

Proposition A.3 (C^2-function) *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b^X, c^X, F^X, A). Suppose that $f : U \to \mathbb{R}^n$ is twice continuously differentiable on some open subset $U \subset \mathbb{R}^d$ such that X,X_- are U-valued. Then the $\mathbb{R}^n$-valued semimartingale $f(X)$ has local characteristics $(b^{f(X)}, c^{f(X)}, F^{f(X)}, A)$, where*

$$\begin{aligned} b_t^{f(X),i} &= \sum_{k=1}^d \partial_k f^i(X_{t-}) b_t^{X,k} + \frac{1}{2} \sum_{k,l=1}^d \partial_{kl} f^i(X_{t-}) c_t^{X,kl} \\ &\quad + \int \left(\widetilde{h}^i(f(X_{t-}+x) - f(X_{t-})) - \sum_{k=1}^d \partial_k f^i(X_{t-}) h^k(x) \right) F_t^X(dx), \\ c_t^{f(X),ij} &= \sum_{k,l=1}^d \partial_k f^i(X_{t-}) c_t^{X,kl} \partial_l f^j(X_{t-}), \\ F_t^{f(X)}(G) &= \int 1_G(f(X_{t-}+x) - f(X_{t-})) F_t^X(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G. \end{aligned}$$

Here, ∂_k etc. denote partial derivatives and $\widetilde{h}$ again the truncation function on $\mathbb{R}^n$.

PROOF. Follows immediately from [10, Corollary A.6]. □

Let $P^\star \overset{\text{loc}}{\sim} P$ be a probability measure with density process Z. Since $P^\star \overset{\text{loc}}{\sim} P$, the processes Z, Z_- are strictly positive by [16, I.2.27]. Hence the *stochastic logarithm* $N := \mathscr{L}(Z) = \frac{1}{Z_-} \bullet Z$ is a well-defined semimartingale. For an $\mathbb{R}^d$-valued semimartingale X we now have the following result, which relates the local $P^\star$-characteristics of (X,N) to the local characteristics of (X,N) under P.

Proposition A.4 (Equivalent change of measure) *Local $P^\star$-characteristics of the process (X,N) are given by $(b^{(X,N)\star}, c^{(X,N)\star}, F^{(X,N)\star}, A)$, where*

$$\begin{aligned} b^{(X,N)\star} &= b^{(X,N)} + c^{(X,N),N} + \int h(x) x_{d+1} F^{(X,N)}(dx), \\ c^{(X,N)\star} &= c^{(X,N)}, \\ F^{(X,N)\star} &= \int 1_G(x)(1+x_{d+1}) F^{(X,N)}(dx) \quad \forall G \in \mathscr{B}^{d+1} \text{ with } 0 \notin G. \end{aligned}$$

PROOF. [17, Lemma 5.1]. □

The following observation is needed in the proof of Theorem 4.4.

Lemma A.5 *Let $Q \overset{\text{loc}}{\ll} P$ with density process Z. Then for any increasing, predictable process A with $A_0 = 0$ we have*

$$E_Q(A_T) = E_P(Z_- \bullet A_T).$$

PROOF. Since Z is a P-martingale and A is predictable and of finite variation, $A \bullet Z$ is a local P-martingale by [16, I.3.10 and I.4.34]. If $(T_n)_{n\in\mathbb{N}}$ denotes a localizing sequence, $A \bullet Z_{T\wedge T_n}$ is a martingale starting at 0. By [16, III.3.4 and I.4.49], this implies

$$\begin{aligned} E_Q(A_{T\wedge T_n}) &= E_P(Z_{T\wedge T_n} A_{T\wedge T_n}) \\ &= E_P(Z_- \bullet A_{T\wedge T_n} + A \bullet Z_{T\wedge T_n}) \\ &= E_P(Z_- \bullet A_{T\wedge T_n}). \end{aligned}$$

Hence monotone convergence yields $E_Q(A_T) = E_P(Z_- \bullet A_T)$ as claimed. □

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