ALMOST REGULAR INVOLUTORY AUTOMORPHISMS OF UNIQUELY 2-DIVISIBLE GROUPS

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Abstract. We prove that a uniquely 2-divisible group that admits an almost regular involutory automorphism is solvable.

1. Introduction

Recall that an automorphism ν of a group H is called *involutory* if $\nu \neq id$ and $\nu^2 = id$. The automorphism ν is called *almost regular*, if $C_H(\nu)$ is finite. Recall that a group U is *uniquely* 2*-divisible* if for each $u \in U$ there exists a unique $v \in U$ such that $v^2 = u$. Note that in particular a uniquely 2-divisible group contains no involutions (i.e. elements of order 2).

The purpose of this note is to use the techniques introduced in the impressive paper [Sh] of Shunkov, where he proves that a periodic group that admits an almost regular involutory automorphism is virtually solvable (i.e. it has a solvable subgroup of finite index). We prove.

Theorem 1.1. *Let U be a uniquely 2-divisible group. If U admits an involutory almost regular automorphism, then U is solvable.*

Our main motivation for dealing with automorphisms of uniquely 2-divisible groups comes from questions about the root groups of special Moufang sets, and those tend to be uniquely 2-divisible, see, e.g., [S]. Indeed, using Theorem 1.1 it immediately follows that

Corollary 1.2. *Let $\mathbb{M}(U, \tau)$ be a special Moufang set. If the Hua subgroup contains an involution ν such that $C_U(\nu)$ is finite, then U is abelian.*

Proof. If U contains involutions, then U is abelian by [DST, Theorem 5.5, p. 782]. If U does not contain involutions, then by [DS, Proposition 4.6, p. 5840], U is uniquely 2-divisible, and then by Theorem 1.1 and by the main theorem of [SW], U is abelian. □

The proof of Theorem 1.1 is obtained as follows. First note that if U is finite, then U has odd order, so by the Feit-Thompson theorem U is solvable. Hence we may assume that U is infinite.

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We let A be a maximal abelian subgroup of U (with respect to inclusion) inverted by ν, (i.e. each element of A is inverted by ν). In Lemma 3.1(2) we show that we can take A to be infinite. We then show that for elements $u_1, \dots, u_n \in U$, the involutions $u_1\nu u_1^{-1}, \dots, u_n\nu u_n^{-1}$ in the semi-direct product $U \rtimes \langle\nu\rangle$ invert a subgroup $D \leq A$ with $|A : D| < \infty$ (Proposition 3.5). The next step is to show that $C_U(D)/D$ is finite and solvable (Lemma 3.3). Since $K := \langle \nu u_1\nu u_1^{-1}, \dots, \nu u_n\nu u_n^{-1}\rangle \leq C_U(D)$, the subgroup K is solvable and $K/Z(K)$ is finite.

Next let $S := \{x \in U \mid x^\nu = x^{-1}\}$. It is easy to see that an element $y \in U$ is in S iff $y = \nu u\nu u^{-1}$, for some $u \in U$, so by the above each finitely generated subgroup H of $R := \langle S\rangle$ is solvable and satisfies: $H/Z(H)$ is finite. Hence R' is periodic (Proposition 3.6). Using the above mentioned result of Shunkov, we see that R' is solvable, so R is solvable.

As is well known (see [K]) $U = RC_U(\nu)$ and $R \trianglelefteq U$. Since $C_U(\nu)$ is finite and uniquely 2-divisible it has odd order. By the Feit-Thomson theorem, $C_U(\nu)$ is solvable and this at last shows that U is solvable.

We remark that it is possible that with the aid of the Theorem on page 286 of [HM], one can get even more delicate information on U, but we do not need that, so we do not pursue this avenue further.

2. Notation and preliminary results

Notation 2.1. (1) Throughout this note U is an infinite uniquely 2-divisible group and $\nu \in \mathrm{Aut}(U)$ is an involutory automorphism which is almost regular.

(2) We denote by G the semi-direct product of U by ν and we indentify U and ν with their images in G. We let $\mathrm{Inv}(G)$ denote the set of involutions of G.

(3) We let $S := \{x \in U \mid x^\nu = x^{-1}\}$.

(4) The letter A always denotes a fixed infinite maximal (with respect to inclusion) abelian subgroup of U which is inverted by ν (i.e. all of whose elements are inverted by ν). The existence of A is guaranteed by Lemma 3.1(2) and by Zorn's lemma.

(5) For each $u \in U$ we denote by A_u the subgroup of A inverted by $u\nu u^{-1}$.

Remarks 2.2. (1) Notice that A is uniquely 2-divisible.

(2) Note also that for any $u \in U$, the subgroup A_u is uniquely 2-divisible.

(3) It is easy to check that $S = \{\nu\nu^x \mid x \in U\}$.

Lemma 2.3 ([N], Lemma 4.1, p. 239)**.** *Let the group H be the union of finitely many, let us say n, cosets of subgroups $C_1, C_2, \dots, C_n$:*

$$H = \bigcup_{i=1}^{n} C_i g_i,$$

Then the index of (at least) one of these subgroups in H does not exceed n.

Corollary 2.4. *Let the group H be the union of finitely many, let us say n, subsets $S_1, S_2, \dots, S_n$:*

$$H = \bigcup_{i=1}^{n} S_i.$$

For each i set $C_i := \langle ab^{-1} \mid a, b \in S_i \rangle$. Then the index of (at least) one of the subgroups $C_1, \dots, C_n$ in H does not exceed n.

Proof. For each $i = 1, \dots, n$, pick an arbitrary $g_i \in S_i$. Notice that $S_i \subseteq C_i g_i$ for all i, so $H = \bigcup_{i=1}^{n} C_i g_i$ and the Corollary follows from Lemma 2.3. □

Lemma 2.5. (1) *All involutions in G are conjugate;*
(2) $S = \{\nu\tau \mid \tau \in \mathrm{Inv}(G)\}$.

Proof. Let $\tau \in \mathrm{Inv}(G)$. Then $\tau = x\nu$, for some $x \in U$. Since τ is an involution $x \in S$. Let $y \in U$ be the unique element with $y^2 = x$, then $y \in S$ and $\tau = x\nu = y^2\nu = y\nu y^{-1}$. This shows (1). Part (2) is Remark 2.2(3). □

Lemma 2.6. *Let D be an abelian uniquely 2-divisible subgroup of U. Then*

(1) *$C_U(D)/D$ is a uniquely 2-divisible group.*
(2) *If D is inverted by ν, then νD is an almost regular involutory automorphism of $C_U(D)/D$.*
(3) *Assume that D is inverted by ν and let E/D be a subgroup of $C_U(D)/D$ which is inverted by νD. Then E is inverted by ν, so, in particular, E is abelian.*

Proof. (1): Set $C := C_U(D)$. Assume that $a, b \in C$ and $a^2 D = b^2 D$. Let $x, y \in D$ with $a^2 x = b^2 y$ and let $u, v \in D$ with $u^2 = x$ and $v^2 = y$. Then $a^2 u^2 = b^2 v^2$ and since a, b commute with u, v we see that $(au)^2 = (bv)^2$, hence $au = bv$ so $aD = bD$.

Furthermore let $aD \in C/D$. Let $b \in U$ with $b^2 = a$. Then $b \in C$ and bD is the square root of aD in C/D.

(2): Clearly νD is an involutory automorphism of C/D (acting via conjugation). Assume that $aD \in C/D$ centralizes νD. Then $\nu^a = \nu d$, for some $d \in D$. Let $x \in D$ with $x^2 = d$. Then ν inverts x and we see that $\nu^a = \nu^x$ and $ax^{-1} \in C_U(\nu)$. It follows that $C_{C/D}(\nu D) = C_C(\nu)D/D$, and since ν is almost regular, so is νD.

(3): Let $xD \in C/D$ be an element inverted by νD. Then $x^\nu = x^{-1}d$, for some $d \in D$, and conjugating by ν we see that $x = x^{-\nu}d^{-1}$ which implies that $x^\nu = x^{-1}d^{-1}$. Thus $d = d^{-1}$ so $d = 1$.

Now let $e \in E$. Then, by hypothesis, eD is inverted by νD, so $e^\nu = e^{-1}$. □

3. The proof of Theorem 1.1

Lemma 3.1. *Let D be an abelian subgroup of U (we allow $D = 1$) such that D is inverted by ν and such that $C_U(D)$ is infinite. Assume that*

$$(S \cap C_U(D)) \smallsetminus D \neq \emptyset.$$

Then

(1) *there exists an element* $w \in C_U(D) \smallsetminus D$ *which is inverted by* ν *and such that* $C_U(\langle D, w\rangle)$ *is infinite;*
(2) *There exists an infinite abelian subgroup of* U *which is inverted by* ν.

Proof. (1): Set $V := C_U(D)$. Then V is an infinite uniquely 2-divisible group, and ν acts on V, so without loss we may assume that $U = V$ and that $D \leq Z(U)$.

Pick $b \in S \smallsetminus D$ (note that b exists by hypothesis), and write $b = \nu\tau$ with $\tau \in \mathrm{Inv}(G)$. Let

$$u \in U \text{ with } u^{-2} = \nu\tau,$$

and note that since u is inverted by both ν and τ, we have

$$\nu = \tau^u.$$

We now find an element $h \in C_U(\tau)$ such that hu is inverted by infinitely many involutions of G. Note that $hu \notin D$; indeed, if $h = 1$, then $hu = u$ and since $b \notin D$ also $u \notin D$. Otherwise if $hu \in D$ and $h \neq 1$, then

$$u^{-1}h^{-1} = (hu)^\tau = h^\tau u^\tau = hu^{-1},$$

and it follows that u inverts h which is not possible in a uniquely 2-divisible group.

Since all involutions in G are conjugate, conjugating hu by an appropriate element we may assume that ν inverts hu and since hu is inverted by infinitely many involutions we see that $C_U(hu)$ is infinite and taking $w = hu$ we are done.

It remains to show the existence of h. For each $a \in S$, let

$$s_a := \nu\tau^a \text{ and } \ell_a^{-2} = s_a.$$

It is easy to check that since ℓ_a is inverted by ν and τ^a, we have $\tau^{a\ell_a} = \nu$. Hence

$$\tau^{a\ell_a} = \tau^u, \text{ and hence } h_a := a\ell_a u^{-1} \in C_U(\tau).$$

It follows that $\ell_a = a^{-1}h_a u$. Since both ℓ_a and a are inverted by ν we get after conjugating by ν that $\ell_a^{-1} = a(h_a u)^\nu = (h_a u)^{-1}a$. Notice now that $a\nu \in \mathrm{Inv}(G)$ and it follows that

$$(h_a u)^{a\nu} = (h_a u)^{-1}.$$

By hypothesis the set $\{h_a \mid a \in S\}$ is finite since it is contained in $C_U(\tau)$. Further, the set S is infinite. This implies the existence of $h \in C_U(\tau)$ such that the number of involutions $a\nu$ that invert hu is infinite. This proves (1).

(2): If D is finite and $C_U(D)$ is infinite, then $(S \cap C_U(D)) \smallsetminus D \neq \emptyset$. Hence part (2) follows from (1) by starting with $D = 1$ and iterating the process as long as the subgroup $\langle D, w\rangle$ is finite. □

Lemma 3.2. *Let B be a finitely generated abelian subgroup of U which is inverted by ν. Then A contains a subgroup A_1 of finite index such that $\langle A_1, B\rangle$ is abelian.*

Proof. Let $\mathcal{B}$ be a finite set of generators for B and set $A_1 := \bigcap_{b\in\mathcal{B}} A_b$. By Proposition 3.5 and since $\mathcal{B}$ is finite $|A : A_1| < \infty$. Further, for each $b \in \mathcal{B}$, ν and $b\nu b^{-1}$ invert A_1, so $b^2 = b\nu b^{-1}\nu \in C_U(A_1)$ (recall that ν inverts b). Since U is uniquely 2-divisible, $b \in C_U(A_1)$. Hence $\mathcal{B} \le C_U(A_1)$ and the lemma holds. □

Lemma 3.3. *Let D be a uniquely 2-divisible subgroup of A of finite index. Then $C_U(D)/D$ is finite and solvable.*

Proof. Set $C := C_U(D)$ and $\overline{C} := C/D$. Assume that $\overline{C}$ is infinite. By Lemma 2.6(1), $\overline{C}$ is uniquely 2-divisible, and by hypothesis $\overline{A} := A/D$ is a finite subgroup of $\overline{C}$.

Let $\overline{\mathcal{A}}$ be an infinite maximal abelian subgroup of $\overline{C}$ inverted by νD. The existence of $\overline{\mathcal{A}}$ is guaranteed by Lemma 2.6(2) and by Lemma 3.1(2) (with $\overline{C}$ in place of U). By Lemma 3.2 (with $\overline{C}$ in place of U and $\overline{A}$ in place of B), there exists an finite index $\overline{\mathcal{A}_1} \le \overline{\mathcal{A}}$ such that $\overline{\mathcal{A}_2} := \langle\overline{\mathcal{A}_1}, \overline{A}\rangle$ is abelian. Note that $\overline{\mathcal{A}_2}$ is inverted by νD, so by Lemma 2.6(3), the inverse image $\mathcal{A}_2$ of $\overline{\mathcal{A}_2}$ in $C_U(D)$ is an abelian subgroup inverted by ν. Clearly $\mathcal{A}_2$ properly contains A. This contradicts the maximality of A and shows that $\overline{C}$ is finite.

Let $\mathcal{D} \le C$ be a maximal central subgroup of C which is inverted by ν. Of course $\mathcal{D} \ge D$. Further, it is clear that $\mathcal{D}$ is a uniquely 2-divisible group. Suppose $t\mathcal{D}$ is an involution in $C/\mathcal{D}$. Then $t^2 \in \mathcal{D}$, so also $t \in \mathcal{D}$ and we see that $C/\mathcal{D}$ has odd order. By the Feit-Thompson theorem, $C/\mathcal{D}$ is solvable, and the proof of the lemma is complete. □

Lemma 3.4. *Let $x \in U$ and let $s \in U$ be the unique element such that $s^{-2} = \nu x^{-1}\nu x$. Then $xs \in C_U(\nu)$.*

Proof. Notice that s is inverted by ν and ν^x. Hence

$$1 = s^2\nu\nu^x = \nu s^{-2}\nu^x = \nu s^{-1}\nu^x s,$$

so the lemma holds. □

Proposition 3.5. *Let A be as in Notation 2.1(4) and let $u \in U$. Let A_u be as in Notation 2.1(5). Then $|A : A_u| < \infty$.*

Proof. Fix $a \in A$ and consider the element

$$\nu\nu^{au}, \quad a \in A.$$

This element is in U. Let $s \in U$ with $s^{-2} = \nu\nu^{au}$. By Lemma 3.4 we get that

$$v_a := aus \in C_U(\nu). \tag{3.1}$$

Now set

$$\mathcal{M}_a := \{b \in A \mid v_b = v_a\}.$$

Notice that since $|C_U(\nu)| < \infty$,

$$\text{the set } \{\mathcal{M}_c \mid c \in A\} \text{ is finite and } A = \textstyle\bigcup_{c\in A}\mathcal{M}_c. \tag{3.2}$$

By equation (3.1) we get $s^{-1} = v_a^{-1}au$ and conjugating by ν noticing that ν inverts a and s and centralizes v_a we see that $s^{-1} = u^{-\nu}av_a$. So we get the equality

$$v_a^{-1}au = u^{-\nu}av_a,$$

from which it follows that

$$u^{-1}\nu bv_a u^{-1} = \nu v_a^{-1}b, \quad \forall b \in \mathcal{M}_a. \tag{3.3}$$

Let $c \in \mathcal{M}_a$, then as in equation (3.3) we get that $u^{-1}\nu cv_a u^{-1} = \nu v_a^{-1}c$ and this together with equation (3.3) yields

$$uv_a^{-1}c^{-1}bv_a u^{-1} = c^{-1}b, \quad \forall b, c \in \mathcal{M}_a.$$

Since ν inverts $c^{-1}b \in A$, it follows that $uv_a^{-1}\nu v_a u^{-1} = u\nu u^{-1}$ inverts $c^{-1}b$. We thus can conclude that

$$u\nu u^{-1} \text{ inverts } \langle bc^{-1} \mid b, c \in \mathcal{M}_a\rangle, \quad \forall a \in A. \tag{3.4}$$

By equation (3.2) and by Corollary 2.4 one of the groups $\langle bc^{-1} \mid b, c \in \mathcal{M}_a\rangle$ has finite index in A, so $|A : A_u| < \infty$ as asserted. □

Proposition 3.6. *Let $R := \langle S\rangle$, then*

(1) *R' is a periodic group;*
(2) *R is solvable.*

Proof. (1): We first show that for elements $u_1, \dots, u_n \in U$ the subgroup $K := \langle \nu u_1\nu u_1^{-1}, \dots, \nu u_n\nu u_n^{-1}\rangle$ is solvable, and $K/Z(K)$ is finite. By Remark 2.2(3), this will show that

(*) if H is a f.g. subgroup of R, then H is solvable, and $H/Z(H)$ is finite.

Let $D := \bigcap_{i=1}^n A_{u_i}$. By the definition of A_{u_i} and by Proposition 3.5, $|A : D| < \infty$ and D is inverted by $\nu, u_1\nu u_1^{-1}, \dots, u_n\nu u_n^{-1}$. Also, by Remark 2.2(2), D is uniquely 2-divisible. By Lemma 3.3, $C_U(D)/D$ is finite and solvable, so since $K \le C_U(D)$, we see that $K/Z(K)$ is finite and solvable. Hence (*) holds.

Next let $g \in R'$. Then there exists a finitely generated subgroup H of R such that $g \in H'$. By (*) and by [A, (33.9), p. 168], H' is finite, so the order of g is finite. This completes the proof of part (1).

(2): By (1), R' is a periodic group and since R is ν-invariant, ν is an almost regular automorphism of R'. By the main result of Shunkov in [Sh], R' is virtually solvable. But by (*), R' is also locally solvable, so this shows that R' is solvable and hence so is R. □

Proof of Theorem 1.1. By Proposition 3.6, $\langle S\rangle$ is solvable. By [K, (3.4), p. 281] (see also [S, Lemma 2.1(1) and Lemma 2.2(1)]), $U = \langle S\rangle C_U(\nu)$ and $\langle S\rangle \trianglelefteq U$. Since $C_U(\nu)$ is a finite uniquely 2-divisible group, so it has odd order. By the Feit-Thompson theorem it is solvable. Hence U is solvable. □

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