On the Performance of Delta Hedging Strategies in Exponential Lévy Models

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Abstract

We consider the performance of non-optimal hedging strategies in exponential Lévy models. Given that both the payoff of the contingent claim and the hedging strategy admit suitable integral representations, we use the Laplace transform approach of Hubalek et al. [8] to derive semi-explicit formulas for the resulting mean squared hedging error in terms of the cumulant generating function of the underlying Lévy process. In a numerical example, we apply these results to compare the efficiency of the Black-Scholes delta hedge to the mean-variance optimal hedge in a normal inverse Gaussian Lévy model.

Keywords: Laplace transform approach, mean-variance hedging, delta hedging, Lévy processes, model misspecification

1 Introduction

A basic problem in Mathematical Finance is how the issuer of an option can hedge the resulting exposure by trading in the underlying. In complete markets, the risk can be offset completely by purchasing the replicating portfolio. In incomplete markets, however, additional criteria are necessary to determine reasonable hedging strategies. A popular approach

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studied intensively in the literature over the last two decades is *mean-variance hedging*. Expressed in discounted terms, the idea is to minimize the *mean squared hedging error*

$$E\left(\left(H - c - \int_0^T \vartheta_t dS_t\right)^2\right)$$

over all initial endowments $c \in \mathbb{R}$ and all in some sense admissible trading strategies ϑ, where H is the payoff of the option and S is the price process of the underlying. Overviews on the topic can be found in [16, 20]. For more recent publications, the reader is referred to [5] and the references therein. In particular, semi-explicit representations of the minimal mean squared hedging error have been obtained in [4, 8] for exponential Lévy models and in [11, 12] for affine stochastic volatility models by making use of an integral representation of the option under consideration.

In practice though, Black-Scholes- or, more generally, delta hedging is still prevalent. Therefore it seems desirable to measure the performance of such a non-optimal hedging strategy in terms of the mean squared hedging error. For exponential Lévy models, this has been done in the unpublished thesis [7] for continuous time under the restriction that the asset price process is a martingale and, more recently, by [1] in a discrete time setup, both using the approach of [8]. In the present study, we extend the results of [7] to the case of a general exponential Lévy process and a larger class of hedging strategies.

This article is organized as follows. In Section 2, we describe the setup for the price process of the underlying and for the integral representation of the payoff function. Subsequently, we introduce the class of Δ-*strategies* that can be dealt with using our approach. At this point, we also discuss the most important examples, namely the Black-Scholes hedge resp., more generally, delta hedges in exponential Lévy models. In Section 4, we then state the main theorem on the hedging error of Δ-strategies. Before giving a rigorous proof, we provide a sketch of it on an intuitive basis. Finally, we illustrate our formulas by comparing the performance of the Black-Scholes hedge with the mean-variance optimal one in a normal inverse Gaussian Lévy model.

For stochastic background and terminology, we refer to the monograph of Jacod and Shiryaev [10]. For a semimartingale X, we denote by $L(X)$ the set of X-integrable predictable processes and write $\varphi \bullet X$ for the stochastic integral of a process $\varphi \in L(X)$ with respect to X. By $\langle X, Y \rangle$, we denote the predictable compensator of the quadratic covariation process $[X, Y]$ of two semimartingales X and Y, provided that $[X, Y]$ is a special semimartingale (cf. [9, comment after Théorème 2.30]).

2 Model and preliminaries

In this section, we state our assumptions on the asset price process and the payoff. Note that we use the same setup as [8] for mean-variance hedging.

2.1 Asset price process

Let $T > 0$ be a fixed time horizon, and denote by $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ a filtered probability space. The discounted price process S of a non-dividend paying stock is assumed to be of the form

$$S_t = S_0 e^{X_t}, \quad S_0 \in \mathbb{R}_+, \quad t \in [0,T],$$

for a Lévy process X. We demand

$$E(S_1^2) < \infty, \tag{2.1}$$

which is a natural requirement when using the second moment of the hedging error as a risk criterion. The entire distribution of the process X is already determined by the law of X_1, which can be characterized in terms of the *cumulant generating function* $\kappa : D \to \mathbb{C}$, i.e. the unique continuous function satisfying

$$E\left(e^{zX_t}\right) = e^{t\kappa(z)}$$

for $z \in D := \left\{z \in \mathbb{C} : E(e^{\mathrm{Re}(z)X_1}) < \infty\right\}$ and $t \in \mathbb{R}_+$. Note that Condition (2.1) implies

$$\{z \in \mathbb{C} : 0 \leq \mathrm{Re}\,(z) \leq 2\} \subset D.$$

Moreover, we exclude the degenerate case of deterministic S by demanding

$$\kappa(2) - 2\kappa(1) \neq 0.$$

2.2 Laplace transform approach

In order to derive semi-explicit formulas in concrete models, we employ the *Laplace transform approach*, which is widely used in option pricing (cf. e.g. [3, 18]) and applied by [8] in the context of mean-variance hedging. The key assumption is the existence of an integral representation of the payoff function in the following sense.

Assumption 2.1. Let the payoff H of a contingent claim be of the form $H = f(S_T)$ for some measurable function $f : (0,\infty) \to \mathbb{R}$, which admits the representation

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z\, p(z)\, dz \tag{2.2}$$

for $p : \mathbb{C} \to \mathbb{C}$ and $R \in \mathbb{R}$ such that $x \mapsto p(R+ix)$ is integrable and

$$E\left(e^{2RX_1}\right) < \infty. \tag{2.3}$$

Note that Condition (2.3) implies $H \in L^2(P)$, which is again a natural assumption in view of the problem at hand.

Example 2.2. Most European options allow for an integral representation as in Assumption 2.1. For example, the payoff function $f(s) = (s - K)^+$ of a European call with strike $K > 0$ can be written as

$$f(s) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} \, dz$$

for arbitrary $R > 1$, cf. [8, Lemma 4.1]. In general, representations of this kind can be obtained by inverting the bilateral Laplace transform of the mapping $x \mapsto f(e^x)$ via the *Bromwich inversion formula*. More details and examples can be found in [8, Section 4].

Henceforth, we consider a fixed contingent claim $H = f(S_T)$ satisfying Assumption 2.1.

3 Δ-strategies

We now introduce the class of strategies for which we will compute the mean squared hedging error in Section 4. We also discuss the two most prominent examples, namely the delta hedge and the mean-variance optimal hedge.

Following the approach of [1], the hedging strategies under consideration are supposed to reflect the integral representation of the payoff function (cf. Section 2.2) in the sense of

Definition 3.1. A real-valued process φ is called Δ-*strategy* if it is of the form

$$\varphi_t = \int_{R-i\infty}^{R+i\infty} \varphi(z)_t \, p(z) \, dz$$

with

$$\varphi(z)_t = S_{t-}^{z-1} g(z,t)$$

for a measurable function $g : (R + i\mathbb{R}) \times [0, T] \to \mathbb{C}$ such that

1. $t \mapsto g(z,t)$ is continuous for all $z \in R + i\mathbb{R}$,
2. there exists some $b \in \mathbb{R}_+$ with $\int_0^T |g(z,s)|^2 \, ds \leq b$ for all $z \in R + i\mathbb{R}$.

Condition 1 is required to ensure the existence of a unique solution of an ordinary differential equation (ODE) with right-hand side $g(z, \cdot)$. Condition 2 ascertains that φ is integrable with respect to S and that the cumulative gains $\int_0^T \varphi_t dS_t$ of the strategy possess a second moment (cf. Lemma 4.5). As a side remark, it also implies that any Δ-strategy is admissible in the sense of [8, Section 3].

Example 3.2. (**Delta hedge**) Computing the derivative of a price process with respect to the underlying in an exponential Lévy model leads to a Δ-strategy. To see why this holds, denote by $\widetilde{S}_t = \widetilde{S}_0 e^{\widetilde{X}_t}$ an exponential Lévy process with driver $\widetilde{X}$ and associated cumulant generating function $\widetilde{\kappa}$ under some martingale measure Q. Note that due to e.g. model misspecification, the hedge may be derived in a model differing from the one where it is eventually applied, which is why we distinguish between the processes S and $\widetilde{S}$. Using Fubini's Theorem and the independence of the increments of $\widetilde{X}$ with respect to Q, the price process of the contingent claim with payoff function f in the model $\widetilde{S}$ is given by

$$E_Q\left(f(\widetilde{S}_T)\middle|\mathscr{F}_t\right) = E_Q\left(\int_{R-i\infty}^{R+i\infty} \widetilde{S}_T^z\, p(z)\,dz\,\middle|\,\mathscr{F}_t\right) = \int_{R-i\infty}^{R+i\infty} E_Q\left(\widetilde{S}_T^z\middle|\mathscr{F}_t\right) p(z)\,dz$$
$$= \int_{R-i\infty}^{R+i\infty} \widetilde{S}_t^z e^{\widetilde{\kappa}(z)(T-t)}\, p(z)\,dz.$$

The *delta hedge* of the contingent claim in this pricing model is then given by $\varphi^{\Delta}(\widetilde{S}_{t-}, t)$ for

$$\varphi^{\Delta}(s,t) := \frac{\partial}{\partial s}\int_{R-i\infty}^{R+i\infty} s^z e^{\widetilde{\kappa}(z)(T-t)}\, p(z)\,dz = \int_{R-i\infty}^{R+i\infty} s^{z-1} z e^{\widetilde{\kappa}(z)(T-t)}\, p(z)\,dz,$$

provided that the derivative exists and that integration and differentiation can be interchanged. This is the case if either the payoff function is smooth enough or the distribution of the driver $\widetilde{X}$ is sufficiently regular, cf. e.g. [6]. Using the resulting hedge in the model S yields the strategy

$$\varphi^{\Delta}(S_{t-}, t) = \int_{R-i\infty}^{R+i\infty} S_{t-}^{z-1} z e^{\widetilde{\kappa}(z)(T-t)}\, p(z)\,dz,$$

which is a Δ-strategy if Condition 2 of Definition 3.1 is satisfied.

Example 3.3. (**Mean-variance optimal hedge in the martingale case**) By [8, Theorem 3.1] the mean-variance optimal hedging strategy in an exponential Lévy model is a Δ-strategy, provided that the corresponding asset price is a martingale. In this case, one can therefore use the results of the present paper to quantify the effect of model misspecification arising from using a mean-variance optimal hedge in another model. If the asset price process fails to be a martingale, the corresponding hedge contains a feedback term and therefore is not a Δ-strategy. Nevertheless, the results from the martingale case should typically serve as a good proxy, because numerical experiments using [8, Theorems 3.1 and 3.2] supply compelling evidence that the effect of a moderate drift rate is rather small for mean-variance hedging.

Notice that the delta hedge and the mean-variance optimal strategy coincide in the Black-Scholes model. Moreover, the regularity conditions in Definition 3.1 are automatically satisfied in this case.

Lemma 3.4. *Let $\widetilde{S}$ be a geometric Brownian motion without drift, i.e.*

$$\widetilde{S}_t = \widetilde{S}_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}, \quad \widetilde{S}_0 \in \mathbb{R}_+, \quad t \in [0,T],$$

for a constant $\sigma \in (0,\infty)$ and a standard Brownian motion W. Then the delta hedge and the mean-variance optimal hedge in the model $\widetilde{S}$ coincide and are given by the Δ-strategy

$$\varphi_t^{BS} = \int_{R-i\infty}^{R+i\infty} \widetilde{S}_{t-}^{z-1} z e^{\frac{1}{2}\sigma^2 z(z-1)(T-t)}\, p(z)\,dz.$$

PROOF. We follow the lines of Example 3.2 and show that the necessary regularity conditions hold in the Black-Scholes case. First, the cumulant generating function of the driver of $\widetilde{S}$ is given by

$$\widetilde{\kappa}(z) = \frac{1}{2}\sigma^2 z(z-1)$$

for $z \in \mathbb{C}$. Using the arguments of Example 3.2, we obtain the price process of H in the model $\widetilde{S}$ as

$$E\left(f(\widetilde{S}_T)\middle|\mathscr{F}_t\right) = \int_{R-i\infty}^{R+i\infty} \widetilde{S}_t^z e^{\widetilde{\kappa}(z)(T-t)}\, p(z)\, dz. \tag{3.4}$$

Note that the conditional expectation exists and Fubini's Theorem can be applied because all exponential moments of the normal distribution are finite. For the further considerations, note that

$$\mathrm{Re}\,(\widetilde{\kappa}(z)) = \frac{1}{2}\sigma^2\left(\mathrm{Re}\,(z)^2 - \mathrm{Re}\,(z) - \mathrm{Im}\,(z)^2\right),$$

and define

$$\widetilde{g}(z,t) := z e^{\widetilde{\kappa}(z)(T-t)}$$

for $z \in R + i\mathbb{R}$ and $t \in [0,T]$. Since the mapping $x \mapsto xe^{-ax^2}$, $a > 0$, is bounded on $\mathbb{R}_+$,

$$z \mapsto |\widetilde{g}(z,t)| \leq \left(e^{\frac{1}{2}\sigma^2(R^2-R)T} \vee 1\right)\left(|R| + |\mathrm{Im}\,(z)|\, e^{-\frac{1}{2}\sigma^2 \mathrm{Im}(z)^2(T-t)}\right)$$

is bounded on $R + i\mathbb{R}$ for fixed $t \in [0,T)$. Hence, the integral $\int_{R-i\infty}^{R+i\infty} s^{z-1}\widetilde{g}(z,t)\, p(z)\, dz$ is well-defined for $s \in (0,\infty)$ and $t \in [0,T)$, and dominated convergence yields

$$\frac{\partial}{\partial s}\int_{R-i\infty}^{R+i\infty} s^z e^{\widetilde{\kappa}(z)(T-t)}\, p(z)\, dz = \int_{R-i\infty}^{R+i\infty} s^{z-1}\widetilde{g}(z,t)\, p(z)\, dz.$$

Therefore the delta hedge of the price process in (3.4) is given by

$$\varphi_t^\Delta := \int_{R-i\infty}^{R+i\infty} \widetilde{S}_{t-}^{z-1}\widetilde{g}(z,t)\, p(z)\, dz = \varphi_t^{BS}.$$

Note that whereas this integral is not well-defined for $t = T$, the cumulated gains process $\varphi^\Delta \bullet \widetilde{S}_T$ does not depend on the value of φ_T^Δ. Since delta hedging leads to perfect replication in the Black-Scholes model, φ^Δ is clearly mean-variance optimal. Let us now verify that φ^Δ is indeed a Δ-strategy. Obviously, $\widetilde{g}(z,\cdot)$ is continuous for fixed $z \in R + i\mathbb{R}$. Moreover, we have

$$\int_0^T |\widetilde{g}(z,s)|^2\, ds \leq \left(e^{\sigma^2(R^2-R)T} \vee 1\right)\left(TR^2 + \int_0^T \mathrm{Im}\,(z)^2\, e^{-\sigma^2 \mathrm{Im}(z)^2(T-s)}\, ds\right).$$

Elementary integration yields that the right-hand side is uniformly bounded for $z \in R + i\mathbb{R}$. Thus Conditions 1 and 2 of Definition 3.1 are satisfied and we are done. □

Remark 3.5. Note that φ^{BS} is also a replicating strategy and in particular mean-variance optimal in the model

$$\widetilde{S}_t = \widetilde{S}_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

for $\mu \in (0,\infty)$.

4 Performance of a Δ-strategy

We measure the performance of a strategy in terms of the resulting mean squared hedging error, i.e. the objective function used in mean-variance hedging.

Definition 4.1. For an initial endowment $c \in \mathbb{R}$ and a strategy $\varphi \in L(S)$ such that $\varphi \bullet S_T \in L^2(P)$, we define the *mean squared hedging error* of the endowment/strategy pair (c, φ) as

$$E\left((H - c - \varphi \bullet S_T)^2\right).$$

4.1 Main result

In this section, we state the main result of this paper. For better readability, the proof is deferred to Section 4.3.

Theorem 4.2. *Consider an initial endowment $c \in \mathbb{R}$ and a Δ-strategy φ of the form*

$$\varphi_t = \int_{R-i\infty}^{R+i\infty} \varphi(z)_t \, p(z) \, dz$$

with

$$\varphi(z)_t = S_{t-}^{z-1} g(z, t).$$

Then the mean squared hedging error of the endowment/strategy pair (c, φ) is given by

$$E\left((H - c - \varphi \bullet S_T)^2\right) = (w - c)^2 + \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} J(y, z) \, p(y) p(z) \, dy dz, \tag{4.5}$$

where

$$\alpha(z, t) := \left(1 - \kappa(1) \int_t^T e^{\kappa(z)(s-T)} g(z, s) \, ds\right) e^{\kappa(z)(T-t)}, \tag{4.6}$$

$$w := \int_{R-i\infty}^{R+i\infty} S_0^z \alpha(z, 0) \, p(z) \, dz, \tag{4.7}$$

$$\begin{aligned} J(y, z) := {} & \big(\kappa(y+z) - \kappa(y) - \kappa(z)\big) \int_0^T S_0^{y+z} e^{\kappa(y+z)s} \alpha(y, s) \alpha(z, s) \, ds \\ & - \big(\kappa(y+1) - \kappa(y) - \kappa(1)\big) \int_0^T S_0^{y+z} e^{\kappa(y+z)s} \alpha(y, s) g(z, s) \, ds \\ & - \big(\kappa(z+1) - \kappa(z) - \kappa(1)\big) \int_0^T S_0^{y+z} e^{\kappa(y+z)s} \alpha(z, s) g(y, s) \, ds \\ & + \big(\kappa(2) - 2\kappa(1)\big) \int_0^T S_0^{y+z} e^{\kappa(y+z)s} g(y, s) g(z, s) \, ds. \end{aligned} \tag{4.8}$$

Remark 4.3. The cumulant generating function κ is often known explicitly, e.g. for normal inverse Gaussian [2] and variance gamma [14] processes or for the models introduced by Merton [15] and Kou [13]. Moreover, the time integrals in (4.6) and in (4.8) can typically be calculated in closed form, which means that the evaluation of the hedging error (4.5) usually amounts to numerical integration of a double integral with known integrand as in [8, Theorem 3.2] for the mean-variance optimal hedge.

4.2 Sketch of the proof

Before giving a rigorous proof of Theorem 4.2, we present our approach on an intuitive level. The notation of Section 4.3 is anticipated here, but it will be defined precisely in the corresponding places later on.

To calculate

$$E\left((H - c - \varphi \bullet S_T)^2\right), \tag{4.9}$$

we look for a martingale L with $L_T = H - c - \varphi \bullet S_T$. Then we can rewrite (4.9) as

$$E\left((H - c - \varphi \bullet S_T)^2\right) = E(L_0^2) + E(\langle L, L\rangle_T) \tag{4.10}$$

by means of the predictable quadratic variation of the process L, cf. [10, I.4.2 and I.4.50(b)]. Using the integral structure of payoff and strategy, we obtain by a stochastic Fubini argument that

$$H - c - \varphi \bullet S_T = \int_{R-i\infty}^{R+i\infty} S_T^z\, p(z)\, dz - c - \int_{R-i\infty}^{R+i\infty} \left(\varphi(z) \bullet S_T\right)\, p(z)\, dz. \tag{4.11}$$

The key idea is to identify a family of martingales $l(z)$, $z \in R + i\mathbb{R}$, such that

$$l(z)_T = S_T^z - \varphi(z) \bullet S_T.$$

Then the process

$$\int_{R-i\infty}^{R+i\infty} l(z)\, p(z)\, dz - c \tag{4.12}$$

is the canonical candidate for the martingale L, and the bilinearity of the predictable covariation $\langle \cdot, \cdot \rangle$ suggests that

$$\begin{aligned} \langle L, L\rangle &= \left\langle \int_{R-i\infty}^{R+i\infty} l(y)\, p(y)\, dy, \int_{R-i\infty}^{R+i\infty} l(z)\, p(z)\, dz \right\rangle \\ &= \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \langle l(y), l(z)\rangle\; p(y)p(z)\, dydz. \end{aligned} \tag{4.13}$$

In this case, Fubini's Theorem yields that

$$E\left(\langle L, L\rangle_T\right) = \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} E\left(\langle l(y), l(z)\rangle_T\right) p(y)p(z)\, dydz.$$

We now consider how to determine $l(z)$. If the stock price S is a martingale, the definition of the cumulant generating function and the martingale property of $\varphi(z) \bullet S$ yield that

$$E\left(S_T^z - \varphi(z) \bullet S_T | \mathscr{F}_t\right) = S_t^z e^{\kappa(z)(T-t)} - \varphi(z) \bullet S_t$$

is the appropriate martingale. Motivated by this fact, we make the ansatz

$$l(z)_t = S_t^z \alpha(z,t) - \varphi(z) \bullet S_t \tag{4.14}$$

for deterministic functions $\alpha(z) : [0,T] \to \mathbb{C}$ with $\alpha(z,T) = 1$ in the general case. The drift rate of (4.14) can be calculated using integration by parts, and setting it to zero yields a linear ODE for the mapping $t \mapsto \alpha(z,t)$. The solution then leads to the desired candidates for $l(z)$ and L via (4.14) resp. (4.12).

4.3 Proof of the main result

We now turn to the proof of Theorem 4.2, which is split up into several intermediate statements. An essential tool for the forthcoming considerations are the special semimartingale decomposition and the predictable covariation of complex powers of S, provided by the following

Lemma 4.4. *For $z \in R + i\mathbb{R}$, the process S^z is a special semimartingale whose canonical decomposition $S^z = S_0^z + M(z) + A(z)$ is given by*

$$M(z)_t = \int_0^t e^{\kappa(z)s}\, dN(z)_s, \quad A(z)_t = \kappa(z) \int_0^t S_{s-}^z\, ds,$$

where

$$N(z)_t := e^{-\kappa(z)t} S_t^z.$$

Moreover, for $y, z \in R + i\mathbb{R}$ and continuously differentiable functions $\beta, \gamma : [0,T] \to \mathbb{C}$, the process $[S^y\beta, S^z\gamma]$ is a special semimartingale with compensator $\langle S^y\beta, S^z\gamma\rangle$ given by

$$\langle S^y\beta, S^z\gamma\rangle_t = (\kappa(y+z) - \kappa(y) - \kappa(z)) \int_0^t S_{s-}^{y+z} \beta_s \gamma_s\, ds. \tag{4.15}$$

PROOF. This follows along the lines of the proof of [8, Lemma 3.2]. □

With the special semimartingale composition of S at hand, we can now establish that the mean squared hedging error of a Δ-strategy is well-defined.

Lemma 4.5. *The Δ-strategy φ satisfies $\varphi \in L(S)$ and $\varphi \bullet S_T \in L^2(P)$.*

PROOF. Fubini's Theorem yields the predictability of φ. The assertion then follows from [8, Lemma 3.1], Fubini's Theorem, Hölder's inequality and Condition 2 of Definition 3.1 since $t \mapsto E(|S_t^z|^2) = S_0^{2R} e^{t\kappa(2R)}$ is bounded on [0,T] for $z \in R + i\mathbb{R}$ by (2.3). □

The following proposition ascertains that deterministic integration in the representation of φ and stochastic integration with respect to S can be interchanged, compare (4.11).

Proposition 4.6. *We have*

$$\varphi \bullet S = \int_{R-i\infty}^{R+i\infty} (\varphi(z) \bullet S)\, p(z)\, dz.$$

PROOF. By the definition of $L(S)$ in [10, III.6.6(c)], Lemma 4.4 and since $E(S_{t-}^2) = S_0^2 e^{t\kappa(2)}$ and the locally bounded process S_- are bounded resp. pathwise bounded on $[0,T]$, a deterministic process $(H_t)_{t\in[0,T]}$ belongs to $L(S)$ if $\int_0^T H_t^2 dt < \infty$. Hence Condition 2 of Definition 3.1 implies

$$\left(\int_{R-i\infty}^{R+i\infty} |g(z,\cdot)|^2 |p(z)|\, dz\right)^{1/2} \in L(S).$$

Since S_-^{R-1} is locally bounded, [10, III.6.6.19(e)] yields

$$\left(\int_{R-i\infty}^{R+i\infty} |\varphi(z)|^2 |p(z)|\, dz\right)^{1/2} = S_-^{R-1} \left(\int_{R-i\infty}^{R+i\infty} |g(z,\cdot)|^2 |p(z)|\, dz\right)^{1/2} \in L(S).$$

The assertion now follows from Fubini's Theorem for stochastic integrals, cf. e.g. [17, Theorems 63 and 65]. □

The following theorem shows that our Ansatz (4.14) indeed works.

Theorem 4.7. *Let $z \in R + i\mathbb{R}$ and define $\alpha(z) : [0,T] \to \mathbb{C}$ as*

$$\alpha(z,t) := \left(1 - \kappa(1) \int_t^T e^{\kappa(z)(s-T)} g(z,s)\, ds\right) e^{\kappa(z)(T-t)}. \tag{4.16}$$

Then α solves the terminal value problem

$$\begin{cases} \alpha'(z,t) + \kappa(z)\alpha(z,t) - \kappa(1) g(z,t) &= 0, \quad t \in [0,T], \\ \alpha(z,T) &= 1, \end{cases} \tag{4.17}$$

and the process $l(z)$ defined by

$$l(z)_t := S_t^z \alpha(z,t) - \varphi(z) \bullet S_t \tag{4.18}$$

is a local martingale with $l(z)_T = S_T^z - \varphi(z) \bullet S_T$.

PROOF. Let $z \in R + i\mathbb{R}$. Differentiation shows that α solves (4.17). To prove the second part of the assertion, we decompose $l(z)$ into a local martingale and a drift, and then conclude that the latter vanishes due to the choice of α. Integration by parts and [10, I.4.49(d)] yield

$$l(z) = S_0^z \alpha(z,0) + \int_0^\cdot S_{s-}^z \alpha'(z,s)\, ds + \alpha(z,\cdot) \bullet S^z - \varphi(z) \bullet S.$$

It now follows from Lemma 4.4 that

$$\begin{aligned} l(z) = S_0^z\alpha(z,0) &+ \int_0^\cdot S_{s-}^z \alpha'(z,s)\,ds + \alpha(z,\cdot)\bullet A(z) + \alpha(z,\cdot)\bullet M(z) \\ &- \varphi(z)\bullet A(1) - \varphi(z)\bullet M(1) \\ = S_0^z\alpha(z,0) &+ \alpha(z,\cdot)\bullet M(z) - \varphi(z)\bullet M(1) \\ &+ \int_0^\cdot S_{s-}^z\Big(\alpha'(z,s) + \kappa(z)\alpha(z,s) - \kappa(1)g(z,s)\Big)\,ds\,. \end{aligned} \tag{4.19}$$

Since α satisfies (4.17), the last integral on the right-hand side of (4.19) vanishes. The remaining terms are local martingales by [10, I.4.34(b)], because $\alpha(z,\cdot)$ and $\varphi(z)$ are locally bounded. Since $\alpha(z,T)=1$, this proves the second part of the assertion. □

Lemma 4.4 now allows us to compute the predictable quadratic covariations $\langle l(y), l(z)\rangle$.

Proposition 4.8. *For all $y,z\in R+i\mathbb{R}$, the process $[l(y),l(z)]$ is a special semimartingale with compensator $\langle l(y),l(z)\rangle$ given by*

$$\begin{aligned} \langle l(y),l(z)\rangle_t = \big(\kappa(y+z)-\kappa(y)-\kappa(z)\big) &\int_0^t S_{s-}^{y+z}\alpha(y,s)\alpha(z,s)\,ds \\ - \big(\kappa(y+1)-\kappa(y)-\kappa(1)\big) &\int_0^t S_{s-}^{y+z}\alpha(y,s)g(z,s)\,ds \\ - \big(\kappa(z+1)-\kappa(z)-\kappa(1)\big) &\int_0^t S_{s-}^{y+z}\alpha(z,s)g(y,s)\,ds \\ + \big(\kappa(2)-2\kappa(1)\big) &\int_0^t S_{s-}^{y+z}g(y,s)g(z,s)\,ds. \end{aligned} \tag{4.20}$$

PROOF. Let $y,z\in R+i\mathbb{R}$. By the definition of $l(\cdot)$ in (4.18), the bilinearity of the quadratic covariation $[\cdot,\cdot]$ and [10, I.4.54], we obtain that

$$\begin{aligned} [l(y),l(z)] = [S^y\alpha(y), S^z\alpha(z)] &- \varphi(z)\bullet[S^y\alpha(y),S] \\ &- \varphi(y)\bullet[S,S^z\alpha(z)] + (\varphi(y)\varphi(z))\bullet[S,S], \end{aligned}$$

because $\varphi(y)$ and $\varphi(z)$ are locally bounded. Recall that by Lemma 4.4, the square bracket processes on the right-hand side are special semimartingales with compensators given by (4.15). Again using that $\varphi(z)$ is locally bounded, it then follows from [10, I.4.34(b)] that

$$\begin{aligned} [l(y),l(z)] - \langle S^y\alpha(y), S^z\alpha(z)\rangle &- \varphi(z)\bullet\langle S^y\alpha(y),S\rangle \\ &- \varphi(y)\bullet\langle S,S^z\alpha(z)\rangle + (\varphi(y)\varphi(z))\bullet\langle S,S\rangle \end{aligned}$$

is a local martingale. By inserting the explicit representations (4.15), we obtain that our candidate for $\langle l(y),l(z)\rangle$ indeed compensates $[l(y),l(z)]$. Since it is also predictable and of finite variation, this completes the proof. □

The following technical lemma provides bounds for several expressions that are needed to apply Fubini arguments in the proofs of Propositions 4.11 and 4.12.

Lemma 4.9. *There exist constants $b_1, b_2, b_3 \geq 0$ and, for almost all $\omega \in \Omega$, $b_4(\omega) \geq 0$ such that*

1. $\mathrm{Re}\,(\kappa(z)) \leq b_1$,
2. $|\alpha(z,t)| \leq b_2$,
3. $E\left(|\langle l(y), l(z)\rangle_t|\right) \leq b_3$,
4. $|\langle l(y), l(z)\rangle_t(\omega)| \leq b_4(\omega)$,

for all $y, z \in R + i\mathbb{R}$ and $t \in [0,T]$, where α and $l(z)$ are defined as in (4.16) and (4.18).

PROOF. Let $y, z \in R + i\mathbb{R}$ and $t \in [0,T]$.

1. The definition of the cumulant generating function and Jensen's inequality yield

$$e^{\mathrm{Re}(\kappa(z))} = \left|E\left(e^{zX_1}\right)\right| \leq E\left(\left|e^{zX_1}\right|\right) = E\left(e^{\mathrm{Re}(z)X_1}\right) = e^{\kappa(\mathrm{Re}(z))} = e^{\kappa(R)} \leq e^{b_1}$$

for $b_1 := \kappa(R) \vee 0$.

2. By Hölder's inequality, we have

$$\begin{aligned}|\alpha(z,t)| &\leq e^{\mathrm{Re}(\kappa(z))(T-t)} + |\kappa(1)| \int_t^T e^{\mathrm{Re}(\kappa(z))(s-t)}\,|g(z,s)|\;ds \\ &\leq e^{b_1 T} + |\kappa(1)|\, e^{b_1(T-t)}(T-t)^{\frac{1}{2}} \left(\int_t^T |g(z,s)|^2\;ds\right)^{1/2}.\end{aligned}$$

Assertion 2 now follows, because the last integral is uniformly bounded for all $z \in R+i\mathbb{R}$ since φ is a Δ-strategy.

For the proof of Assertions 3 and 4, first note that the bilinearity of $\langle\cdot,\cdot\rangle$ yields

$$\pm\mathrm{Re}\,(\langle l(y), l(z)\rangle) = \frac{1}{2}\Big(\left\langle l(y) \pm \overline{l(z)}, \overline{l(y) \pm \overline{l(z)}}\right\rangle - \left\langle l(y), \overline{l(y)}\right\rangle - \left\langle l(z), \overline{l(z)}\right\rangle\Big) \quad (4.21)$$

and

$$\begin{aligned}\left\langle l(y) \pm \overline{l(z)}, \overline{l(y) \pm \overline{l(z)}}\right\rangle &\leq \left\langle l(y) \pm \overline{l(z)}, \overline{l(y) \pm \overline{l(z)}}\right\rangle + \left\langle l(y) \mp \overline{l(z)}, \overline{l(y) \mp \overline{l(z)}}\right\rangle \\ &= 2\Big(\left\langle l(y), \overline{l(y)}\right\rangle + \left\langle l(z), \overline{l(z)}\right\rangle\Big).\end{aligned}$$

Applying an analogous polarization argument to $\mathrm{Im}\,(\langle l(y), l(z)\rangle)$ by replacing $\overline{l(z)}$ with $i\overline{l(z)}$, we see that it is sufficient to consider only the covariation of the form $\left\langle l(z), \overline{l(z)}\right\rangle$ in order to show 3 and 4. Since this process is real-valued and increasing, we can restrict ourselves to $t = T$.

3. By applying the arguments from the proof of Proposition 4.8 to $\left[l(z), \overline{l(z)}\right]$ instead of $[l(y), l(z)]$ and taking absolute values, we obtain

$$\begin{aligned}\left\langle l(z), \overline{l(z)}\right\rangle_T \leq & \int_0^T S_{s-}^{2\mathrm{Re}(z)} |\alpha(z,s)|^2 |\kappa(2\mathrm{Re}\,(z)) - 2\mathrm{Re}\,(\kappa(z))| \, ds \\ & + \int_0^T S_{s-}^{2\mathrm{Re}(z)} |g(z,s)|^2 |\kappa(2) - 2\kappa(1)| \, ds \\ & + \int_0^T 2S_{s-}^{2\mathrm{Re}(z)} |\alpha(z,s)| \, |g(z,s)| \, |\kappa(z+1) - \kappa(z) - \kappa(1)| \, ds.\end{aligned} \tag{4.22}$$

Using

$$E\left(S_{s-}^{2\mathrm{Re}(z)}\right) = E\left(S_{s}^{2\mathrm{Re}(z)}\right) = S_0^{2\mathrm{Re}(z)} e^{s\kappa(2\mathrm{Re}(z))} \leq S_0^{2R}\left(e^{T\kappa(2R)} \vee 1\right)$$

and Fubini's Theorem, we obtain from (4.22) that

$$\begin{aligned}E\left(\left\langle l(z), \overline{l(z)}\right\rangle_T\right) \leq \left(S_0^{2R}\left(e^{T\kappa(2R)} \vee 1\right)\right) & \left(\int_0^T |\alpha(z,s)|^2 |\kappa(2\mathrm{Re}\,(z)) - 2\mathrm{Re}\,(\kappa(z))| \, ds\right. \\ & + \int_0^T |g(z,s)|^2 |\kappa(2) - 2\kappa(1)| \, ds \\ & \left. + \int_0^T 2 |\alpha(z,s)| \, |g(z,s)| \, |\kappa(z+1) - \kappa(z) - \kappa(1)| \, ds\right).\end{aligned} \tag{4.23}$$

To prove Assertion 3, it therefore suffices to show that all integrals in (4.23) are uniformly bounded with respect to $z \in R + i\mathbb{R}$. For the first one, we have

$$\begin{aligned}&\int_0^T |\alpha(z,s)|^2 |\kappa(2\mathrm{Re}\,(z)) - 2\mathrm{Re}\,(\kappa(z))| \, ds \\ &\qquad \leq |\kappa(2R)| \, T b_2^2 + 2\int_0^T |\alpha(z,s)|^2 |\mathrm{Re}\,(\kappa(z))| \, ds\end{aligned} \tag{4.24}$$

with the bound $b_2 \geq 0$ for α derived in the proof of Assertion 2. Inserting the representation (4.16) for α, we obtain for the integral in (4.24)

$$\begin{aligned}&\int_0^T |\mathrm{Re}\,(\kappa(z))| \, |\alpha(z,s)|^2 \, ds \\ &\leq \int_0^T |\mathrm{Re}\,(\kappa(z))| \, e^{2\mathrm{Re}(\kappa(z))(T-s)} \, ds \\ &\quad + 2\int_0^T |\mathrm{Re}\,(\kappa(z))| \, e^{\mathrm{Re}(\kappa(z))(T-s)} \, |\kappa(1)| \left(\int_s^T e^{\mathrm{Re}(\kappa(z))(\tau-s)} \, |g(z,\tau)| \, d\tau\right) ds \\ &\quad + \int_0^T |\mathrm{Re}\,(\kappa(z))| \, |\kappa(1)|^2 \left(\int_s^T e^{\mathrm{Re}(\kappa(z))(\tau-s)} \, |g(z,\tau)| \, d\tau\right)^2 ds\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T |\mathrm{Re}\,(z)|\, e^{2\mathrm{Re}(\kappa(z))(T-s)}\, ds \\
&\quad + 2\int_0^T |\mathrm{Re}\,(\kappa(z))|\, e^{\mathrm{Re}(\kappa(z))(T-s)}\, |\kappa(1)| \left(\int_s^T e^{2\mathrm{Re}(\kappa(z))(\tau-s)}\, d\tau\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_s^T |g(z,\tau)|^2\, d\tau\right)^{\frac{1}{2}} ds \\
&\quad + \int_0^T |\mathrm{Re}\,(\kappa(z))|\, |\kappa(1)|^2 \left(\int_s^T e^{2\mathrm{Re}(\kappa(z))(\tau-s)}\, d\tau\right) \left(\int_s^T |g(z,\tau)|^2\, d\tau\right) ds,
\end{aligned}
$$

where we applied Hölder's inequality twice in the last step. Using that $\mathrm{Re}\,(\kappa(z))$, $z \in R + i\mathbb{R}$, is bounded from above by the constant $b_1 \geq 0$, it is easily seen by elementary integration that the integrals of the form

$$
\int_a^T |\mathrm{Re}\,(\kappa(z))|\, e^{m\mathrm{Re}(\kappa(z))s}\, ds, \quad 0 \leq a \leq T, \quad m \in \{1, 2\},
$$

are uniformly bounded on $R + i\mathbb{R}$. Moreover, the integrals over $s \mapsto |g(z,s)|^2$ are uniformly bounded as well, because φ is a Δ-strategy. This also yields that the second integral in (4.23) is bounded. To deal with the third one, we use the inequality

$$
|\kappa(z+1) - \kappa(z) - \kappa(1)|^2 \leq (\kappa(2\mathrm{Re}\,(z)) - 2\mathrm{Re}\,(\kappa(z)))\,(\kappa(2) - 2\kappa(1))
$$

established in [8, Lemma 3.4] and apply Hölder's inequality to conclude that

$$
\begin{aligned}
&\int_0^T |\alpha(z,s)|\, |g(z,s)|\, |\kappa(z+1) - \kappa(z) - \kappa(1)|\, ds \\
&\leq \left(\int_0^T |g(z,s)|^2\, ds\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^T |\alpha(z,s)|^2\, |\kappa(2\mathrm{Re}\,(z)) - 2\mathrm{Re}\,(\kappa(z))|\, |\kappa(2) - 2\kappa(1)|\, ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

This proves Assertion 3, because both integrals on the right-hand side have already been shown to be bounded on $R + i\mathbb{R}$.

4. Note that in (4.22) the only stochastic terms are given by $S_-^{2\mathrm{Re}(z)} = S_-^{2R}$. Since this process is locally bounded, almost all of its paths are bounded on $[0,T]$. The estimates from the proof of Assertion 3 therefore also show that Assertion 4 holds. □

The estimates in Lemma 4.9 immediately yield the following

Corollary 4.10. *For all $z \in R+i\mathbb{R}$, the process $l(z)$ defined in (4.18) is a square-integrable martingale.*

PROOF. Recall that $l(z)$ has already shown to be a local martingale in Theorem 4.7. Since $[l(z), \overline{l(z)}] - \langle l(z), \overline{l(z)}\rangle$ is a local martingale as well, the assertion follows from [10, I.4.50(c)] by localization, monotone convergence and Lemma 4.9(3). □

The next two propositions show that the candidates proposed in Equations (4.12) and (4.13) indeed coincide with the desired martingale and its quadratic variation of Ansatz (4.10).

Proposition 4.11. *The process L defined by*

$$L_t := \int_{R-i\infty}^{R+i\infty} l(z)_t \, p(z) \, dz - c \tag{4.25}$$

is a real-valued square-integrable martingale with $L_T = H - c - \varphi \bullet S_T$.

PROOF. First note that by Lemma 4.9(2) and Proposition 4.6, the integral in (4.25) is well-defined. Fubini's Theorem and dominated convergence show that $\int_{R-i\infty}^{R+i\infty} S_t^z \alpha(z,t) \, p(z) \, dz$ is an adapted càdlàg process. For $t \in [0,T]$, Hölder's inequality and another application of Fubini's Theorem yield

$$\begin{aligned} E\left(\left| \int_{R-i\infty}^{R+i\infty} l(z)_t \, p(z) \, dz \right|^2 \right) &\leq E\left(\int_{R-i\infty}^{R+i\infty} \left| l(z)_t \right|^2 \, \left| p(z) \right| \, dz \right) \left(\int_{R-i\infty}^{R+i\infty} \left| p(z) \right| \, dz \right) \\ &= \left(\int_{R-i\infty}^{R+i\infty} E\left(\left| l(z)_t \right|^2 \right) \, \left| p(z) \right| \, dz \right) \left(\int_{R-i\infty}^{R+i\infty} \left| p(z) \right| \, dz \right). \end{aligned}$$

Since $l(z)$ is a square-integrable martingale by Corollary 4.10, [10, I.4.2 and I.4.50(b)] implies

$$E\left(\left| l(z)_t \right|^2 \right) = \left| l(z)_0 \right|^2 + E\left(\left\langle l(z), \overline{l(z)} \right\rangle_t \right) \leq S_0^{2R} b_2^2 + b_3 \tag{4.26}$$

for constants b_2, $b_3 \geq 0$ independent of $z \in R + i\mathbb{R}$ and $t \in [0,T]$ given by Lemma 4.9. Since $x \mapsto p(R+ix)$ is integrable by Assumption 2.1, we conclude that

$$\sup_{t \in [0,T]} E\left(\left| L_t \right|^2 \right) < \infty.$$

In order to show the martingale property of L, consider arbitrary $0 \leq s \leq t \leq T$ and $F \in \mathscr{F}_s$. By Fubini's Theorem and the martingale property of $l(z)$, we have

$$\begin{aligned} E\left((L_t - L_s) 1_F \right) &= E\left(\int_{R-i\infty}^{R+i\infty} \left(l(z)_t - l(z)_s \right) 1_F \, p(z) \, dz \right) \\ &= \int_{R-i\infty}^{R+i\infty} E\left(\left(l(z)_t - l(z)_s \right) 1_F \right) \, p(z) \, dz = 0, \end{aligned}$$

and hence $E(L_t | \mathscr{F}_s) = L_s$. Since $\alpha(z,T) = 1$, it follows from Assumption 2.1 and Proposition 4.6 that L_T is given by the asserted, real-valued random variable. The martingale property of L then yields that L_t is real-valued for all $t \in [0,T]$, which completes the proof. □

Proposition 4.12. *The predictable quadratic variation $\langle L, L \rangle$ of the process L defined in (4.25) is given by*

$$\langle L, L \rangle_t = \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \langle l(y), l(z) \rangle_t \, p(y) p(z) \, dy dz. \tag{4.27}$$

PROOF. By Proposition 4.11 and [10, I.4.2 and I.4.50(b)], it suffices to prove that the candidate $C := \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \langle l(y), l(z) \rangle \ p(y)p(z)\, dydz$ is a predictable process of finite variation such that $L^2 - C$ is a local martingale. First, note that the integral in (4.27) is well-defined by Lemma 4.9(4). Moreover, since $t \mapsto \langle l(y), l(z) \rangle_t$ is continuous by Proposition 4.8 and hence predictable for all $y, z \in R + i\mathbb{R}$, the process C is predictable as well by Fubini's Theorem. To see that C is of finite variation, note that it is a linear combination of expressions of the form

$$\int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \operatorname{Re}\left(\langle l(y), l(z) \rangle\right) \ h(y)j(z)\, dydz$$

or

$$\int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \operatorname{Im}\left(\langle l(y), l(z) \rangle\right) \ h(y)j(z)\, dydz$$

for $h, j \in \left\{\operatorname{Re}(p)^+, \operatorname{Re}(p)^-, \operatorname{Im}(p)^+, \operatorname{Im}(p)^-\right\}$. In view of (4.21), we obtain that

$$\begin{aligned}
&\int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \operatorname{Re}\left(\langle l(y), l(z) \rangle\right) \ h(y)j(z)\, dydz \\
&= \frac{1}{2} \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \left\langle l(y) + \overline{l(z)}, \overline{l(y) + \overline{l(z)}} \right\rangle \ h(y)j(z)\, dydz \\
&\quad - \frac{1}{2} \left(\int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \left(\left\langle l(y), \overline{l(y)} \right\rangle + \left\langle l(z), \overline{l(z)} \right\rangle \right) \ h(y)j(z)\, dydz \right),
\end{aligned}$$

which is the difference of two increasing adapted processes and hence of finite variation. The argument applies analogously to $\operatorname{Im}\left(\langle l(y), l(z) \rangle\right)$. To show the martingale property of $L^2 - C$, we can assume without loss of generality that $c = 0$. Observe that

$$\begin{aligned}
E\left(\left|L_t^2 - C_t\right|\right) &= E\left(\left| \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \left(l(y)_t l(z)_t - \langle l(y), l(z) \rangle_t \right) p(y)p(z)\, dydz \right| \right) \\
&\leq \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \left(\left(E\left(|l(y)_t l(z)_t|\right) + E\left(|\langle l(y), l(z) \rangle_t|\right) \right) \ |p(y)|\, |p(z)| \ dydz.
\end{aligned}$$

Moreover, $E(|l(y)_t l(z)_t|) \leq E(|l(y)_t|^2)^{1/2} E(|l(z)_t|^2)^{1/2}$ and $E(|\langle l(y), l(z) \rangle_t|)$ are uniformly bounded with respect to $y, z \in R + i\mathbb{R}$ and $t \in [0, T]$ by (4.26) and Lemma 4.9(3). Since p is integrable, this shows that $L_t^2 - C_t \in L^1(P)$. For $0 \leq s \leq t \leq T$ and $F \in \mathscr{F}_s$, we can therefore apply Fubini's Theorem to obtain

$$\begin{aligned}
E\left(\left(L_t^2 - C_t\right) 1_F\right) &= \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} E\left(\left(l(y)_t l(z)_t - \langle l(y), l(z) \rangle_t\right) 1_F\right) \ p(y)p(z)\, dydz \\
&= \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} E\left(\left(l(y)_s l(z)_s - \langle l(y), l(z) \rangle_s\right) 1_F\right) \ p(y)p(z)\, dydz \\
&= E\left(\left(L_s^2 - C_s\right) 1_F\right),
\end{aligned}$$

where we used in the second step that $l(y)l(z) - \langle l(y), l(z) \rangle$ is a martingale by Corollary 4.10 and [10, I.4.2 and I.4.50(b)]. Hence, $E\left(L_t^2 - C_t \middle| \mathscr{F}_s\right) = L_s^2 - C_s$, which completes the proof. □

Now we can finally prove our main Theorem 4.2 by combining the preceding results.

PROOF OF THEOREM 4.2. In view of Lemma 4.5, the mean squared hedging error corresponding to the endowment/strategy pair (c, φ) is well-defined. By Assumption 2.1, Proposition 4.6, the definition of $l(z)$ in (4.18) and Proposition 4.11, we have

$$E((H - c - \varphi \bullet S_T)^2) = E(L_T^2) = E(L_0^2) + E(\langle L, L\rangle_T),$$

where the second equality follows from [10, I.4.2 and I.4.50(b)]. Now notice that by definition,

$$L_0^2 = \left(\int_{R-i\infty}^{R+i\infty} S_0^z \alpha(z,0)\, p(z)\, dz - c\right)^2 .$$

In view of Lemma 4.9(3), Fubini's Theorem and Proposition 4.12 yield

$$E(\langle L, L\rangle_T) = \int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty} E(\langle l(y), l(z)\rangle_T)\, p(y)p(z)\, dydz.$$

Since $E\left(S_{t-}^{y+z}\right) = S_0^{y+z} e^{t\kappa(y+z)}$, and because the continuous functions $t \mapsto |g(z,t)|$, $t \mapsto |\alpha(z,t)|$ and

$$t \mapsto E\left(\left|S_{t-}^{y+z}\right|\right) \leq S_0^{2R} e^{t\kappa(2R)}$$

are bounded on $[0, T]$, Proposition 4.8 and another application of Fubini's Theorem complete the proof. □

5 Numerical illustration

In this section, we illustrate our formulas by examining the performance of the Black-Scholes strategy in the *normal inverse Gaussian* (henceforth NIG) Lévy model (cf. [2]). The cumulant generating function of the NIG Lévy process is given by

$$\kappa(z) = \mu z + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}\right)$$

for $\mu \in \mathbb{R}$, $\delta > 0$, $0 \leq |\beta| \leq \alpha$ and $z \in \{y \in \mathbb{C} : |\beta + \mathrm{Re}\,(y)\,| \leq \alpha\}$ (cf. e.g. [8, Section 5.3.2]). As for parameters, we use

$$\alpha = 75.49,\ \beta = -4.089,\ \delta = 3.024,\ \mu = -0.04,$$

which corresponds to the annualized daily estimates from [19] for Deutsche Bank, assuming 252 trading days per year, and discounting by an annual deterministic rate of 4%. It is easily verified that this market satisfies the prerequisites of Section 2. The volatility parameter σ for the Black-Scholes strategy (cf. Lemma 3.4) is set to 0.2 so that the log-returns in the corresponding Black-Scholes market and in the NIG Lévy market exhibit the same variance.

Now consider a European call option with maturity $T = 0.25$ years and discounted strike $K = 99$. The integral representation of the corresponding payoff function is given in Example 2.2. For the numerical computations we use $R = 1.1$.

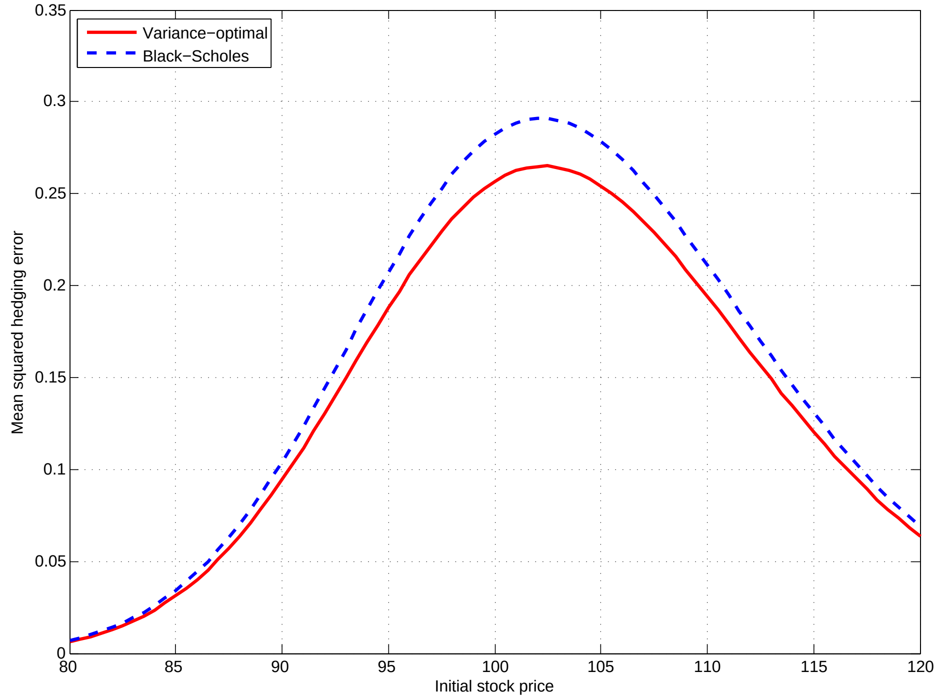

Figure 1: Mean squared hedging error for varying initial stock price

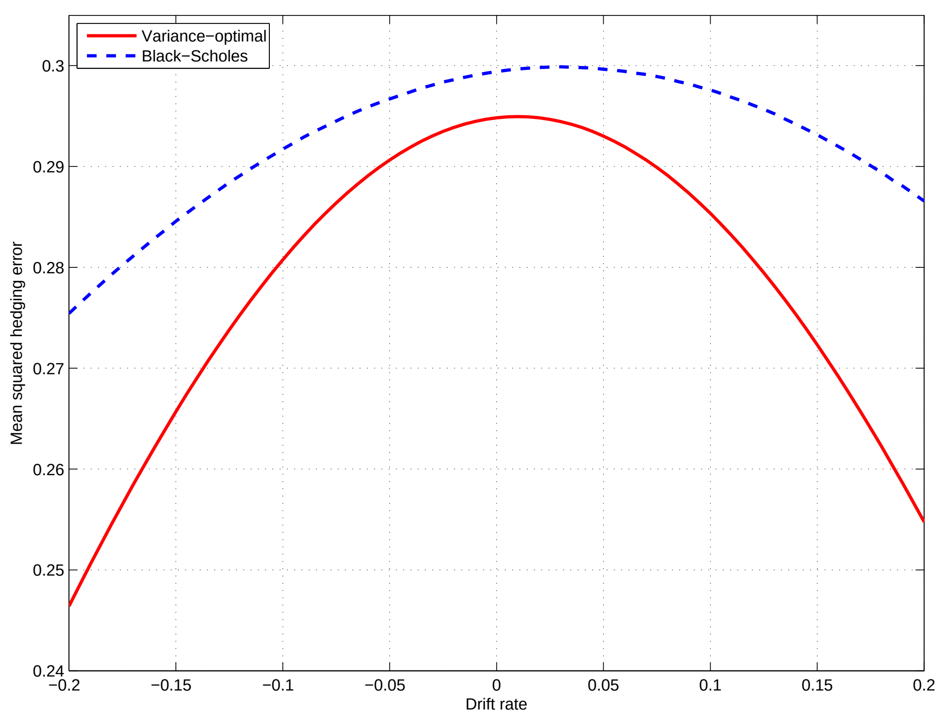

Figure 2: Mean squared hedging error for varying drift rate $\kappa(1)$

Figure 1 shows the mean squared hedging errors of the Black-Scholes strategy (using the Black-Scholes price of the option as initial capital) and the mean-variance optimal strategy for varying initial stock price. To evaluate the mean-variance optimal error, we used the formulas in [8]. For an at-the-money option the standard deviations of the errors differ by 4.73% and amount to 11.3% (mean-variance optimal) and 11.8% (Black-Scholes) of the respective initial capital.

Figure 2 illustrates how the two strategies react to different drift rates of the underlying. More specifically, the figure shows the mean squared hedging errors for an at-the-money call option with strike $K = 100$ and maturity $T = 0.25$ for varying drift rate $\kappa(1)$, controlled by varying the location parameter μ of the NIG process. Since the Black-Scholes strategy does not incorporate such systematic drifts directly, the two errors differ least in the martingale case $\kappa(1) = 0$. Altogether, the Black-Scholes strategy seems to be a surprisingly good proxy for the mean-variance optimal hedge, particularly for moderate drift rates.

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