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On the Existence of Shadow Prices in Finite Discrete Time

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Abstract

A *shadow price* is a process $\widetilde{S}$ lying within the bid/ask prices $\underline{S}, \overline{S}$ of a market with proportional transaction costs, such that maximizing expected utility from consumption in the frictionless market with price process $\widetilde{S}$ leads to the same maximal utility as in the original market with transaction costs. For finite Ω, this note provides an elementary proof for the existence of such a shadow price.

Key words: transactions costs, portfolio optimization, shadow price

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1 Introduction

When considering problems in Mathematical Finance, one classically works with a *frictionless* market, i.e. one assumes that securities can be purchased and sold for the same price S. This is clearly a strong modeling assumption, since in reality one usually has to pay a higher *ask price* when purchasing securities, whereas one only receives a lower *bid price* when selling them. In addition, the introduction of even miniscule transaction costs often fundamentally changes the structure of the problem at hand (cf. e.g. [4, 7, 2]). Therefore models with transaction costs have been extensively studied in the literature.

Optimization problems involving transaction costs are usually tackled by one of two different approaches. Whereas the first method employs methods from stochastic control theory, the second reformulates the task at hand as a similar problem in a frictionless market. This second approach goes back to the pioneering paper of [8]. They showed that under suitable conditions, a market with bid/ask prices $\underline{S}, \overline{S}$ is arbitrage free if and only if there exists a *shadow price* $\widetilde{S}$ lying within the bid/ask bounds, such that the frictionless market

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with price process $\widetilde{S}$ is arbitrage free. The same idea has since been employed extensively leading to various other versions of the fundamental theorem of asset pricing in the presence of transaction costs (cf. e.g. [6, 16, 7] and the references therein). It has also found its way into other branches of Mathematical Finance. For example, [12] have shown that bid/ask prices can be replaced by a shadow price in the context of local risk-minimization, whereas [1, 3, 13, 11] prove that the same is true for portfolio optimization in certain Itô process settings.

In this article we establish that in finite discrete time, this *general principle* holds true literally for investment/consumption problems. We first introduce our finite market model with proportional transaction costs in Section 2. Subsequently, we state our main result concerning the existence of shadow prices and prove it using elementary convex analysis.

For a vector $x = (x^1, \dots, x^d)$, we write $x^+ = (\max\{x^1, 0\}, \dots, \max\{x^d, 0\})$ and $x^- = (\max\{-x^1, 0\}, \dots, \max\{-x^d, 0\})$. Likewise, inequalities and equalities are understood to be componentwise in a vector-valued context. Moreover, for any stochastic process X we write $\Delta X_t := X_t - X_{t-1}$.

2 Utility maximization with transaction costs in finite discrete time

We study the problem of maximizing expected utility from consumption in a finite market model with proportional transaction costs. Our general framework is as follows. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\{0,1,\dots,T\}}, P)$ be a filtered probability space, where $\Omega = \{\omega_1, \dots, \omega_K\}$ and the time set $\{0, 1, \dots, T\}$ are finite. In order to avoid lengthy notation, we let $\mathscr{F} = \mathscr{P}(\Omega)$, $\mathscr{F}_0 = \{\varnothing, \Omega\}$, and assume that $P(\{\omega_k\}) > 0$ for all $k \in \{1, \dots, K\}$. However, one can show that all following statements remain true without these restrictions.

The financial market we consider consists of a risk-free asset 0 (also called *bank account*) with price process S^0 normalized to $S^0_t = 1$, $t = 0, \dots, T$, and risky assets $1, \dots, d$ whose prices are expressed in multiples of S^0. More specifically, they are modelled by their (discounted) *bid price process* $\underline{S} = (\underline{S}^1, \dots, \underline{S}^d)$ and their (discounted) *ask price process* $\overline{S} = (\overline{S}^1, \dots, \overline{S}^d)$, where we naturally assume that $\overline{S}, \underline{S}$ are adapted and satisfy $\overline{S} \geq \underline{S} > 0$. Their meaning should be obvious: if one wants to purchase security i at time t, one must pay the higher price $\overline{S}^i_t$ whereas one receives only $\underline{S}^i_t$ for selling it.

Remark 2.1 This setup amounts to assuming that the risk-free asset can be purchased and sold without incurring any transaction costs. This assumption is commonly made in the literature dealing with optimal portfolios in the presence of transaction costs (cf. e.g. [4]), and seems reasonable when thinking of security 0 as a bank account. For foreign exchange markets where it appears less plausible, a numeraire free approach has been introduced by [9]. This approach would, however, require the use of multidimensional utility functions as in [5] in our context.

Definition 2.2 A *trading strategy* is an $\mathbb{R}^{d+1}$-valued predictable stochastic process $(\varphi^0, \varphi) = (\varphi^0, (\varphi^1, \ldots, \varphi^d))$, where φ^i_t, $t = 0, \ldots, T+1$ denotes the number of shares held in security i until time t after rearranging the portfolio at time $t-1$. A (discounted) *consumption process* is an $\mathbb{R}$-valued, adapted stochastic process c, where c_t, $t = 0, \ldots, T$ represents the amount consumed at time t. A pair $((\varphi^0, \varphi), c)$ of a trading strategy (φ^0, φ) and a consumption process c is called *portfolio/consumption pair*.

To capture the notion of a self-financing strategy, we use the intuition that no funds are added or withdrawn. More specifically, this means that the proceeds of selling stock must be added to the bank account while the expenses from consumption and the purchase of stock have to be deducted from the bank account whenever the portfolio is readjusted from φ_t to φ_{t+1} and an amount c_t is consumed at time $t \in \{0, \ldots, T\}$. Defining purchase and sales processes $\Delta\varphi^\uparrow, \Delta\varphi^\downarrow$ as

$$\Delta\varphi^\uparrow := (\Delta\varphi)^+, \quad \Delta\varphi^\downarrow := (\Delta\varphi)^-, \tag{2.1}$$

this leads to the following

Definition 2.3 A portfolio/consumption pair (φ, c) is called *self-financing* (or $\varphi\, c$*-financing*) if

$$\Delta\varphi^0_{t+1} = \underline{S}_t^\top \Delta\varphi^\downarrow_{t+1} - \overline{S}_t^\top \Delta\varphi^\uparrow_{t+1} - c_t, \quad t = 0, \ldots, T. \tag{2.2}$$

Remark 2.4 For $i = 1, \ldots, d$, define the cumulated purchases $\varphi^\uparrow$ and sales $\varphi^\downarrow$ as

$$\varphi^\uparrow := (\varphi_0)^+ + \sum_{t=1}^{\cdot} \Delta\varphi^\uparrow_t, \quad \varphi^\downarrow := (\varphi_0)^- + \sum_{t=1}^{\cdot} \Delta\varphi^\downarrow_t.$$

Then the self-financing condition (2.2) implies that $\big((\varphi^0, \varphi^\uparrow, -\varphi^\downarrow), c\big)$ is self-financing in the usual sense for a frictionless market with $2d+1$ securities $(1, \overline{S}, \underline{S})$. Moreover, note that for $\underline{S} = \overline{S}$, we recover the usual self-financing condition for frictionless markets.

We consider an investor who disposes of an *initial endowment* $(\eta_0, \eta) \in \mathbb{R}^{d+1}_+$, referring to the initial number of securities of type i, $i = 0, \ldots, d$, respectively.

Definition 2.5 A self-financing portfolio/consumption pair $((\varphi^0, \varphi), c)$ is called *admissible* if $(\varphi^0_0, \varphi_0) = (\eta_0, \eta)$ and $(\varphi^0_{T+1}, \varphi_{T+1}) = (0, 0)$. An admissible portfolio/consumption pair $((\varphi^0, \varphi), c)$ is called *optimal* if it maximizes

$$\kappa \mapsto E\left(\sum_{t=0}^{T} u_t(\kappa_t)\right) \tag{2.3}$$

over all admissible portfolio/consumption pairs $((\psi^0, \psi), \kappa)$, where the *utility process* u is a mapping $u : \Omega \times \{0, \ldots, T\} \times \mathbb{R} \to [-\infty, \infty)$, such that $(\omega, t) \mapsto u_t(\omega, x)$ is predictable for any $x \in \mathbb{R}$ and $x \mapsto u_t(\omega, x)$ is a proper, upper-semicontinuous, concave function for any $(\omega, t) \in \Omega \times \{0, \ldots, T\}$, which is strictly increasing on its effective domain $\{x \in \mathbb{R} : u_t(\omega, x) > -\infty\}$.

Consequently, we only deal with portfolio/consumption pairs where the *entire* liquidation wealth of the portfolio is consumed at time T. Note that this can be done without loss of generality, because the utility process is strictly increasing in consumption.

Remark 2.6 Since we allow the utility process to be random, assuming $S_t^0 = 1$, $t = 0, \ldots, T$ also does not entail a loss of generality in the present setup. More specifically, let S^0 be an arbitrary strictly positive, predictable process. In this undiscounted case a portfolio/consumption pair (φ, c) should be called *self-financing* if

$$\Delta\varphi_{t+1}^0 S_t^0 = \underline{S}_t^\top \Delta\varphi_{t+1}^\downarrow - \overline{S}_t^\top \Delta\varphi_{t+1}^\uparrow - c_t,$$

for $t = 0, \ldots, T$. *Admissibility* is defined as before. By direct calculations, one easily verifies that $((\varphi^0, \varphi), c)$ is self-financing resp. admissible if and only if $((\varphi^0, \varphi), \hat{c}) = ((\varphi^0, \varphi), c/S^0)$ is self-financing resp. admissible relative to the discounted processes $\hat{S}^0 := S^0/S^0 = 1$, $\hat{\overline{S}} := \overline{S}/S^0$ and $\hat{\underline{S}} := \underline{S}/S^0$. In view of

$$E\left(\sum_{t=0}^{T} u_t(c_t)\right) = E\left(\sum_{t=0}^{T} \hat{u}_t(\hat{c}_t)\right)$$

for the utility process $\hat{u}_t(x) = u_t(S^0 x)$, the problem of maximizing undiscounted utility with respect to u is equivalent to maximizing discounted expected utility with respect to $\hat{u}$.

3 Existence of shadow prices

We now introduce the central concept of this paper.

Definition 3.1 We call an adapted process $\widetilde{S}$ *shadow price process* if

$$\underline{S} \leq \widetilde{S} \leq \overline{S}$$

and if the maximal expected utilities in the market with bid/ask-prices $\underline{S}, \overline{S}$ and in the market with price process $\widetilde{S}$ *without* transaction costs coincide.

The following theorem shows that in our finite market model, shadow price processes always exist, except in the trivial case where all admissible portfolio/consumption pairs lead to expected utility $-\infty$.

Theorem 3.2 *Suppose an optimal portfolio/consumption pair (φ, c) exists for the market with bid/ask prices $\underline{S}, \overline{S}$. Then if $E(\sum_{t=0}^{T} u_t(c_t)) > -\infty$, a shadow price process $\widetilde{S}$ exists.*

PROOF. *Step 1*: As the utility process in strictly increasing, allowing for sales and purchases at the same time does not increase the maximal expected utility. More precisely, since $x \mapsto u_t(x)$ is strictly increasing for fixed t, maximizing (2.3) over all admissible portfolio/consumption pairs yields the same maximal expected utility as maximizing (2.3)

over all $((\psi^0, \psi^\uparrow, \psi^\downarrow), \kappa)$, where $(\psi^0(t))_{t=0,\dots,T+1}$ is an $\mathbb{R}$-valued predictable process with $\psi^0_0 = \eta_0$ and $\psi^0_{T+1} = 0$, the increasing, $\mathbb{R}^d$-valued predictable processes $(\psi^\uparrow_t)_{t=0,\dots,T+1}$ and $(\psi^\downarrow_t)_{t=0,\dots,T+1}$ satisfy $\psi^\uparrow_0 = \eta^+$, $\psi^\downarrow_0 = \eta^-$, $\psi^\uparrow_{T+1} - \psi^\downarrow_{T+1} = 0$ and $(\kappa_t)_{t=0,\dots,T}$ is a consumption process such that (2.2) holds for $t = 0, \dots, T$ and (ψ, κ) instead of (φ, c). Moreover, if we define $\Delta\varphi^\uparrow$ and $\Delta\varphi^\downarrow$ as in (2.1) above and set

$$\varphi^\uparrow := \eta^+ + \sum_{t=1}^{\cdot} \Delta\varphi^\uparrow_t, \quad \varphi^\downarrow := \eta^- + \sum_{t=1}^{\cdot} \Delta\varphi^\downarrow_t,$$

then $((\varphi^0, \varphi^\uparrow, \varphi^\downarrow), c)$ is an optimal strategy in this set.

Step 2: Denote by $F^1_t, \dots, F^{m_t}_t$ the partition of Ω that generates $\mathscr{F}_t$, $t \in \{0, \dots, T\}$. Since a mapping is $\mathscr{F}_t$-measurable if and only if it is constant on the sets F^j_t, $j = 1, \dots, m_t$, we can identify the set of all processes $((\psi^0, \psi^\uparrow, \psi^\downarrow), \kappa)$, where $(\psi^0_t)_{t=0,\dots,T+1}$ is $\mathbb{R}$-valued and predictable with $\psi^0_0 = \eta_0$, $(\psi^\uparrow_t)_{t=0,\dots,T+1}$ and $(\psi^\downarrow_t)_{t=0,\dots,T+1}$ are increasing, $\mathbb{R}^d$-valued and predictable with $\psi^\uparrow_0 = \eta^+$, $\psi^\downarrow_0 = \eta^-$ and $(\kappa_t)_{t=0,\dots,T}$ is a consumption process such that (2.2) holds for $t = 0, \dots, T$ with

$$\mathbb{R}^{2dn}_+ \times \mathbb{R}^n := (\mathbb{R}^{m_0 d}_+ \times \dots \times \mathbb{R}^{m_T d}_+) \times (\mathbb{R}^{m_0 d}_+ \times \dots \times \mathbb{R}^{m_T d}_+) \times (\mathbb{R}^{m_0} \times \dots \times \mathbb{R}^{m_T}),$$

and vice versa, namely with

$$(\Delta\psi^\uparrow, \Delta\psi^\downarrow, c) := (\Delta\psi^{\uparrow,1,1}_1, \dots, \Delta\psi^{\uparrow,m_T,d}_{T+1}, \Delta\psi^{\downarrow,1,1}_1, \dots, \Delta\psi^{\downarrow,m_T,d}_{T+1}, c^1_0, \dots, c^{m_T}_T),$$

where we use the notation $\Delta\psi^{\uparrow,j,i}_t := \Delta\varphi^{\uparrow,i}_t(\omega)$ for $i = 1, \dots, d$, $t = 0, \dots, T$, $j = 1, \dots, m_t$ and $\omega \in F^j_t$ (and analogously for $\Delta\psi^\downarrow$, c, $\underline{S}$, $\overline{S}$). Using this identification, we can define mappings $f : \mathbb{R}^{2dn}_+ \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $h^j_0 : \mathbb{R}^{2dn}_+ \times \mathbb{R}^n \to \mathbb{R}$ and $h^j : \mathbb{R}^{2dn}_+ \times \mathbb{R}^n \to \mathbb{R}^d$ (for $j = 1, \dots, m_T$) by

$$\begin{aligned}
f(\Delta\psi^\uparrow, \Delta\psi^\downarrow, c) &:= -E\left(\sum_{t=1}^T u_t(c_t)\right), \\
h^j_0(\Delta\psi^\uparrow, \Delta\psi^\downarrow, c) &:= \eta_0 + \sum_{t=1}^T \left((\underline{S}^j_{t-1})^\top \Delta\psi^{\downarrow,j}_t - (\overline{S}^j_{t-1})^\top \Delta\psi^{\uparrow,j}_t\right) - \sum_{t=0}^T c^j_t, \\
h^j(\Delta\psi^\uparrow, \Delta\psi^\downarrow, c) &:= \eta + \sum_{t=1}^{T+1} \left(\Delta\psi^{\uparrow,j}_t - \Delta\psi^{\downarrow,j}_t\right).
\end{aligned}$$

With this notion, $(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c)$ is optimal if and only if it minimizes f over $\mathbb{R}^{2dn}_+ \times \mathbb{R}^n$ subject to the constraints $h^j_0 = 0$ and $h^j = 0$ for $j = 1, \dots, m_T$. Since all mappings are actually convex functions on $\mathbb{R}^{(2d+1)n}$, this is equivalent to $(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c)$ minimizing f over $\mathbb{R}^{(2d+1)n}$ subject to the constraints $h^j_0 = 0$, $h^j = 0$ (for $j = 1, \dots, m_T$) and $g^{\uparrow,j}_t, g^{\downarrow,j}_t \le 0$ (for $t = 0, \dots, T$ and $j = 1, \dots, m_t$), where the convex mappings $g^{\uparrow,j}_t, g^{\downarrow,j}_t : \mathbb{R}^{(2d+1)n} \to \mathbb{R}^d$ are given by

$$g^{\uparrow,j}_t(\Delta\psi^\uparrow, \Delta\psi^\downarrow, c) := -\Delta\psi^{\uparrow,j}_{t+1}, \quad g^{\downarrow,j}_t(\Delta\psi^\uparrow, \Delta\psi^\downarrow, c) := -\Delta\psi^{\downarrow,j}_{t+1}.$$

In view of [14, Theorems 28.2 and 28.3], $(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c)$ is therefore optimal if and only if there exists a *Lagrange multiplier*, i.e. real numbers ν^j, $\mu^{j,i}$ (for $i = 1, \dots, d$ and $j = 1, \dots, m_T$) and $\lambda_t^{\uparrow,j,i}, \lambda_t^{\downarrow,j,i}$ (for $t = 0, \dots, T$, $i = 1, \dots, d$ and $j = 1, \dots, m_t$) such that the following holds.

1. For $t = 0, \dots, T$, $j = 1, \dots, m_t$ and $i = 1, \dots, d$, we have $\lambda_t^{\uparrow,j,i}, \lambda_t^{\downarrow,j,i} \geq 0$ as well as $g_t^{\uparrow,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c), g_t^{\downarrow,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) \leq 0$ and $\lambda_t^{\uparrow,j,i} g_t^{\uparrow,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ as well as $\lambda_t^{\downarrow,j,i} g_t^{\downarrow,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$.

2. $h_0^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ and $h^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) = 0$ for $j = 1, \dots, m_T$.

3. $$\begin{aligned} 0 \in & \partial f(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{j=1}^{m_T} \nu^j \partial h_0^j(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{i=1}^{d} \sum_{j=1}^{m_T} \mu^{j,i} \partial h^{j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) \\ & + \sum_{t=0}^{T} \sum_{i=1}^{d} \sum_{j=1}^{m_t} \lambda_t^{\uparrow,j,i} \partial g_t^{\uparrow,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) + \sum_{t=0}^{T} \sum_{i=1}^{d} \sum_{j=1}^{m_t} \lambda_t^{\downarrow,j,i} \partial g_t^{\downarrow,j,i}(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c). \end{aligned}$$

Here, ∂ denotes the subdifferential of a convex mapping (cf. [14] for more details).

Step 3: By [15, Proposition 10.5] we can split Statement 3 into many similar statements where the subdifferentials on the right-hand side are replaced with partial subdifferentials relative to $\Delta\varphi_1^{\uparrow,1,1}, \dots, \Delta\varphi_{T+1}^{\uparrow,m_T,d}$, $\Delta\varphi_1^{\downarrow,1,1}, \dots, \Delta\varphi_{T+1}^{\downarrow,m_T,d}$, $c_t^1, \dots, c_T^{m_T}$, respectively. In particular, for c_T^j, $j \in \{1, \dots, m_T\}$, we obtain

$$0 \in \partial_{c_T^j} f(\Delta\varphi^{\uparrow}, \Delta\varphi^{\downarrow}, c) - \nu^j, \tag{3.1}$$

where ∂_x denotes the partial subdifferential of a convex function relative to a vector x. Hence $\nu^j < 0$, $j = 1, \dots, m_T$, because f is strictly decreasing in c_T^j. Furthermore, since the mappings $g_t^{\uparrow,j,i}, g_t^{\downarrow,j,i}$ (for $t = 0, \dots, T$, $j = 1, \dots, m_t$ and $i = 1, \dots, d$) and $h_0^j, h^{j,i}$ (for $j = 1, \dots, m_T$ and $i = 0, \dots, d$) are differentiable, their partial subdifferentials coincide with the respective partial derivatives by [14, Theorem 25.1]. Hence, taking partial derivatives with respect to $\Delta\varphi_{t+1}^{\uparrow,j,i}$ resp. $\Delta\varphi_{t+1}^{\downarrow,j,i}$, $t \in \{0, \dots, T\}$, $j \in \{1, \dots, m_t\}$, $i \in \{0, \dots, d\}$, Statement 3 above implies that

$$\begin{aligned} 0 &= \sum_{k:\omega_k \in F_t^j} \mu^{j,i} - \left(\sum_{k:\omega_k \in F_t^j} \nu^k \right) \overline{S}_t^{j,i} - \lambda_t^{\uparrow,j,i} \\ &= \sum_{k:\omega_k \in F_t^j} \mu^{j,i} - \left(\sum_{k:\omega_k \in F_t^j} \nu^k \right) \left(1 + \frac{\lambda_t^{\uparrow,j,i}}{\overline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k} \right) \overline{S}_t^{j,i}, \end{aligned} \tag{3.2}$$

and likewise

$$0 = \sum_{k:\omega_k \in F_t^j} \mu^{j,i} - \left(\sum_{k:\omega_k \in F_t^j} \nu^k \right) \left(1 - \frac{\lambda_t^{\downarrow,j,i}}{\underline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k} \right) \underline{S}_t^{j,i}. \tag{3.3}$$

In particular we have, for $t = 0, \ldots, T$, $j = 1, \ldots, m_t$, $i = 1, \ldots, d$,

$$\left(1 + \frac{\lambda_t^{\uparrow,j,i}}{\overline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k}\right) \overline{S}_t^{j,i} = \left(1 - \frac{\lambda_t^{\downarrow,j,i}}{\underline{S}_t^{j,i} \sum_{k:\omega_k \in F_t^j} \nu^k}\right) \underline{S}_t^{j,i} =: \widetilde{S}_t^{j,i}.$$

Since $\widetilde{S} := (\widetilde{S}^1, \ldots, \widetilde{S}^d)$ is constant on F_t^j by definition, this defines an adapted process. Furthermore, we have $\underline{S} \leq \widetilde{S} \leq \overline{S}$, by Statement 1 above and because $\nu^k < 0$ for $k = 1, \ldots, m_T$. Moreover, Statement 1 above also implies that

$$\widetilde{S}^i = \overline{S}^i \text{ on } \{\Delta\varphi^{\uparrow,i} > 0\}, \quad \widetilde{S}^i = \underline{S}^i \text{ on } \{\Delta\varphi^{\downarrow,i} > 0\}. \tag{3.4}$$

Set $\widetilde{\mu}^{j,i} := \mu^{j,i}$ (for $j = 1, \ldots, m_T$, $i = 1, \ldots, d$), $\widetilde{\nu}^j := \nu^j$ (for $j = 1, \ldots, m_T$) and $\widetilde{\lambda}_t^{\uparrow,j,i}, \widetilde{\lambda}_t^{\downarrow,j,i} := 0$ (for $t = 0, \ldots, T$, $j = 1, \ldots, m_t$ and $i = 1, \ldots, d$). Statements 1, 2 and 3 above, Equations (3.2), (3.3), (3.4) and the definition of $\widetilde{S}$ then yield the following.

1. For $t = 0, \ldots, T$, $i = 1, \ldots, d$ and $j = 1, \ldots, m_t$ we have $\widetilde{\lambda}_t^{\uparrow,j,i}, \widetilde{\lambda}_t^{\downarrow,j,i} \geq 0$ as well as $\widetilde{g}_t^{\uparrow,j,i}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c), \widetilde{g}_t^{\downarrow,j,i}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) \leq 0$ and $\widetilde{\lambda}_t^{\uparrow,j,i}\widetilde{g}_t^{\uparrow,j,i}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) = 0$ as well as $\widetilde{\lambda}_t^{\downarrow,j,i}\widetilde{g}_t^{\downarrow,j,i}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) = 0$

2. $\widetilde{h}_0^j(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) = 0$ and $\widetilde{h}^j(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) = 0$ for $j = 1, \ldots, m_T$,

3. $$\begin{aligned} 0 \in & \partial\widetilde{f}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) + \sum_{j=1}^{m_T} \widetilde{\nu}^j \partial\widetilde{h}_0^j(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) + \sum_{i=1}^{d}\sum_{j=1}^{m_T} \widetilde{\mu}^{j,i} \partial\widetilde{h}^{j,i}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) \\ & - \sum_{t=0}^{T}\sum_{i=1}^{d}\sum_{j=1}^{m_t} \widetilde{\lambda}_t^{\uparrow,j,i} \partial\widetilde{g}_t^{\uparrow,j,i}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c) - \sum_{t=0}^{T}\sum_{i=1}^{d}\sum_{j=1}^{m_t} \widetilde{\lambda}_t^{\downarrow,j,i} \partial\widetilde{g}_t^{\downarrow,j,i}(\Delta\varphi^\uparrow, \Delta\varphi^\downarrow, c), \end{aligned}$$

where the mappings $\widetilde{f}$, $\widetilde{h}_0^j$, $\widetilde{h}^j$, $\widetilde{g}_t^{\uparrow,j}$, $\widetilde{g}_t^{\downarrow,j}$ are defined by setting $\underline{S} = \overline{S} = \widetilde{S}$ in the definition of the mappings f, h_0^j, h^j, $g_t^{\uparrow,j}$, $g_t^{\downarrow,j}$ above. In view of [14, Theorem 28.3] and Steps 1 and 2 above, (φ, c) is therefore not only optimal in the market with bid/ask prices $\underline{S}, \overline{S}$, but in the market with bid-ask prices $\widetilde{S}, \widetilde{S}$ (i.e. in the frictionless market with price process $\widetilde{S}$) as well. Hence $\widetilde{S}$ is a shadow price process and we are done. □

Remark 3.3 An analogue of Theorem 3.2 for utility from terminal wealth can be obtained by considering the objective function $f((\Delta\underline{\varphi}, \Delta\overline{\varphi}), c)) := -E(u_T(c_T))$ subject to the additional constraints $c_1 = \ldots = c_{T-1} = 0$.

Corollary 3.4 (Fundamental Theorem of Utility Maximization with transaction costs) *Let (φ, c) be an admissible portfolio consumption pair for the market with bid/ask prices $\underline{S}, \overline{S}$ satisfying $E(\sum_{t=0}^T u_t(c_t)) > -\infty$. Then we have equivalence between:*

1. *(φ, c) is optimal in the market with bid/ask prices $\underline{S}, \overline{S}$.*

2. *There exists an adapted process $\widetilde{S}$ with $\underline{S} \leq \widetilde{S} \leq \overline{S}$, a number $\alpha \in (0, \infty)$ and a probability measure $Q \sim P$ such that $\widetilde{S}$ is a Q-martingale and*

$$E\left(\frac{dQ}{dP}\middle|\mathscr{F}_t\right) \in \frac{1}{\alpha}\partial u_t(c_t), \quad t = 0, \ldots, T.$$

PROOF. $1 \Rightarrow 2$: This follows from Theorem 3.2 combined with [10, Theorem 3.5, Remark 3 after Theorem 3.7 and Definition 2.3].

$2 \Rightarrow 1$: By [10, Theorem 3.5], Statement 2 above is equivalent to (φ, c) being optimal in the frictionless market with price process $\widetilde{S}$. Let (ψ, κ) be any admissible portfolio consumption pair in the market with bid/ask prices $\underline{S}, \overline{S}$. For $t = 1, \ldots, T+1$, define $\Delta\psi_t^{\uparrow} := (\Delta\psi_t)^+$, $\Delta\psi_t^{\downarrow} := (\Delta\psi_t)^-$, as above and let

$$\widetilde{\kappa}(t) := \kappa(t) + (\Delta\psi_t^{\uparrow})^\top(\overline{S}_t - \widetilde{S}_t) + (\Delta\psi_t^{\downarrow})^\top(\widetilde{S}_t - \underline{S}_t).$$

Then $\widetilde{\kappa} \geq \kappa$ since $\underline{S} \leq \widetilde{S} \leq \overline{S}$ and $(\psi, \widetilde{\kappa})$ is a self-financing portfolio/consumption pair in the frictionless market with price process $\widetilde{S}$, i.e. with bid/ask-prices $\widetilde{S}, \widetilde{S}$. Since (φ, c) is optimal in this market, we have

$$E\left(\sum_{t=0}^{T} u_t(\kappa_t)\right) \leq E\left(\sum_{t=0}^{T} u_t(\widetilde{\kappa}_t)\right) \leq E\left(\sum_{t=0}^{T} u_t(c_t)\right).$$

Therefore (φ, c) is optimal in the market with bid/ask prices $\underline{S}, \overline{S}$ as well. □

Remarks.

1. If, for fixed $(\omega, t) \in \Omega \times \mathbb{R}_+$, $x \mapsto u_t(\omega, x)$ is differentiable on its effective domain with derivative u', $E(\frac{dQ}{dP}|\mathscr{F}_t) \in \frac{1}{\alpha}\partial u_t(c_t)$ reduces to $E(\frac{dQ}{dP}|\mathscr{F}_t) = \frac{1}{\alpha}u_t'(c_t)$.

2. The pair $(\widetilde{S}, Q)$ consisting of the shadow price process $\widetilde{S}$ and the corresponding dual martingale measure Q is called a *consistent price system* by [6]. Using this terminology, Corollary 3.4 can be rephrased as follows: An admissible portfolio/consumption pair is optimal in the market with bid/ask prices $\underline{S}, \overline{S}$ if and only if there exists a consistent price system $(\widetilde{S}, Q)$ such that $E(\frac{dQ}{dP}|\mathscr{F}_t) \in \frac{1}{\alpha}\partial u_t(c_t)$ for some $\alpha \in (0, \infty)$.

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