DVORETZKY TYPE THEOREMS FOR MULTIVARIATE POLYNOMIALS AND SECTIONS OF CONVEX BODIES

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Abstract. In this paper we prove the Gromov–Milman conjecture (the Dvoretzky type theorem) for homogeneous polynomials on $\mathbb{R}^n$, and improve bounds on the number $n(d,k)$ in the analogous conjecture for odd degrees d (this case is known as the Birch theorem) and complex polynomials.

We also consider a stronger conjecture on the homogeneous polynomial fields in the canonical bundle over real and complex Grassmannians. The latter conjecture is much stronger and false in general, but it is proved in the cases of $d = 2$ (for k's of certain type), odd d, the complex Grassmannian (for odd and even d and any k). Corollaries for the John ellipsoid of projections or sections of a convex body are deduced from the case $d = 2$ of the polynomial field conjecture.

1. Introduction

The following theorem was conjectured in [20] (see also [21]), it is known as the Gromov–Milman conjecture. This theorem resembles the famous theorem of Dvoretzky [8] on near-elliptical sections of convex bodies. It considers polynomials instead of convex bodies, and unlike the Dvoretzky theorem, it gives strict "roundness" rather than approximate "roundness".

Theorem 1. *For an even positive integer d and a positive integer k there exists $n(d,k)$ such that for any homogeneous polynomial f of degree d on $\mathbb{R}^n$, where $n \ge n(d,k)$, there exists a linear k-subspace $V \subseteq \mathbb{R}^n$ such that $f|_V$ is proportional to the $d/2$-th power of the standard quadratic function*

$$Q = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Remark. Actually, the conjecture in [20] was stated in a bit different way: the restriction $f|_V$ was required to be proportional to the $d/2$-th power of *some* quadratic form. But a straightforward argument (using the diagonal form in an orthonormal basis) shows that $n(2,k) = 2k-1$, i.e. any quadratic form on $\mathbb{R}^{2k-1}$ is proportional to the standard form

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on a subspace of dimension k. Hence these two versions are equivalent modulo the precise values of $n(d,k)$, and this equivalence is used in the proof below.

Remark. In [20] it was also conjectured that $n(d,k)$ is of order k^d. We do not have results of this kind here because of using a topological Borsuk-Ulam type theorem without explicit bound, see Section 3 and the remark at the end of Section 8.

Besides the trivial case $d = 2$, there were other partial results in this conjecture. Theorem 1 was proved in [20, 18, 19] (the essential idea goes back to M. Gromov) for $k = 2$ by topological methods (actually, the stronger Conjecture 1 was proved for $k = 2$), and with good bounds for $n(d,2)$. In case of special polynomials of the form $f = x_1^d + x_2^d + \dots + x_n^d$ this theorem was proved in [23], see also [20] for a short proof with the averaging trick. If we let d be odd, this theorem is known as the Birch theorem and holds in a stronger form with good estimates on $n(d,k)$, see [5, 1] and Theorems 4 and 5 below.

In this paper we combine the topological technique with the averaging method of [20] to prove Theorem 1. Let us state a more general conjecture, that would imply Theorem 1, if it were true.

Definition 1. Denote G_n^k the Grassmannian of linear k-subspaces in $\mathbb{R}^n$, denote by $\gamma_n^k : E(\gamma_n^k) \to G_n^k$ its canonical bundle.

Definition 2. For a vector bundle $\xi : E(\xi) \to X$ denote $\Sigma^d(\xi)$ its fiberwise symmetric d-th power. We consider every vector bundle ξ along with some Riemannian metric on its fibers, i.e. a nonzero section $Q(\xi)$ of $\Sigma^2(\xi)$.

Conjecture 1. *Suppose d and k are even positive integers. Then there exists $n(d,k)$ such that for every section of the bundle $\Sigma^d(\gamma_n^k)$ over G_n^k with $n \geq n(d,k)$, there exists a subspace $V \in G_n^k$ such that this section is a multiple of $(Q(\gamma_n^k))^{d/2}$ over V.*

This conjecture would imply Theorem 1, because every polynomial of degree d defines a section of $\Sigma^d(\gamma_n^k)$ tautologically.

Unfortunately, there already exist some negative results on Conjecture 1. It is shown in [12, Ch. IV, § 1 (A)] (with reference to [11]) that this conjecture fails for odd k. The counterexample is for $d = 2$ and the oriented Grassmannian (the space $BSO(k)$), but it seems like the case of the Grassmannian $G_\infty^k = BO(k)$ is handled in the same way. The counterexamples for even $d > 2$ are obtained by taking the $d/2$-th power of the counterexample for $d = 2$. In [6] a counterexample to Conjecture 1 is given for $k = 4$ and $d \geq 4$.

Of course, these counterexamples do not give a counterexample to the original Theorem 1. Moreover, it would be sufficient to prove Conjecture 1 for some infinite sequence of k's in order to deduce Theorem 1 for all k's.

As it was noted, Conjecture 1 is known for $k = 2$, see [20, 18, 19]. Here we prove another its particular case.

Theorem 2. *Conjecture 1 is true for $d = 2$ and $k = 2p^\alpha$ for a prime p. In the first nontrivial case we have a particular estimate*

$$n(2,4) \leq 12.$$

Note that this theorem does not add anything new to Theorem 1 (the quadratic forms are not interesting there), but it has some applications to sections and projections of convex bodies, given in Section 9. Theorem 2 has the following generalization for several sections.

Theorem 3. *Suppose $k = 2p^\alpha$ for a prime p, m is a positive integer. Then there exists $n(2, k, m)$ such that for every m sections $s_1, \dots, s_m$ of the bundle $\Sigma^d(\gamma_n^k)$ over G_n^k with $n \ge n(2, k, m)$, there exists a subspace $V \in G_n^k$ such that all the sections s_i are multiples of $(Q(\gamma_n^k))^{d/2}$ over V.*

The topological proof of Theorems 2 and 3 cannot be applied directly to the cases of Conjecture 1 with $d \ge 4$. But returning to Theorem 1 for multivariate polynomials, we shall see that the similar topological technique is essentially used, along with some averaging and combinatorics.

Now let us turn to the case of odd d in Theorem 1 and Conjecture 1. This version of Theorem 1 was known even before the formulation of this conjecture for even-degree polynomials in [20]. In [5] it was shown to be true in an (obviously) stronger from, i.e. $f = 0$ on a k-dimensional subspace. In [1] the bound on $n(d, k)$ was improved. The following theorem improves the bound in [1], at least by a factor of $k!$. The topological technique in our proof was also used in [6] to study Dvoretzky type theorems over Grassmannians.

Theorem 4. *Suppose d and k are positive integers, d being odd. Then there exists*

$$n(d, k) = k + \binom{d + k - 1}{d}$$

such that every section of the bundle $\Sigma^d(\gamma_n^k)$ over G_n^k (with $n \ge n(d, k)$) has a zero.

Simple dimension considerations show that $n(d, k)$ cannot be made less than

$$k + \frac{1}{k}\binom{d + k - 1}{d}$$

in this theorem. We may conclude that the bound in Theorem 4 is quite satisfactory, but still may be improved.

The topological approach with Grassmannians also allows to prove the following version of Theorem 1 for odd polynomial maps. Of course, the corresponding version of Conjecture 1 is also true, but its statement would be too complicated, so we formulate the statement without the Grassmannian and bundles here.

Theorem 5. *Suppose d, k, m are positive integers, d being odd. Then there exists*

$$n(d, k, m) = k + m \sum_{1 \le \delta \le d,\ \delta \equiv 1 \mod 2} \binom{\delta + k - 1}{\delta}$$

with the following property. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is an odd polynomial map, such that $n \ge n(d, k, m)$, and the coordinate functions of f have degrees $\le d$. Then f maps some k-dimensional linear subspace $L \subset \mathbb{R}^n$ to zero.

Similar theorems are also true for complex polynomials, Grassmannians, and bundles. In this case the degree does not have to be odd, it can be arbitrary. The following result is the complex analogue of Conjecture 1.

Theorem 6. *Suppose d and k are positive integers. Then there exists*

$$n(d,k) = k + \binom{d+k-1}{d}$$

such that every section of the bundle $\Sigma^d(\mathbb{C}\gamma_n^k)$ over $\mathbb{C}G_n^k$ (with $n \ge n(d,k)$) has a zero.

The following result is the stronger complex analogue of Theorem 1.

Theorem 7. *Suppose d, k, m are positive integers. Then there exists*

$$n(d,k,m) = k + m\binom{d+k}{d}$$

with the following property. Suppose $f : \mathbb{C}^n \to \mathbb{C}^m$ is a polynomial map, such that $n \ge n(d,k,m)$, and the coordinate functions of f have degrees $\le d$. Then f maps some k-dimensional linear subspace $L \subset \mathbb{C}^n$ to zero.

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2. Proof of theorems on odd and complex polynomials

We start with proofs of the theorems concerning odd and complex polynomials, because their proofs are simple and give a good idea of the topological machinery. The reader can find standard topological facts about characteristic classes of vector bundles in the textbooks [10, 22, 17], if it is needed.

First consider the infinite Grassmannian G_∞^k and the canonical bundle γ_∞^k over it. The cohomology $H^*(G_\infty^k, Z_2)$ is a subalgebra of $Z_2[t_1, \dots, t_k]$, consisting of symmetrical polynomials, the Stiefel-Whitney class of γ_∞^k is

$$w(\gamma_\infty^k) = \prod_{i=1}^k (1 + t_i).$$

Hence, the Stiefel-Whitney class of $\Sigma^d(\gamma_\infty^k)$ is

$$w(\Sigma^d(\gamma_\infty^k)) = \prod_{i_1,\dots,i_k \ge 0}^{i_1+\dots+i_k=d} (1 + i_1 t_i + \dots i_k t_k).$$

Since d is odd, then none of the expressions $i_1 t_i + \dots i_k t_k$ is zero $\mod 2$, and we obtain that the topmost Stiefel-Whitney class of $\Sigma^d(\gamma_\infty^k)$ of dimension $\binom{d+k-1}{d}$ is nonzero. By the standard reasoning it means that $\Sigma^d(\gamma_\infty^k)$ cannot have a section without zeros.

Now we have to go back to finite Grassmannians. The kernel of the natural map $H^*(G_\infty^k, Z_2) \to H^*(G_n^k, Z_2)$ is generated by the dual Stiefel-Whitney classes of γ_∞^k of dimensions $> n-k$. If $n \ge k + \binom{d+k-1}{d}$ the topmost Stiefel-Whitney class of γ_n^k turns out to be nonzero from the dimension considerations.

The proof of Theorem 5 proceeds in the same way, considering the bundle

$$\bigoplus_{1\le\delta\le d,\ \delta\equiv 1 \mod 2} \Sigma^\delta(\gamma),$$

and taking its m-fold Whitney power. Obviously, the topmost Stiefel-Whitney class of the resulting bundle over G_∞^k is nonzero and has dimension

$$n_0 = m \sum_{1\le\delta\le d,\ \delta\equiv 1 \mod 2} \binom{\delta+k-1}{\delta}.$$

Then we can pass to the finite Grassmannian $G_{n_0+k}^k$ as above.

The proof of Theorems 6 and 7 is the same with Chern classes in $H^*(\mathbb{C}G_n^k, \mathbb{Z})$ instead of Stiefel-Whitney classes. In this case the topmost Chern class of $\Sigma^d(\mathbb{C}\gamma_\infty^k)$ is always nonzero. Besides, in Theorem 7 we use the well-known formula

$$\sum_{\delta=0}^{d} \binom{\delta+k-1}{k-1} = \binom{d+k}{k}.$$

3. Borsuk-Ulam property for p-toral groups

Before proving Theorems 1 and 2 we need to consider the following Borsuk-Ulam type problem, see the books [12, 4] for needed facts and definitions, concerning the continuous group actions. By EG we denote a homotopy trivial G-CW-complex with free action of G.

Problem 1. Suppose G is a compact Lie group, V its representation. Determine whether the vector bundle

$$EG \times V \to EG$$

has a G-equivariant nonzero section.

Equivalently, there exists a G-equivariant map $f : EG \to S(V)$ to the sphere space of the bundle.

Or equivalently, the vector bundle

$$(EG \times V)/G \to BG = EG/G$$

has a nonzero section.

In this section it is convenient to use the statement if this problem in the second version, with equivariant map $f : EG \to S(V)$. This version is obviously a generalization of the Borsuk-Ulam theorem.

Of course, if the representation V has a nonzero fixed point set V^G, the map f obviously exists, we can map EG to any point in $V^G \setminus \{0\}$. The following result from [2, 3, 4, 7] gives an inverse statement for a special class of groups.

Definition 3. Suppose

$$0 \to T \to G \to F \to 0$$

is an exact sequence of groups, where T is a torus, F is a p-group. In this case G is called *p-toral.*

Lemma 1. *Suppose G is a p-toral group and V its representation. Then the image of an equivariant map $f : EG \to V$ intersects V^G.*

From the standard reasoning in obstruction theory the following also holds. There exists $n(G, V)$ such that if a free G-space X is $n-1$-connected ($n \geq n(G, V)$) then the image of an equivariant map $f : X \to V$ intersects V^G.

4. The rational obstructions to nonzero sections of vector bundles

In order to give a particular bound in Theorem 2 for $k = 4$, we also need the following expression of the rational obstruction to a nonzero section of some vector bundles $\xi : E(\xi) \to X$. Suppose that ξ is oriented. In case $\dim \xi$ is even the first obstruction is the Euler class $e(\xi)$. Consider the case $\dim \xi = 2m + 1$. In this case the rational Euler class is zero, but if ξ has a nonzero section, then we have $\xi = \eta \oplus \varepsilon$, where

$$\varepsilon : X \times \mathbb{R} \to X$$

is the trivial bundle. The bundle η is naturally oriented, and we have (we index the Pontryagin classes by their dimension)

$$p_{4m}(\eta) = e(\eta)^2.$$

Since $p_{4m}(\xi) = p_{4m}(\eta)$ we see that the nonexistence of the square root $\sqrt{p_{4m}(\xi)}$ in $H^*(X, \mathbb{Z})$ is an obstruction to a nonzero section of ξ.

Considering the fiberwise Postnikov tower for the sphere bundle $S(\xi)$ it can be shown that this is the only rational obstruction for a nonzero section of ξ, but we do not need this fact here.

5. Proof of Theorems 2 and 3

First let us prove the existence of $n(2, k, m)$ in Theorem 3 using Lemma 1. We can consider G_∞^k, the existence of $n(2, k, m)$ follows from the obstruction theory as in Lemma 1. Considering the infinite Stiefel variety V_∞^k, we have

$$G_\infty^k = V_\infty^k / O(k),$$

in other words V_∞^k is a realization of $EO(k)$. Now let us decompose $\mathbb{R}^k$ into p^α 2-dimensional spaces $L_1 \oplus \cdots \oplus L_{p^\alpha}$. Let the p^α-dimensional torus T act on $\mathbb{R}^k$ by independent rotations of L_i. Let the group $F = (Z_p)^\alpha$ permute the spaces L_i transitively. In this case we obtain an action of the p-toral group $G = T \rtimes F$ on $\mathbb{R}^k$.

Now consider the section s_i of $\Sigma^2(\gamma_\infty^k)$ as an $O(k)$-equivariant map $f_i : V_\infty^k \to \Sigma^2(\mathbb{R}^k)$. Restricting the group action to G we see that the product map $f = (f_1, \dots, f_m)$ is a G-equivariant map to a linear representation space. Hence f should map some frame $x \in V_\infty^k$

to an array of m quadratic forms on $\mathbb{R}^k$, all being G-invariant. Now it remains to note that T-invariant forms on $\mathbb{R}^k$ are those of the form

$$Q = \sum_{i=1}^{p^\alpha} a_i(x_{2i-1}^2 + x_{2i}^2),$$

and if we require them to be F-invariant, then we obtain

$$a_1 = a_2 = \cdots = a_{p^\alpha},$$

such a quadratic form is proportional to the standard one. This completes the proof of existence of $n(2, 2p^\alpha, m)$.

Now consider the particular case $k = 4$ in Theorem 2, and suppose that the Grassmannians are oriented in the reasonings below, it is needed to apply the results of Section 4. Consider a simpler case of the bundle $\Sigma^2(\gamma_\infty^4)$ instead of $\Sigma^2(\gamma_{12}^4)$ first. The cohomology $H^*(G_\infty^4, \mathbb{Q})$ (see [17]) is a subalgebra of $\mathbb{Q}[a, b]$ (a and b are two-dimensional generators), generated by ab (the Euler class) and $a^2 + b^2$ (the 4-dimensional Pontryagin class). Since there exists an $SO(4)$-equivariant quadratic form, we can decompose the bundle of quadratic forms

$$\Sigma^2(\gamma_\infty^4) = \xi \oplus \varepsilon.$$

Now we have to find an obstruction to a nonzero section of the 9-dimensional bundle ξ. From the standard calculation it follows that

$$p_{16}(\xi) = p_{16}(\Sigma^2(\gamma_\infty^4)) = 16a^2b^2(a^2 - b^2).$$

Hence $\sqrt{p_{16}(\xi)} = 4ab(a^2 - b^2)$, which does not belong to $H^*(G_\infty^4, \mathbb{Q})$. If we consider G_{12}^4 instead of the infinite Grassmannian, we see that the kernel of the natural map

$$H^*(G_\infty^4, \mathbb{Q}) \to H^*(G_{12}^4, \mathbb{Q})$$

is generated by the dual Pontryagin classes of γ_∞^4 (i.e. the Pontryagin classes of its complementary bundle γ_{12}^8) of dimension ≥ 20. Such relations do not affect taking a square root of $p_{16}(\xi)$ by the dimension considerations. Besides the image of $H^*(G_\infty^4, \mathbb{Q})$, the cohomology $H^*(G_{12}^4, \mathbb{Q})$ has another generator: the Euler class of the complementary bundle $e(\gamma_{12}^8)$ of dimension 8, along with the relation

$$e(\gamma_{12}^4)e(\gamma_{12}^8) = 0.$$

It is easy to see that this does not help to take a square root of $p_{16}(\xi)$ either, and the proof is complete.

Remark. Note that this way of reasoning may work for larger d if we find a p-toral subgroup $G \subset O(k)$ which is dense enough in $O(k)$. Unfortunately, by the well-known theorem of Jordan [14], for any finite subgroup $G \subset O(k)$ the index of intersection with the maximal torus $[G : G \cap T]$ is bounded by some constant $J(k)$. Hence, for large enough d there exist nontrivial polynomials of degree d in k variables that are invariant under G.

6. Proof of Theorem 1 for $d = 4$. The topological part

Now we have all prerequisites to prove Theorem 1. For the reader's convenience we first outline the proof in the case $d = 4$, the final proof is given in Section 8. In this particular case, as well as in the general case we combine the topological technique based on the Borsuk-Ulam theorem for p-groups with the averaging argument [20].

First, we apply Lemma 1 and show that it suffices to prove the theorem for a very special type of homogeneous polynomials of degree 4.

Let $m = 2^\alpha$. Consider the group $G = (Z_2)^m \rtimes \Sigma_m^{(2)}$, acting on a m-dimensional space $\mathbb{R}^m$ as follows. Let $(Z_2)^m$ act by changing signs of the coordinates, and let $\Sigma_m^{(2)}$ (the 2-Sylow subgroup of the symmetric group Σ_m) act by permuting the coordinates.

Denote $[m] = \{1, 2, \dots, m\}$. The group $\Sigma_m^{(2)}$ is generated by permutations of two consecutive blocks in $[m]$: $[a2^l + 1, a2^l + 2^{l-1}]$ and $[a2^l + 2^{l-1} + 1, (a + 1)2^l]$, where $1 \le l \le \alpha$ and $0 \le a \le 2^{\alpha-l} - 1$. This permutation can be also described as follows. Consider α-digit binary numbers, possibly with leading zeros (call them *strings*), every index $i \in [m]$ will correspond to the binary representation of $i - 1$. Then we fix some start substring s, and permute as follows (for every possible end substring e)

$$s0e \mapsto s1e, s1e \mapsto s0e,$$

such permutations generate $\Sigma_m^{(2)}$.

Another description of $\Sigma_m^{(2)}$ is as follows: represent the numbers $1, 2, \dots, m$ as leafs of a full ordered binary tree of depth α. Then $\Sigma_m^{(2)}$ is generated by the permutations that transpose two children of some tree node, keeping all other children orders.

Lemma 1 tells that if n is large enough, than every homogeneous polynomial of degree 4 becomes G-invariant after restricting to some m-dimensional subspace. Now let us describe G-invariant polynomials f on $\mathbb{R}^m$. The invariance w.r.t. $(Z_2)^m$ is equivalent to the fact that

$$f = \sum_{1 \le i,j \le m} a_{ij} y_i^2 y_j^2,$$

where a_{ij} is a symmetric $m \times m$ matrix, y_i are the coordinates in $\mathbb{R}^m$. The $\Sigma_m^{(2)}$-invariance implies more relations on a_{ij}, which can be described as follows. If we write $i - 1$ and $j - 1$ as α-digit binary numbers with leading zeros, then a_{ij} depends only on the first from the left digit position where $i - 1$ and $j - 1$ differ. In this case there are at most $\alpha + 1$ different values of a_{ij}.

7. Proof of Theorem 1 for $d = 4$. The geometrical part

Now we are going to use an averaging argument, similar to what is given in [20]. Using the remark after the statement of Theorem 1, the proof of its particular case for $f = \sum_{i=1}^n x_i^d$ in [20], and the result of the above section, we note the following. In order to prove the theorem, we have to find large enough $m = 2^\alpha$ for every given k such that

$$(x_1^2 + x_2^2 + \dots + x_k^2)^2 \sim \sum_{1 \le i,j \le m} a_{ij} l_i(x)^2 l_j(x)^2, \qquad (1)$$

where a_{ij} is a given $\Sigma_m^{(2)}$-symmetrical matrix, and

$$l_i(x) = l_i(x_1, \dots, x_k)$$

are some linear forms that we have to find. These forms would give a map $\mathbb{R}^k \to \mathbb{R}^m$ such that its image V is the required subspace, because the form $\sum a_{ij} l_i^2 l_j^2$ becomes a square after restriction to V. Note that these forms have to span $(\mathbb{R}^k)^*$ to give a map with zero kernel.

We are going to find the forms $l_i(x)$ using the following procedure. Let s be the least power of two that is greater or equal to k, let $m = m's$. Choose s linear forms $\lambda_1(x), \dots, \lambda_s(x)$, with the only restriction that

$$x_1^2 + \dots + x_k^2 \sim \lambda_1(x)^2 + \dots + \lambda_s(x)^2. \tag{2}$$

In this case these s forms already span $(\mathbb{R}^k)^*$. Let us partition all the forms l_i ($i = 1, \dots, m's$) into consecutive s-tuples, and let the s-tuple number $t = 1, \dots, m'$ be obtained from the $(\lambda_1, \dots, \lambda_s)$ by a transform $\sigma_t \in SO(k) \times \mathbb{R}^+$ (a rotation with a positive homothety), i.e.

$$l_{st-i}(x) = \lambda_{s-i}(\sigma_t x),$$

where $i = 0, \dots, s-1$. Every σ_t multiplies the quadratic form $x_1^2 + \dots + x_k^2$ by a positive number, hence Equation 2 holds for every considered s-tuple of l_i's.

Note that the right hand part of Equation 1 can be rewritten using Equation 2 (and the symmetry of a_{ij}) as follows

$$(x_1^2 + x_2^2 + \dots + x_k^2)^2 \sim \left(\sum_{t=1}^{m'} \sum_{1 \le i,j \le s} a_{ij} \lambda_i(\sigma_t x)^2 \lambda_j(\sigma_t x)^2 \right) + B(x_1^2 + \dots + x_k^2)^2.$$

The first summand is formed by $s \times s$ cells on the diagonal of a_{ij}. Each of the non-diagonal $s \times s$ cells of a_{ij} consists of a single constant (from the $\Sigma_m^{(2)}$-symmetry condition). Hence, the non-diagonal $s \times s$ cells give a summand proportional to $(x_1^2 + \dots + x_k^2)^2$ (from Equation 2).

Now denote

$$g(x) = \sum_{1 \le i,j \le s} a_{ij} \lambda_i(x)^2 \lambda_j(x)^2,$$

we have to prove that for some $\sigma_1, \dots, \sigma_{m'} \in SO(k) \times \mathbb{R}^+$

$$(x_1^2 + x_2^2 + \dots + x_k^2)^2 \sim \sum_{t=1}^{m'} g(\sigma_t x). \tag{3}$$

The rest of the proof is similar to the proof in [20]. If we substitute the right hand part of Equation 3 by an integral over every possible rotation $\rho \in SO(k)$, we surely obtain an $SO(k)$-invariant 4-from, which has to be proportional to $(x_1^2 + \dots + x_k^2)^2$. Then we use the Carathéodory theorem to show that if

$$m' \ge \binom{k+3}{4},$$

then some m' rotations give a symmetric convex combination

$$\sum_{t=1}^{m'} w_t g(\rho_t x) \sim (x_1^2 + x_2^2 + \ldots + x_k^2)^2,$$

now it suffices to denote $\sigma_t = \rho_t \sqrt[4]{w_t}$ and use the fact that $g(x)$ is 4-homogeneous.

Remark. The total estimate on $n(4, k)$ in this proof is not very good. The averaging argument gives $m \sim k^5$, after that n is determined by m using the Borsuk-Ulam property. The latter estimate is not known directly, because it uses asymptotic facts on the equivariant cohomotopy of classifying spaces. From a detailed analysis of the proof it is clear that the group $\Sigma_m^{(2)}$ can be replaced by a smaller subgroup, but anyway, the group depends on k and the estimate on $n(m)$ is not known. See also the discussion at the end of Ch. 3 in [4], the results conjectured there would imply a polynomial bound for $n(m)$.

8. Proof of Theorem 1 for arbitrary d

First, let us apply the Borsuk-Ulam theorem for 2-groups to the group $G = (Z_2)^m \rtimes \Sigma_m^{(2)}$, m is to be defined later. If the initial dimension n is large enough, then the restriction of f to some m-dimensional subspace equals (put $d/2 = \delta$)

$$f = \sum_{1 \le i_1, i_2, \ldots, i_\delta \le m} a_{i_1, \ldots, i_\delta} y_{i_1}^2 y_{i_2}^2 \ldots y_{i_\delta}^2, \tag{4}$$

where the numbers $a_{i_1, \ldots, i_\delta}$ are invariant under the component-wise action of $\Sigma_m^{(2)}$ on the indexes $i_1, \ldots, i_\delta$.

We are going to use the averaging argument from [20] several times and for several polynomials simultaneously, so we describe the averaging procedure in the following lemma. Denote the group of rotations composed with a homothety by $S(k) = SO(k) \times \mathbb{R}^+$, call its elements *similarity transforms*. Sometimes we consider the zero transform as a similarity transform too, denote $S_0(k) = S(k) \cup \{0\}$.

Lemma 2. *Suppose $f_1, \ldots, f_l$ are even homogeneous polynomials of degree $\le d$ in k variables,*

$$n \ge l \binom{k + d - 1}{d}.$$

We can find n similarity transforms $\sigma_1, \ldots, \sigma_\in S_0(k)$ (not all zero) such that all the polynomials

$$\overline{f}_j(x) = \sum_{i=1}^{n} f_j(\sigma_i x)$$

are proportional to $Q^{\deg f_j/2} = (x_1^2 + \ldots + x_k^2)^{\deg f_j/2}$.

Proof. Note that if a polynomial $f_j(x)$ has degree $d' < d$, we can multiply it by $Q^{\frac{d-d'}{2}}$ and assume that all $f_j(x)$ has the same degree d.

As in [20], the polynomials

$$\overline{f}_j(x) = \int_{\rho\in SO(k)} f_j(\rho x) d\rho$$

are $SO(k)$-invariant, and therefore proportional to $Q^{d/2}$. Note that the linear space of l-tuples of polynomials of degree d in k variables has dimension $l\binom{k+d-1}{d}$, and by the Carathéodory theorem the vector $(\overline{f}_1, \dots, \overline{f}_l)$ is proportional to a convex combination

$$(\overline{f}_1(x), \dots, \overline{f}_l(x)) = \sum_{i=1}^{n} w_i(f_1(\rho_i x), \dots, \overline{f}_l(\rho_i x))$$

for some $\rho_1, \dots, \rho_n \in SO(k)$. Putting $\sigma_i = w_i^{1/d}\rho_i \in S_0(k)$, we obtain the required formula. □

Remark. Note that the averaging procedure is linear in the polynomials f_j. We can take $l = \binom{k+d-1}{d}$ and average *all* the even polynomials of degree $\le d$ by the same sequence $\sigma_1, \dots, \sigma_n \in S_0(k)$ for $n = \binom{k+d-1}{d}^2$.

Let us describe the structure if the G-invariant polynomial f in Equation 4 in more detail. Denote $H = \Sigma_m^{(2)}$ for brevity. Let us describe the H-orbits of the multisets $(i_1, \dots, i_\delta)$, the coefficients $a_{i_1,\dots,i_\delta}$ of f, corresponding to the orbit, should be equal. Let us identify the set $[m]$ with the leafs of the full binary tree T of height $h = \log_2 m$. The group H is the group of automorphisms of T preserving the grading. We choose the grading so that the leafs are of grading zero, what is above them is of grading 1, and so on.

For every multiset $S = (i_1, \dots, i_\delta)$ consider the subtree T_S, consisting of all ancestors of any $i_j \in S$. Let us describe distinct orbits of such $T_S \subseteq T$ under the action of H. The orbit of S is fully characterized by the corresponding orbit of T_S and assignment of multiplicities (of the multiset S) to the leafs of T_S. Consider the nodes of T_S that have ≥ 2 childs, call then the *branching* nodes. The number of such nodes $\le \delta - 1$, and T_S is fully described by gradings of these nodes, and the parent-child relation between them. In this description the only value that depends on h is the gradings, therefore the number of H-orbits of trees T_S (and the number of H-orbits of index multisets S) is

$$\le C(\delta)h^{\delta-1}.$$

In the sequel we will use the above tree description for different heights h, so denote $H_h = \Sigma_{2^h}^{(2)}$ the 2-Sylow permutation group, and the corresponding full binary tree T_h. Suppose U is a multiset of cardinality $\delta' \le \delta$ in $[2^h]$ (the leafs of T_h), denote by

$$g_U(y_1, \dots, y_{2^h}) = \sum_{\substack{\sigma\in H_h \\ \sigma(U)=(i_1,\dots,i_{\delta'})}} y_{i_1}^2 \dots y_{i'_\delta}^2$$

the H_h-invariant polynomials. The number of such distinct polynomials g_U is $\le C(\delta)h^\delta$, and the considered polynomial f is a linear combination of such polynomials in m variables for $|U| = \delta$.

We are going to find m linear forms $l_t(x)$ on $\mathbb{R}^k$ such that every polynomial $g_U(l_1(x), \dots, l_m(x))$ (after substituting $y_t = l_t(x)$) is proportional to Q^δ, this will imply the same claim about $f(l_1(x), \dots, l_m(x))$ by linearity.

Take some nonzero linear function $l_1(x_1, \dots, k)$ in k variables. Then build the other forms $l_i(x)$ by the following procedure. Define the sequence of the powers of two $s_0 = 1, s_1 = 2^{h_1}, \dots, s_\delta = 2^{h_\delta}$ that satisfies the following inequality

$$s_{i+1} \geq C(\delta){h_i}^\delta \binom{k+d-1}{d} s_i.$$

Then suppose we have already chosen the linear functions $l_1(x), \dots, l_{s_i}(x)$. Consider all the polynomials $g_U(y_1, \dots, y_{s_i})$, corresponding to multisets U in $[s_i]$ of cardinality $\leq \delta$, the number of distinct such polynomials is at most $C(\delta){h_i}^\delta$. Put

$$\phi_U(x) = g_U(l_1(x), \dots, l_{s_i}(x))$$

and apply Lemma 2 to the polynomials $\phi_U(x)$ to obtain $n \leq C(\delta){h_i}^\delta\binom{k+d-1}{d}$ similarity transforms $\sigma_1, \dots, \sigma_n \in S_0(k)$ (not all zero) such that all the expressions

$$\sum_{j=1}^n g_U(l_1(\sigma_j x), \dots, l_{s_i}(\sigma_j x))$$

are proportional to $Q^{|U|}$. Denote for $t = s_i(j-1) + r$ $(1 \leq j \leq n,\ 1 \leq r \leq s_i)$

$$l_t(x) = l_r(\sigma_j x).$$

Now we have $s_{i+1} = ns_i$ linear functions, consisting of n similar copies of the previous set of s_i linear functions.

Finally we define $m = s_\delta$, note that here m is roughly of order $C(d)(k \log k)^{d^2/2}$. We could also take $n \leq 2k\binom{k+d-1}{d}^d$ using the remark after Lemma 2, this bound is worse but does not contain unknown functions of d. Note again that the explicit bound in this theorem depends on the (unknown) explicit bound on n in terms of m and k in the Borsuk-Ulam theorem for 2-groups.

Consider a polynomial g_S, corresponding to a multiset S in $[m]$ of cardinality δ. Denote for brevity for a multiset $S = (i_1, \dots, i_\delta)$

$$\overline{y}^{2S} = (y_1, \dots, y_m)^{2S} = y_{i_1}^2 y_{i_2}^2 \dots y_{i_\delta}^2.$$

The corresponding tree T_S has no branching with gradings in $(h_i, h_{i+1}]$ for some i by the pigeonhole principle (it has $\leq \delta - 1$ branching nodes). If we fix the part T_0 of this tree with gradings $> h_{i+1}$, and cut it off, then we obtain several subtrees $T_1, \dots, T_r$ of height h_{i+1}, with branching nodes no higher than h_i, denote their corresponding leaf multisets $S_1, \dots, S_r$. If we decompose $[m]$ into segments $I_1, \dots, I_{m/s_{i+1}}$ of length s_{i+1} each, then we see that the leaf multisets S_j are intersections of S with the corresponding segments I_j. Consider the orbit of S under the group

$$F_1 \times \dots \times F_{m/s_{i+1}} \subseteq H_h,$$

where $F_j = \Sigma^{(2)}_{s_{i+1}}$ is the group of 2-Sylow permutations in every I_j. Denote $r = m/s_{i+1}$ The sum of the corresponding monomials

$$\sum_{\gamma_1\times\dots\times\gamma_r\in F_1\times\dots\times F_r} \overline{y}^{2\gamma_1\times\dots\times\gamma_r(S)}$$

can be rewritten as the product

$$\prod_{\substack{1\le j\le r\\ S\cap I_j\ne\emptyset}} \sum_{\gamma_j\in F_j} \overline{y}^{2\gamma_j(S\cap I_j)}.$$

Every expression $\sum_{\gamma_j\in F_j} \overline{y}^{2\gamma_j(S_j)}$ (we denote $S_j = I_j \cap S$), corresponding to a particular I_j, is a homogeneous polynomial of the form g_{S_j} in s_{i+1} variables y_t ($t \in I_j$). Consider the subgroup $G_j \in F_j$, consisting of elements of the form $\alpha \times \dots \times \alpha$, where $\alpha \in \Sigma^{(2)}_{s_i}$ is a permutation of size $[s_i]$, i.e. G_j permutes all the s_i-blocks of I_j in the same way. Consider also the subgroup $K \subset F_j$ (isomorhic to $(Z_2)^{h_{i+1}/h_i}$) that permutes the whole s_i-blocks in I_j transitively, this is the group, generated by applying the same transposition of childs at a particular binary tree level. Note that these two groups generate a Cartesian product subgroup $G_j \times K \in F_j$.

Note that the corresponding to S_j tree T_j has no branching higher than h_i. The sum

$$\sum_{\gamma\times\kappa\in G_j\times K} \overline{y}^{2\gamma\times\kappa(S_j)} \tag{5}$$

can be rewritten as summation over $\gamma \in G_j$, and then on $\kappa \in K$. The first summation gives a polynomial of type g_{S_j} in s_i variables y_t, corresponding to the s_i-block of I_j, where all elements S_j are contained (because T_j has no branching higher than h_i and S_j is contained in a single s_i-block).

If we substitute $y_t = l_t(x)$ and sum such polynomials g_{S_j} over K, we obtain an expression in $x_1, \dots, x_k$ proportional to $Q^{|S_j|}$ by the construction of the linear functions $l_t(x)$. Suppose I_j is the segment I_1 of the first s_{i+1} variables y_t, the first s_i functions $l_t(x)$ of this segment were transformed into s_{i+1} functions $l_t(\sigma_u(x))$ by the construction, so that the summation over $u = 1, \dots, s_{i+1}/s_i$ of the expressions

$$g_{S_j}(l_1(\sigma_u x), \dots, l_{s_i}(\sigma_u x))$$

makes them proportional to $Q^{|S_j|}$. The same holds for every (not only the first) s_{i+1}-segment I_j, because all the corresponding linear functions $\{l_t(x) : t \in I_j\}$ are obtained from the linear functions $\{l_t(x) : t \in I_1\}$ by substituting $l_t(\tau x)$ with the same similarity transform τ, which appears in the construction of $s_{i+2}, \dots, s_\delta$. In this case we make the summation of

$$g_{S_j}(l_1(\sigma_u \tau x), \dots, l_{s_i}(\sigma_u \tau x))$$

over $u = 1, \dots, s_{i+1}/s_i$ (i.e. over the group K), and by the construction we obtain a polynomial proportional to $Q(\tau x)^{|S_j|}$, which is proportional to $Q(x)^{|S_j|}$, since τ is a similarity transform.

If we pass to summation in Equation 5 over a larger group $F_j \supseteq G_j \times K$, then we again obtain a sum similar to $Q^{|S_j|}$ after substituting $y_t = l_t(x)$. The same argument is valid for summation of the monomials $\overline{y}^{2S}$ over the entire group $H_h \supseteq F_1 \times \cdots \times F_r$, so every such sum will be proportional to Q^δ after substituting $y_t = l_t(x)$, as required.

Note that the case $i + 1 = 1$ is done in the same manner assuming $s_0 = 1$.

Remark. Note that if we use the "universal averaging", according to the remark after Lemma 2, then the only needed averaging in the proofs is averaging over the groups K_i (denoted simply K in the proof). These groups are 2-tori of size $2^{h_{i+1}/h_i}$, and the total required group is the wreath product $K_1 \wr K_2 \wr \cdots \wr K_\delta$. This observation may help if some explicit bounds in the Borsuk-Ulam theorem for wreath products of 2-tori are found.

9. Sections and projections of convex bodies

Now we return to the case of quadratic forms, and deduce some corollaries for sections or projections of convex bodies from Theorems 2 and 3. We use the standard approach (e.g. see [25]) of taking geometric consequences of the results over Grassmanians.

Corollary 8. *Suppose k is an integer of the form $2p^\alpha$, $m \geq 1$, $n \geq n(2, k, m)$ from Theorem 3 or 2. Let $K_1, \dots, K_m$ be convex bodies in $\mathbb{R}^n$. Then there exists a k-dimensional linear subspace $L \subseteq \mathbb{R}^n$ such that the orthogonal projections of any K_i onto L has a Euclidean ball as its John ellipsoid.*

Proof. Consider all possible choices of L, they form the Grassmannian G_n^k. The John ellipsoid [13] of the projection $\pi_L(K_i)$ depends continuously on L, its homogeneous component of degree 2 is a quadratic form on L, hence it gives a section s_i of $\Sigma^2(\gamma_n^k)$. By Theorem 3 these quadratic forms are simultaneously proportional to the standard quadratic form over some L. ☐

The following corollary is proved in the same way.

Corollary 9. *Suppose k is an integer of the form $2p^\alpha$, $m \geq 1$, $n \geq n(2, k, m)$ from Theorem 3 or 2. Let $K_1, \dots, K_m$ be convex bodies in $\mathbb{R}^n$, and x be a point inside $\bigcap_{i=1}^m K_i$. Then there exists a k-dimensional affine subspace $x \in L \subseteq \mathbb{R}^n$ such that for any i the section $K_i \cap L$ has a Euclidean ball as its John ellipsoid.*

It is easy to see that instead of the John ellipsoid we can consider the second moment matrix of the projection (or the section), or some other quadratic form, depending continuously on the convex body. Note that some "approximate" version of these theorems follows form the original Dvoretzky theorem, e.g. we can state that the John ellipsoid is ε-close to a ball.

10. The weak form of the Knaster conjecture

Let us state the weak form of the Knaster conjecture [16].

Conjecture 2. *There exists* $n = n(l)$ *such that for any* l *points* $X = \{x_1, \dots, x_l\}$ *on the unit sphere* S^{n-1} *and any continuous function* $f : S^{n-1} \to \mathbb{R}$ *there exists a rotation* $\rho \in O(n)$ *such that*

$$f(\rho x_1) = f(\rho x_2) = \dots = f(\rho x_l).$$

Originally Knaster conjectured that $n(l) = l$, but counterexamples to his conjecture were found in [15]. In [9] it was proved that $n(3) = 3$, but already the value $n(4)$ is not known and not shown to be finite. Known results in this conjecture either consider sets X, distributed along a two-dimensional vector subspace of $\mathbb{R}^n$ (see [20, 18]), or require very specific symmetry conditions, e.g. require X to be an (almost) orthogonal frame (see [24]).

In [20] it was shown that the original Knaster conjecture would imply the Dvoretzky theorem with good estimates on $n(k, \varepsilon)$, it would also imply Theorem 1. The weak form of the Knaster conjecture would also give some bounds in the Dvoretzky theorem, as well as explicit bounds in Theorem 1. In order to prove Dvoretzky type results we have to consider sets X distributed densely enough in a sphere S^{k-1}.

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