FACTORIZATION AND DILATION PROBLEMS FOR COMPLETELY POSITIVE MAPS ON VON NEUMANN ALGEBRAS

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ABSTRACT. We study factorization and dilation properties of Markov maps between von Neumann algebras equipped with normal faithful states, i.e., completely positive unital maps which preserve the given states and also intertwine their automorphism groups. The starting point for our investigation has been the question of existence of non-factorizable Markov maps, as formulated by C. Anantharaman-Delaroche. We provide simple examples of non-factorizable Markov maps on $M_n(\mathbb{C})$ for all $n \geq 3$, as well as an example of a one-parameter semigroup $(T(t))_{t \geq 0}$ of Markov maps on $M_4(\mathbb{C})$ such that $T(t)$ fails to be factorizable for all small values of $t > 0$. As applications, we solve in the negative an open problem in quantum information theory concerning an asymptotic version of the quantum Birkhoff conjecture, as well as we sharpen the existing lower bound estimate for the best constant in the noncommutative little Grothendieck inequality.

1. INTRODUCTION

Motivated by the study of ergodic actions of free groups on noncommutative spaces, C. Anantharaman-Delaroche investigated in [2] the noncommutative analogue of G.-C. Rota's "Alternierende Verfahren" theorem from classical probability, asserting that if T is a measure-preserving Markov operator on the probability space (Ω, μ), then for every $p \geq 1$ and $f \in L^p(\Omega, \mu)$, the sequence $T^n(T^*)^n(f)$ converges almost everywhere, as $n \to \infty$. In the noncommutative setting, the probability space is replaced by a von Neumann algebra M, equipped with a normal faithful state ϕ, and T is now a unital, completely positive map on M such that $\phi \circ T = \phi$. However, in this setting the existence of the adjoint map T^* is not automatic. It turns out (see more precise references below) that it is equivalent to the fact that T commutes with the modular automorphism group of ϕ. This motivated the following definition considered in [2] (cf. Definition 2.6), where we have chosen to follow the slightly modified notation from [31]:

Definition 1.1. *Let (M, ϕ) and (N, ψ) be von Neumann algebras equipped with normal faithful states ϕ and ψ, respectively. A linear map $T \colon M \to N$ is called a (ϕ, ψ)-Markov map if*

(1) *T is completely positive*
(2) *T is unital*
(3) *$\psi \circ T = \phi$*
(4) *$T \circ \sigma_t^\phi = \sigma_t^\psi \circ T$, for all $t \in \mathbb{R}$, where $(\sigma_t^\phi)_{t \in \mathbb{R}}$ and $(\sigma_t^\psi)_{t \in \mathbb{R}}$ denote the automorphism goups of the states ϕ and ψ, respectively.*

In particular, when $(M, \phi) = (N, \psi)$, we say that T is a ϕ-Markov map.

Note that a linear map $T \colon M \to N$ satisfying conditions $(1) - (3)$ above is automatically normal. If, moreover, condition (4) is satisfied, then it was proved in [1] (see also Lemma 2.5 in [2]) that there exists

a unique completely positive, unital map $T^*\colon N \to M$ such that

$$(1.1) \qquad\qquad \phi(T^*(y)x) = \psi(yT(x))\,, \quad x \in M\,, y \in N\,.$$

It is easy to show that T^* is a $(\psi\,,\phi)$-Markov map.

Remark 1.2. A special case of interest is the one of a (ϕ,ψ)-Markov map $J\colon M \to N$ which is a $*$-monomorphism. In this case $J(M)$ is a σ^ψ-invariant sub-von Neumann algebra of N. Hence, by [34], there is a unique ψ-preserving normal faithful conditional expectation $\mathbb{E}_{J(M)}$ of N onto $J(M)$, and thus the adjoint J^* is given by $J^* = J^{-1} \circ \mathbb{E}_{J(M)}\,.$

C. Anantharaman-Delaroche proved in [2] that the noncommutative analogue of Rota's theorem holds for Markov maps which are factorizable in the following sense (cf. Definition 6.2 in [2]):

Definition 1.3. *A (ϕ,ψ)-Markov map $T\colon M \to N$ is called factorizable if there exists a von Neumann algebra P equipped with a faithful normal state χ, and $*$-monomorphisms $J_0\colon M \to P$ and $J_1\colon N \to P$ such that J_0 is (ϕ,χ)-Markov and J_1 is (ψ,χ)-Markov, satisfying, moreover, $T = J_0^* \circ J_1$.*

Remark 1.4. (a) Note that if both ϕ and ψ are *tracial* states on M and N, respectively, and $T\colon M \to N$ is factorizable, then the factorization can be chosen through a von Neumann algebra with a faithful normal *tracial* state, as well. This can be achieved by replacing (P,χ) whose existence is ensured by the definition of factorizability by $(P_\chi, \chi_{|P_\chi})$, where P_χ denotes the centralizer of the state χ, since $J_0(M) \subseteq P_\chi$ and $J_1(N) \subseteq P_\chi$.

(b) The class of factorizable (ϕ,ψ)-Markov maps is known to be closed under composition, the adjoint operation, taking convex combinations and w^*-limits (see Proposition 2 in [31]).

C. Anantharaman-Delaroche raised in [2] the question whether all Markov maps are factorizable. This was the starting point of investigation for our paper. The class of maps which are known to be factorizable includes all Markov maps between abelian von Neumann algebras (as it was explained in [2], Remark 6.3 (a)), the trace-preserving Markov maps on $M_2(\mathbb{C})$ (due to a result of B. Kümmerer from [21]), as well as Schur multipliers associated to positive semi-definite real matrices having diagonal entries all equal to 1 (as shown by E. Ricard in [31]). It is therefore natural to further study the problem of factorizability of τ_n-Markov maps on $M_n(\mathbb{C})$, for $n \geq 3$, where τ_n denotes the unique normalized trace on the $n \times n$ complex matrices.

In Section 2 we give a general characterization of factorizable τ_n-Markov maps on $M_n(\mathbb{C})$, as well as a characterization of those τ_n-Markov maps which lie in the convex hull of $*$-automorphisms of $M_n(\mathbb{C})$. We also discuss the case of Schur multipliers.

As an application, we construct in Section 3 several examples of non-factorizable τ_n-Markov maps on $M_n(\mathbb{C})$, $n \geq 3$ (cf. Examples 3.1 and 3.2), an example of a factorizable Schur multiplier on $(M_6(\mathbb{C}),\tau_6)$ which does not lie in the convex hull of $*$-automorphisms of $M_6(\mathbb{C})$ (cf. Example 3.3), as well as an example of a one-parameter semigroup $(T(t))_{t\geq 0}$ of τ_4-Markov maps on $M_4(\mathbb{C})$ such that $T(t)$ fails to be factorizable for all small values of $t > 0$ (see Theorem 3.4). This latter example is to be contrasted with a result of B. Kümmerer and H. Maasen from [24], asserting that if $(T(t))_{t\geq 0}$ is a one-parameter semigroup of τ_n-Markov maps on $M_n(\mathbb{C})$, $n \geq 1$, such that each $T(t)$ is self-adjoint, then $T(t)$ is factorizable, for all $t \geq 0$. We have been informed of recent work of M. Junge, E. Ricard and D. Shlyakhtenko, where they have generalized Kümmerer and Maasen's result to the case of a strongly continuous one-parameter

semigroup of self-adjoint Markov maps on an arbitrary finite von Neumann algebra. This result has been independently obtained by Y. Dabrowski (see the preprint [9]).

In Section 4 we discuss the connection between the notion of factorizability and Kümmerer's notions of dilation, respectively, of Markov dilation, that he introduced in [22]. The starting point for our analysis was a private communication by C. Koestler [20], who informed us in the Spring of 2008 that for a ϕ-Markov map on a von Neumann algebra M, factorizability is equivalent to the existence of a dilation (in the sense of [22]), and that Kümmerer in his unpublished Habilitationsschrift [23] had constructed examples of τ_n-Markov maps on $M_n(\mathbb{C})$, $n \geq 3$, having no dilations, and hence being non-factorizable. The equivalence between factorizability and the existence of a dilation is based on an inductive limit argument also from Kümmerer's unpublished work [23]. In Theorem 4.4 and its proof, we provide the details of the argument communicated to us by C. Koestler. Moreover, we show that the existence of a dilation is equivalent to the–seemingly stronger–condition of existence of a Markov dilation in the sense of Kümmerer.

Section 5 is devoted to the study of the so-called *Rota dilation property* of a Markov map, introduced by M. Junge, C. LeMerdy and Q. Xu in [16]. This notion has proven to be very fruitful for the development of semigroup theory in the noncommutative setting, and applications to noncommutative L_p-spaces and noncommutative harmonic analysis (see, e.g., [26], [15]). The Rota dilation property of a Markov map implies its factorizability, but it is more restrictive, as it forces the map to be self-adjoint. As a consequence of Theorem 6.6 in [2], the square of any factorizable self-adjoint Markov map has the Rota dilation property. Our main result in this section is that there exists a self-adjoint τ_n-Markov map T on $M_n(\mathbb{C})$, for some positive integer n, such that T^2 does not have the Rota dilation property (see Theorem 5.4), and therefore the analogue of Rota's classical dilation theorem for Markov operators does not hold, in general, in the noncommutative setting.

The existence of non-factorizable Markov maps turned out to have an interesting application to an open problem in quantum information theory, known as *the asymptotic quantum Birkhoff conjecture*. The conjecture, originating in joint work of A. Winter, J. A. Smolin and F. Verstraete (cf. [33]), asserts that if $T \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a τ_n-Markov map, $n \geq 1$, then T satisfies the following *asymptotic quantum Birkhoff property*:

$$\lim_{k \to \infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T \,, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C}))) \right) = 0\,.$$

We would like to thank V. Paulsen for bringing this problem to our attention. In Section 6 we solve the conjecture in the negative (see Theorem 6.1)), by showing that every non-factorizable τ_n-Markov map on $M_n(\mathbb{C})$, $n \geq 3$, fails the above asymptotic quantum Birkhoff property.

Finally, in Section 7, as an application of some of the techniques developed in the previous sections, we prove that the best constant in the noncommutative little Grothendieck inequality (cf. [30] and [14]) is strictly greater than 1, thus sharpening the existing lower bound estimate for it.

2. Factorizability of τ_n-Markov maps on $M_n(\mathbb{C})$

Let P be a von Neumann algebra and n a positive integer. Recall (see, e.g., [18] (Vol. II, Sect. 6.6) that a family $(f_{ij})_{1 \leq i,j \leq n}$ of elements of P is a set of *matrix units* in P if the following conditions are satisfied: $f_{ij}f_{kl} = \delta_{jk}f_{il}$, $1 \leq i,j,k,l \leq n$, $f_{ij}^* = f_{ji}$, $1 \leq i,j \leq n$ and $\sum_{i=1}^{n} f_{ii} = 1_P$. If this is the case, then $F \colon = \mathrm{Span}\{f_{ij} : 1 \leq i,j \leq n\}$ is a $*$-subalgebra of P isomorphic to $M_n(\mathbb{C})$ and $1_P \in F$.

The following result is well-known, but we include a proof for the convenience of the reader.

Lemma 2.1. *Let P be a von Neumann algebra, n a positive integer, and $(f_{ij})_{1 \le i,j \le n}$, $(g_{ij})_{1 \le i,j \le n}$ two sets of matrix units in P. Then there exists a unitary operator $u \in P$ such that*

$$u f_{ij} u^* = g_{ij}, \quad 1 \le i, j \le n.$$

Proof. By hypothesis, $(f_{ii})_{1 \le i \le n}$ and $(g_{ii})_{1 \le i \le n}$ are two sets of pairwise orthogonal projections in P with $\sum_{i=1}^{n} f_{ii} = 1_P = \sum_{i=1}^{n} g_{ii}$, satisfying, moreover, $f_{11} \sim f_{22} \sim \ldots \sim f_{nn}$ and $g_{11} \sim g_{22} \sim \ldots \sim g_{nn}$, respectively, where $\sim$ denotes the relation of equivalence of projections. By, e.g., [18] (Vol. II, Ex. 6.9.14), it follows that $f_{11} \sim g_{11}$, i.e., there exists a partial isometry $v \in P$ such that $v^*v = f_{11}$ and $vv^* = g_{11}$. Set now $u := \sum_{i=1}^{n} g_{i1} v f_{1i}$. It is elementary to check that u is a unitary in P. Moreover, for $1 \le k, l \le n$, $u f_{kl} u^* = \sum_{i,j=1}^{n} g_{i1} v f_{1i} f_{kl} f_{j1} v^* g_{1j} = g_{k1} v f_{11} v^* f_{1l} = g_{k1} g_{11} g_{1l} = g_{kl}$, which proves the result. $\qquad\square$

By a result of M.-D. Choi (see [7]), a linear map $T \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is completely positive if and only if T can be written in the form

$$(2.1) \qquad\qquad T x = \sum_{i=1}^{d} a_i^* x a_i, \quad x \in M_n(\mathbb{C}),$$

for some $a_1, \ldots, a_d \in M_n(\mathbb{C})$. The condition that T is unital is then equivalent to $\sum_{i=1}^{d} a_i^* a_i = 1_n$, while the condition that T is trace-preserving, i.e., $\tau_n \circ T = \tau_n$, is equivalent to $\sum_{i=1}^{d} a_i a_i^* = 1_n$. Here 1_n denotes the identity matrix in $M_n(\mathbb{C})$.

Theorem 2.2. *Let $T \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map, written in the form (2.1) where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ are chosen to be linearly independent and satisfy $\sum_{i=1}^{d} a_i^* a_i = \sum_{i=1}^{d} a_i a_i^* = 1_n$. Then the following conditions are equivalent:*

  (i) *T is factorizable.*
  (ii) *There exists a finite von Neumann algebra N equipped with a normal faithful tracial state τ_N and a unitary operator $u \in M_n(N) = M_n(\mathbb{C}) \otimes N$ such that*

$$(2.2) \qquad\qquad T x = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

  (iii) *There exists a finite von Neumann algebra N equipped with a normal faithful tracial state τ_N and $v_1, \ldots, v_d \in N$ such that $u := \sum_{i=1}^{d} a_i \otimes v_i$ is a unitary operator in $M_n(\mathbb{C}) \otimes N$ and*

$$\tau_N(v_i^* v_j) = \delta_{ij}, \quad 1 \le i, j \le d.$$

Proof. We first show that $(i) \Rightarrow (ii)$. Assume that T is factorizable. By Remark 1.4 (a), there exists a finite von Neumann algebra P with a normal faithful tracial state τ_P and two unital $*$-monomorphisms $\alpha, \beta \colon M_n(\mathbb{C}) \to P$ such that $T = \beta^* \circ \alpha$. Note that α and β are automatically (τ_n, τ_P)-Markov maps, since τ_n is the unique normalized trace on $M_n(\mathbb{C})$. Let $\{e_{ij}\}_{1 \le i,j \le n}$ be the standard matrix units in $M_n(\mathbb{C})$ and set $f_{ij} := \alpha(e_{ij})$, respectively, $g_{ij} := \beta(e_{ij})$, for all $1 \le i, j \le n$. Choose now a unitary operator $u \in P$ as in Lemma 2.1. Then $\beta(x) = u\alpha(x)u^*$, for all $x \in M_n(\mathbb{C})$. Equivalently, $\alpha(x) = u^*\beta(x)u$, for all $x \in M_n(\mathbb{C})$. Consider now the relative commutant

$$N := (\beta(M_n(\mathbb{C})))' \cap P = \{g_{ij} : 1 \le i, j \le n\}' \cap P,$$

and let τ_N be the restriction of τ_P to N. Since the map $\sum_{i,j=1}^{n} e_{ij} \otimes x_{ij} \mapsto \sum_{i,j=1}^{n} g_{ij} x_{ij}$, where $x_{ij} \in N$, $1 \le i, j \le n$, defines a $*$-isomorphism of $M_n(\mathbb{C}) \otimes N$ onto P (see, e.g., [18], Vol. II, Sect. 6.6.), we can

make the identifications $P = M_n(\mathbb{C}) \otimes N$, $\tau_P = \tau_n \otimes \tau_N$ and $\beta(x) = x \otimes 1_N$, $x \in M_n(\mathbb{C})$. This implies that $\alpha(x) = u^*(x \otimes 1_N)u$, $x \in M_n(\mathbb{C})$. Since $T = \beta^* \circ \alpha = \beta^{-1} \circ \mathbb{E}_{\beta(M_n(\mathbb{C}))} \circ \alpha$ (see Remark 1.2), then

$$Tx \otimes 1_N = \mathbb{E}_{M_n(\mathbb{C}) \otimes 1_N}(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}),$$

where $\mathbb{E}_{M_n(\mathbb{C}) \otimes 1_N}$ is the unique $\tau_P = \tau_n \otimes \tau_N$-preserving conditional expectation of $M_n(\mathbb{C}) \otimes N$ onto $M_n(\mathbb{C}) \otimes 1_N$. Then (2.2) follows and the implication is proved.

Conversely, assume that (ii) holds. Define maps α, $\beta \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$ by $\alpha(x) := u^*(x \otimes 1_N)u$, respectively, $\beta(x) := x \otimes 1_N$, for all $x \in M_n(\mathbb{C})$. Then α and β are $(\tau_n, \tau_n \otimes \tau_N)$-Markov $*$-monomorphisms of $M_n(\mathbb{C})$ into $M_n(\mathbb{C}) \otimes N$ satisfying $T = \beta^* \circ \alpha$, which proves that T is factorizable.

Next we prove the implication $(ii) \Rightarrow (iii)$. Assume that (ii) holds and choose a von Neumann algebra N with a normal faithful tracial state τ_N and a unitary operator $u \in M_n(\mathbb{C}) \otimes N$ satisfying (2.2). Since $a_1, \ldots, a_d \in M_n(\mathbb{C})$ are linearly independent, we can extend the set $\{a_1, \ldots, a_d\}$ to an algebraic basis $\{a_1, \ldots, a_{n^2}\}$ for $M_n(\mathbb{C})$. Then u has a representation of the form $u = \sum_{i=1}^{n^2} a_i \otimes v_i$, where $v_1, \ldots, v_{n^2} \in N$. By (2.1) and (2.2) we deduce that

$$(2.3) \qquad \sum_{i=1}^{d} a_i^* x a_i = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N) \left(\sum_{i,j=1}^{n^2} a_i^* x a_j \otimes v_i^* v_j \right) = \sum_{i,j=1}^{n^2} \tau_N(v_i^* v_j) a_i^* x a_j.$$

For $a \in M_n(\mathbb{C})$ we let L_a and R_a denote, respectively, the operators of left and right multiplication by a on $M_n(\mathbb{C})$, i.e., $L_a x = ax$, $R_a x = xa$, $x \in M_n(\mathbb{C})$. It is well-known that the map $\sum_{k=1}^{r} a_k \otimes b_k \mapsto \sum_{k=1}^{r} L_{a_k} \otimes R_{b_k}$ defines a vector space isomorphism of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ onto $\mathcal{B}(M_n(\mathbb{C}))$. By (2.3),

$$(2.4) \qquad \sum_{i=1}^{d} a_i^* \otimes a_i = \sum_{i,j=1}^{n^2} \tau_N(v_i^* v_j) a_i^* \otimes a_j.$$

Moreover, the set $\{a_i^* \otimes a_j : 1 \le i, j \le n\}$ is an algebraic basis for $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, so in particular, this set is linearly independent. Therefore (2.4) implies that

$$\tau_N(v_i^* v_j) = \begin{cases} \delta_{ij} & \text{if } 1 \le i, j \le d \\ 0 & \text{else}. \end{cases}$$

In particular, $\tau_N(v_i^* v_i) = 0$ for $i > d$, which by the faithfulness of τ_N implies that $v_i = 0$, for all $i > d$. Hence $u = \sum_{i=1}^{d} a_i \otimes v_i$, which proves (iii).

It remains to prove that (iii) implies (ii). Choose (N, τ_N) and operators $v_1, \ldots, v_d \in N$ as in (iii). Then $u := \sum_{i=1}^{d} a_i \otimes v_i$ is a unitary operator in $M_n(\mathbb{C}) \otimes N$ and $\tau(v_i^* v_j) = \delta_{ij}$, for $1 \le i, j \le d$. Thus

$$T(x) = \sum_{i=1}^{d} a_i^* x a_i = \sum_{i,j=1}^{n} \tau_N(v_i^* v_j) a_i^* x a_j = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u),$$

which gives (ii) and the proof is complete. $\qquad \square$

Corollary 2.3. *Let $T \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map of the form (2.1), where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ and $\sum_{i=1}^{d} a_i^* a_i = \sum_{i=1}^{d} a_i a_i^* = 1_n$. If $d \ge 2$ and the set $\{a_i^* a_j : 1 \le i, j \le d\}$ is linearly independent, then T is not factorizable.*

Proof. Assume that T is factorizable. Since the linear independence of the set $\{a_i^* a_j : 1 \le i, j \le d\}$ implies that the set $\{a_i : 1 \le i \le d\}$ is linearly independent, as well, Theorem 2.2 applies. Hence, by

the equivalence $(ii) \Leftrightarrow (iii)$ therein, there exists a finite von Neumann algebra N with a normal, faithful tracial state τ_N and operators $v_1, \ldots, v_d \in N$ such that $u := \sum_{i=1}^{d} a_i \otimes v_i \in M_n(\mathbb{C}) \otimes N = M_n(N)$ is a unitary operator and $\tau_N(a_i^* a_j) = \delta_{ij}$, for $1 \leq i, j \leq d$. Then

$$\sum_{i,j=1}^{d} a_i^* a_j \otimes (v_i^* v_j - \delta_{ij} 1_N) = u^* u - \left(\sum_{i=1}^{d} a_i^* a_i \right) \otimes 1_N = 1_{M_n(N)} - 1_n \otimes 1_N = 0_{M_n(N)} \, .$$

By the linear independence of the set $\{ a_i^* a_j : 1 \leq i, j \leq d \}$ it follows that for every functional $\phi \in N^*$, we have $\phi(v_i^* v_j - \delta_{ij} 1_N) = 0$, for all $1 \leq i, j \leq d$. Hence

$$v_i^* v_j = \delta_{ij} 1_N \, , \quad 1 \leq i, j \leq d \, .$$

Since $d \geq 2$, we infer in particular that $v_1^* v_2 = 0_N$ and $v_1^* v_1 = v_2^* v_2 = 1_N$. The latter condition ensures that v_1 and v_2 are unitary operators, since N is a finite von Neumann algebra. But this leads to a contradiction with the fact that $v_1^* v_2 = 0_N$. This proves that T is not factorizable. $\qquad \square$

Let $\mathrm{Aut}(M)$ denote the set of $*$-automorphisms of a von Neumann algebra M. If $M = M_n(\mathbb{C})$, for some $n \in \mathbb{N}$, then $\mathrm{Aut}(M_n(\mathbb{C})) = \{ \mathrm{ad}(u) : u \in \mathcal{U}(n) \}$, where $\mathrm{ad}(u)x = uxu^*$, $x \in M_n(\mathbb{C})$, and $\mathcal{U}(n)$ denotes the unitary group of $M_n(\mathbb{C})$. Since $\mathcal{B}(M_n(\mathbb{C}))$ is finite dimensional, the convex hull of $\mathrm{Aut}(M_n(\mathbb{C}))$, denoted by $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))$, is closed in the norm topology on $\mathcal{B}(M_n(\mathbb{C}))$. Further, let us denote by $\mathcal{FM}(M_n(\mathbb{C}))$ the set of factorizable τ_n-Markov maps on $M_n(\mathbb{C})$. By Remark 1.4 (b),

(2.5) $$\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})) \subseteq \mathcal{FM}(M_n(\mathbb{C})) \, .$$

Note that for $n = 2$ the two sets above are equal, as they are further equal to the set of τ_2-Markov maps on $M_2(\mathbb{C})$, as shown by Kümmerer in [21].

Proposition 2.4. *Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map written in the form (2.1), where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ are linearly independent and $\sum_{i=1}^{d} a_i^* a_i = \sum_{i=1}^{d} a_i a_i^* = 1_n$. Then the following conditions are equivalent:*

(1) *$T \in conv(Aut(M_n(\mathbb{C}))$.*
(2) *T satisfies condition (ii) of Theorem 2.2 with N abelian.*
(3) *T satisfies condition (iii) of Theorem 2.2 with N abelian.*

Proof. We first show that $(1) \Rightarrow (2)$. Assume that $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))$. Then there exist $u_1, \ldots, u_s \in \mathcal{U}(n)$ and positive real numbers $c_1, \ldots, c_s$ with sum equal to 1, for some positive integer s, so that

$$Tx = \sum_{i=1}^{s} c_i u_i^* x u_i \, , \quad x \in M_n(\mathbb{C}) \, .$$

Next, consider the abelian von Neumann algebra $N := l^\infty(\{1, \ldots, s\})$ with faithful tracial state τ_N given by $\tau_N(a) := \sum_{i=1}^{s} c_i a_i$, $a = (a_1, \ldots, a_s) \in N$. Set $u := (u_1, \ldots, u_s) \in l^\infty(\{1, \ldots, s\}, M_n(\mathbb{C})) = M_n(\mathbb{C}) \otimes N$. Then u is unitary and relation (2.2) is satisfied.

We now show that $(2) \Rightarrow (1)$. Assume that (2) holds, i.e., there exists an abelian von Neumann algebra N with a normal faithful tracial state τ_N and a unitary operator $u \in M_n(N)$ such that (2.2) is satisfied. Let $\widehat{N}$ denote the spectrum of N (i.e., the set of non-trivial multiplicative linear functionals on N). Then

6

$\widehat{N}$ is compact in the w*-topology, $N \simeq C(\widehat{N})$ and τ_N corresponds to a regular Borel probability measure μ on $\widehat{N}$. By identifying N with $C(\widehat{N})$, we have $u \in M_n(C(\widehat{N})) = C(\widehat{N}, M_n(\mathbb{C}))$ and

$$Tx = \int_{\widehat{N}} u(t)^* x u(t) d\mu(t), \quad x \in M_n(\mathbb{C}).$$

Thus T lies in the norm-closure of $\mathrm{conv}(\mathrm{ad}(u(t)^*) : t \in \widehat{N})$. Since $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))$ is a closed set in $\mathcal{B}(M_n(\mathbb{C}))$, condition (1) follows.

The implication $(2) \Rightarrow (3)$ follows immediately from the proof of the corresponding implication $(ii) \Rightarrow (iii)$ in Theorem 2.2. $\qquad\square$

Corollary 2.5. *Let $T \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map of the form (2.1), where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ are self-adjoint, $\sum_{i=1}^d a_i^2 = 1_n$ and $a_i a_j = a_j a_i$, for all $1 \leq i, j \leq d$. Then the following hold:*

(a) *T is factorizable.*

(b) *If $d \geq 3$ and the set $\{a_i a_j : 1 \leq i \leq j \leq d\}$ is linearly independent, then $T \notin \mathrm{conv}(Aut(M))$.*

Proof. The proof of (a) is inspired by the proof of Theorem 3 in [31]. Let N be the CAR-algebra over a d-dimensional Hilbert space H with orthonormal basis $e_1, \ldots, e_d$, and let $a(e_i)$, $1 \leq i \leq d$ be the corresponding annihilation operators. Then $N \simeq M_{2^d}(\mathbb{C})$ and the operators defined by

$$v_i \colon = a(e_i) + a(e_i)^*, \quad 1 \leq i \leq d$$

form a set of anti-commuting self-adjoint unitaries (see, e.g., [6]). Set now $u \colon = \sum_{i=1}^d a_i \otimes v_i \in M_n(\mathbb{C}) \otimes N$. Then u is unitary since

$$u^* u = \sum_{i,j=1}^d a_i a_j \otimes v_i v_j = \frac{1}{2} \sum_{i,j=1}^d (a_i a_j + a_j a_i) \otimes v_i v_j = \frac{1}{2} \sum_{i,j=1}^d a_i a_j \otimes (v_i v_j + v_j v_i)$$

$$= \sum_{i=1}^d a_i^2 \otimes 1_N = 1_{M_n(N)}.$$

Moreover, $\tau_N(v_i^* v_j) = \tau_N((v_i v_j + v_j v_i)/2) = \delta_{ij}$, $1 \leq i, j \leq d$. Hence, by the implication $(iii) \Rightarrow (i)$ of Theorem 2.2, we deduce that T is factorizable.

We now prove (b). Assume that $d \geq 3$ and that $\{a_i a_j : 1 \leq i \leq j \leq d\}$ is linearly independent. In particular, the set $\{a_i : 1 \leq i \leq d\}$ is linearly independent. If $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))$, then by Proposition 2.4, there exists an abelian von Neumann algebra N with normal faithful tracial state τ_N and operators $v_1, \ldots, v_d \in N$ such that the operator $u \colon = \sum_{i=1}^d a_i \otimes v_i \in M_n(\mathbb{C}) \otimes N = M_n(N)$ is unitary. Therefore, $1_{M_n(N)} = u^* u = \sum_{i,j=1}^d a_i a_j \otimes v_i^* v_j$. Using the fact that $a_i a_j = a_j a_i$, for all $1 \leq i, j \leq d$, and that $\sum_{i=1}^d a_i^2 = 1_n$, we infer that

$$\sum_{i=1}^d a_i^2 \otimes (v_i^* v_i - 1_N) + \sum_{1 \leq i < j \leq d} a_i a_j \otimes (v_i^* v_j + v_j^* v_i) = 0_{M_n(N)}.$$

By the linear independence of the set $\{a_i a_j : 1 \leq i \leq j \leq d\}$, it follows that $v_i^* v_i = 1_N$, for $1 \leq i \leq d$, and, respectively, that $v_i^* v_j + v_j^* v_i = 0_N$, for $1 \leq i < j \leq d$. Since $N \simeq C(\widehat{N})$, we deduce that

$$|v_i(t)| = 1, \quad t \in \widehat{N}, \quad 1 \leq i \leq d,$$

and, respectively,

$$\mathrm{Re}(\overline{v_i(t)} v_j(t)) = 0, \quad t \in \widehat{N}, \quad 1 \leq i \neq j \leq d,$$

Since $d \geq 3$, it follows that $\overline{v_1(t)}v_2(t)$, $\overline{v_2(t)}v_3(t)$ and $\overline{v_3(t)}v_1(t)$ are purely imaginary complex numbers, for all $t \in \widehat{N}$. Hence the product of these numbers is also purely imaginary. On the other hand, this product equals $|v_1(t)|^2|v_2(t)|^2|v_3(t)|^2 = 1$, which gives rise to a contradiction. We conclude that $T \notin \operatorname{conv}(\operatorname{Aut}(M))$. $\qquad \square$

We now discuss the case of Schur multipliers. The following fact is probably well-known, but we include a proof for completeness.

Proposition 2.6. *Let $B = (b_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$ and let $T_B \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be its corresponding Schur multiplier, i.e., $T_B(x) = (b_{ij}x_{ij})_{i,j=1}^n$, for all $x = (x_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$. The following conditions are equivalent:*

    (1) *T_B is positive*
    (2) *T_B is completely positive*
    (3) *There exist diagonal matrices $a_1, \ldots, a_d \in M_n(\mathbb{C})$ such that*

$$(2.6) \qquad\qquad T_B(x) = \sum_{i=1}^d a_i^* x a_i, \quad x \in M_n(\mathbb{C}).$$

    (4) *There exist linearly independent diagonal matrices $a_1, \ldots, a_d \in M_n(\mathbb{C})$ such that (2.6) holds.*
    (5) *B is a positive semi-definite matrix, i.e., $B = B^*$ and all eigenvalues of B are non-negative.*

Proof. The series of implications $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ is trivial, so we only have to prove that $(1) \Rightarrow (5) \Rightarrow (4)$. Assume that T_B is positive, i.e., $T_B((M_n(\mathbb{C}))_+) = (M_n(\mathbb{C}))_+$, where $(M_n(\mathbb{C}))_+$ denotes the set of positive semi-definite $n \times n$ complex matrices. Clearly, the matrix x_0 whose entries are all equal to 1 belongs to $(M_n(\mathbb{C}))_+$, and therefore $B = T_B(x_0) \in (M_n(\mathbb{C}))_+$. This shows that $(1) \Rightarrow (5)$.

To prove $(5) \Rightarrow (4)$, assume that $B = B^*$ with non-negative eigenvalues. Then $B = C^*DC$, where C is unitary and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal entries are the eigenvalues of B repeated according to multiplicity. In particular, $\lambda_i \geq 0$, for $1 \leq i \leq n$. Let $d \colon = \operatorname{rank}(B) = \operatorname{rank}(D)$. We may assume that $\lambda_1, \ldots, \lambda_d > 0$ and $\lambda_{d+1} = \ldots = \lambda_n = 0$. For any $1 \leq i \leq d$ consider now the diagonal $n \times n$ matrix given by $a_i \colon = \sqrt{\lambda_i}\operatorname{diag}(c_{i1}, \ldots c_{in})$, where $(c_{i1}, \ldots, c_{in})$ is the i-th row of C. Then $a_1, \ldots, a_d$ are linearly independent and one checks easily that (2.6) holds. $\qquad \square$

Let $B \in M_n(\mathbb{C})$. By Proposition 2.6, the Schur multiplier T_B associated to the matrix B is a τ_n-Markov map if and only if B is positive semi-definite and $b_{11} = b_{22} = \ldots = b_{nn} = 1$, because the latter condition is equivalent to having $T_B(1_n) = 1_n$ and $\tau_n \circ T_B = \tau_n$.

Remark 2.7. In [31] E. Ricard proved that if $B = (b_{ij})_{i,j=1}^n \in M_n(\mathbb{R})$ is a positive semi-definite matrix whose diagonal entries are all equal to 1, then the associated Schur multiplier T_B is always factorizable. This result can also be obtained from Corollary 2.5 (a). Indeed, under the above hypotheses, $B = C^t DC$, where C is an orthogonal matrix and $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ is a diagonal matrix with $\lambda_i \geq 0$. Let $d := \operatorname{rank}(D)$. Then, following the proof of the implication $(5) \Rightarrow (4)$ in Proposition 2.6, we deduce that $T_B(x) = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, where $a_1, \ldots, a_d$ are linearly independent diagonal matrices with $a_i = a_i^*$, $1 \leq i \leq d$ (since the entries of C are real numbers). Moreover, $\sum_{i=1}^d a_i^2 = \sum_{i=1}^d a_i^* a_i = T_B(1_n) = 1_n$, and $a_i a_j = a_j a_i$, for $1 \leq i, j \leq d$. It then follows from Corollary 2.5 (a) that T_B is factorizable.

We end this section with a general characterization of factorizable Schur multipliers, which turns out to be useful for applications.

Lemma 2.8. *Let $B = (b_{ij})_{i,j=1}^n$ be a positive semi-definite $n \times n$ complex matrix having all diagonal entries equal to 1. Then the associated Schur multiplier T_B is a factorizable τ_n-Markov map if and only if there exists a finite von Neumann algebra N with normal faithful tracial state τ_N and unitaries $u_1, \dots, u_n \in N$ such that*

$$(2.7) \qquad b_{ij} = \tau_N(u_i^* u_j), \quad 1 \leq i, j \leq n.$$

Proof. Assume that T_B is factorizable. Then by Theorem 2.2, there exists a finite von Neumann algebra N with normal faithful tracial state τ_N, and a unitary $u \in M_n(N) = M_n(\mathbb{C}) \otimes N$ such that

$$(2.8) \qquad T_B(x) = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

It follows that

$$(2.9) \qquad \tau_n(yT_B(x)) = (\tau_n \otimes \tau_N)((y \otimes 1_N)u^*(x \otimes 1_N)u), \quad x, y \in M_n(\mathbb{C}).$$

Let $(e_{jk})_{1 \leq j,k \leq n}$ be the standard matrix units in $M_n(\mathbb{C})$. Then $u = \sum_{i,k=1}^n e_{jk} \otimes u_{jk}$, where $u_{jk} \in N$, $1 \leq j, k \leq n$, and $u^* = \sum_{i,k=1}^n e_{kj} \otimes u_{jk}^*$. Consider now $j, k \in \{1, \dots, n\}$, $j \neq k$. By applying (2.9) to $x = e_{jj}$ and $y = e_{kk}$, we get $\tau_n(e_{kk}T_B(e_{jj})) = b_{jj}\tau_n(e_{kk}e_{jj}) = 0$. Therefore,

$$
\begin{aligned}
0 = (\tau_n \otimes \tau_N)((e_{kk} \otimes 1_N)u^*(e_{jj} \otimes 1_N)u) &= (\tau_n \otimes \tau_N)((e_{kk} \otimes 1_N)u^*(e_{jj} \otimes 1_N)u(e_{kk} \otimes 1_N)) \\
&= (\tau_n \otimes \tau_N)(e_{kk} \otimes u_{jk}^* u_{jk}) \\
&= (1/n)\tau_N(u_{jk}^* u_{jk}).
\end{aligned}
$$

By the faithfulness of τ_N, $u_{jk} = 0_N$ for $j \neq k$. Thus $u = \sum_{j=1}^n e_{jj} \otimes u_{jj}$. For $1 \leq j, k \leq n$ we then get

$$
\begin{aligned}
b_{jk} = b_{jk}n\tau_n(e_{kj}e_{jk}) = n\tau_n(e_{kj}T_B(e_{jk})) &= n(\tau_n \otimes \tau_N)((e_{kj} \otimes 1_N)u^*(e_{jk} \otimes 1_N)u) \\
&= n(\tau_n \otimes \tau_N)(e_{kk} \otimes u_{jj}^* u_{kk}) \\
&= \tau_N(u_{jj}^* u_{kk}).
\end{aligned}
$$

Hence (2.7) holds with $u_j = u_{jj}$, for $1 \leq j \leq n$.

Conversely, if (2.7) holds for a set of n unitaries $u_1, \dots, u_n$ in a finite von Neumann algebra N with normal, faithful, tracial state τ_N, then the operator $u := \sum_{j=1}^n e_{jj} \otimes u_j$ is a unitary in $M_n(\mathbb{C}) \otimes N$ and one checks easily that (2.8) holds. Hence, by Theorem 2.2, T_B is factorizable. $\qquad\square$

3. Examples

We begin by exhibiting an example of a non-factorizable τ_3-Markov maps on $M_3(\mathbb{C})$.

Example 3.1. Set

$$
a_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Then $\sum_{i=1}^3 a_i^* a_i = \sum_{i=1}^3 a_i a_i^* = 1_3$, and hence the operator T defined by $Tx := \sum_{i=1}^3 a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$ is a τ_3-Markov map. If T were factorizable, then by the implication $(i) \Rightarrow (iii)$ in Theorem

2.2, there would exist a finite von Neumann algebra N with a normal faithful tracial state τ_N and elements $v_1, v_2, v_3 \in N$ such that the operator

$$u: = \sum_{i=1}^{3} a_i \otimes v_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

is unitary, but as observed in [13] (pp. 282-283), this is impossible. Indeed, since $u^*u = 1_N$, we have

(3.1) $$v_1^*v_1 + v_2^*v_2 = v_2^*v_2 + v_3^*v_3 = v_3^*v_3 + v_1^*v_1 = 2\,1_N$$

and, respectively,

(3.2) $$v_1^*v_2 = v_2^*v_3 = v_3^*v_1 = 0_N\,.$$

Note that (3.1) implies that $v_1^*v_1 = v_2^*v_2 = v_3^*v_3 = 1_N$, and since N is finite, it follows that v_1, v_2 and v_3 are unitary operators, which contradicts (3.2). This shows that T is not factorizable.

Alternatively, one can check that $\{a_i^*a_j : 1 \leq i, j \leq 3\}$ is a linearly independent set in $M_3(\mathbb{C})$ and then use Corollary 2.3 to prove that T is not factorizable.

We now present some concrete examples of non-factorizable Schur multipliers.

Example 3.2. Following an example constructed in [8], for $0 \leq s \leq 1$ set

$$B(s): = \begin{pmatrix} 1 & \sqrt{s} & \sqrt{s} & \sqrt{s} \\ \sqrt{s} & s & s & s \\ \sqrt{s} & s & s & s \\ \sqrt{s} & s & s & s \end{pmatrix} + (1-s) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \omega & \overline{\omega} \\ 0 & \overline{\omega} & 1 & \omega \\ 0 & \omega & \overline{\omega} & 1 \end{pmatrix},$$

where $\omega: = e^{i2\pi/3} = -1/2 + i\sqrt{3}/2$ and $\overline{\omega}$ is the complex conjugate of ω. Note that $B(s)$ is positive semi-definite, since

$$B(s) = x_1(s)^*x_1(s) + x_2(s)^*x_2(s),$$

where $x_1(s) = (1, \sqrt{s}, \sqrt{s}, \sqrt{s})$ and $x_2(s) = \sqrt{1-s}(0, 1, \omega, \overline{\omega})$. Moreover, $b_{11} = b_{22} = b_{33} = b_{44} = 1$. Thus $T_{B(s)}$ is a τ_4-Markov map for all $0 \leq s \leq 1$.

We claim that for $0 < s < 1$, the map $T_{B(s)}$ is not factorizable. To prove this, we will use Corollary 2.3. Let $0 < s < 1$ and observe that

$$T_{B(s)}(x) = a_1(s)^*xa_1(s) + a_2(s)^*xa_2(s), \quad x \in M_4(\mathbb{C}),$$

where $a_1(s)$ and $a_2(s)$ are the diagonal 4×4 matrices

$$a_1(s): = \text{diag}(1, \sqrt{s}, \sqrt{s}, \sqrt{s}), \quad a_2(s): = \sqrt{1-s}\,\text{diag}(0, 1, \omega, \overline{\omega})\,.$$

It is elementary to check that the following four matrices: $a_1(s)^*a_1(s) = \text{diag}(1, s, s, s)$, $a_2(s)^*a_2(s) = (1-s)\,\text{diag}(0, 1, 1, 1)$, $a_1(s)^*a_2(s) = \sqrt{s}\,\text{diag}(0, 1, \omega, \overline{\omega})$ and $a_2(s)^*a_1(s) = \sqrt{s}\,\text{diag}(0, 1, \overline{\omega}, \omega)$ are linearly independent. Hence, by the above-mentioned corollary, $T_{B(s)}$ is not factorizable.

Note that for $s = 1/3$, $B(s)$ has a particularly simple form, namely,

$$B(1/3) = \begin{pmatrix} 1 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 1 & i/\sqrt{3} & -i/\sqrt{3} \\ 1/\sqrt{3} & -i/\sqrt{3} & 1 & i/\sqrt{3} \\ 1/\sqrt{3} & i/\sqrt{3} & -i/\sqrt{3} & 1 \end{pmatrix}$$

The above 4×4 matrix examples can easily be generalized to examples of non-factorizable Schur multipliers on $(M_n(\mathbb{C}), \tau_n)$, for all $n \geq 4$, by setting

$$B(s) := x_1(s)^* x_1(s) + x_2(s)^* x_2(s),$$

where $0 < s < 1$ and x_1, x_2 are the column vectors in $\mathbb{C}^n$ given by

$$x_1(s) := (1, \sqrt{s}, \sqrt{s}, \ldots, \sqrt{s}), \quad x_2(s) := \sqrt{1-s}(0, 1, \rho, \rho^2, \ldots, \rho^{n-2}),$$

where $\rho = e^{i\, 2\pi/(n-1)}$. Further, let $a_1(s)$ and $a_2(s)$ be the corresponding $n \times n$ diagonal matrices. Then the linear independence of the set $\{(a_j(s))^* a_k(s) : 1 \leq j, k \leq 2\}$ follows from the computation

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho & \rho^2 \\ 1 & \bar{\rho} & \bar{\rho}^2 \end{pmatrix} = \bar{\rho}^2(\rho+1)(\rho-1)^3 \neq 0,$$

where $\bar{\rho}$ is the conjugate of ρ. Then, an application of Corollary 2.3 shows that the Schur multiplier $T_{B(s)}$ is not factorizable.

Example 3.3. Let $\beta = 1/\sqrt{5}$ and set

$$B := \begin{pmatrix} 1 & \beta & \beta & \beta & \beta & \beta \\ \beta & 1 & \beta & -\beta & -\beta & -\beta \\ \beta & \beta & 1 & \beta & -\beta & -\beta \\ \beta & -\beta & \beta & 1 & \beta & -\beta \\ \beta & -\beta & -\beta & \beta & 1 & \beta \\ \beta & \beta & -\beta & -\beta & \beta & 1 \end{pmatrix}.$$

We claim that T_B is a factorizable τ_6-Markov map on $M_6(\mathbb{C})$, but $T_B \notin \operatorname{conv}(\operatorname{Aut}(M_6(\mathbb{C})))$. To prove this, observe first that since $\cos(2\pi/5) = \cos(8\pi/5) = (-1+\sqrt{5})/4$ and $\cos(4\pi/5) = \cos(6\pi/5) = (-1-\sqrt{5})/4$, then B can be written in the form

$$B = x_1^* x_1 + x_2^* x_2 + x_3^* x_3,$$

where $x_1 := (1, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5})$, $x_2 := \sqrt{2/5}(0, 1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5})$ and $x_3 := \overline{x_2} = \sqrt{2/5}(0, 1, e^{-i2\pi/5}, e^{-i4\pi/5}, e^{-i6\pi/5}, e^{-i8\pi/5})$. Hence B is positive semi-definite. By Remark 2.7, T_B is a factorizable τ_6-Markov map on $M_6(\mathbb{C})$. Moreover,

$$T_B(x) = \sum_{i=1}^{3} b_i^* x b_i, \quad x \in M_6(\mathbb{C}),$$

where $b_1 := \operatorname{diag}(1, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5})$, $b_2 := \sqrt{2/5}\operatorname{diag}(0, 1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5})$ and $b_3 := b_2{}^*$. Set now

$$a_1 := b_1, \quad a_2 := \frac{1}{\sqrt{2}}(b_2 + b_3), \quad a_3 := \frac{1}{i\sqrt{2}}(b_2 - b_3).$$

Then $T_B(x) = \sum_{i=1}^{3} a_i x a_i$, for all $x \in M_6(\mathbb{C})$. Note that a_1, a_2, a_3 are commuting self-adjoint (diagonal) matrices with $\sum_{i=1}^{3} a_i^2 = 1_6$. Thus, if we knew that the set $\{a_i a_j : 1 \leq i \leq j \leq 3\}$ is linearly independent, then, by Corollary 2.5 (b) we could conclude that $T \notin \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C})))$. Note that the linear independence of the above set is equivalent to the linear independence of the set

$$(3.3) \qquad\qquad \{b_i b_j : 1 \leq i \leq j \leq 3\}.$$

Set $\gamma := e^{i2\pi/5}$. Then the following relations hold:

(3.4) $\quad b_1^2 = \dfrac{1}{5}\operatorname{diag}(5,1,1,1,1,1)\,, b_1 b_2 = \dfrac{\sqrt{2}}{5}\operatorname{diag}(0,1,\gamma,\gamma^2,\gamma^3,\gamma^4)\,, b_1 b_3 = \dfrac{\sqrt{2}}{5}\operatorname{diag}(0,1,\gamma^4,\gamma^8,\gamma^{12},\gamma^{16})$

$\quad b_2^2 = \dfrac{2}{5}\operatorname{diag}\left(0,1,\gamma^2,\gamma^4,\gamma^6,\gamma^8\right)\,, b_2 b_3 = \dfrac{2}{5}\operatorname{diag}(0,1,1,1,1,1)\,, b_3^2 = \dfrac{2}{5}\operatorname{diag}\left(0,1,\gamma^3,\gamma^6,\gamma^9,\gamma^{12}\right)\,.$

Now let

$$H := \left(\exp\left(i\frac{2\pi kl}{5}\right)\right)_{0\le k,l\le 4} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \gamma^2 & \gamma^4 & \gamma^6 & \gamma^8 \\ 1 & \gamma^3 & \gamma^6 & \gamma^9 & \gamma^{12} \\ 1 & \gamma^4 & \gamma^8 & \gamma^{12} & \gamma^{16} \end{pmatrix}.$$

Then H is a complex 5×5 Hadamard matrix, i.e., $|h_{i,j}|^2 = 1$, for all $0\le i,j\le 4$ and $H^* H = HH^* = 5\,1_5$. It follows that the rows of H are linearly independent. This fact, combined with the relations (3.4), shows that the set in (3.3) is linearly independent. Hence the assertion is proved.

Theorem 3.4. *Let $L = (L_{jk})_{1\le j,k\le 4}$ be the 4×4 complex matrix given by*

(3.5) $$L := \begin{pmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1-\omega & 1-\overline{\omega} \\ 1/2 & 1-\overline{\omega} & 0 & 1-\omega \\ 1/2 & 1-\omega & 1-\overline{\omega} & 0 \end{pmatrix}$$

where $\omega := e^{i2\pi/3}$ and $\overline{\omega}$ is the complex conjugate of ω. Let $(C(t))_{t\ge 0}$ denote the one-parameter family of 4×4 complex matrices

(3.6) $$C(t) := (e^{-tL_{jk}})_{1\le j,k\le 4}\,, \quad t\ge 0\,.$$

Then the corresponding Schur multipliers

(3.7) $$T_t := T_{C(t)}\,, \quad t\ge 0$$

form a continuous one-parameter semigroup of τ_4-Markov maps on $M_4(\mathbb{C})$ starting at $T(0) = id_{M_4(\mathbb{C})}$. Moreover, there exists $t_0 > 0$ such that $T(t)$ is not factorizable for any $0 < t < t_0$.

Proof. Let $((B(s))_{1\le s\le 1}$ be the positive semi-definite 4×4 complex matrices considered in Example 3.2. In particular, the matrix $B(1)$ has all entries equal to 1. Note that the matrix L given by (3.5) is the first derivative of $B(s)$ at $s = 1$, i.e.,

$$L = \frac{dB(s)}{ds}\Big|_{s=1} = \lim_{s\nearrow 1}\frac{B(s) - B(1)}{s-1}\,.$$

Since $B(s)$ is positive semi-definite for all $0\le s\le 1$, we have that

$$\sum_{j,k=1}^{4}(B(s) - B(1))c_j\bar c_k \ge 0\,,$$

whenever $c_1,\dots,c_4 \in \mathbb{C}$ and $c_1 + \dots + c_4 = 0$. This implies that $\sum_{j,k=1}^{4} L_{jk}c_j\bar c_k \ge 0$, i.e., L is a conditionally negative definite matrix. By Schoenberg's theorem (see, e.g., [3]), the matrices $C(t)$, $t\ge 0$ given by (3.6) are all positive semi-definite. Moreover, since $L_{11} = L_{22} = L_{33} = L_{44} = 0$, we also have $C(t)_{11} = C(t)_{22} = C(t)_{33} = C(t)_{44} = 0$, for all $t\ge 0$. Hence the Schur multipliers $T(t) = T_{C(t)}$, $t\ge 0$ are

all τ_4-Markov maps. Clearly, the family $(T(t))_{t\geq 0}$ forms a continuous one-parameter semigroup of Schur multipliers starting at $T(0) = \mathrm{id}_{M_4(\mathbb{C})}$. Now set

$$F:\ = \{t > 0 : T_{C(t)} \text{ is factorizable}\}\,.$$

We will show that there exists $t_0 > 0$ such that $F \cap (0, t_0) = \emptyset$.

By Lemma 2.8, for each $t \in F$ we can find a finite von Neumann algebra $N(t)$ with normal faithful tracial state $\tau_{N(t)}$ and four unitary operators $u_1(t), \dots, u_4(t) \in N(t)$ such that

$$C(t)_{jk} = \tau_{N(t)}(u_j(t)^* u_k(t)), \quad 1 \leq j, k \leq 4\,.$$

Since $\omega + \overline{\omega} + 1 = 0$ and $\omega^2 = \overline{\omega}$, we can express $u_1(t), \dots, u_4(t)$ in the form $u_1(t) = x(t) + w(t)$, $u_2(t) = x(t) + y(t) + z(t)$, $u_3(t) = x(t) + \omega y(t) + \overline{\omega} z(t)$, $u_4(t) = x(t) + \overline{\omega} y(t) + \omega z(t)$, where $x(t):\ = (u_2(t) + u_3(t) + u_4(t))/3$, $y(t):\ = (u_2(t) + \overline{\omega} u_3(t) + \omega u_4(t))/3$, $z(t):\ = (u_2(t) + \omega u_3(t) + \overline{\omega} u_4(t))/3$, $w(t) = x(t) - u_1(t)$. Note that for all $t \in F$,

$$(3.8) \qquad \|x(t)\| \leq 1, \quad \|y(t)\| \leq 1, \quad \|z(t)\| \leq 1\,.$$

We prove next that

$$(3.9) \qquad \|y(t)\|_2 \leq 2\|z(t)\|_2 + \|w(t)\|_2\,.$$

For this, observe first that $x(t)^* y(t) + y(t)^* z(t) + z(t)^* x(t) = 0$, which implies that

$$(3.10) \qquad \begin{aligned} \|x(t)^* y(t)\|_2 \ &\leq\ \|y(t)^* z(t)\|_2 + \|z(t)^* x(t)\|_2 \\ &\leq\ \|y(t)\| \|z(t)\|_2 + \|z(t)\|_2 \|x(t)\| \\ &\leq\ 2\|z(t)\|_2\,, \end{aligned}$$

wherein we used (3.8). Now recall that $x(t) = u_1(t) + w(t)$, where $u_1(t)$ is a unitary. Thus, by (3.10),

$$\begin{aligned} \|y(t)\|_2 \ =\ \|u_1(t)^* y(t)\|_2 \ &=\ \|x(t)^* y(t) - w(t)^* y(t)\|_2 \\ &\leq\ \|x(t)^* y(t)\|_2 + \|w(t)\|_2 \|y(t)\|_2 \\ &\leq\ 2\|z(t)\|_2 + \|w(t)\|_2\,, \end{aligned}$$

which proves (3.9). Next, observe that the 2-norms of $y(t)$, $z(t)$ and $w(t)$ can be expressed in terms of the entries of the matrix L, because for $c := (c_1, c_2, c_3, c_4) \in \mathbb{C}^4$ and $t \in F$ we have

$$\left\| \sum_{j=1}^4 c_j u_j(t) \right\|_2^2 = \sum_{j,k=1}^4 \bar{c}_j c_k \tau_N(u_j(t)^* u_k(t)) = \sum_{j,k=1}^4 \bar{c}_j c_k e^{-t L_{jk}} = f(c, t)\,,$$

where the function $t \mapsto f(c, t)$ is actually defined for all $t \geq 0$ and satisfies

$$(3.11) \qquad f(c, t) = |c_1 + c_2 + c_3 + c_4|^2 - \left(\sum_{j,k=1}^4 \bar{c}_j c_k L_{jk} \right) t + O(t^2), \quad \text{as } t \searrow 0\,,$$

in Landau's O-notation. Consider now the three special cases where $c:\ = (c_1, c_2, c_3, c_4) \in \mathbb{C}^4$ is equal to $(0, 1/3, \overline{\omega}/3, \omega/3)$, $(0, 1/3, \omega/3, \overline{\omega}/3)$ and $(-1, 1/3, 1/3, 1/3)$, respectively. For $t \geq 0$, let us denote the function $f(c, t)$ in (3.11) by $g(t)$, $h(t)$ and $k(t)$, respectively, in each of the corresponding case. Then,

$$(3.12) \qquad g(t) = \|y(t)\|_2^2, \quad h(t) = \|z(t)\|_2^2, \quad k(t) = \|z(t)\|_2^2, \qquad t \in F\,.$$

Moreover, by (3.11),

$$(3.13) \qquad g(t) = t + O(t^2), \quad h(t) = O(t^2), \quad k(t) = O(t^2), \qquad \text{as } t \searrow 0.$$

Assume now that $\inf(F) = 0$. Then there exists a sequence $(t_n)_{n \geq 1}$ in F such that $t_n \to 0$ as $n \to \infty$. By (3.9) and (3.12) we have

$$g(t_n)^{1/2} \leq 2h(t_n)^{1/2} + k(t_n)^{1/2}, \quad n \geq 1.$$

However, by (3.13), $2h(t_n)^{1/2} + k(t_n)^{1/2} = O(t_n)$, while $g(t_n)^{1/2} = t_n^{1/2} + O(t_n^{3/2})$, both for large enough n. This gives rise to a contradiction. Hence $\inf(F) > 0$, i.e., there exists $t_0 > 0$ such that $(0, t_0) \cap F = \emptyset$. The proof is complete. $\qquad \square$

Remark 3.5. The above theorem is to be contrasted with a result of Kümmerer and Maassen (cf. [24]), showing that if $(T(t))_{t \geq 0}$ is a one-parameter semigroup of τ_n-Markov maps on $M_n(\mathbb{C})$ satisfying $T(t)^* = T(t)$, for all $t \geq 0$, then $T(t) \in \text{conv}(\text{Aut}(M_n(\mathbb{C})))$, for all $t \geq 0$. In particular, $T(t)$ is factorizable, for all $t \geq 0$.

In very recent work, Junge, Ricard and Shlyakhtenko [17] have generalized Kümmerer and Maassen's result to the case of a strongly continuous one-parameter semigroup $(T(t))_{t \geq 0}$ of self-adjoint Markov maps on an arbitrary von Neumann algebra with a faithful, normal tracial state by proving that also in this case $T(t)$ is factorizable, for all $t \geq 0$. This result has been independently obtained by Y. Dabrowski (see [9]).

Remark 3.6. In the recent preprint [10], K. Dykema and K. Juschenko have indirectly exhibited an example of a τ_4-Markov map on $M_4(\mathbb{C})$ which is not factorizable.

More precisely, for every $n \geq 1$ they considered the sets $\mathcal{F}_n$, defined as the closure of the union over $k \geq 1$ of sets of $n \times n$ complex matrices $(b_{ij})_{1 \leq i,j \leq n}$ such that $b_{ij} = \tau_k(u_i u_j^*)$, where $u_1, \ldots, u_n \in \mathcal{U}(k)$, respectively, $\mathcal{G}_n$, consisting of all $n \times n$ complex matrices $(b_{ij})_{1 \leq i,j \leq n}$ such that $b_{ij} = \tau_M(u_i u_j^*)$, where $u_1, \ldots, u_n$ are unitaries in some von Neumann algebra M equipped with normal faithful tracial state τ_M (where M varies). By a refinement of Kirchberg's deep results from [19], they concluded that Connes' embedding problem whether every II_1-factor with separable predual embeds in the ultrapower of the hyperfinite II_1 factor has an affirmative answer, if and only if $\mathcal{F}_n = \mathcal{G}_n$, for all $n \geq 1$. Further, they pointed out that $\mathcal{F}_n \subseteq \mathcal{G}_n \subseteq \Theta_n$, for all $n \geq 1$, where Θ_n is the set of $n \times n$ (complex) correlation matrices, i.e., positive semi-definite matrices whose entries on the main diagonal are all equal to 1. A natural question to consider is whether $\mathcal{F}_n = \Theta_n$, for all $n \geq 1$. One of the results of [10] is that the answer to this question is negative, as soon as $n \geq 4$. More precisely, Dykema and Juschenko showed that $\mathcal{G}_4$ has no extreme points of rank 2, while there are extreme points of rank 2 in Θ_4. Hence $\mathcal{G}_4 \neq \Theta_4$. In view of Lemma 2.8 above, any element of $\Theta_4 \setminus \mathcal{G}_4$ is an example of a non-factorizable τ_4-Markov map on $M_4(\mathbb{C})$.

4. KÜMMERER'S NOTIONS OF DILATION AND THEIR CONNECTION TO FACTORIZABILITY

The following definitions are due to Kümmerer (see [22], Definitions 2.1.1 and 2.2.4, respectively):

Definition 4.1. *Let M be a von Neumann algebra with a normal faithful state ϕ and let $T: M \to M$ be a ϕ-Markov map. A dilation of T is a quadruple (N, ψ, α, ι), where N is a von Neumann algebra*

with a normal faithful state ψ, $\alpha \in Aut(N, \psi)$, i.e., α is an automorphism of N leaving ψ invariant and $\iota \colon M \to N$ is a (ϕ, ψ)-Markov $*$-monomorphism, satisfying

$$(4.14) \qquad T^n = \iota^* \circ \alpha^n \circ \iota, \quad n \geq 1.$$

Furthermore, we say that that (N, ψ, α, ι) is a dilation of T of order 1 if (4.14) holds for $n = 1$ but not necessarily for $n \geq 2$.

Definition 4.2. A dilation (N, ψ, α, ι) of a ϕ-Markov map $T \colon M \to M$ is called a Markov dilation if

$$(4.15) \qquad \mathbb{P}_{\{0\}}(x) = \mathbb{P}_{(-\infty, 0]}(x), \quad x \in \big(\bigcup_{k \geq 0} \alpha^k \circ \iota(M)\big)'',$$

where for $I \subseteq \mathbb{Z}$, $\mathbb{P}_I$ denotes (cf. Lemma 2.1.3 of [22]) the unique ψ-preserving normal faithful conditional expectation of N onto its subalgebra $\big(\bigcup_{k \in I} \alpha^k \circ \iota(M)\big)''$.

Remark 4.3. The condition (4.15) is equivalent to

$$\mathbb{P}_{\{0\}}\mathbb{P}_{[0,\infty)} = \mathbb{P}_{(-\infty, 0]}\mathbb{P}_{[0,\infty)}.$$

Clearly, $\mathbb{P}_{[0,\infty)}\mathbb{P}_{\{0\}} = \mathbb{P}_{\{0\}}$, and since both $\mathbb{P}_{\{0\}}$ and $\mathbb{P}_{[0,\infty)}$ extend uniquely to self-adjoint projections on $L^2(N, \psi)$, it also follows that $\mathbb{P}_{\{0\}}\mathbb{P}_{[0,\infty)} = \mathbb{P}_{\{0\}}$. Hence (4.15) is further equivalent to

$$(4.16) \qquad \mathbb{P}_{\{0\}} = \mathbb{P}_{(-\infty, 0]}\mathbb{P}_{[0,\infty)}.$$

In the Spring of 2008, C. Koestler informed us in a private correspondence that he was aware of the fact that for a given Markov map, the existence of a dilation (in the sense of Definition 4.1) is actually equivalent to C. Anantharaman-Delaroche's condition of factorizability of the map. The proof relies on a construction of inductive limit of von Neumann algebras naturally associated to a Markov $*$-monomorphism, studied by Kümmerer in his unpublished Habilitationsschrift [23]. We are very grateful to C. Koestler for sharing all this information with us and for kindly providing us with a copy of [23].

We were further able to show that the existence of a dilation for a given Markov map is equivalent to the existence of a Markov dilation (in the sense of Definition 4.2) for it. For completeness, we collect together all these equivalent statements in the following theorem.

Theorem 4.4. Let M be a von Neumann algebra with normal faithful state ϕ and let $T \colon M \to M$ be a ϕ-Markov map. The following statements are equivalent:

(1) T is factorizable.
(2) T has a dilation.
(3) T has a Markov dilation.

For convenience, we include the details of the above-mentioned inductive limit construction for von Neumann algebras, that will be used in the proof of Theorem 4.4.

Lemma 4.5. Suppose that for each positive integer k, we are given a von Neumann algebra M_k with a normal faithful state ψ_k and a unital $*$-monomorphism $\beta_k \colon M_k \to M_{k+1}$ such that $\psi_{k+1} \circ \beta_k = \psi_k$, satisfying, moreover,

$$(4.17) \qquad \sigma_t^{\psi_{k+1}} \circ \beta_k = \beta_k \circ \sigma_t^{\psi_k}, \quad t \in \mathbb{R}.$$

Then, there exists a von Neumann algebra M with a normal faithful state ψ and unital $*$-monomorphisms $\mu_k \colon M_k \to M$ such that $\mu_{k+1} \circ \beta_k = \mu_k$, $\psi \circ \mu_k = \psi_k$ and $\sigma_t^\psi \circ \mu_k = \mu_k \circ \sigma_t^{\psi_k}$, $t \in \mathbb{R}$, for all $k \geq 1$, and such that $\bigcup_{k=1}^\infty \mu_k(M_k)$ is weakly dense in M.

Moreover, if we are given another von Neumann algebra N with a normal faithful state φ and (normal) $*$-monomorphisms $\lambda_k \colon M_k \to N$ such that $\lambda_{k+1} \circ \beta_k = \lambda_k$ and $\varphi \circ \lambda_k = \psi_k$ for all $k \geq 1$, then there exists a unique $*$-monomorphism $\lambda \colon M \to N$ such that $\lambda \circ \mu_k = \lambda_k$ for all $k \geq 1$ and $\varphi \circ \lambda = \psi$.

Proof. As a first step towards the existence of M, let M_∞ be the C^*-algebra inductive limit of the sequence $M_1 \to M_2 \to M_3 \to \cdots$. This is a C^*-algebra equipped with $*$-monomorphisms $\widetilde{\mu}_k \colon M_k \to M_\infty$ (which are unital when the connecting mappings β_k all are unital) satisfying $\widetilde{\mu}_{k+1} \circ \beta_k = \widetilde{\mu}_k$ for all $k \geq 1$, and which contains $\bigcup_{k=1}^\infty \widetilde{\mu}_k(M_k)$ as a norm-dense sub-algebra. The states $\widetilde{\psi}_k$ on $\widetilde{\mu}_k(M_k)$, defined by $\widetilde{\psi}_k \circ \widetilde{\mu}_k = \psi_k$, are coherent, i.e., the restriction of $\widetilde{\psi}_{k+1}$ to $\widetilde{\mu}_k(M_k)$ is equal to $\widetilde{\psi}_k$, and so they extend to a state $\widetilde{\psi}$ on M_∞. Let M be the weak closure of M_∞ in the GNS-representation of M_∞ with respect to the state $\widetilde{\psi}$, and let $\mu_k \colon M_k \to M$ be the composition of $\widetilde{\mu}_k$ with the inclusion mapping of M_∞ into M. The state $\widetilde{\psi}$ extends to a normal state ψ on M. It is clear that

$$\mu_{k+1} \circ \beta_k = \mu_k, \quad \psi \circ \mu_k = \psi_k, \qquad k \geq 1.$$

We prove next that ψ is faithful. For simplicity, we will now identify M_k with $\mu_k(M_k)$, $k \geq 1$. Then we have the inclusions $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ and $\bigcup_{k=1}^\infty M_k$ is weakly dense in M. Moreover, these inclusions extend to isometric embeddings of the corresponding GNS Hilbert spaces for (M_k, ψ_k), $k \geq 1$, i.e., $L^2(M_1, \psi_1) \subseteq L^2(M_2, \psi_2) \subseteq L^2(M_3, \psi_3) \subseteq \ldots$ and $\bigcup_{k=1}^\infty L^2(M_k, \psi_k)$ is dense in $L^2(M, \psi)$. After these identifications, for every $k \geq 1$, the condition $\sigma_t^{\psi_{k+1}} \circ \beta = \beta \circ \sigma_t^{\sigma_k}$, $t \in \mathbb{R}$, is equivalent to

$$(4.18) \qquad \sigma_t^{\psi_{k+1}}(x) = \sigma_t^{\psi_k}(x), \quad x \in M_k, \quad t \in \mathbb{R}.$$

Hence, by [34] there exist unique normal faithful conditional expectations $\mathbb{E}_k \colon M_{k+1} \to M_k$ such that $\psi_k \circ \mathbb{E}_k = \psi_{k+1}$, $k \geq 1$. Also from [34] it follows that the isometric modular conjugation operator J_{ψ_k} on $L^2(M_k, \psi_k)$ is the restriction of $J_{\psi_{k+1}}$ to $L^2(M_k, \psi_k)$. Hence, there exists a conjugate-linear isometric involution J on $L^2(M, \psi)$ which extends al the J_k's. Since, moreover, $J_{\psi_k} \pi_{\psi_k}(M_k) J_{\psi_k} = \pi_{\psi_k}(M_k)'$, for all $k \geq 1$, it follows that

$$(4.19) \qquad J\pi_\psi(M)J \subseteq \pi_\psi(M)'.$$

Let ξ_ψ be the cyclic vector in $L^2(M, \psi)$ corresponding to the unit operator 1 in M. Then $J\xi_\psi = \xi_\psi$, and therefore by (4.19), ξ_ψ is also cyclic for $\pi_\psi(M)'$. Hence ξ_ψ is separating for $\pi_\psi(M)$, which proves that ψ is faithful. Finally, in order to prove that $\sigma_t^\psi \circ \mu_k = \mu_k \circ \sigma_t^{\psi_k}$, $t \in \mathbb{R}$, $k \geq 1$, we have to show that under the above identifications,

$$(4.20) \qquad \sigma_t^\psi(x) = \sigma_t^{\psi_k}(x), \quad x \in M_k, \quad t \in \mathbb{R}, \quad k \geq 1.$$

By (4.18), together with the fact that $\psi_k = \psi_{k+1|M_k}$, $k \geq 1$, it follows that the modular automorphism groups $(\sigma_t^{\psi_k})_{t \in \mathbb{R}}$, $k \geq 1$ have a unique extension to a strongly continuous one-parameter group of automorphisms $(\sigma_t)_{t \in \mathbb{R}}$ on M. Moreover, ψ satisfies the KMS condition with respect to $(\sigma_t)_{t \in \mathbb{R}}$, since each ψ_k is a $(\sigma_t^{\psi_k})_{t \in \mathbb{R}}$-KMS state on M_k (see Theorem 1.2, Chap. VIII, in [35]). Therefore $\sigma_t = \sigma_t^\psi$, $t \in \mathbb{R}$, which proves (4.20).

To prove the second part of the lemma we can without loss of generality assume that $\bigcup_{k=1}^{\infty} \lambda_k(M_k)$ is weakly dense in N (otherwise replace N by the weak closure of $\bigcup_{k=1}^{\infty} \lambda_k(M_k)$). Consider the GNS-representations of M and N on Hilbert spaces H and H' with respect to the normal faithful states ψ and φ, respectively. Then there exist cyclic and separating vectors $\xi \in H$ and $\xi' \in H'$ for M and N, respectively, such that

$$\psi(x) = \langle x\xi, \xi \rangle, \quad \varphi(y) = \langle y\xi', \xi' \rangle, \qquad x \in M, y \in N.$$

By the universal property of the C^*-algebra inductive limit, there is a $*$-monomorphism $\widetilde{\lambda} \colon M_\infty \to N$ satisfying $\widetilde{\lambda} \circ \mu_k = \lambda_k$ for all $k \geq 1$. Observe that $\varphi \circ \widetilde{\lambda}(x) = \psi(x)$, for all $x \in M_\infty$. It follows that the map $u_0 \colon M_\infty \xi \to H'$ defined by $u_0 x\xi = \widetilde{\lambda}(x)\xi'$, $x \in M_\infty$, is isometric and has dense range in H'. Hence it extends to a unitary $u \colon H \to H'$. We see that $uxu^*\xi' = \widetilde{\lambda}(x)\xi'$, and hence that $uxu^* = \widetilde{\lambda}(x)$, for all $x \in M_\infty$. The map $\lambda \colon M \to N$ defined by $\lambda(x) = uxu^*$, for $x \in M$, has the desired properties. $\qquad\square$

Remark 4.6. We would like to draw the reader's attention upon the subtle fact that condition (4.17) is crucial for guaranteeing that the canonical state $\widetilde{\psi}$ on the C^*-algebra inductive limit M_∞ extends to a *faithful* state ψ on the von Neumann algebra M, obtained via the GNS representation of M_∞ with respect to $\widetilde{\psi}$.

The von Neumann algebra (M, ψ) is said to be the von Neumann algebra inductive limit of the sequence

$$(M_1, \psi_1) \xrightarrow{\ \beta_1\ } (M_2, \psi_2) \xrightarrow{\ \beta_2\ } (M_3, \psi_3) \xrightarrow{\ \beta_3\ } \cdots .$$

The following result is a reformulation of Proposition 2.1.7 in [23].

Proposition 4.7. *Let N be a von Neumann algebra with a normal faithful state ψ, and let $\beta \colon N \to N$ be a ψ-Markov $*$-monomorphism. Then there exists a von Neumann algebra $\widetilde{N}$ with a normal faithful state $\widetilde{\psi}$, a $(\psi, \widetilde{\psi})$-Markov embedding $\iota \colon N \to \widetilde{N}$ and an $\alpha \in \mathrm{Aut}(\widetilde{N})$ for which $\widetilde{\psi} \circ \alpha = \widetilde{\psi}$ such that $\beta = \iota^* \circ \alpha \circ \iota$.*

Proof. Let $(\widetilde{N}, \widetilde{\psi})$ be the von Neumann algebra inductive limit of the sequence

$$(N, \psi) \xrightarrow{\ \beta\ } (N, \psi) \xrightarrow{\ \beta\ } (N, \psi) \xrightarrow{\ \beta\ } \cdots ,$$

and for every $k \geq 1$, let $\mu_k \colon N \to \widetilde{N}$ be the associated $*$-monomorphism from the kth copy of N into $\widetilde{N}$. By the second part of Lemma 4.5 applied to the $*$-monomorphisms $\lambda_k \colon N \to \widetilde{N}$ given by $\lambda_k = \mu_k \circ \beta$, there exists a $*$-monomorphism α on $\widetilde{N}$ such that $\alpha \circ \mu_k = \mu_k \circ \beta$, for all $k \geq 1$. It follows that

$$\bigcup_{k=1}^{\infty} \mu_k(N) = \bigcup_{k=1}^{\infty} \mu_{k+1} \circ \beta(N) = \bigcup_{k=1}^{\infty} \alpha \circ \mu_{k+1}(N) \subseteq \mathrm{Im}(\alpha).$$

As $\bigcup_{k=1}^{\infty} \mu_k(N)$ is dense in $\widetilde{N}$ and the image of α is a von Neumann subalgebra of $\widetilde{N}$, this shows that α is onto, and hence an automorphism.

Take $\iota \colon N \to \widetilde{N}$ to be μ_1. Then $\alpha \circ \iota = \iota \circ \beta$. Moreover, by Lemma 4.5, ι is a ψ-Markov map. Since

$$\widetilde{\psi} \circ \alpha \circ \mu_k = \widetilde{\psi} \circ \mu_k \circ \beta = \psi \circ \beta = \psi = \widetilde{\psi} \circ \mu_k, \quad k \geq 1,$$

$\widetilde{\psi} \circ \alpha$ and $\widetilde{\psi}$ coincide on $\bigcup_{k \geq 1}^{\infty} \mu_k(N)$. Therefore, $\widetilde{\psi} = \widetilde{\psi} \circ \alpha$. The existence of the adjoint map $\iota^* \colon \widetilde{N} \to N$ follows from Remark 1.2. As $\iota^* \circ \iota$ is the identity on N we get that $\beta = \iota^* \circ \alpha \circ \iota$ using the previously obtained identity $\iota \circ \beta = \alpha \circ \iota$. This completes the proof. $\qquad\square$

Proof of Theorem 4.4: The implication $(3) \Rightarrow (2)$ is trivial. Also, the implication $(2) \Rightarrow (1)$ follows immediately, since if (N, ψ, α, ι) is a dilation of T, then $T = \iota^* \circ (\alpha \circ \iota)$ is a factorization of T through (N, ψ) in the sense of Definition 1.3, because $\alpha \in \mathrm{Aut}(N, \psi)$ implies that $\alpha \circ \sigma_t^\psi = \sigma_t^\psi \circ \alpha$, $t \in \mathbb{R}$.

Next we prove that $(1) \Rightarrow (3)$. Assume that T is factorizable. Then, by Theorem 6.6 in [2], there exists a von Neumann algebra N with a normal faithful state ψ, a ψ-Markov normal $*$-endomorphism $\beta \colon N \to N$ and a (ϕ, ψ)-Markov $*$-monomorphism $j \colon M \to N$ such that $T^n = j^* \circ \beta^n \circ j$, for all $n \geq 1$. By Proposition 4.7, we can find a dilation $(\tilde{N}, \tilde{\psi}, \alpha, \iota)$ of β, where $\alpha \in \mathrm{Aut}(\tilde{N}, \tilde{\psi})$. We may (and will) consider N as a subalgebra of $\tilde{N}$. In this way, ι is just the inclusion map, ι^* is the $\tilde{\psi}$-preserving normal faithful conditional expectation of $\tilde{N}$ onto N, and $\beta = \alpha_{|_N}$. Then it is clear that with $\tilde{j} \colon = \iota \circ j$, the quadruple $(\tilde{N}, \tilde{\psi}, \alpha, \tilde{j})$ is a dilation of T (which actually proves $(1) \Rightarrow (2)$). To complete the proof of the implication $(1) \Rightarrow (3)$, we will show that by the construction of β from [2], the quadruple $(\tilde{N}, \tilde{\psi}, \alpha, \tilde{j})$ becomes a Markov dilation of T.

For $J \subseteq \{n \in \mathbb{Z} : n \geq 0\}$, let $\mathbb{E}_J$ denote the unique ψ-preserving conditional expectation of N onto its subalgebra $\mathcal{B}_J \colon = (\bigcup_{k \in J} \beta^k \circ j(M))''$. Then, by condition (6.2) in Theorem 6.6 of [2], we get

$$(4.21) \qquad \mathbb{E}_{[0, n+k]} \circ \beta^k = \beta^k \circ \mathbb{E}_{[0, n]}, \quad n, k \geq 0.$$

Moreover, with $\mathcal{B}$ and $(\mathcal{B}_n, \phi_n)$, $n \geq 0$ defined as in the proof of the above-mentioned theorem, we have $\mathcal{B}_n = \mathcal{B}_{[0, n]}$ and $N = \mathcal{B} = \mathcal{B}_{[0, \infty)}$. Hence $\mathbb{E}_{[0, n]} = \mathbb{E}_{\mathcal{B}_n}$, the unique ψ-invariant conditional expectation of N onto $\mathcal{B}_n$, and $\mathbb{E}_{[0, \infty)} = \mathbb{E}_N = \mathrm{id}_N$. Set now $H \colon = L^2(N, \psi)$, $H_n \colon = L^2(\mathcal{B}_n, \phi_n)$, $n \geq 0$, and let $V \colon = \tilde{\beta}$, be the unique extension of β to an isometry on $L^2(N, \psi)$. By (4.21),

$$P_{H_{n+k}} V^k = V^k P_{H_n}, \quad n, k \geq 0,$$

where $P_K \in \mathcal{B}(H)$ denotes the orthogonal projection onto a closed subspace K of H. Hence,

$$(4.22) \qquad P_{H_{n+k}} P_{V^k(H)} = P_{H_{n+k}} V^k (V^*)^k = V^k P_{H_n} (V^*)^k = P_{V^k(H_n)}, \quad n, k \geq 0.$$

By the definition, it is clear that $\beta^k(\mathcal{B}_J) = \mathcal{B}_{J+k}$, for all $k \geq 0$ and all $J \subseteq \{n \in \mathbb{Z} : n \geq 0\}$. In particular,

$$\beta^k(N) = \beta^k(\mathcal{B}_{[0, \infty)}) = \mathcal{B}_{[k, \infty)}, \quad \beta^k(\mathcal{B}_n) = \mathcal{B}_{[k, k+n]}, \qquad n, k \geq 0.$$

Thus, by restricting (4.22) to $N \subseteq L^2(N, \psi)$, we get $\mathbb{E}_{[0, n+k]} \mathbb{E}_{[k, \infty)} = \mathbb{E}_{[k, k+n]}$, for all $n, k \geq 0$. In particular, we have

$$(4.23) \qquad \mathbb{E}_{[0, k]} \mathbb{E}_{[k, \infty)} = \mathbb{E}_{\{k\}}, \quad k \geq 0.$$

Since $\tilde{j} = \iota \circ j$, we have from Proposition 4.7 that

$$\alpha^k \circ \tilde{j}(M) = \beta^k \circ j(M), \quad k \geq 0.$$

Hence, by composing (4.23) from the right with the $\tilde{\psi}$-preserving conditional expectation ι^* of $\tilde{N}$ onto N, we get (following the notation set forth in Definition 4.2) that

$$(4.24) \qquad \mathbb{P}_{[0, k]} \mathbb{P}_{[k, \infty)} = \mathbb{P}_{\{k\}}, \quad k \geq 0.$$

Note that for every $I \subseteq \mathbb{Z}$, by the definition of $\mathbb{P}_I$ one has $\alpha^n \mathbb{P}_I \alpha^{-n} = \mathbb{P}_{I+n}$, for all $n \in \mathbb{Z}$. Hence, from (4.24) we get that

$$\mathbb{P}_{[-n, 0]} \mathbb{P}_{[0, \infty)} = \mathbb{P}_{\{0\}}, \quad n \geq 0.$$

In the limit as $n \to \infty$, this yields

$$\mathbb{P}_{(-\infty, 0]} \mathbb{P}_{[0, \infty)} = \mathbb{P}_{\{0\}},$$

a condition which, by Remark 4.3, ensures that $(\tilde{N}, \tilde{\psi}, \alpha, \tilde{j})$ is a Markov dilation of T. $\qquad\square$

Note that by the proof of Theorem 4.4 it follows that a ϕ-Markov map $T: M \to M$ admits a dilation if and only if it has a dilation of order 1 (see Definition 4.1), since in order to show that $(2) \Rightarrow (1)$ above we have only used the existence of a dilation of order 1 for the given map.

Remark 4.8. Kümmerer constructed in [23] examples of τ_n-Markov maps on $M_n(\mathbb{C})$, $n \geq 3$, having no dilation, as follows:

(1) Let $a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and $a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then the map given by $Tx := \sum_{i=1}^{3} a_i^* x a_i$, $x \in M_3(\mathbb{C})$, is a τ_3-Markov map having no dilation.

(2) Let $n \geq 4$ and consider the $n \times n$ diagonal matrices $a_1 = \mathrm{diag}(1, 1/\sqrt{2}, 1/\sqrt{2}, 0, \dots, 0)$ and $a_2 = \mathrm{diag}(0, 1/\sqrt{2}, i/\sqrt{2}, 1, \dots, 1)$. Then the map given by $Tx := \sum_{i=1}^{2} a_i^* x a_i$, $x \in M_n(\mathbb{C})$, is a τ_n-Markov Schur multiplier with no dilation.

In view of Koestler's communication, these are all examples of non-factorizable Markov maps. In [23], Kümmerer also constructed an example of a τ_6-Markov Schur multiplier on $M_6(\mathbb{C})$ which admits a dilation, hence it is factorizable, but does not lie in $\mathrm{conv}(\mathrm{Aut}(M_6(\mathbb{C})))$. However, he did not consider the one-parameter semigroup case.

We were also informed by B. V. R. Bhat and A. Skalski [4] that, unaware of Kümmerer's examples and their connection with Anantharaman-Delaroche's problem, as well as of our already existing work, they have also constructed examples of a non-factorizable τ_3-Markov map on $M_3(\mathbb{C})$, respectively, of a τ_4-Markov Schur multiplier on $M_4(\mathbb{C})$ which is not factorizable.

5. The noncommutative Rota dilation property

The following was introduced in [16] (see Definition 10.2 therein):

Definition 5.1. *Let M be a von Neumann algebra equipped with a normalized (normal and faithful) trace τ. We say that a bounded operator $T: M \to M$ satisfies the Rota dilation property if there exists a von Neumann algebra N equipped with a normalized (normal and faithful) trace τ_N, a normal unital faithful $*$-representation $\pi: M \to N$ which preserves the traces (i.e., $\tau_N \circ \pi = \tau$), and a decreasing sequence $(N_m)_{m \geq 1}$ of von Neumann subalgebras of N such that*

$$T^m = Q \circ \mathbb{E}_m \circ \pi, \quad m \geq 1.$$

Here $\mathbb{E}_m$ denotes the canonical (trace-preserving) conditional expectation of N onto N_m, and $Q: N \to M$ is the conditional expectation associated to π, that is, $Q := \pi^* = \pi^{-1} \circ \mathbb{E}_{\pi(M)}$, where $\mathbb{E}_{\pi(M)}$ is the trace-preserving conditional expectation of N onto $\pi(M)$.

Remark 5.2. If $T: M \to M$ has the Rota dilation property, then T is completely positive, unital and trace-preserving. Since in the tracial setting condition (4) in Definition 1.1 is trivially satisfied, it follows that T is automatically a τ-Markov map. Moreover, since $\mathbb{E}_1$ (viewed as an operator from N into N) can be written as $\mathbb{E}_1 = j_1^* \circ j_1$, where $j_1 : N_1 \hookrightarrow N$ is the inclusion map, then

(5.25) $$T = Q \circ \mathbb{E}_1 \circ \pi = (j_1^* \circ \pi)^* \circ (j_1^* \circ \pi).$$

Hence, T is positive as an operator on the pre-Hilbert space M with inner product $\langle x, y \rangle := \tau(y^*x)$, $x, y \in M$. This also implies that $T = T^*$, where $T^*: M \to M$ is the adjoint of T in the sense of (1.1). (This observation is also stated in [16] (cf. Remark 10.3 therein)).

Furthermore, equalities (5.25) show that the Rota dilation property implies factorizability of T, in view of Remark 1.4 (b).

The following is a consequence of Theorem 6.6 in [2]:

Theorem 5.3. *If $T: M \to M$ is a factorizable τ-Markov map with $T = T^*$, then T^2 has the Rota dilation property.*

Proof. Since T is factorizable, Theorem 6.6 in [2] ensures the existence of a von Neumann algebra N with a normal faithful state ψ, a normal unital endomorphism $\beta : N \to N$ which is ψ-Markov and a normal unital $*$-homomorphism $J_0 : M \to N$ which is (ϕ, ψ)-Markov such that, if we set $J_n := \beta^n \circ J_0$ and $\mathbb{E}_{[n}$ denotes the conditional expectation of N onto its von Neumann subalgebra generated by $\bigcup_{k \geq n} J_k(M)$ for all $n \geq 0$, while $\mathbb{E}_{0]}$ is the conditional expectation of N onto $J_0(M)$, then

$$(5.26) \qquad \begin{aligned} \mathbb{E}_{0]} \circ J_n &= J_0 \circ T^n, & n \geq 1, \\ \mathbb{E}_{[n} \circ J_0 &= J_n \circ (T^*)^n, & n \geq 1. \end{aligned}$$

It follows that $J_0 \circ T^n \circ (T^*)^n = \mathbb{E}_{0]} \circ J_n \circ (T^*)^n = \mathbb{E}_{0]} \circ \mathbb{E}_{[n} \circ J_0$, $n \geq 1$. Since $T^* = T$, we infer that

$$(5.27) \qquad (T^2)^n = T^n \circ (T^*)^n = J_0^{-1} \circ \mathbb{E}_{0]} \circ \mathbb{E}_{[n} \circ J_0 = J_0^* \circ \mathbb{E}_{[n} \circ J_0, \quad n \geq 1.$$

Observing that $(\mathbb{E}_{[n})_{n \geq 1}$ is a sequence of conditional expectations with decreasing ranges, (5.27) shows that T^2 has the Rota dilation property. $\qquad \square$

Note that if M is abelian and $T : M \to M$ is factorizable, then, following the construction in [2], one can choose an abelian dilation N for T. Therefore, Theorem 5.3 is a noncommutative analogue of Rota's classical dilation theorem for Markov operators. The next result shows that the factorizability condition cannot be removed from the hypothesis of Theorem 5.3, thus the Rota dilation theorem does not hold in general in the noncommutative setting.

Theorem 5.4. *There exists a τ_n-Markov map $T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$, for some $n \geq 1$, such that $T = T^*$, but T^2 is not factorizable. In particular, T^2 does not have the Rota dilation property.*

To prove the theorem, we start with the following

Lemma 5.5. *Let $n, d \in \mathbb{N}$ and consider $a_1, \ldots, a_d \in M_n(\mathbb{C})$ be self-adjoint with $\sum_{i=1}^{d} a_i^2 = 1_n$. Set*

$$(5.28) \qquad T(x) := \sum_{i=1}^{d} a_i x a_i, \quad x \in M_n(\mathbb{C}).$$

Suppose that

    (i) $a_i^2 a_j = a_j a_i^2$, $1 \leq i, j \leq d$.
    (ii) $A := \{a_i a_j : 1 \leq i, j \leq d\}$ is linearly independent.
    (iii) $B := \cup_{i=1}^{6} B_i$ is linearly independent, where

$B_1 := \{a_i a_j a_k a_l : 1 \leq i \neq j \neq k \neq l \leq d\}$, $B_2 := \{a_i a_j a_k^2 : 1 \leq i \neq j \neq k \neq k \leq d\}$, $B_3 := \{a_i^3 a_j : 1 \leq i \neq j \leq d\}$, $B_4 := \{a_i a_j^3 : 1 \leq i \neq j \leq d\}$, $B_5 := \{a_i^2 a_j^2 : 1 \leq i < j \leq d\}$, $B_6 := \{a_i^4 : 1 \leq i \leq d\}$. Furthermore, it is assumed that B is the disjoint union of the sets B_i, $1 \leq i \leq 6$, and that the elements listed in each B_i are distinct.

Assume further that

(iv) $d \geq 5$.

Then T is a self-adjoint τ_n-Markov map, for which T^2 is not factorizable.

Proof. We have $T^2 x = \sum_{i,j=1}^d (a_i a_j)^* x(a_i a_j)$, for all $x \in M_n(\mathbb{C})$. It is clear that T^2 is a τ_n-Markov map, for which Theorem 2.2 can be applied, due to condition (ii). Hence, if T^2 were factorizable, it would then follow that there exists a finite von Neumann algebra N with a norma faithful tracial state τ_N, and a unitary $u \in M_n(N)$ such that

$$T^2 x \otimes 1_N = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

Moreover, since T^2 is self-adjoint, an easy argument shows that u can be chosen to be self-adjoint. Namely, one can replace N by $M_2(N)$ and u by $\tilde{u} := \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \in M_2(M_n(N)) = M_n(M_2(N))$. Moreover, Theorem 2.2 ensures that u is of the form

(5.29)
$$u = \sum_{i,j=1}^d a_i a_j \otimes v_{ij},$$

where $v_{ij} \in N$, for all $1 \leq i, j \leq d$, and

(5.30)
$$\tau_N(v_{ij}^* v_{kl}) = \delta_{ik} \delta_{jl}.$$

By (i), the elements v_{ij}, $1 \leq i, j \leq d$ are uniquely determined from (5.29). Since, moreover, $u = u^*$, we deduce that

(5.31)
$$v_{ij}^* = v_{ji}, \quad 1 \leq i, j \leq d.$$

Now, by condition (i), we obtain the following set of relations for all $i \neq j \neq k \neq i$,

(5.32)
$$a_i a_j a_k^2 = a_i a_k^2 a_j = a_k^2 a_i a_j$$

and, respectively, for all $i \neq j$,

(5.33)
$$a_i^3 a_j = a_i a_j a_i^2, \quad a_i a_j^3 = a_j^2 a_i a_j, \quad a_i^2 a_j^2 = a_i a_j^2 a_i = a_j^2 a_i^2 = a_j a_i^2 a_j.$$

These conditions imply that every matrix of the form $a_i a_j a_k a_l$, $1 \leq i, j, k, l \leq d$ occurs precisely once the set $B = \cup_{i=1}^6 B_i$. Moreover, two elements of the form $b = a_i a_j a_k a_l$, $b' = a_{i'} a_{j'} a_{k'} a_{l'}$, where $(i, j, k, l) \neq (i', j', k', l')$ are equal if and only if one of the four cases listed in (5.32) and (5.33) holds.

Furthermore, since $u^2 = u^* u = 1_{M_n(N)}$, we have

(5.34)
$$1_{M_n(N)} = \sum_{i,j,k,l}^d a_i a_j a_k a_l \otimes v_{ij} v_{kl}.$$

By applying $\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N$ on both sides of (5.34), we get $1_n = \sum_{i,j,k,l=1}^d \tau_N(v_{ij} v_{kl}) a_i a_j a_k a_l$, and therefore $0_{M_n(N)} = \sum_{i,j,k,l=1}^d a_i a_j a_k a_l \otimes (v_{ij} v_{kl} - \tau_N(v_{ij} v_{kl}) 1_N)$. By (5.30) and (5.31), this can further be reduced

to

$$(5.35) \qquad 0_{M_n(N)} = \sum_{i,j,k,l=1}^{d} a_i a_j a_k a_l \otimes (v_{ij}v_{kl} - \delta_{il}\delta_{jk}1_N).$$

Using the remark following (5.33), the equation (5.35) can be rewritten as $0_{M_n(N)} = \sum_{b \in B} b \otimes w_b$, where $w_b \in N$. Since B is a linearly independent set, this implies that $w_b = 0_N$, for all $b \in B$. Hence, if $b \in B_1$, i.e., $b = a_i a_j a_k a_l$, where $i \neq j \neq k \neq l$, we infer that $0_N = w_b = v_{ij}v_{kl} - \delta_{il}\delta_{jk}1_N$, which implies that

$$(5.36) \qquad v_{ij}v_{kl} = 0_N, \quad i \neq j \neq k \neq l.$$

Similarly, if $b \in B_6$, i.e., $b = a_i^4$, for some $1 \leq i \leq d$, then the same argument applies, and we obtain $0_N = w_b = v_{ii}^2 - \delta_{ii}^2 1_N$, i.e.,

$$(5.37) \qquad v_{ii}^2 = 1_N, \quad 1 \leq i \leq d.$$

On the other hand, if $b \in B_2$, then by (5.33) it follows that $b = a_i a_j a_k^2 = a_i a_k^2 a_j = a_i a_j a_k^2$, for some $1 \leq i \neq j \neq k \neq i \leq d$, and therefore $w_b = v_{ij}v_{kk}^2 + v_{ik}v_{kj} + v_{kk}^2 v_{ij} - (\delta_{ik}\delta_{jk} + \delta_{ij}\delta_{kk} + \delta_{kj}\delta_{ki})1_N$. Hence

$$(5.38) \qquad v_{ij}v_{kk}^2 + v_{ik}v_{kj} + v_{kk}^2 v_{ij} = 0_N, \quad 1 \leq i \neq j \neq k \neq i \leq d.$$

Similarly, using $w_b = 0_N$ for all $b \in B_m$, where $3 \leq m \leq 5$, we obtain that the following relations hold for all $1 \leq i \neq j \leq d$:

$$(5.39) \qquad v_{ii}v_{ij} + v_{ij}v_{ii} = 0_N$$

$$(5.40) \qquad v_{ij}v_{jj} + v_{jj}v_{ij} = 0_N$$

$$(5.41) \qquad v_{ii}v_{jj} + v_{ij}v_{ji} + v_{jj}v_{ii} + v_{ji}v_{ij} = 21_N.$$

Now, recall that $v_{ij}^* = v_{ji}$, $1 \leq i, j \leq d$, so by (5.37), we deduce that $\{v_{ii}, 1 \leq i \leq d\}$ is a set of self-adjoint unitaries. Thus, by (5.41) we have

$$\|v_{ij}v_{ij}^* + v_{ij}^* v_{ij}\| = \|21_N - v_{ii}v_{jj} - v_{jj}v_{ii}\| \leq 4, \quad 1 \leq i \neq j \leq d,$$

which implies that $\|v_{ij}\| \leq 2$, for all $1 \leq i \neq j \leq d$. Now, for every $1 \leq j \leq d$ set $p_j := \bigvee_{i \neq j} s(v_{ij}^* v_{ij})$, where $s(v_{ij}^* v_{ij})$ denotes the support projection of $v_{ij}^* v_{ij}$. By (5.36) it follows that p_j and p_k are orthogonal projections, whenever $1 \leq j \neq k \leq d$, and hence

$$(5.42) \qquad \sum_{j=1}^{d} \tau_N(p_j) \leq \tau_N(1) = 1.$$

On the other hand, by (5.30), $\tau_N(v_{ij}^* v_{ij}) = 1$, for all $1 \leq i, j \leq d$. Moreover, for $i \neq j$,

$$v_{ij}^* v_{ij} \leq \|v_{ij}^* v_{ij}\| p_j \leq 4p_j.$$

Thus $\tau_N(p_j) \geq (\tau_N(v_{ij}^* v_{ij}))/4 = 1/4$, for all $1 \leq i \neq j \leq d$. This implies that $\sum_{j=1}^{d} \tau_N(p_j) \geq d/4$, and since $d \geq 5$, this contradicts (5.42). The proof is complete. $\qquad \square$

The condition $d \geq 5$ is essential in the statement of Lemma 5.5 above, as it can be seen from the following remark.

Remark 5.6. Assume that $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is of the form (5.28), where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ are self-adjoint with $\sum_{i=1}^{d} a_i^2 = 1_n$. If $d \leq 4$, then T^2 is factorizable.

Proof. We can assume without loss of generality that $d = 4$ (otherwise add zero-terms). Set

$$u: = \sum_{i,j=1}^{4} a_i a_j \otimes (2e_{ij} - \delta_{ij} 1_4),$$

where $(e_{ij})_{1 \leq i,j \leq 4}$ are the standard matrix units in $M_4(\mathbb{C})$. Then $u = u^* \in M_n(\mathbb{C}) \otimes M_4(\mathbb{C}) = M_{4n}(\mathbb{C})$. We first show that u is a unitary. We have

$$
\begin{aligned}
u^* u \;=\; uu^* = u^2 \;&=\; \sum_{i,j,k,l=1}^{4} a_i a_j a_k a_l \otimes (2e_{ij} - \delta_{ij} 1_4)(2e_{kl} - \delta_{kl} 1_4) \\
&=\; \sum_{i,j,k,l=1}^{4} a_i a_j a_k a_l \otimes (4\delta_{jk} e_{il} - 2\delta_{ij} e_{kl} - 2\delta_{kl} e_{ij} + \delta_{ij}\delta_{kl} 1_4) \\
&=\; 4s_1 - 2s_2 - 2s_3 + s_4\,,
\end{aligned}
$$

where $s_1: = \sum_{i,j,l=1}^{4} a_i a_j^2 a_l \otimes e_{il} = \sum_{i,l=1}^{4} a_i a_l \otimes e_{il}$, $s_2: = \sum_{i,k,l=1}^{4} a_i^2 a_k a_l \otimes e_{kl} = \sum_{k,l=1}^{4} a_k a_l \otimes e_{kl}$, $s_3: = \sum_{i,j,k=1}^{4} a_i a_j a_k^2 \otimes e_{ij} = \sum_{i,j=1}^{4} a_i a_j \otimes e_{ij}$ and $s_4: = \sum_{i,j=1}^{4} a_i^2 a_j^2 \otimes 1_4 = 1_{4n}$. We have repeatedly used the fact that $\sum_{i=1}^{d} a_i^2 = 1_n$. Hence $s_1 = s_2 = s_3$, and therefore

$$u^* u = uu^* = s_4 = 1_{4n}\,.$$

Next we prove that T^2 is factorizable by showing that

$$\mathbb{E}_{M_n(\mathbb{C}) \otimes 1_4}(u^*(x \otimes 1_4)u) = T^2 x\,, \quad x \in M_n(\mathbb{C})\,.$$

Since $u = u^*$, the left hand side above becomes

$$
\begin{aligned}
\mathbb{E}_{M_n(\mathbb{C}) \otimes 1_4}(u^*(x \otimes 1_4)u) \;&=\; \mathbb{E}_{M_n(\mathbb{C}) \otimes 1_4}\left(\sum_{i,j,k,l=1}^{4} a_i a_j \otimes (2e_{ij} - \delta_{ij} 1_4)(x \otimes 1_4)(a_k a_l \otimes (2e_{kl} - \delta_{kl} 1_4)) \right) \\
&=\; \sum_{i,j,k,l=1}^{4} a_i a_j x a_k a_l \tau_4((2e_{ij} - \delta_{ij} 1)(2e_{kl} - \delta_{kl} I)) \\
&=\; \sum_{i,j=1}^{4} a_i a_j x a_j a_i \\
&=\; T^2 x\,,
\end{aligned}
$$

wherein we have used that fact that $\tau_4((2e_{ij} - \delta_{ij} 1_4)(2e_{kl} - \delta_{kl} 1_4)) = (4/4)\delta_{il}\delta_{jk} - (2/4)\delta_{ij}\delta_{kl} - (2/4)\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{kl} = \delta_{il}\delta_{jk}$. The proof is complete. $\qquad \square$

Lemma 5.7. *Let $d \geq 5$, $b_1, \ldots, b_d$ be self-adjoint matrices in $M_m(\mathbb{C})$, and $u_1, \ldots, u_d$ be self-adjoint unitary matrices in $M_r(\mathbb{C})$, where m and r are positive integers. Assume that*

(a) *$\sum_{i=1}^{d} b_i^2 = 1_m$.*
(b) *$b_i b_j = b_j b_i$, for all $1 \leq i, j \leq d$.*
(c) *$b_i b_j b_k b_l \neq 0_m$, for all $1 \leq i, j, k, l \leq d$.*
(d) *For every $1 \leq i \neq j \leq d$, the set $\{b_i b_j b_k^2 : 1 \leq k \leq d\}$ is linearly independent in $M_m(\mathbb{C})$.*
(e) *The set $\{b_i^2 b_j^2 : 1 \leq i < j \leq d\}$ is linearly independent in $M_m(\mathbb{C})$.*
(f) *The set $\{1_r\} \cup \{u_i u_j : 1 \leq i \neq j \leq d\} \cup \{u_i u_j u_k u_l : 1 \leq i \neq j \neq k \neq l \leq d\}$ is linearly independent in $M_r(\mathbb{C})$.*

Then $a_i\colon = b_i \otimes u_i$, $1 \le i \le d$ are self-adjoint matrices in $M_{mr}(\mathbb{C}) = M_m(\mathbb{C}) \otimes M_r(\mathbb{C})$ which satisfy $\sum_{i=1}^d a_i^2 = 1_{mr}$, as well as the conditions $(i)-(iv)$ in Lemma 5.5 with $n = mr$.

Proof. Note first that conditions (a) and (b), together with the fact that $u_i^2 = 1_r$, for all $1 \le i \le d$, imply that $\sum_{j=1}^d a_j^2 = 1_{mr}$ and $a_i^2 a_j = a_j a_i^2$, $1 \le i, j \le d$, i.e., condition (i) in Lemma 5.5 is satisfied. Further, the set $A\colon = \{a_i a_j : 1 \le i, j \le d\}$ is equal to

$$\{b_i^2 \otimes 1_r : 1 \le i \le d\} \cup \{b_i b_j \otimes u_i u_j : 1 \le i \ne j \le d\}.$$

By (e), the set $\{1_r\} \cup \{u_i u_j : 1 \le i \ne j \le d\}$ is linearly independent. Hence A is linearly independent if and only if $b_i b_j \ne 0_m$, whenever $1 \le i \ne j \le d$. The linear independence of $b_1^2, \ldots, b_d^2$ follows from (d), and by (c) we get that $b_i b_j \ne 0_m$, for all $1 \le i, j \le d$. This proves condition (ii) in Lemma 5.5.

Next, consider the set $B\colon = B_1 \cup \ldots \cup B_6$, where $B_1, \ldots, B_6$ are defined as in (iii) in the above mentioned lemma. Since $u_i^2 = 1_r$, for all $1 \le i \le d$, the sets $B_1, \ldots, B_6$ can be rewritten as: $B_1 = \{b_i b_j b_k b_l \otimes u_i u_j u_k u_l : 1 \le i \ne j \ne k \ne l \le d\}$, $B_2 = \{b_i b_j b_k^2 \otimes u_i u_j : 1 \le i \ne j \ne k \ne i \le d\}$, $B_3 = \{b_i^3 b_j \otimes u_i u_j : 1 \le i \ne j \le d\}$, $B_4 = \{b_i b_j^3 \otimes u_i u_j : 1 \le i \ne j \le d\}$, $B_5 = \{b_i^2 b_j^2 \otimes 1_l : 1 \le i < j \le d\}$ and $B_6 = \{b_i^4 \otimes 1_l : 1 \le i \le d\}$. By (e), B is a linearly independent set if and only if the following three conditions hold:

(1) $b_i b_j b_k b_l \ne 0_m$, whenever $1 \le i \ne j \ne k \ne l \le d$.
(2) For every $1 \le i \ne j \le d$, the set $\{b_i b_j b_k^2 : 1 \le k \le d, k \ne i, k \ne j\} \cup \{b_i^3 b_j : 1 \le i, j \le d\} \cup \{b_i b_j^3 : 1 \le i, j \le d\}$ is linearly independent.
(3) The set $\{b_i^2 b_j^2 : 1 \le i < j \le d\} \cup \{b_i^4 : 1 \le i \le d\}$ is linearly independent.

Clearly, (c) implies (1), (e) implies (3), and by (b), condition (d) implies (2). Hence (iii) in Lemma 5.5 holds, and since $d \ge 5$, condition (iv) holds, as well, thus completing the proof. $\qquad \square$

Proof of Theorem 5.4: It remains to be proved that for $d \ge 5$, there exist positive integers m, r and matrices $b_1, \ldots, b_d \in M_m(\mathbb{C})$, $u_1, \ldots, u_d \in M_r(\mathbb{C})$ satisfying the hypotheses of Lemma 5.7.

Let $S^{d-1} = S(\mathbb{R}^d)$ denote the unit sphere in $\mathbb{R}^d$, and let $\phi_1, \ldots, \phi_d : S^{d-1} \to \mathbb{R}$ be the coordinate functions. It is not difficult to check that these functions in $C(S^{d-1})$ satisfy conditions $(a)-(e)$ in Lemma 5.7. Conditions $(a)-(c)$ are, indeed, obvious. To prove (e), note that $\phi_d^2 = 1 - \phi_1^2 - \ldots - \phi_{d-1}^2$. Hence, the linear independence of the set $\{\phi_i^2 \phi_j^2 : 1 \le i < j \le d\}$ is equivalent to the linear independence of the set of polynomials

$$\mathcal{P} := \{x_i^2 x_j^2 : 1 \le i \le j \le d-1\} \cup \{x_i^2 : 1 \le i \le d-1\} \cup \{1\}$$

in $C(B(\mathbb{R}^{d-1}))$, where $B(\mathbb{R}^{d-1})$ is the closed unit ball in $\mathbb{R}^{d-1}$. But $\mathcal{P}$ is clearly a linearly independent set, because if a polynomial in $\mathbb{R}^{d-1}$ vanishes in a neighborhood of 0, then all its coefficients are 0. This shows that $\phi_1, \ldots, \phi_d$ satisfy (e). The same method gives that $\{\phi_1^2, \ldots, \phi_d^2\}$ is a linearly independent set, and since for $1 \le i \ne j \le d$, the set $\{x \in S^{d-1} : \phi_i(x)\phi_j(x) \ne 0\}$ is dense in S^{d-1}, it follows that also condition (d) holds for $\phi_1, \ldots, \phi_d$.

Next we show that $(a)-(d)$ hold for the restriction of $(\phi_1, \ldots, \phi_d)$ to some finite subset of S^{d-1}. For this, assume that (e) fails for the restriction of $(\phi_1, \ldots, \phi_d)$ to any finite subset F of S^{d-1}. Then, for each such F, we can find coefficients c_{ij}^F, $1 \le i \le j \le d$, not all equal to zero, such that

$$\sum_{1 \le i \le j \le d} c_{ij}^F \phi_i^2(x)\phi_j^2(x) = 0, \quad x \in F.$$

Moreover, we can assume that $\sum_{1 \leq i \leq j \leq d} |c_{ij}^F|^2 = 1$. Take now a weak*-limit point $c = (c_{ij})_{1 \leq i \leq j \leq d}$ of the net $((c_{ij}^F)_{1 \leq i \leq j \leq d})_F$, where the finite subsets $F \subseteq S^{d-1}$ are ordered by inclusion. Then

$$\sum_{1 \leq i \leq j \leq d} c_{ij} \phi_i^2(x) \phi_j^2(x) = 0, \quad x \in S^{d-1},$$

and not all coefficients c_{ij} above vanish. This contradicts the fact that $\phi_1, \ldots, \phi_d$ satisfy (e). Using this type of argument, it is easy to see that one can choose a finite subset F of S^{d-1} such that not only (e), but also (d) and (c) hold for the restrictions of $\phi_1, \ldots, \phi_d$ to F. Of course, conditions (a) and (b) also hold for these restrictions. Set now $m := |F|$, and let $F = \{p_1, \ldots, p_m\}$. Then the diagonal matrices

$$b_i : = \mathrm{diag}\{\phi_i(p_1), \ldots, \phi_i(p_m)\}, \quad 1 \leq i \leq d$$

in $M_m(\mathbb{C})$ satisfy $(a) - (d)$.

It remains to be proved that we can find self-adjoint unitaries $u_1, \ldots, u_d$ in some matrix algebra $M_r(\mathbb{C})$ satisfying (f). For this, consider the free product group $G : = \mathbb{Z}_2 * \ldots * \mathbb{Z}_2$ (d copies), and let $g_1, \ldots, g_d$ be its generators. Then $g_i^2 = 1$, for all $1 \leq i \leq d$ and $g_{i_1} g_{i_2} \ldots g_{i_s} \neq 0$, whenever s is a positive integer and $i_1 \neq i_2 \neq \ldots \neq i_s$. Since G is residually finite (cf. [11]), by passing to a quotient of G we can find a finite group Γ generated by $\gamma_1, \ldots, \gamma_d$ such that $\gamma_i^2 = 1$, for all $1 \leq i \leq d$, and $\gamma_{i_1} \gamma_{i_2} \ldots \gamma_{i_s} \neq 0$, whenever $1 \leq s \leq d$ and $i_1 \neq i_2 \neq \ldots \neq i_s$. This implies that the group elements listed in the set $\{1\} \cup \{\gamma_i \gamma_j : 1 \leq i \neq j \leq d\} \cup \{\gamma_i \gamma_j \gamma_k \gamma_l : 1 \leq i \neq j \neq k \neq l \leq d\}$ are all distinct. Set now $r : = |\Gamma|$ and let $u_1, \ldots, u_d$ be the ranges of $\gamma_1, \ldots, \gamma_d$ by the left regular representation λ_Γ of Γ in $\mathcal{B}(l^2(\Gamma)) \simeq M_r(\mathbb{C})$. Then $u_1, \ldots, u_d$ are self-adjoint unitaries. Moreover, the set $\{\lambda_\Gamma(\gamma) : \gamma \in \Gamma\}$ is linearly independent, because $\lambda_\Gamma(\gamma)\delta_e = \delta_\gamma$, where $\delta_\gamma \in l^2(\Gamma)$ is defined by $\delta_\gamma(\gamma') = 1$, if $\gamma' = \gamma$ and $\delta_\gamma(\gamma') = 0$, else. Hence $u_1, \ldots, u_d$ satisfy (f) and the proof is complete. $\square$

Remark 5.8. The above proof does not provide explicit numbers m and r for which $b_1, \ldots, b_d$ and $u_1, \ldots u_d$ can be realized, but it is easy to find lower bounds. For $d = 5$, the set in (e) has 15 linearly independent elements, and since the b_i's can be simultaneously diagonalized, it follows that $m \geq 15$. Also, for $d = 5$, the set in (f) has $1 + 5 \times 4 + 5 \times 4^3 = 341$ linearly independent elements in $M_r(\mathbb{C})$. Hence $r^2 \geq 341$, which implies that $r \geq 19$. Altogether, we conclude that $n : = mr \geq 15 \times 19 = 285$.

We will end this section by giving a characterization of those τ_M-Markov maps $S : M \to M$ which admit a Rota dilation.

Theorem 5.9. *Let M be a finite von Neumann algebra with normal, faithful, tracial state τ_M, and let $S : M \to M$ be a linear operator. Then the following statements are equivalent:*

  (i) *S satisfies the Rota dilation property.*
  (ii) *$S = T^*T$, for some factorizable (τ_M, τ_N)-Markov map $T : M \to N$ taking values in a von Neumann algebra N with a normal, faithful, tracial state τ_N.*

Proof. The implication $(i) \Rightarrow (ii)$ follows immediately from Remark 5.2 (see (5.25) therein). We now prove that $(ii) \Rightarrow (i)$. Suppose that there exists a factorizable (τ_M, τ_N)-Markov map $T : M \to N$, where N is a von Neumann algebra with a normal, faithful, tracial state τ_N such that

$$S = T^*T.$$

Since T is factorizable, it follows by Remark 1.4 (a) that there exists a finite von Neumann algebra P with a normal, faithful tracial state τ_P such that $T = \beta^* \circ \alpha$, where $\alpha : M \to P$ and $\beta : N \to P$ are unital $*$-monomorphisms satisfying $\tau_M = \tau_P \circ \alpha$ and $\tau_N = \tau_P \circ \beta$. Consider now the von Neumann algebras $M \oplus N$ and $P \oplus P$, equipped with the normal, faithful, tracial states defined by $\tau_{M \oplus N} : = (\tau_M \oplus \tau_N)/2$ and $\tau_{P \oplus P} : = (\tau_P \oplus \tau_P)/2$, respectively. Further, define an operator $\tilde{T}$ on $M \oplus N$ by

$$\tilde{T}(x,y) : = (T^*y, Tx), \quad x \in M, y \in N.$$

Then $\tilde{T}$ is a $\tau_{M \oplus N}$-Markov map on $M \oplus N$ and $\tilde{T}^* = \tilde{T}$. Moreover, $\tilde{T}$ is factorizable, since $\tilde{T} = \delta^* \circ \gamma$, where $\delta, \gamma : M \oplus N \to P \oplus P$ are the $*$-monomorphisms given by $\gamma(x,y) : = (\alpha(x), \beta(y))$, respectively, $\delta(x,y) : = (\beta(y), \alpha(x))$, $x \in M$, $y \in N$, and $\tau_{M \oplus N} = (\tau_{P \oplus P}) \circ \gamma = (\tau_{P \oplus P}) \circ \delta$.

Hence, by Theorem 5.3, $\tilde{T}^2$ has a Rota dilation, i.e., there exists a finite von Neumann algebra Q with a normal, faithful, tracial state τ_Q, a unital $*$-monomorphism $i : M \oplus N \to Q$ for which $\tau_{M \oplus N} = \tau_Q \circ i$, and a decreasing sequence $(Q_n)_{n \geq 1}$ of von Neumann subalgebras of Q such that

$$\tilde{T}^{2n} = i^* \circ \mathbb{E}_{Q_n} \circ i, \quad n \geq 1,$$

where $\mathbb{E}_{Q_n}$ is the unique τ_Q-preserving normal conditional expectation of Q onto Q_n. Note that

$$\tilde{T}^{2n}(x,y) = ((T^*T)^n x, (TT^*)^n y), \quad (x,y) \in M \oplus N.$$

In particular, $\tilde{T}^{2n}(1_M, 0_N) = (1_M, 0_N)$. Set $e : = i((1_M, 0_N))$. Then e is a projection in Q. We will show next that $e \in Q_n$, for all $n \geq 1$. For simplicity of notation, set $z : = (1_M, 0_N)$ and $w : = 1_{M \oplus N} - z$. For all $n \geq 1$, $w^* \tilde{T}^{2n}(z) = w^* z = 0_{M \oplus N}$, and therefore

$$
\begin{aligned}
0 &= \langle \tilde{T}^{2n}(z), w \rangle_{L^2(M \oplus N)} \\
&= \langle (i^* \circ \mathbb{E}_{Q_n} \circ i)(z), w \rangle_{L^2(M \oplus N)} \\
&= \langle \mathbb{E}_{Q_n}(i(z)), i(w) \rangle_{L^2(Q)} \\
&= \tau_Q((1_Q - e) \mathbb{E}_{Q_n}(e)) \\
&= \tau_Q((1_Q - e) \mathbb{E}_{Q_n}(1_Q - e)).
\end{aligned}
$$

Since τ_Q is faithful and $\mathbb{E}_{Q_n}(e) \geq 0$, it follows that $\mathbb{E}_{Q_n}(e) \in eQe$. Similarly, we obtain that $\mathbb{E}_{Q_n}(1-e) \in (1_Q - e)Q(1_Q - e)$. Since $\mathbb{E}_{Q_n}(e) - e = (1_Q - e) - \mathbb{E}_{Q_n}(1_Q - e)$, we deduce that

$$\mathbb{E}_{Q_n}(e) - e \in eQe \cap (1_Q - e)Q(1_Q - e) = \{0_Q\}, \quad n \geq 1,$$

which proves the claim. Further, note that $\tau_Q(e) = \tau_{M \oplus N}((1_N, 0_N)) = 1/2$. Set $R : = eQe$, $\tau_R : = 2(\tau_Q)_{|R}$, and for $x \in M$, let $j(x) : = i(x, 0_N)$. Then it is easy to check that R is a von Neumann algebra with normal, faithful tracial state τ_R and that the map $j : M \to R$ above defined is a unital $*$-monomorphism for which $\tau_M = j \circ \tau_R$. Moreover, for all $n \geq 1$,

$$(T^*T)^n = j^* \circ \mathbb{E}_{R_n} \circ j, \quad n \geq 1,$$

where $R_n : = eQ_n e$, $n \geq 1$, form a decreasing sequence of von Neumann subalgebras of R, and $\mathbb{E}_{R_n} : = (\mathbb{E}_{Q_n})_{|R}$ is the unique τ_R-preserving conditional expectation of R onto R_n. It follows by the definition that $S = T^*T$ has a Rota dilation. $\qquad \square$

Note that from the proof of Theorem 5.9 it follows right-away that in order for a linear map $T\colon M \to M$ to satisfy the Rota dilation property, it suffices that it satisfies the conditions set forth in Definition 5.1 for $m = 1$, only.

6. On the asymptotic quantum Birkhoff conjecture

In 1946 G. Birkhoff [5] proved that every doubly stochastic matrix is a convex combination of permutation matrices. Note that if one considers the abelian von Neumann algebra $D\colon = l^\infty(\{1, 2, \ldots, n\})$ with trace given by $\tau(\{i\}) = 1/n$, $1 \le i \le n$, then the positive unital trace-preserving maps on D are those linear operators on D which are given by doubly stochastic $n \times n$ matrices. Since every automorphism of D is given by a permutation of $\{1, 2, \ldots, n\}$, this led naturally to the question whether Birkhoff's classical result extends to the quantum setting. The statement that every completely positive, unital trace-preserving map $T\colon (M_n(\mathbb{C}), \tau_n) \to (M_n(\mathbb{C}), \tau_n)$ lies in $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ turned out to be false for $n \ge 3$. For the case $n \ge 4$, this was shown by Kümmerer and Maasen (cf. [24]), while the case $n = 3$ was settled by Kümmerer in [23] (see Remark 4.8). In [25], Landau and Streater gave a more elementary proof of Kümmerer and Maasen's result, and also constructed another counterexample to the quantum Birkhoff conjecture in the case $n = 3$.

Recently, V. Paulsen brought to our attention the following asymptotic version of the quantum Birkhoff conjecture, listed as Problem 30 on R. Werner's web page of open problems in quantum information theory (see [36]):

The asymptotic quantum Birkhoff conjecture: Let $n \ge 1$. If $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a τ_n-Markov map, then T satisfies the following *asymptotic quantum Birkhoff property*:

$$(6.43) \qquad \lim_{k \to \infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) = 0.$$

As mentioned in the introduction, this conjecture originates in joint work of A. Winter, J. A. Smolin and F. Verstraete. The main results obtained in [33] motivated its formulation. We would like to thank M.-B. Ruskai for kindly providing us with the Report of the workshop on Operator structures in quantum information theory that took place at BIRS, February 11-16, 2007, where A. Winter discussed the conjecture, as well as for pointing out related very recent work of C. Mendl and M. Wolf (cf. [27]).

Using the existence of non-factorizable Markov maps, we prove the following:

Theorem 6.1. *For every $n \ge 3$, there exist τ_n-Markov maps on $M_n(\mathbb{C})$ which do not satisfy the asymptotic quantum Birkhoff property (6.43).*

Proof. We will show that any non-factorizable τ_n-Markov map on $M_n(\mathbb{C})$ does not satisfy (6.43). Such maps do exist for every $n \ge 3$, as it was shown in Section 3.

The key point in our argument is to prove that any τ_n-Markov map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $n \ge 3$, satisfies the following inequality:

$$(6.44) \qquad d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T, \mathcal{FM}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)\right) \ge d_{\mathrm{cb}}(T, \mathcal{FM}(M_n(\mathbb{C}))), \quad k \ge 1.$$

Then, since $\mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C}))) \subset \mathcal{FM}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)$, for all $k \ge 1$, the desired conclusion will follow immediately, using the fact that the set of factorizable maps on $M_n(\mathbb{C})$ is closed in the norm-topology, cf. Remark 1.4 (b). (Note that in our concrete finite-dimensional setting, this latter fact can also be obtained directly from Theorem 2.2 using a simple ultraproduct argument.)

Now, in order to prove (6.44), we show that given $m, l \geq 3$, then for any τ_m-Markov map T on $M_m(\mathbb{C})$ and any τ_l-Markov map S on $M_l(\mathbb{C})$, we have

(6.45) $$d_{\mathrm{cb}}(T \otimes S, \mathcal{FM}(M_m(\mathbb{C}) \otimes M_l(\mathbb{C}))) \geq d_{\mathrm{cb}}(T, \mathcal{FM}(M_m(\mathbb{C}))).$$

Let $\iota \colon M_m(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_l(\mathbb{C})$ be defined by $\iota(x) := x \otimes 1_l$, for all $x \in M_m(\mathbb{C})$. Then its adjoint map $\iota^* \colon M_m(\mathbb{C}) \otimes M_l(\mathbb{C}) \to M_m(\mathbb{C})$ is given by $\iota^*(z) = (1_m \otimes \tau_l)(z)$, for all $z \in M_m(\mathbb{C}) \otimes M_l(\mathbb{C})$. It is easily checked that $\iota^*(T \otimes S)\iota = T$. Since $\|\iota\|_{\mathrm{cb}} = \|\iota^*\|_{\mathrm{cb}} = 1$, we then obtain

(6.46) $$d_{\mathrm{cb}}(T \otimes S, \mathcal{FM}(M_m(\mathbb{C}) \otimes M_l(\mathbb{C}))) \geq d_{\mathrm{cb}}(T, \iota^*\mathcal{FM}(M_m(\mathbb{C}) \otimes M_l(\mathbb{C}))\iota).$$

By the permanence properties of factorizability, $\iota^*\mathcal{FM}(M_m(\mathbb{C}) \otimes M_m(\mathbb{C}))\iota \subset \mathcal{FM}(M_m(\mathbb{C}))$. Together with (6.46), this completes the proof of (6.45), which, in turn, yields (6.44). $\qquad\square$

It is now a natural question whether every factorizable τ_n-Markov map on $M_n(\mathbb{C})$ does satisfy the asymptotic quantum Birkhoff property, for all $n \geq 3$. It turns out that this question has a tight connection to Connes' embedding problem, which is known to be equivalent to a number of different fundamental problems in operator algebras (for references, see, e.g., Ozawa's excellent survey paper [28]).

Theorem 6.2. *If for any $n \geq 3$, every factorizable τ_n-Markov map on $M_n(\mathbb{C})$ satisfies the asymptotic quantum Birkhoff property, then Connes' embedding problem has a positive answer.*

Proof. Assume by contradiction that Connes's embedding problem has a negative answer. Then, by Dykema and Juschenko's results from [10] (as explained in Remark 3.6), there exists a positive integer n such that $\mathcal{G}_n \setminus \mathcal{F}_n \neq \emptyset$. Choose $B \in \mathcal{G}_n \setminus \mathcal{F}_n$. It follows that the associated Schur multiplier T_B is factorizable.

We will prove that T_B does not satisfy the asymptotic quantum Birkhoff property. Suppose by contradiction that T_B does satisfy (6.43). For every positive integer k, let $\iota_k \colon M_n(\mathbb{C}) \to M_{n^k}(\mathbb{C})$ be the map defined by $\iota_k(x) := x \otimes 1_n \otimes \ldots \otimes 1_n$, for all $x \in M_n(\mathbb{C})$. We deduce that

$$\lim_{k \to \infty} d_{\mathrm{cb}}(T, \ \mathrm{conv}(\iota_k^* \circ \mathrm{Aut}(M_{n^k}(\mathbb{C})) \circ \iota_k)) = 0.$$

For $k \geq 1$, choose operators $T_k \in \mathrm{conv}(\iota_k^* \circ \mathrm{Aut}(M_{n^k}(\mathbb{C})) \circ \iota_k)$ such that

(6.47) $$\lim_{k \to \infty} \|T - T_k\|_{\mathrm{cb}} = 0.$$

Each T_k is of the form $T_k = \sum_{i=1}^{s_k} c_i^{(k)} \iota_k \circ \mathrm{ad}(u_i^{(k)}) \circ \iota_k$, for some positive integer s_k, unitaries $u_i^{(k)} \in \mathcal{U}(n^k)$ and positive numbers $c_i^{(k)}$, $1 \leq i \leq s_k$, with $\sum_{i=1}^{s_k} c_i^{(k)} = 1$.

Set $u_k := (u_1^{(k)}, \ldots, u_{s_k}^{(k)})$. Then $u_k \in \bigoplus_{i=1}^{s_k} M_{n^k}(\mathbb{C}) := A_k$. Equip A_k with the trace given by $\tau((a_1, \ldots, a_{s_k})) := \sum_{i=1}^{s_k} c_i^{(k)} \tau_{n^k}(a_i)$, for all $(a_1, \ldots, a_{s_k}) \in A_k$. Finally, define $j_k \colon M_n(\mathbb{C}) \to A_k$ by $j_k(x) := (\iota_k(x), \ldots, \iota_k(x))$ (s_k terms), for all $x \in M_n(\mathbb{C})$. It can be checked that the adjoint j_k^* of j_k is given by $j_k^*((a_1, \ldots, a_{s_k})) = \iota_k^*(\sum_{i=1}^{s_k} c_i^{(k)} a_i)$, for all $(a_1, \ldots, a_{s_k}) \in A_k$. Then T_k can be rewritten as

(6.48) $$T_k = j_k^* \circ \mathrm{ad}(u_k) \circ j_k.$$

Since A_k admits naturally a τ_k-preserving embedding into the hyperfinite II$_1$-factor R, equipped with its trace τ_R, we can replace in the above formula A_k by R, and view u_k as a unitary in R, respectively view j_k as a unital embedding of $M_n(\mathbb{C})$ into R. By taking ultraproducts, and using (6.47) we obtain that

$$T = j^* \circ \mathrm{ad}(u) \circ j,$$

where u is a unitary in the ultrapower R^ω of R and $j\colon M_n(\mathbb{C}) \to R^\omega$ is a unital embedding. Using the identification

$$R^\omega = j(M_n(\mathbb{C})) \otimes (j(M_n(\mathbb{C}))' \cap R^\omega) \simeq M_n(j(M_n(\mathbb{C}))' \cap R^\omega),$$

we obtain from the proof of Lemma 2.8 applied to the factorizable Schur multiplier T_B that $u = \mathrm{diag}(u_1, \dots, u_n)$, where $u_i \in j(M_n(\mathbb{C}))' \cap R^\omega \subset R^\omega$, for all $1 \le i \le n$, and that the (k,l)-th entry B_{kl} of B is given by $B_{kl} = \tau_{R^\omega}(u_k^* u_l)$, for all $1 \le k, l \le n$. Here τ_{R^ω} denotes the trace on R^ω.

By a standard ultraproduct argument, for every $1 \le k \le n$, we can find a sequence $(u_k^{(m)})_{m \ge 1}$ of unitaries in R representing u, and we conclude that

$$\lim_{m \to \infty} \tau_R \left(\left(u_k^{(m)} \right)^* u_l^{(m)} \right) = \tau_{R^\omega}(u_k^* u_l), \quad 1 \le k, l \le n.$$

Since $\mathcal{F}_n$ is a closed set, this shows that $B \in \mathcal{F}_n$, which contradicts the assumption on B. Therefore, the proof is complete. $\qquad\square$

7. ON THE BEST CONSTANT IN THE NONCOMMUTATIVE LITTLE GROTHENDIECK INEQUALITY

Let $OH(I)$ denote Pisier's operator Hilbert space based on $l^2(I)$, for a given index set I. Further, let A be a C*-algebra and $T\colon A \to OH(I)$ a completely bounded map. Then, by the refinement of the second part of Corollary 3.4 of [30] obtained in [14], there exist states f_1, f_2 on A such that

$$(7.49) \qquad \|Tx\| \le \sqrt{2}\, \|T\|_{\mathrm{cb}} f_1(xx^*)^{1/4} f_2(x^*x)^{1/4}, \quad x \in A.$$

Definition 7.1. *For a completely bounded map $T\colon A \to OH(I)$ we denote by $C(T)$ the smallest constant $C > 0$ for which there exist states f_1, f_2 on A such that*

$$(7.50) \qquad \|Tx\| \le C f_1(xx^*)^{1/4} f_2(x^*x)^{1/4}, \quad x \in A.$$

The existence of a smallest constant $C(T)$ as above follows from a simple compactness argument using the fact that the set $Q(A) := \{f \in A_+^* : \|f\| \le 1\}$ is w*-compact. From [30] we know that $\|T\|_{\mathrm{cb}} \le C(T)$. Hence, by (7.49), we infer that

$$(7.51) \qquad \|T\|_{\mathrm{cb}} \le C(T) \le \sqrt{2}\, \|T\|_{\mathrm{cb}}.$$

In the following we will prove that for suitable choices of A, I and T, $C(T) > \|T\|_{\mathrm{cb}}$, i.e., the constant $\sqrt{2}$ in (7.49) cannot be reduced to 1. It would be interesting to know what is the best constant C_0 in the noncommutative little Grothendieck inequality (7.49), i.e., what is the smallest constant C_0 for which $C(T) \le C_0 \|T\|_{\mathrm{cb}}$, for arbitrary choices of A, I and T.

Theorem 7.2. *There exist linear maps $T_1\colon M_3(\mathbb{C}) \to OH(\{1, 2, 3\})$ and $T_2\colon l^\infty(\{1, \dots, 4\}) \to OH(\{1, 2\})$ such that*

$$\|T_i\|_{cb} < C(T_i), \quad i = 1, 2.$$

In particular, the best constant C_0 in the noncommutative little Grothendieck inequality (7.49) is strictly larger than 1.

The key result that will be used in the proof of the above theorem is the following:

Theorem 7.3. *Let (A, τ) be a finite dimensional (unital) C*-algebra with a faithful trace τ. Furthermore, let d be a positive integer and let $a_1, \dots, a_d$ be elements in A satisfying*

29

(i) $\tau(a_i^* a_j) = \delta_{ij}$, $1 \le i,j \le d$,

(ii) $\sum_{i=1}^{d} a_i^* a_i = \sum_{i=1}^{d} a_i a_i^* = d1_A$.

Consider the map $T \colon A \to OH(d) \colon = OH(\{1 \ldots, d\})$ *given by*

(iii) $Tx \colon = (\tau(a_1^* x), \ldots, \tau(a_d^* x))$, $a \in A$.

Then $C(T) = 1$. *If, furthermore,*

(iv) $d \ge 2$,

(v) $\{a_i^* a_j : 1 \le i,j \le d\}$ *is linearly independent,*

then $\|T\|_{cb} < 1$.

We will first prove a number of intermediate results.

Lemma 7.4. *Let* A , τ , $a_1, \ldots, a_d$ *and* $T \colon A \to OH(d)$ *be as in Theorem 7.3* (i) , (ii) *and* (iii) . *Then* $C(T) = 1$.

Proof. By (i) , $a_1, \ldots, a_d$ is an orthonormal set in $L^2(A, \tau)$. Moreover, $\tau(a_1^* x), \ldots, \tau(a_d^* x)$ are the coordinates of the orthogonal projection P of $x \in A = L^2(A, \tau)$ onto $E \colon = \mathrm{span}\{a_1 \ldots, a_d\}$ with respect to the basis $\{a_1, \ldots, a_d\}$. Thus

$$\|Tx\| = \sum_{i=1}^{d} |\tau(a_i^* x)|^2 = \|Px\|_2 \le \|x\|_2 = \tau(x^* x)^{1/2} = \tau(x^* x)^{1/4} \tau(x^* x)^{1/4} .$$

Hence $C(T) \le 1$. Conversely, assume that

$$\|Tx\| \le K f_1(xx^*)^{1/4} f_2(x^* x)^{1/4} , \quad x \in A ,$$

for a constant $K > 0$ and states f_1 , f_2 on A . By (i) and (ii) , it follows that for every $1 \le i \le d$,

$$Ta_i = (0, \ldots, 1, 0, \ldots, 0) ,$$

where the number 1 is at the ith coordinate. Therefore, $1 = \|Ta_i\|^2 \le K^2 f_1(a_i a_i^*)^{1/2} f_2(a_i^* a_i)^{1/2}$. By the Cauchy-Schwarz inequality and (ii) , we infer that

$$d = \sum_{i=1}^{d} \|Ta_i\|^2 \le K^2 \left(\sum_{i=1}^{2} f_1(a_i a_i^*) \right)^{1/2} \left(\sum_{i=1}^{2} f_2(a_i^* a_i) \right)^{1/2} = K^2 d .$$

Hence $K \ge 1$, which proves that $C(T) \ge 1$, and the conclusion follows. $\square$

Lemma 7.5. *Let* A , τ , $a_1, \ldots, a_d$ *be as in Theorem 7.3* (i) *and* (ii) , *set* $r \colon = \dim(A)$ *and choose* $a_{d+1}, \ldots, a_r$ *such that the set* $\{a_1, \ldots, a_r\}$ *is an orthonormal basis for* A . *Let* B *be a unital* C^*-*algebra. Then every element* $u \in A \otimes B$ *has a unique representation of the form*

(7.52)
$$u = \sum_{i=1}^{r} a_i \otimes u_i ,$$

where $u_i \in B$, $1 \le i \le d$. *Moreover, if* u *is unitary, then*

(7.53)
$$\sum_{i=1}^{r} u_i^* u_i = \sum_{i=1}^{r} u_i u_i^* = 1_B .$$

Proof. Existence and uniqueness of $u_1, \ldots, u_r \in B$ in (7.52) is obvious. To prove (7.53), note that if $u^*u = uu^* = 1_{A \otimes B}$, then by (i),

$$1_B = (\tau_A \otimes \mathrm{id}_B)(u^*u) = \sum_{i,j=1}^{d} \tau_A(a_i^* a_j) u_i^* u_j = \sum_{i=1}^{d} u_i^* u_i \,,$$

and similarly, by the trace property of τ_A and (i),

$$1_B = (\tau_A \otimes \mathrm{id}_B)(uu^*) = \sum_{i,j=1}^{d} \tau_A(a_i a_j^*) u_i u_j^* = \sum_{i=1}^{d} u_i u_i^* \,,$$

which completes the proof. $\qquad\square$

Lemma 7.6. *Let A, τ, $a_1, \ldots, a_d$ and T be as in Theorem 7.3 (i), (ii) and (iii), and choose $a_{d+1}, \ldots, a_r$ as in Lemma 7.5. Assume further that $\|T\|_{cb} = 1$. Then*

(a) *For every $\varepsilon > 0$, there exists a unital C^*-algebra $B(\varepsilon)$, a Hilbert $B(\varepsilon)$-bimodule $\mathcal{H}(\varepsilon)$, elements $u_1, \ldots, u_r \in B(\varepsilon)$ and unit vectors $\xi, \eta \in \mathcal{H}(\varepsilon)$ such that the operator $u: = \sum_{i=1}^{r} a_i \otimes u_i \in A \otimes B(\varepsilon)$ is unitary and the following inequalities are satisfied:*

(7.54)
$$\sum_{i=1}^{d} \|u_i \xi - \eta u_i\|^2 + \sum_{i=d+1}^{r} (\|u_i \xi\|^2 + \|\eta u_i\|^2) < \varepsilon\,,$$

(7.55)
$$\sum_{i=1}^{d} \|u_i^* \eta - \xi u_i^*\|^2 + \sum_{i=d+1}^{r} (\|u_i^* \eta\|^2 + \|\xi u_i^*\|^2) < \varepsilon\,.$$

(b) *There exist a unital C^*-algebra B, a Hilbert bimodule $\mathcal{H}$, elements $u_1, \ldots, u_r \in B$ and unit vectors $\xi, \eta \in \mathcal{H}$ such that the operator $u: = \sum_{i=1}^{r} a_i \otimes u_i \in A \otimes B$ is unitary and the following identities are satisfied:*

(7.56)
$$u_i \xi = \eta u_i\,, \quad u_i^* \eta = \xi u_i^*\,, \qquad 1 \le i \le d\,,$$

(7.57)
$$u_i \xi = \eta u_i = u_i^* \eta = \xi u_i^* = 0\,, \quad d+1 \le i \le r\,.$$

Proof. Let $\varepsilon > 0$. Since $\|T\|_{cb} = 1$, there exists a positive integer n such that $\|T \otimes \mathrm{id}_{M_n(\mathbb{C})}\|_{cb} > 1 - \varepsilon/4$. Since $\dim(A) < \infty$, the unit ball of $M_n(A) = A \otimes M_n(\mathbb{C})$ is the convex hull of its unitary operators. Hence there exists a unitary operator $u \in A \otimes M_n(\mathbb{C})$ such that

$$\|(T \otimes \mathrm{id}_{M_n(\mathbb{C})})(u)\|_{M_n(OH)} > 1 - \varepsilon/4\,.$$

By Lemma 7.5, u has the form $u = \sum_{i=1}^{r} a_i \otimes u_i$ for a unique set of elements $u_1, \ldots, u_r \in M_n(\mathbb{C})$ satisfying

(7.58)
$$\sum_{i=1}^{r} u_i^* u_i = \sum_{i=1}^{r} u_i u_i^* = 1_n\,.$$

By condition (iii) in Theorem 7.3,

$$T(a_i) = \begin{cases} e_i, & 1 \le i \le d \\ 0, & d+1 \le i \le r, \end{cases}$$

where e_i is the ith vector in the standard unit vector basis of $l^2(\{1,\dots,d\}) = OH(d)$. Hence

$$(T \otimes \mathrm{id}_{M_n(\mathbb{C})})(u) = \sum_{i=1}^{d} e_i \otimes u_i.$$

It then follows (cf. [29]) that

$$\left\|\sum_{i=1}^{d} u_i \otimes \bar{u_i}\right\|_{M_n(\mathbb{C}) \otimes \overline{M_n(\mathbb{C})}} = \left\|\sum_{i=1}^{d} e_i \otimes u_i\right\|^2_{M_n(OH)} = \left\|(T \otimes \mathrm{id}_{M_n(\mathbb{C})})(u)\right\|_{M_n(OH)} > \left(1 - \frac{\varepsilon}{4}\right)^2 > 1 - \frac{\varepsilon}{2}.$$

We can identify $M_n(\mathbb{C}) \otimes \overline{M_n(\mathbb{C})}$ isometrically with the bounded operators on $HS(n) = L^2(M_n(\mathbb{C}), \mathrm{Tr})$, where Tr denotes the standard non-normalized trace on $M_n(\mathbb{C})$, by letting $a \otimes \bar{b}$ correspond to $L_a R_{b^*}$, i.e., $(a \otimes \bar{b})\xi = a\xi b^*$, for all $\xi \in HS(n)$. In particular,

$$\sum_{i=1}^{d}(u_i \otimes \overline{u_i})\xi = \sum_{i=1}^{r} u_i \xi u_i^*, \quad \xi \in HS(n).$$

Let $HS(n)_{\mathrm{sa}}$ denote the self-adjoint part of $HS(n)$. Since $HS(n) = HS(n)_{\mathrm{sa}} + iHS(n)_{\mathrm{sa}}$, and the operator $\sum_{i=1}^{d} u_i \otimes \overline{u_i} \in \mathcal{B}(HS(n))$ leaves $HS(n)_{\mathrm{sa}}$ invariant, one checks easily that the norm of $\sum_{i=1}^{d} u_i \otimes \overline{u_i}$ is the same as the norm of $\sum_{i=1}^{d} u_i \otimes \overline{u_i}$ restricted to $HS(n)_{\mathrm{sa}}$. By compactness of the unit ball in $HS(n)_{\mathrm{sa}}$ we deduce that there exist vectors $\xi, \eta \in HS(n)_{\mathrm{sa}}$ such that

$$(7.59) \qquad \qquad \|\xi\|_2 = \|\eta\|_2 = 1,$$

satisfying, moreover,

$$\left|\left\langle \sum_{i=1}^{d} u_i \xi u_i^*, \eta \right\rangle_{HS(n)}\right| = \left\|\sum_{i=1}^{d} u_i \otimes \overline{u_i}\right\| > 1 - \frac{\varepsilon}{2}.$$

Furthermore, since $\left\langle \sum_{i=1}^{d} u_i \xi u_i^*, \eta \right\rangle_{HS(n)}$ is a real number, we infer that

$$(7.60) \qquad \sum_{i=1}^{d}\langle u_i \xi, \eta u_i \rangle = \left\langle \sum_{i=1}^{d} u_i \xi u_i^*, \eta \right\rangle_{HS(n)} > 1 - \frac{\varepsilon}{2},$$

by replacing η with $-\eta$, if needed. Hence,

$$(7.61) \qquad \sum_{i=1}^{d} \left(\|u_i \xi\|_2^2 + \|\eta u_i\|_2^2 - \|u_i \xi - \eta u_i\|_2^2\right) = 2\mathrm{Re}\sum_{i=1}^{d}\langle u_i \xi, \eta u_i \rangle_{HS(n)}$$

$$= 2\sum_{i=1}^{d}\langle u_i \xi u_i^*, \eta \rangle_{HS(n)} > 2 - \varepsilon.$$

Moreover, by (7.58) and (7.59),

$$(7.62) \qquad \sum_{i=1}^{r}\|u_i \xi\|_2^2 + \sum_{i=1}^{r}\|\eta u_i\|_2^2 = 2.$$

Subtracting (7.62) from (7.61), we get

$$\sum_{i=1}^{d}\|u_i \xi - \eta u_i\|_2^2 + \sum_{i=d+1}^{r}\left(\|u_i \xi\|_2^2 + \|\eta u_i\|_2^2\right) < \varepsilon.$$

Furthermore, since $\xi = \xi^*$ and $\eta = \eta^*$, we get by taking adjoints that

$$\sum_{i=1}^{d} \|u_i^* \xi - \eta u_i^*\|_2^2 + \sum_{i=d+1}^{r} \left(\|u_i \xi\|_2^2 + \|\eta u_i\|_2^2 \right) < \varepsilon.$$

This proves (a) with $B(\varepsilon) = M_n(\mathbb{C})$ and $\mathcal{H}(\varepsilon) = HS(n)$.

We now prove (b). Given a positive integer n, let $\varepsilon_n := 1/n^2$ and set $B_n := B(\varepsilon_n)$ and $\mathcal{H}_n := \mathcal{H}(\varepsilon_n)$. Then B_n is a unital C*-algebra and $\mathcal{H}_n$ a B_n-Hilbert bimodule. Moreover, there exist elements $u_1^{(n)}, \ldots, u_r^{(n)} \in B_n$ and unit vectors $\xi^{(n)}, \eta^{(n)} \in \mathcal{H}(n)$ such that the operator $u^{(n)} := \sum_{i=1}^{r} a_i \otimes u_i^{(n)} \in A \otimes B_n$ is unitary and the following inequalities hold:

$$\left\| u_i^{(n)} \xi_n - \eta_n u_i^{(n)} \right\| < 1/n, \quad \left\| \left(u_i^{(n)} \right)^* \eta_n - \xi_n \left(u_i^{(n)} \right)^* \right\| < 1/n, \qquad 1 \leq i \leq d$$

respectively,

$$\left\| \left(u_i^{(n)} \right)^* \xi_n \right\| < 1/n, \quad \left\| \eta_n u_i^{(n)} \right\| < 1/n, \quad \left\| \left(u_i^{(n)} \right)^* \eta_n \right\| < 1/n, \quad \left\| \xi_n \left(u_i^{(n)} \right)^* \right\| < 1/n, \qquad d+1 \leq i \leq r.$$

Now (b) follows from (a) by a standard ultraproduct construction (see, e.g., [12]). $\qquad \square$

Lemma 7.7. *Let A, τ, $a_1, \ldots, a_d$ and T be as in Theorem 7.3 (i), (ii) and (iii), and assume that $\|T\|_{cb} = 1$. Then*

(a) *There exist a finite von Neumann algebra N with a normal, faithful tracial state τ_N, a projection $p \in N$ and elements $v_1, \ldots, v_d \in (1-p)Np$ such that the operator $v := \sum_{i=1}^{d} a_i \otimes v_i \in A \otimes N$ is a partial isometry satisfying*

$$v^* v = 1_A \otimes p, \quad vv^* = 1_A \otimes (1-p).$$

(b) *There exist a finite von Neumann algebra P with a normal, faithful tracial state τ_P and elements $w_1, \ldots, w_d \in P$ such that the operator $w := \sum_{i=1}^{d} a_i \otimes w_i \in A \otimes P$ is unitary.*

Proof. Let $r := \dim(A)$ and choose $a_1, \ldots, a_r \in A$ as in Lemma 7.5. Further, let B, $\mathcal{H}$, $u_1, \ldots, u_r$ and ξ, η be as in Lemma 7.6 (b). In particular, the operator $u := \sum_{i=1}^{r} a_i \otimes u_i$ is a unitary in $A \otimes B$.

Note that $M_2(\mathcal{H})$ is an $M_2(B)$-bimodule by standard matrix multiplication, and $M_2(\mathcal{H})$ is a Hilbert space with norm

$$\left\| \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right\|_{M_2(\mathcal{H})}^2 = \sum_{i,j=1}^{2} \|\sigma_{ij}\|^2, \quad \sigma_{ij} \in \mathcal{H}, 1 \leq i,j \leq 2.$$

Set $\zeta := \frac{1}{\sqrt{2}} \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \in M_2(\mathcal{H})$ and $s_i := \begin{pmatrix} 0 & 0 \\ u_i & 0 \end{pmatrix} \in M_2(B)$, for all $1 \leq i \leq r$. Then $\|\zeta\| = 1$, and by (7.56) and (7.57) it follows that

(7.63) $$s_i \zeta = \zeta s_i, \ s_i^* \zeta = \zeta s_i^*, \quad 1 \leq i \leq d,$$

(7.64) $$s_i \zeta = s_i^* \zeta = 0, \qquad d+1 \leq i \leq r.$$

Further, set $e := \begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix} \in M_2(B)$. Then e is a projection and by (7.53),

(7.65) $$\sum_{i=1}^{r} s_i^* s_i = e, \quad \sum_{i=1}^{r} s_i s_i^* = 1_{M_2(B)} - e.$$

33

Next, set $s := \sum_{i=1}^{r} a_i \otimes s_i$. Since u is a unitary operator, it follows that

$$(7.66) \qquad s^* s = 1_A \otimes e, \quad ss^* = 1_A \otimes (1_{M_2(B)} - e).$$

Let C denote the C*-algebra generated by $s_1, \ldots, s_d$ in $M_2(B)$. By (7.65), both e and $1_{M_2(B)} - e$ belong to C, and hence $1_{M_2(B)} \in C$. Moreover, by (7.63) and (7.64),

$$(7.67) \qquad c\zeta = \zeta c, \quad c \in C.$$

Define now a state ϕ on C by $\phi(c) := \langle c\zeta, \zeta \rangle_{M_2(\mathcal{H})}$, for all $c \in C$. Note that ϕ is tracial, since

$$\phi(c^* c) = \langle c\zeta, c\zeta \rangle_{M_2(\mathcal{H})} = \langle \zeta c, \zeta c \rangle_{M_2(\mathcal{H})} = \langle \zeta c c^*, \zeta \rangle_{M_2(\mathcal{H})} = \langle cc^* \zeta, \zeta \rangle_{M_2(\mathcal{H})} = \phi(cc^*), \quad c \in C.$$

Let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS-representation of C with respect to ϕ. Then $N := \pi_\phi(C)''$ is a finite von Neumann algebra with normal, faithful tracial state τ_N given by $\tau_N(x) := \langle x\xi_\phi, \xi_\phi \rangle_{H_\phi}$, for all $x \in N$. Moreover, $\phi(c) = \tau_N(\pi_\phi(c))$, for all $c \in C$.

Now set $v_i := \pi_\phi(s_i)$, for all $1 \le i \le r$. By (7.64), it follows that $\tau_N(v_i^* v_i) = \phi(s_i^* s_i) = 0$, for all $d+1 \le i \le r$, and hence

$$(7.68) \qquad v_i = 0, \quad d+1 \le i \le r.$$

Next set $p := \pi_\phi(e)$. Then p is a projection in N. By (7.65) and (7.68) we infer that

$$(7.69) \qquad \sum_{i=1}^{d} v_i^* v_i = p, \quad \sum_{i=1}^{d} v_i v_i^* = 1_N - p.$$

Finally, set $v := (\mathrm{id}_A \otimes \pi_\phi)(s) = \sum_{i=1}^{r} a_i \otimes v_i = \sum_{i=1}^{d} a_i \otimes v_i$. Then, by (7.66) it follows that $v^* v = 1_A \otimes p$ and $vv^* = 1_A \otimes (1_N - p)$. This proves part (a).

To prove (b), we note first that by (7.69), p and $1_N - p$ have the same central valued trace, and therefore they are equivalent (as projections in N) (see, e.g., [18] (Vol. II, Chap. 8)). In particular, $\tau_N(p) = \tau_N(1_N - p) = 1/2$. Choose now $t \in N$ such that $t^* t = p$ and $tt^* = 1_N - p$, and set $w_i := t^* v_i$, for all $1 \le i \le d$. Then

$$\sum_{i=1}^{d} w_i^* w_i = \sum_{i=1}^{d} w_i w_i^* = p,$$

and the operator $w := \sum_{i=1}^{d} a_i \otimes w_i$ satisfies

$$w^* w = ww^* = 1_A \otimes p.$$

Hence (b) follows from (a) by setting $P := pNp$ and defining $\tau_P(x) := 2\tau_N(x)$, for all $x \in P$. $\qquad \square$

Proof of Theorem 7.3: By (7.51) and Lemma 7.4 we have that $\|T\|_{\mathrm{cb}} \le C(T) = 1$. If we assume by contradiction that $\|T\|_{\mathrm{cb}} = 1$, then by Lemma 7.7 (b), there exist a finite von Neumann algebra P with a normal, faithful tracial state τ_P and elements $w_1, \ldots, w_d \in P$ such that the operator $w := \sum_{i=1}^{d} a_i \otimes w_i$ is unitary in $A \otimes P$. By the hypothesis of Theorem 7.3 (cf. (iv) and (v)), the additional assumptions that $d \ge 2$ and the set $\{a_i^* a_j : 1 \le i, j \le d\}$ is linearly independent do hold. Therefore, we can proceed almost as in the proof of Corollary 2.3. Namely, we have $w^* w = \sum_{i,j=1}^{d} a_i^* a_j \otimes w_i^* w_j$. Therefore, using (ii) we deduce that

$$0_{A \otimes P} = w^* w - 1_A \otimes 1_P = \sum_{i,j=1}^{d} a_i^* a_j \otimes \left(w_i^* w_j - \frac{1}{d} \delta_{ij} 1_P \right).$$

Hence, by (v) we conclude that

$$w_i^* w_i = \frac{1}{d}\, \delta_{ij} 1_P \,, \quad 1 \le i, j \le d \,.$$

This implies that $\sqrt{d}w_1$ and $\sqrt{d}w_2$ are two isometries in the finite von Neumann algebra P, having orthogonal ranges. This is impossible. Therefore $\|T\|_{\mathrm{cb}} < 1$ and the proof is complete. $\qquad\square$

Proof of Theorem 7.2:

(1) Consider $A = M_3(\mathbb{C})$, $\tau = \tau_3$, $d = 3$, and let $a_1, a_2, a_3 \in M_3(\mathbb{C})$ be given by

$$a_1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a_3 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $T_1 \colon M_3(\mathbb{C}) \to OH(3)$ by

$$T_1(x) \colon = (\tau(a_1^* x), \tau(a_2^* x), \tau(a_3^* x)), \quad x \in M_3(\mathbb{C}).$$

(2) Consider $A = l^\infty(\{1, \ldots, 4\})$, $\tau(c) = (c_1 + \ldots + c_4)/4$, $c = (c_1, \ldots, c_4) \in A$, $d = 2$, and let $a_1, a_2 \in A$ be given by

$$a_1 \colon = (\sqrt{2}, \sqrt{2}/\sqrt{3}, \sqrt{2}/\sqrt{3}, \sqrt{2}/\sqrt{3}), \quad a_2 \colon = (0, 2/\sqrt{3}, (2/\sqrt{3})\omega, (2/\sqrt{3})\bar{\omega}),$$

where $\omega \colon = e^{i2\pi/3}$, and $\bar{\omega}$ is its complex conjugate. Define $T_2 \colon l^\infty(\{1, \ldots, 4\}) \to OH(2)$ by

$$T_2(x) \colon = (\tau(a_1^* x), \tau(a_2^* x)), \quad x \in l^\infty(\{1, \ldots, 4\}).$$

In each of the cases (1) and (2) it is easily checked that conditions (i), (ii), (iv) and (v) in the hypothesis of Theorem 7.3 are verified. Hence, the maps T_i (defined above according to condition (iii) of Theorem 7.3) will satisfy $C(T_i) = 1 > \|T_i\|_{\mathrm{cb}}$, for $i = 1, 2$. Note that a_1, a_2, a_3 in case (1) are scalar multiples of the matrices considered in Example 3.1, while a_1, a_2 in case (2) correspond, up to a scalar factor, to the diagonal 4×4 matrices in Example 3.2 with $s = 1/3$. $\qquad\square$

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