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TWO DIMENSIONAL MEIXNER RANDOM VECTORS OF CLASS $\mathcal{M}_L$

AUREL I. STAN

Abstract. The paper is divided into two parts. In the first part we lay down the foundation for defining the joint annihilation–preservation–creation decomposition of a finite family of, not necessarily commutative random variables, and show that this decomposition is essentially unique. In the second part we show that any two, not necessarily commutative, random variables X and Y, for which the vector space spanned by their annihilation, preservation, and creation operators equipped with the bracket given by the commutator, forms a Lie algebra, are equivalent, up to an invertible linear transformation to two independent Meixner random variables with mixed preservation operators. In particular if X and Y commute, then they are equivalent, up to an invertible linear transformation to two independent classic Meixner random variables. To show this we start with a small technical condition called "non–degeneracy".

1. Introduction

This paper continues the work from [12], by considering examples in dimensions higher than one. The multi–dimensional case is much more difficult than the one–dimensional one. We will restrict our attention to the case $d = 2$. The multi–dimensional case gives us the opportunity to work also in the non–commutative framework, by considering "random variables" that do not commute. In section 2 we introduce the creation, preservation, and annihilation (APC) operators for a finite family of not necessarily commutative random variables. In section 3, we make the crucial observation that the vector space spanned by the identity and APC operators of most of the Meixner random variables, equipped with the commutator as a bracket, forms a Lie algebra. We use this observation, to define the notion of "Meixner random vector (X, Y) of class $\mathcal{M}_L$". We also give two important examples of such random vectors, one in which X and Y commute, and another in which they do not commute. In section 4, making a simple and natural assumption, that we call "non–degeneracy", we classify all Meixner random vectors (X, Y) of class $\mathcal{M}_L$, by showing that through an invertible linear transformation, they can be reduced to the two examples from Section 3.

[1]*2000 Mathematics Subject Classifications:* 05E35, 60H40, 46L53.

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2. Background

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{R}$. Let $X_1, X_2, \dots, X_d$ be d symmetric densely defined linear operators on H. We denote by $\mathcal{A}$ the algebra generated by $X_1, X_2, \dots, X_d$. We assume that there exists an element ϕ of H, such that ϕ belongs to the domain of g, for any $g \in \mathcal{A}$. We fix such an element ϕ and call it the *vacuum vector*.

Definition 2.1. *We call any element g of $\mathcal{A}$, a* random variable. *For any g in $\mathcal{A}$, we define:*

$$E[g] \quad := \quad \langle g\phi, \phi \rangle \tag{2.1}$$

and call the number $E[g]$ the expectation *of the random variable g. Finally, we call the pair $(\mathcal{A}, \phi)$ a* probability space supported by H.

The above definition follows the classic GNS representation (see [13] and [15]). We restrict ourselves to finitely generated algebras, but this is not necessary. We always work with unital algebras $\mathcal{A}$, that means we assume that $I \in \mathcal{A}$, where I denotes the identity operator of H. It is not hard to see that, for any g in $\mathcal{A}$, there exists a polynomial $p(x_1, x_2, \dots, x_d)$, of d non–commutative variables $x_1, x_2, \dots, x_d$, such that $g = p(X_1, X_2, \dots, X_d)$. We also introduce the following equivalence relation:

Definition 2.2. *Let $(\mathcal{A}, \phi)$ and $(\mathcal{A}', \phi')$ be two probability spaces supported by two Hilbert spaces H and H', and let E and E' denote their expectations. Let $X_1, X_2, \dots, X_d$ be operators from $\mathcal{A}$, and $X_1', X_2', \dots, X_d'$ operators from $\mathcal{A}'$. We say that the vector random variables $(X_1, X_2, \dots, X_d)$ and $(X_1', X_2', \dots, X_d')$ are* moment equal *and denote this fact by $(X_1, X_2, \dots, X_d) \equiv (X_1', X_2', \dots, X_d')$, if for any polynomial $p(x_1, x_2, \dots, x_d)$ of d non–commutative variables, we have:*

$$E\left[p\left(X_1, X_2, \dots, X_d\right)\right] = E'\left[p\left(X_1', X_2', \dots, X_d'\right)\right]. \tag{2.2}$$

For any non–negative integer n, we define the space F_n as being the set of all vectors of the form $p(X_1, X_2, \dots, X_d)\phi \in H$, where p is a polynomial of total degree less than or equal to n. It is clear that F_n is a finite–dimensional subspace of H. Being finite dimensional, F_n is a closed subspace of H, for all $n \geq 0$. We have:

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset H$$

We define $G_0 := F_0$, and for all $n \geq 1$, $G_n := F_n \ominus F_{n-1}$, that means G_n is the orthogonal complement of F_{n-1} into F_n. For any $n \geq 0$, we call G_n the *homogenous chaos space of order* n generated by $X_1, X_2, \dots, X_d$. We also define the space:

$$\mathcal{H} \quad := \quad \oplus_{n=0}^{\infty} G_n,$$

and call $\mathcal{H}$ the *chaos space* generated by X_1, X_2, ..., X_d. It is not hard to see that $\mathcal{H}$ is the closure of the space $\mathcal{A}\phi := \{g\phi \mid g \in \mathcal{A}\}$ in H.

It is also easy to see, based on the fact that X_1, X_2, ..., X_d are symmetric operators, that we have the following lemma:

Lemma 2.3. *For any index $i \in \{1, 2, \ldots, d\}$ and any non–negative integer n:*

$$X_i G_n \quad \perp \quad G_k, \tag{2.3}$$

for all $k \neq n-1$, n, and $n+1$.

See also [1] for a proof. It follows now, mutatis mutandis as in [1], that for any $i \in \{1, 2, \ldots, d\}$, there exist three operators $a^-(i)$, $a^0(i)$, and $a^+(i)$, called the *annihilation*, *preservation*, and *creation* operators, respectively, such that:

$$X_i \quad = \quad a^-(i) + a^0(i) + a^+(i). \tag{2.4}$$

In (2.4), the domain of X_i, $a^-(i)$, $a^0(i)$, and $a^+(i)$ is understood to be $\mathcal{A}\phi$. It is important to remember that, for any $i \in \{1, 2, \ldots, d\}$ and $n \geq 0$, $a^-(i) : G_n \to G_{n-1}$, $a^0(i) : G_n \to G_n$, and $a^+(i) : G_n \to G_{n+1}$, where $G_{-1} := \{0\}$ is the null space.

If Y and Z are two operators, then we define their commutator $[Y, Z]$ as:

$$[Y, Z] \quad := \quad YZ - ZY.$$

It is also not hard to see that the operators X_1, X_2, ..., X_d commute among themselves if and only if the following conditions hold, for any i, $j \in \{1, 2, \ldots, d\}$:

$$\left[a^-(i), a^-(j)\right] \quad = \quad 0, \tag{2.5}$$

$$\left[a^-(i), a^0(j)\right] \quad = \quad \left[a^-(j), a^0(i)\right], \tag{2.6}$$

$$\left[a^-(i), a^+(j)\right] - \left[a^-(j), a^+(i)\right] \quad = \quad \left[a^0(j), a^0(i)\right]. \tag{2.7}$$

These conditions are derived, as in [1] and [2], from the fact that, for all $n \geq 0$:

$$[X_i, X_j] : G_n \to G_{n-2} \oplus G_{n-1} \oplus G_n \oplus G_{n+1} \oplus G_{n+2},$$

by projecting the equality $[X_i, X_j] = 0$ on the spaces G_{n-2}, G_{n-1}, and G_n, respectively. Once we know that these projections are zero, we can conclude that the other two projections (on G_{n+1} and G_{n+2}) are also zero, from duality arguments, since a^0 is a symmetric operator, while a^+ is the adjoint of a^-.

If X_1, X_2, ..., X_d are classic random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ and having finite moments of any order, then we can take $H = L^2(\Omega, \mathcal{F}, P)$ and $\phi = 1$, i.e., the constant random variable equal to 1. We then regard X_1, X_2, ..., X_d as multiplication operators on the space $\mathcal{A}1 \subset H$, where $\mathcal{A}$ is the algebra of the random variables of the form $p(X_1$, X_2,

$\ldots, X_d)$, where p is a polynomial of d variables. It is clear that $X_iX_j = X_jX_i$, for all $1 \le i < j \le d$.

Conditions (2.5), (2.6), and (2.7) separate the Commutative Probability from the Non–Commutative one.

Definition 2.4. *Let H be a Hilbert space, $(\mathcal{A}, \phi)$ a probability space supported by H, and $\{X_i\}_{1\le i\le d}$ elements of $\mathcal{A}$, that are symmetric operators. Let $\{a_{x_i}^-\}_{1\le i\le d}$, $\{a_{x_i}^0\}_{1\le i\le d}$, and $\{a_{x_i}^+\}_{1\le i\le d}$ be three families of linear operators, defined on subspaces of H, such that, ϕ belongs to the domain of $a_{i_1}^{\epsilon_1}a_{i_1}^{\epsilon_2}\cdots a_{i_n}^{\epsilon_n}$, for all $n \ge 1$, $(i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, d\}^n$, and $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{-, 0, +\}^n$. We say that these families of operators form* a joint annihilation-preservation-creation *(APC)* decomposition of $\{X_i\}_{1\le i\le d}$ relative to $\mathcal{A}$, *if the following conditions hold:*

$$X_i = a_{x_i}^- + a_{x_i}^0 + a_{x_i}^+, \tag{2.8}$$

$$\left(a_{x_i}^+\right)^* |\mathcal{A}\phi = a_{x_i}^-|\mathcal{A}\phi, \tag{2.9}$$

$$a_{x_i}^- H_n \subset H_{n-1}, \tag{2.10}$$

$$a_{x_i}^0 H_n \subset H_n, \tag{2.11}$$

for all $1 \le i \le d$ and $n \ge 0$, where $H_{-1} := \{0\}$, $H_0 := \mathbb{R}\phi$, and H_k is the vector space spanned by all vectors of the form $a_{x_{i_1}}^+ a_{x_{i_2}}^+ \cdots a_{x_{i_k}}^+ \phi$, where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\}$, for all $k \ge 1$.
We call $a_{x_i}^-$ an annihilation *operator, $a_{x_i}^0$ a* preservation *operator, and $a_{x_i}^+$ a* creation *operator, for all $1 \le i \le d$.*

For all $1 \le i \le d$, by $\left(a_{x_i}^+\right)^* |\mathcal{A}\phi = a_{x_i}^-|\mathcal{A}\phi$ we mean:

$$\langle a_{x_i}^+ u, v\rangle = \langle u, a_{x_i}^- v\rangle,$$

for all u and v in $\mathcal{A}\phi$, where $\langle\cdot, \cdot\rangle$ denotes the inner product of H. Since, for all $i \in \{1, 2, \ldots, d\}$, X_i is a symmetric operator, we conclude from (2.9), that:

$$\left(a_{x_i}^0\right)^* |\mathcal{A}\phi = a_{x_i}^0|\mathcal{A}\phi. \tag{2.12}$$

Let us observe that for any $X_1, X_2, \ldots, X_d \in \mathcal{A}$, we can consider the unital algebra $\mathcal{A}' \subset \mathcal{A}$ generated by $X_1, X_2, \ldots, X_d$. Doing the construction described before, by considering, for all $n \ge 0$, the space F_n of all vectors of the form $p(X_1, X_2, \ldots, X_d)\phi$, where p is a polynomial of degree at most n, then defining $G_n := F_n \ominus F_{n-1}$, and so on, we can construct the annihilation, preservation, and creation operators $a^-(i)$, $a^0(i)$, and $a^+(i)$ of X_i, respectively, where $1 \le i \le d$. It is now clear that these operators form a joint (APC) decomposition of $\{X_i\}_{1\le i\le d}$ relative to $\mathcal{A}'$. We call this decomposition the *minimal joint (APC) decomposition of* $\{X_i\}_{1\le i\le d}$. We can prove the following lemma about the uniqueness of the (APC) decomposition, which justifies the choice of the word "minimal".

Lemma 2.5. *Let $\{X_i\}_{1\le i\le d}$ be a family of symmetric random variables in a non–commutative probability space $(\mathcal{A},\ \phi)$, and $\{a^-_{x_i}\}_{1\le i\le d}$, $\{a^0_{x_i}\}_{1\le i\le d}$, and $\{a^+_{x_i}\}_{1\le i\le d}$ a joint (APC) decomposition of $\{X_i\}_{1\le i\le d}$ relative to $\mathcal{A}$. Let $\mathcal{A}'$ be the algebra generated by $\{X_i\}_{1\le i\le d}$, and $\{a^-(i)\}_{1\le i\le d}$, $\{a^0(i)\}_{1\le i\le d}$, and $\{a^+(i)\}_{1\le i\le d}$ the minimal joint (APC) decomposition of $\{X_i\}_{1\le i\le d}$. Then for any $i\in\{1,\ 2,\ \ldots,\ d\}$ and any $\epsilon\in\{-,\ 0,\ +\}$, we have:*

$$a^{\epsilon}_{x_i}|\mathcal{A}'\phi \quad = \quad a^{\epsilon}(i)|\mathcal{A}'\phi. \tag{2.13}$$

Moreover, if $\mathcal{A}''$ denotes the algebra generated by $\cup_{i=1}^d\{a^-_{x_i},\ a^0_{x_i},\ a^+_{x_i}\}$, then

$$\mathcal{A}''\phi \quad = \quad \mathcal{A}'\phi \tag{2.14}$$

and $\{a^-_{x_i}\}_{1\le i\le d}$, $\{a^0_{x_i}\}_{1\le i\le d}$, and $\{a^+_{x_i}\}_{1\le i\le d}$ is a joint (APC) decomposition of $\{X_i\}_{1\le i\le n}$ relative to $\mathcal{A}''$.

Proof. As before, for all $n\ge 0$, let F_n be the space of all polynomials of d non–commutative variables of degree at most n (in which the variables x_1, x_2, …, x_n, are replaced by X_1, X_2, …, X_n), $G_n := F_n\ominus F_{n-1}$, and H_n the space spanned by all vectors of the form $a^+_{x_{i_1}}a^+_{x_{i_2}}\cdots a^+_{x_{i_n}}\phi$, where i_1, i_2, …, $i_n\in\{1,\ 2,\ \ldots,\ d\}$.

Claim 1. For all $n\ge 0$, $H_n\subset F_n$.

We prove this claim by induction on n. For $n=0$, we have $H_0=F_0=\mathbb{R}\phi$. Let us assume that for some $n\ge 1$, we have $H_k\subset F_k$, for all $k\le n-1$, and prove that $H_n\subset F_n$. We need to show that $a^+_{x_{i_1}}a^+_{x_{i_2}}\cdots a^+_{x_{i_n}}\phi\in F_n$, for all i_1, i_2, …, $i_n\in\{1,\ 2,\ \ldots,\ d\}$. Indeed, we have:

$$\begin{aligned}
& X_{i_1}X_{i_2}\ldots X_{i_n}\phi \\
= \ & (a^+_{x_{i_1}}+a^0_{x_{i_1}}+a^-_{x_{i_1}})(a^+_{x_{i_2}}+a^0_{x_{i_2}}+a^-_{x_{i_2}})\cdots(a^+_{x_{i_n}}+a^0_{x_{i_n}}+a^-_{x_{i_n}})\phi \\
= \ & a^+_{x_{i_1}}a^+_{x_{i_2}}\cdots a^+_{x_{i_n}}\phi+\sum_{(\epsilon_1,\epsilon_2,\ldots,\epsilon_n)\in\{-,0,+\}^n\setminus\{(+,+,\cdots,+)\}}a^{\epsilon_1}_{x_{i_1}}a^{\epsilon_2}_{x_{i_2}}\cdots a^{\epsilon_d}_{x_{i_n}}\phi.
\end{aligned}$$

Because at least one operator from the terms of the last sum is a preservation or annihilation operator, it follows from (2.11) and (2.10), that:

$$\sum_{(\epsilon_1,\epsilon_2,\ldots,\epsilon_n)\in\{-,0,+\}^n\setminus\{(+,+,\cdots,+)\}}a^{\epsilon_1}_{x_{i_1}}a^{\epsilon_2}_{x_{i_2}}\cdots a^{\epsilon_d}_{x_{i_n}}\phi \quad\in\quad H_0+H_1+\cdots+H_{n-1}.$$

Thus, using the induction hypothesis, we get:

$$\begin{aligned}
& a^+_{x_{i_1}} a^+_{x_{i_2}} \cdots a^+_{x_{i_n}} \phi \\
= \; & X_{i_1} X_{i_2} \ldots X_{i_n} \phi - \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{-,0,+\}^n \setminus \{(+,+,\cdots,+)\}} a^{\epsilon_1}_{x_{i_1}} a^{\epsilon_2}_{x_{i_2}} \cdots a^{\epsilon_d}_{x_{i_n}} \phi \\
\in \; & F_n - (H_0 + H_1 + \cdots + H_{n-1}) \\
\subset \; & F_n - F_{n-1} \\
\subset \; & F_n.
\end{aligned}$$

Claim 2. For all $m \neq n$, $H_m \perp H_n$ (as subspaces of H).

We prove by induction on n, that for all $n \geq 1$, we have: $H_n \perp H_k$, for all $0 \leq k \leq n-1$.

For $n = 1$, we need to prove that $H_1 \perp H_0$, which reduces to showing that for all $1 \leq i \leq d$, $a^+_{x_i}\phi \perp \phi$. If $\langle \cdot, \cdot \rangle$ denotes the inner product of H, then it follows from condition (2.9) and (2.10) that:

$$\begin{aligned}
\langle a^+_{x_i}\phi, \phi \rangle &= \langle \phi, a^-_{x_i}\phi \rangle \\
&= \langle \phi, 0 \rangle \\
&= 0.
\end{aligned}$$

Thus $a^+_{x_i}\phi \perp \phi$.

Let us suppose that $H_n \perp H_k$, for all $0 \leq k \leq n-1$, and prove that $H_{n+1} \perp H_r$, for all $0 \leq r \leq n$. To do this, we need to show that if $r \leq n$, then

$$a^+_{x_{i_1}} a^+_{x_{i_2}} \cdots a^+_{x_{i_{n+1}}} \phi \quad \perp \quad a^+_{x_{j_1}} a^+_{x_{j_2}} \cdots a^+_{x_{j_r}} \phi,$$

for all $i_1, i_2, \ldots, i_{n+1}, j_1, j_2, \ldots, j_r \in \{1, 2, \ldots, d\}$. Using again the condition (2.9), because H_{n+1} and H_r are both contained in F_{n+1}, and $F_{n+1} \subset \mathcal{A}\phi$, we get:

$$\begin{aligned}
& \langle a^+_{x_{i_1}} a^+_{x_{i_2}} \cdots a^+_{x_{i_{n+1}}} \phi, a^+_{x_{j_1}} a^+_{x_{j_2}} \cdots a^+_{x_{j_r}} \phi \rangle \\
= \; & \langle a^+_{x_{i_2}} a^+_{x_{i_3}} \cdots a^+_{x_{i_{n+1}}} \phi, a^-_{x_{i_1}} a^+_{x_{j_1}} a^+_{x_{j_2}} \cdots a^+_{x_{j_r}} \phi \rangle.
\end{aligned}$$

From (2.10), we know that $a^-_{x_{i_1}} a^+_{x_{j_1}} a^+_{x_{j_2}} \cdots a^+_{x_{j_r}} \phi \in H_{r-1}$. Since $r \leq n$, we have $r - 1 \leq n - 1$, and thus, it follows from the induction hypothesis that:

$$\langle a^+_{x_{i_2}} a^+_{x_{i_3}} \cdots a^+_{x_{i_{n+1}}} \phi, a^-_{x_{i_1}} a^+_{x_{j_1}} a^+_{x_{j_2}} \cdots a^+_{x_{j_r}} \phi \rangle \quad = \quad 0.$$

Claim 3. For all $n \geq 0$, $F_n = H_0 \oplus H_1 \oplus \cdots \oplus H_n$, where "$\oplus$" denotes the orthogonal sum.

We know from the previous two claims that $H_0 \oplus H_1 \oplus \cdots \oplus H_n \subset F_n$. On the other hand for any monomial $X_{i_1} X_{i_2} \ldots X_{i_m} \phi$, of degree m, where $m \leq n$, we have:

$$\begin{aligned} X_{i_1}X_{i_2}\ldots X_{i_m}\phi &= \sum_{(\epsilon_1,\epsilon_2,\ldots,\epsilon_m)\in\{-,0,+\}^m} a^{\epsilon_1}_{x_{i_1}}a^{\epsilon_2}_{x_{i_2}}\cdots a^{\epsilon_d}_{x_{i_n}}\phi \\ &\in H_0\oplus H_1\oplus\cdots\oplus H_m. \end{aligned}$$

Since, these monomials are spanning F_n, we get: $F_n\subset H_0\oplus H_1\oplus\cdots\oplus H_n$.

It follows now from Claim 3, that for all $n\geq 0$,

$$\begin{aligned} H_n &= [H_0\oplus H_1\oplus\cdots\oplus H_n]\ominus[H_0\oplus H_1\oplus\cdots\oplus H_{n-1}] \\ &= F_n\ominus F_{n-1} \\ &= G_n. \end{aligned}$$

Let $n\geq 0$ be fixed and let $w\in H_n$, then for any $1\leq i\leq d$, we have:

$$X_iw = a^+_{x_i}w+a^0_{x_i}w+a^-_{x_i}w.$$

Since $a^+_{x_i}w\in H_{n+1}=G_{n+1}$, $a^0_{x_i}w\in H_n=G_n$ (from (2.10)), and $a^-_{x_i}w\in H_{n-1}=G_{n-1}$ (from (2.11)), we conclude that $a^+_{x_i}w=a^+(i)w$, $a^0_{x_i}w=a^0(i)w$, and $a^-_{x_i}w=a^-(i)w$, and the proof of (2.13) is complete.

Moreover,

$$\begin{aligned} \mathcal{A}''\phi &= \cup_{n\geq 0}\left(\oplus^n_{k=0}H_k\right) \\ &= \cup_{n\geq 0}F_n \\ &= \mathcal{A}'\phi. \end{aligned}$$

Let us observe now that, for each $1\leq i\leq d$, $X_i=a^-_{x_i}+a^0_{x_i}+a^+_{x_i}\in\mathcal{A}''$. Thus $\{X_i\}_{1\leq i\leq d}$ can be regarded as random variables in the non–commutative probability space $(\mathcal{A}'',\ \phi)$. It is now clear that $\{a^-_{x_i}\}_{1\leq i\leq d}$, $\{a^0_{x_i}\}_{1\leq i\leq d}$, and $\{a^+_{x_i}\}_{1\leq i\leq d}$ is a joint (APC) decomposition of $\{X_i\}_{1\leq i\leq n}$ relative to $\mathcal{A}''$. □

Using the above notations, we close this section by defining the number operator.

Definition 2.6. *We define the* number operator $\mathcal{N}$ *as the linear operator whose domain is* $\mathcal{A}'\phi$*, defined by the formula:*

$$\mathcal{N}u = nu, \tag{2.15}$$

for all $n\geq 0$ *and all* $u\in H_n$.

3. Meixner random vectors of class $\mathcal{M}_L$

In the one–dimensional case, a *Meixner random variable* X, with infinite support, has the Szegö–Jacobi parameters: $\alpha_n=\alpha n+\alpha_0$ and $\omega_n=\beta n^2+(t-\beta)n$, for all $n\geq 1$, where α, β, and t are real numbers, such that $\beta>0$ and $t>0$. The Meixner random variables (r.v.), with infinite support, are divided, up to a re–scaling, into five sub–classes: Gaussian, Poisson, negative binomial, gamma, and two parameter hyperbolic secant r.v.. A Meixner random variable X, with finite support, has the following Szegö–Jacobi parameters:

$$\alpha_n = \begin{cases} \alpha n + \alpha_0 & \text{if} \quad n \leq k-1 \\ 0 & \text{if} \quad n \geq k \end{cases}$$

and

$$\omega_n = \begin{cases} \beta n^2 + (t-\beta)n & \text{if} \quad n \leq k-1 \\ 0 & \text{if} \quad n \geq k \end{cases},$$

where k is a natural number equal to the number of different values that X takes on with positive probabilities, and α, β, and t are real numbers, such that if $k \geq 2$, then $t > 0$ and $t + \beta(k-1) > 0$. See [4], [9], and [14].

Thus we can write the class $\mathcal{M}$ of all Meixner random variables as:

$$\mathcal{M} := \mathcal{M}_u \cup \mathcal{M}_b, \tag{3.1}$$

where $\mathcal{M}_u$ represents the class of all Meixner random variables with infinite (unbounded) support, and $\mathcal{M}_b$ the class of all Meixner random variables with finite (compact) support.

Let us consider now a Meixner random variable with infinite support, having the Szegö–Jacobi parameters: $\alpha_n = \alpha n + \alpha_0$ and $\omega_n = \beta n^2 + (t-\beta)n$, for all $n \geq 1$. This is equivalent to the fact that the commutators of the terms of its minimal (APC) decomposition $X = a^- + a^0 + a^+$ are: $[a^-, a^+] = 2\beta\mathcal{N} + tI$, and $[a^-, a^0] = \alpha a^-$, where $\mathcal{N}$ denotes the number operator, i.e., the linear operator that maps the $n-$th orthogonal polynomial $f_n(X)$, generated by X, to $nf_n(X)$, for all $n \geq 0$. It also follows that $a^0 = \alpha\mathcal{N} + \alpha_0 I$. Thus, if $\alpha \neq 0$ or $\alpha = \beta = 0$, then we have $[a^-, a^+] = 2(\beta/\alpha)a^0 + [t - 2\alpha_0(\beta/\alpha)]I$ and $[a^-, a^0] = \alpha a^-$, where $\beta/\alpha := 0$ if $\alpha = \beta = 0$. By duality we also obtain;

$$\begin{aligned} [a^0, a^+] &= \left[(a^0)^*, (a^-)^*\right] \\ &= [a^-, a^0]^* \\ &= (\alpha a^-)^* \\ &= \alpha a^+. \end{aligned}$$

This means that the vector space spanned by the operators a^-, a^0, a^+, and I is closed with respect to taking the commutator. Thus this vector space equipped with the commutator $[\cdot, \cdot]$ forms a Lie algebra. If we define $\beta/\alpha := \infty$, for $\alpha = 0$ and $\beta \neq 0$, then we can split the class $\mathcal{M}_u$ into two subclasses: $\mathcal{M}_{u,f}$ and $\mathcal{M}_{u,\infty}$. $\mathcal{M}_{u,f}$ is the class of all Meixner random variables, with infinite support, for which $\alpha \neq 0$ or $\alpha = \beta = 0$. $\mathcal{M}_{u,\infty}$ is the class of all Meixner random variables, with infinite support, for which $\alpha = 0$ and $\beta \neq 0$. Thus an element of $\mathcal{M}_{u,\infty}$ is a random variable having a symmetric (symmetry about α_0) re–scaled two parameter hyperbolic secant distribution. Moreover, the following proposition holds:

Proposition 3.1. *The Meixner random variables of class $\mathcal{M}_{u,f}$, are exactly those random variables $X = a^- + a^0 + a^+$, having finite moments of all orders and infinite support, for which the vector space W spanned by a^-, a^0, a^+, and I, equipped with the commutator $[\cdot,\ \cdot]$, forms a Lie algebra, where I denotes the identity operator. The Gaussian random variables are the only symmetric (symmetric about a number) random variables, for which the vector space spanned by a^-, a^+, and I, equipped with the commutator $[\cdot,\ \cdot]$, forms a Lie algebra.*

Proof. ($\Rightarrow$) If X is a Meixner random variable of class $\mathcal{M}_{u,f}$, then we have already explained why the vector space $(W,\ [\cdot,\ \cdot])$ forms a Lie algebra.

($\Leftarrow$) Let us suppose that $(W,\ [\cdot,\ \cdot])$ is a Lie algebra. Let $\{f_n\}_{n\geq 0}$ be the sequence of orthogonal polynomials generated by X, and let $G_n := \mathbb{R}f_n$, for all $n \geq 0$. Since $(W,\ [\cdot,\ \cdot])$ is a Lie algebra, there exist four constants α, c, d, and e, such that:

$$[a^-, a^0] \quad = \quad \alpha a^- + ca^0 + da^+ + eI. \tag{3.2}$$

Thus for all $n \geq 0$,

$$[a^-, a^0]f_n(X) \quad = \quad \alpha a^- f_n(X) + ca^0 f_n(X) + da^+ f_n(X) + eIf_n(X).$$

This is equivalent to:

$$(\alpha_n - \alpha_{n-1})\omega_n f_{n-1}(X) \quad = \quad \alpha\omega_n f_{n-1}(X) + (c\alpha_n + e)f_n(X) + df_{n+1}(X),$$

for all $n \geq 0$, where $\alpha_{-1} := 0$. Since X has an infinite support, $f_{n-1}(X)$, $f_n(X)$, and $f_{n+1}(X)$ are linearly independent, for all $n \geq 1$. Because $\omega_n > 0$, if we look at the coefficient of f_{n-1} in the last equality, we get $\alpha_n - \alpha_{n-1} = \alpha$, for all $n \geq 1$. Thus:

$$\begin{aligned} \alpha_n &= \alpha_0 + \sum_{k=1}^{n}(\alpha_k - \alpha_{k-1}) \\ &= \alpha_0 + \sum_{k=1}^{n}\alpha \\ &= \alpha_0 + \alpha n, \end{aligned}$$

for all $n \geq 1$.

Because $(W,\ [\cdot,\ \cdot])$ is a Lie algebra, there exist four constants p, q, r, and s, such that:

$$[a^-, a^+] \quad = \quad pa^- + qa^0 + ra^+ + sI. \tag{3.3}$$

Since, for all $n \geq 0$, $[a^-,\ a^+]$ and $qa^0 + sI$ map G_n into G_n, pa^- maps G_n into G_{n-1}, and ra^+ maps G_n into G_{n+1}, by restricting (3.3) to G_n, we get:

$$[a^-, a^+]|G_n \quad = \quad (qa^0 + sI)|G_n, \tag{3.4}$$

for all $n \geq 0$. If we define $\omega_0 := 0$, then for all $n \geq 0$,

$$[a^-, a^+]|G_n = (\omega_{n+1} - \omega_n)I|G_n$$

and

$$\begin{aligned}(qa^0 + sI)|G_n &= (q\alpha_n + s)I|G_n \\ &= (q\alpha n + q\alpha_0 + s)I|G_n.\end{aligned}$$

Hence, it follows from (3.4), that $\omega_{n+1} - \omega_n = q\alpha n + q\alpha_0 + s$, for all $n \geq 0$. Since $\omega_0 = 0$, we obtain:

$$\begin{aligned}\omega_n &= \sum_{k=0}^{n-1}(\omega_{k+1} - \omega_k) \\ &= \sum_{k=0}^{n-1}(q\alpha k + q\alpha_0 + s) \\ &= q\alpha\frac{n(n-1)}{2} + (q\alpha_0 + s)n \\ &= \frac{q\alpha}{2}n^2 + \left(q\alpha_0 + s - \frac{q\alpha}{2}\right)n,\end{aligned}$$

for all $n \geq 1$. Thus X is a Meixner random variable. Moreover, the numbers α and β, of X, satisfy the relation $\beta = (q/2)\alpha$, which means that if $\alpha = 0$, then $\beta = 0$, too. Hence X is of class $\mathcal{M}_{u,f}$. □

We also have the following proposition:

Proposition 3.2. *Let X be a centered random variable, taking on only k different values with positive probabilities. The space W, spanned by the identity and (APC) operators a_x^-, a_x^0, and a_x^+ of X, equipped with the bracket given by the commutator, is a Lie algebra if and only if X is a Meixner random variable, taking on only k different values, such that, the constants t and β, involved in its Szegö–Jacobi parameters, satisfy the following condition:*

$$t = -\beta(k-1), \tag{3.5}$$

and $\alpha \neq 0$. Conditions (3.5) and $\alpha \neq 0$ are satisfied exactly by the non–symmetric binomial distributions, with Krawtchouk orthogonal polynomials.

Proof. We do the same proof as before, except that due to the fact that the last non–zero orthogonal polynomial is f_{k-1}, we have $\omega_k = 0$. Thus $[a^-, a^+]f_{k-1} = -\omega_{k-1}f_{k-1}$. Since:

$$[a^-, a^+]|G_{n-1} = \{2(\beta/\alpha)a^0 + [t - 2\alpha_0(\beta/\alpha)]I\}|G_{n-1},$$

we must have $[a^-, a^+]f_{k-1} = \{2(\beta/\alpha)a^0 + [t - 2\alpha_0(\beta/\alpha)]I\}f_{k-1}$. This means

$$-\omega_{k-1} = 2\beta(k-1) + t.$$

Since $\omega_{k-1} = \beta(k-1)^2 + (t-\beta)(k-1)$ it follows that (3.5) must hold. □

We can split the class of Meixner random variables with finite support as:

$$\mathcal{M} \;=\; \mathcal{M}_{b,f} \cup \mathcal{M}_{b,\infty}, \tag{3.6}$$

where $\mathcal{M}_{b,f}$ are the non–symmetric binomials, while $\mathcal{M}_{b,\infty}$ are the symmetric ones.

Let us now define the following class:

$$\mathcal{M}_L \;:=\; \mathcal{M}_{u,f} \cup \mathcal{M}_{b,f} \tag{3.7}$$

and call it the *Meixner–Lie class*. We can wrap up our discussion now in the following lemma:

Lemma 3.3. *The real vector space spanned by the identity operator, and the annihilation, preservation, and creation operators of a random variable, having finite moments of all orders, equipped with the bracket given by the commutator, forms a Lie algebra if and only if that random variable is of Meixner–Lie class.*

It has been known, see for example [6] and [7], that various Meixner classes give rise to Lie algebras like: $\mathfrak{su}(1,\,1)$, $\mathfrak{sl}(2)$, and others.

We can always assume that X is *centered*, that means $\alpha_0 = E[X] = 0$, since otherwise, we can consider the random variable $X' = X - E[X]$, whose ω–parameters are the same as those of X, while the α–parameters are $\alpha_n' = \alpha_n - \alpha_0$, for all $n \geq 0$ (in the infinite support case).

Let us take Lemma 3.3 as a starting point in defining the notion of Meixner probability distributions on $\mathbb{R}^2$ of class $\mathcal{M}_L$.

Definition 3.4. *Let μ be a probability measure on $\mathbb{R}^2$ having finite moments of all orders. Let us denote by $(x,\,y)$ a generic vector in $\mathbb{R}^2$. Let a_x^-, a_x^0, and a_x^+ be the annihilation, preservation, and creation operators generated by the operator X of multiplication by x, and a_y^-, a_y^0, and a_y^+ the annihilation, preservation, and creation operators generated by the operator Y of multiplication by y. We call μ a* Meixner distribution of class $\mathcal{M}_L$, *if the real vector space W spanned by the operators a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, a_y^+, and I, equipped with the bracket $[\cdot,\,\cdot]$ given by the commutator, forms a Lie algebra.*

We extend this definition to the non–commutative case in the following way:

Definition 3.5. *Let $(\mathcal{A},\,\phi)$ be a non–commutative probability space and X and Y two random variables from $\mathcal{A}$. Let $\{a_u^-,\,a_u^0,\,a_u^+\}_{u\in\{x,y\}}$ be their minimal joint (APC) decomposition. We say that the pair $(X,\,Y)$ is a* Meixner random vector of class $\mathcal{M}_L$ *if the real vector space W spanned by the operators a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, a_y^+, and I, equipped with the bracket $[\cdot,\,\cdot]$ given by the commutator, forms a Lie algebra.*

Definition 3.6. *Let X and Y be two random variables in a non–commutative probability space $(\mathcal{A},\, \phi)$ supported by the Hilbert space H. We say that the random vector $(X,\, Y)$ is* non–degenerate *if the vectors $X\phi$, $Y\phi$, and ϕ are linearly independent in H. In particular if both X and Y are centered, then the random vector $(X,\, Y)$ is non–degenerate if $X\phi$ and $Y\phi$ are linearly independent.*

Proposition 3.7. *If $(X,\, Y)$ is a non–degenerate random vector, then the annihilation operators a_x^- and a_y^-, of X and Y, are linearly independent.*

Proof. Let c and d be two real numbers such that

$$ca_x^- + da_y^- = 0.$$

Taking the adjoint in both sides of this equality we get:

$$ca_x^+ + da_y^+ = 0.$$

Since $a_x^0\phi = E[X]\phi$ and $a_y^0\phi = E[Y]\phi$, we have:

$$\begin{aligned} & cX\phi + dY\phi - (cE[X] + dE[Y])\,\phi \\ = \;& c(a_x^- + a_x^0 + a_x^+)\phi + d(a_y^- + a_y^0 + a_y^+)\phi - \left(ca_x^0 + da_y^0\right)\phi \\ = \;& 0. \end{aligned}$$

Thus, because $X\phi$, $Y\phi$, and ϕ are linearly independent, we conclude that $c = d = 0$. Therefore, a_x^- and a_y^- are linearly independent. □

We present now two fundamental examples.

Example 1. Let X and Y be two independent centered Meixner random variables of class $\mathcal{M}_L$ defined on the same probability space $(\Omega,\, \mathcal{F},\, P)$. Let μ be the joint probability distribution of X and Y. We can identify now X and Y with the multiplication operators on $L^2(\mathbb{R}^2,\, \mu)$, by the coordinates x and y of the generic vector $(x,\, y)$, respectively. In this way $X := a_x^- + a_x^0 + a_x^+$ and $Y := a_y^- + a_y^0 + a_y^+$. Since X and Y are independent, we know from [1], that $[a_x^{\epsilon_1},\, a_y^{\epsilon_2}] = 0$, for all $(\epsilon_1,\, \epsilon_2) \in \{-,\, 0,\, +\}^2$. Moreover, one can see that:

$$\left[a_x^-, a_x^+\right] \in \mathbb{R}I + \mathbb{R}a_x^0, \tag{3.8}$$

$$\left[a_y^-, a_y^+\right] \in \mathbb{R}I + \mathbb{R}a_y^0, \tag{3.9}$$

$$\left[a_x^-, a_x^0\right] \in \mathbb{R}a_x^-, \tag{3.10}$$

$$\left[a_x^0, a_x^+\right] \in \mathbb{R}a_x^+, \tag{3.11}$$

$$\left[a_y^-, a_y^0\right] \in \mathbb{R}a_y^-, \tag{3.12}$$

$$\left[a_y^0, a_y^+\right] \in \mathbb{R}a_y^+. \tag{3.13}$$

Hence $(W,\, [\cdot,\, \cdot])$ is a Lie algebra, where W is the real vector space spanned by I, a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, and a_y^+. Thus μ is a Meixner distribution on $\mathbb{R}^2$ of class $\mathcal{M}_L$, or equivalently $(X,\, Y)$ is a commutative Meixner random vector of class $\mathcal{M}_L$.

Example 2. Let T and Z be two independent centered Meixner random variables of class $\mathcal{M}_L$ defined on the same probability space $(\Omega, \mathcal{F}, P)$, having the same numbers $\alpha = 1$, $t' = 1$ (we are using t' instead of t, since the letter t we will be used later as a subscript), and the β numbers as follows: $\beta_T := (1/2)(cp + dr)$ and $\beta_Z := (1/2)(js' + kv)$, where c, p, d, r, j, s', k, and v are some given real numbers (the choice of these letters will become more clear in the next section), such that:

$$cs' + dv = 0$$

and

$$jp + kr = 0.$$

In other words, using the usual dot product "$\cdot$" on $\mathbb{R}^2$, and the orthogonal relation "$\perp$" generated by it, we have:

$$\beta_T = \frac{1}{2}(c, d) \cdot (p, r), \tag{3.14}$$

$$\beta_Z = \frac{1}{2}(j, k) \cdot (s', v), \tag{3.15}$$

$$(c, d) \perp (s', v), \tag{3.16}$$

$$(j, k) \perp (p, r). \tag{3.17}$$

Relations (3.14)–(3.17) are important, for the following reason. β_T and β_Z must be non–negative. We can make sure that they are not negative by remembering that, the dot product of two vectors is the product of the length of the vectors times the cosine of the angle formed by the vectors. Thus we can start with two non–zero vectors (c, d) and (p, r) forming an acute angle. Then we rotate the semi–lines supporting these vectors by 90°, and choose any two non–zero vectors (s', v) and (j, k) on the rotated semi–lines. In this way we know for sure that β_T and β_Z are strictly positive and their sides are perpendicular, as required by conditions (3.16) and (3.17).

Let μ be the joint probability distribution of T and Z. As before, we can identify now T and Z with the multiplication operators (densely defined) on $H = L^2(\mathbb{R}^2, \mu)$, by the coordinates t and z of the generic vector (t, z). In this way $T = a_t^- + a_t^0 + a_t^+$ and $Z = a_z^- + a_z^0 + a_z^+$. Let us consider the following symmetric operators:

$$X := a_t^- + (pa_t^0 + s'a_z^0) + a_t^+$$

and

$$Y := a_z^- + (ra_t^0 + va_z^0) + a_z^+.$$

Let $\mathcal{A}$ be the unital algebra generated by a_t^-, a_t^0, a_t^+, a_z^-, a_z^0, and a_z^+. Let $\phi := 1$ (the constant random variable, of H, equal to 1). Then $(\mathcal{A}, 1)$ is a non–commutative probability space supported by H, and X and Y are random variables in $(\mathcal{A}, 1)$.

Claim 1. (X, Y) is a Meixner vector of class $\mathcal{M}_L$.

Indeed, a joint (APC) decomposition of X and Y, relative to $\mathcal{A}$ is:

$$a_x^- = a_t^-, \tag{3.18}$$
$$a_x^0 = pa_t^0 + s'a_z^0, \tag{3.19}$$
$$a_x^+ = a_t^+, \tag{3.20}$$
$$a_y^- = a_z^-, \tag{3.21}$$
$$a_y^0 = ra_t^0 + va_z^0, \tag{3.22}$$
$$a_y^+ = a_z^+. \tag{3.23}$$

Using now Lemma 2.5, we conclude that this (APC) decomposition, restricted to the space $\mathcal{A}'1$, where $\mathcal{A}'$ is the unital algebra generated by X and Y, is the minimal (APC) decomposition.

Since T and Z are linearly independent we have:

$$\begin{aligned}\left[a_x^{\pm}, a_y^{\pm}\right] &= \left[a_t^{\pm}, a_z^{\pm}\right] \\ &= 0\end{aligned}$$

and

$$\left[a_x^0, a_y^0\right] = 0.$$

Because T is a centered Meixner random variable of class $\mathcal{M}_L$, we have:

$$\begin{aligned}\left[a_x^-, a_x^+\right] &= \left[a_t^-, a_t^+\right] \\ &= \frac{2\beta_T}{\alpha}a_t^0 + t'I \\ &= (cp + dr)a_t^0 + I.\end{aligned}$$

Multiplying both sides of the relation $a_x^0 = pa_t^0 + s'a_z^0$ by c and both sides of $a_y^0 = ra_t^0 + va_z^0$ by d, and then adding the two relations together, since $cs' + dv = 0$, we get:

$$(cp + dr)a_t^0 = ca_x^0 + da_y^0.$$

Thus, we get:

$$\left[a_x^-, a_x^+\right] = ca_x^0 + da_y^0 + I \in W.$$

Similarly, we have:

$$\left[a_y^-, a_y^+\right] = ja_x^0 + ka_y^0 + I \in W.$$

We also have:

$$\begin{aligned}\left[a_x^-, a_x^0\right] &= \left[a_t^-, pa_t^0 + s'a_z^0\right] \\ &= p\left[a_t^-, a_t^0\right] + s'\left[a_t^-, a_z^0\right] \\ &= p\alpha a_t^- + s'(0) \\ &= pa_t^- \\ &= pa_x^- \in W.\end{aligned}$$

Similarly, we have:

$$\begin{aligned}\left[a_x^-, a_y^0\right] &= ra_x^- \in W, \\ \left[a_y^-, a_x^0\right] &= s'a_y^- \in W, \\ \left[a_y^-, a_y^0\right] &= va_y^- \in W.\end{aligned}$$

By duality (taking the adjoint) we also get that the commutators between the preservation and creation operators belong to W.

Thus the space W, spanned by I, a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, and a_y^+, equipped with the commutator $[\cdot, \cdot]$ is a Lie algebra. Hence, (X, Y) is a Meixner random vector of class $\mathcal{M}_L$.

Claim 2. If T and Z are both not almost surely equal to 0, then the centered random vector (X, Y) is non–degenerate.

Indeed, in this case the vacuum vector is $\phi = 1$. Because $a_t^0 1 = a_z^0 1 = 0$ (since T and Z were assumed to be centered from the beginning), we have $X1 = T1 = T$ and $Y1 = Z1 = Z$. Since T and Z are independent, they must be linearly independent, because if $T = \lambda Z$, for some $\lambda \neq 0$, then:

$$\begin{aligned}0 &= E[T]E[Z] \\ &= E[TZ] \\ &= E[\lambda Z^2] \\ &= \lambda E[Z^2] \\ &\neq 0.\end{aligned}$$

We get a contradiction. Thus $X1$ and $Y1$ are linearly independent and so, the centered random vector (X, Y) is non–degenerate.

Let us observe that X and Y commute if and only if $[a_x^-, a_y^0] = [a_y^-, a_x^0]$ ((2.6) is the only axiom of the commutative probability that is not guaranteed to hold in this example), which means $r = s' = 0$, and hence: $X = a_t^- + pa_t^0 + a_t^+$ and $Y = a_z^- + va_z^0 + a_z^+$ are two independent centered Meixner random variables, with parameters $\alpha_X = p$, $\alpha_Y = v$, $\beta_X = \beta_T$, $\beta_Y = \beta_Z$, and $t' = 1$. Moreover, X is of class $\mathcal{M}_L$, since if $\alpha_X = 0$, then $\beta_X = (1/2)(cp + dr) = 0$, because $p = \alpha_x = 0$ and $r = 0$. Similarly Y is of class $\mathcal{M}_L$. Therefore, if $XY = YX$, then Example 2 reduces to Example 1.

Since the annihilation and creation operators of X and Y are the same as the annihilation and creation operators of T and Z, and the preservation operators of X and Y are superpositions of the preservation operators of T and Z, and because T and Z are independent, we call X and Y, from Example 2, *two independent Meixner random variables of class $\mathcal{M}_L$ with mixed preservation operators.*

Observation 3.8. *If (X, Y) is a non–degenerate Meixner random vector of class $\mathcal{M}_L$, then for any invertible linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(X', Y') := T(X, Y)$ is also a non–degenerate Meixner random vector of class $\mathcal{M}_L$. In particular, if $XY = YX$, then $X'Y' = Y'X'$.*

4. The classification of non–degenerate centered Meixner random vectors of class $\mathcal{M}_L$

In this section we show that every non–degenerate centered Meixner random vector (X, Y), of class $\mathcal{M}_L$, can be reduced, through an invertible linear transformation, to a random vector equivalent to a vector of two independent centered Meixner random variables X' and Y', of class $\mathcal{M}_L$, with mixed preservation operators. In particular if X and Y commute, then X' and Y' are independent.

We present first the following lemma.

Lemma 4.1. *If (X, Y) is a Meixner random vector of class $\mathcal{M}_L$, then the following relations hold:*

(1) *There exist some real numbers b, c, d, e, f, g, h, j, k, p, q, r, s, r', s', u, and v, such that:*

$$\left[a_x^-, a_x^+\right] = bI + ca_x^0 + da_y^0, \tag{4.24}$$

$$\left[a_x^-, a_y^+\right] = eI + fa_x^0 + ga_y^0, \tag{4.25}$$

$$\left[a_y^-, a_x^+\right] = eI + fa_x^0 + ga_y^0, \tag{4.26}$$

$$\left[a_y^-, a_y^+\right] = hI + ja_x^0 + ka_y^0, \tag{4.27}$$

$$\left[a_x^-, a_x^0\right] = pa_x^- + qa_y^-, \tag{4.28}$$

$$\left[a_x^0, a_x^+\right] = pa_x^+ + qa_y^+, \tag{4.29}$$

$$\left[a_x^-, a_y^0\right] = ra_x^- + sa_y^-, \tag{4.30}$$

$$\left[a_y^0, a_x^+\right] = ra_x^+ + sa_y^+, \tag{4.31}$$

$$\left[a_y^-, a_x^0\right] = r' a_x^- + s' a_y^-, \tag{4.32}$$

$$\left[a_x^0, a_y^+\right] = r' a_x^+ + s' a_y^+, \tag{4.33}$$

$$\left[a_y^-, a_y^0\right] = u a_x^- + v a_y^-, \tag{4.34}$$

$$\left[a_y^0, a_y^+\right] = u a_x^+ + v a_y^+. \tag{4.35}$$

(2)

$$\left[a_x^-, a_y^-\right] = 0, \tag{4.36}$$

$$\left[a_x^+, a_y^+\right] = 0, \tag{4.37}$$

$$\left[a_x^0, a_y^0\right] = 0. \tag{4.38}$$

(3) *If $XY = YX$, then we can take $r' = r$ and $s' = s$.*

Proof. Let $\{G_n\}_{n\geq 0}$ be the homogenous chaos spaces generated by X and Y.

1. Since, for each $n \geq 0$, the operators $[a_x^-, a_x^+]$, $[a_x^-, a_y^+]$, and $[a_y^-, a_y^+]$ are mapping G_n into G_n, and because $(W, [\cdot, \cdot])$ is a Lie algebra, each of these commutators must be a linear combination of only three operators from the set $\{I, a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, a_y^+\}$, namely, I, a_x^0, and a_y^0. Similarly since, for each $n \geq 0$, $[a_x^-, a_x^0]$, $[a_x^-, a_y^0]$, $[a_y^-, a_x^0]$, and $[a_y^-, a_y^0]$ map G_n into G_{n-1}, each of these commutators must be a linear combination of a_x^- and a_y^-. Therefore, there must exist some real numbers b, c, d, e, f, h, j, k, p, q, r, s, r', s', u, and v such that:

$$\begin{aligned}
\left[a_x^-, a_x^+\right] &= bI + c a_x^0 + d a_y^0, \\
\left[a_x^-, a_y^+\right] &= eI + f a_x^0 + g a_y^0, \\
\left[a_y^-, a_y^+\right] &= hI + j a_x^0 + k a_y^0, \\
\left[a_x^-, a_x^0\right] &= p a_x^- + q a_y^-, \\
\left[a_x^-, a_y^0\right] &= r a_x^- + s a_y^-, \\
\left[a_y^-, a_x^0\right] &= r' a_x^- + s' a_y^-, \\
\left[a_y^-, a_y^0\right] &= u a_x^- + v a_y^-.
\end{aligned}$$

Taking the adjoint in both sides of (4.25), (4.28), (4.30), (4.32), and (4.34) we get (4.26), (4.29), (4.31), (4.33), and (4.35), respectively.

2. Since, for all $n \geq 0$, $[a_x^+, a_y^+]$ maps G_n into G_{n+2}, and no operator from the set $\{I, a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, a_y^+\}$, spanning W, has this property, we have:

$$[a_x^+, a_y^+] = 0.$$

Similarly, we can see that:

$$[a_x^-, a_y^-] \quad = \quad 0.$$

Since, for all $n \geq 0$, $[a_x^0,\, a_y^0]$ maps G_n into G_n, there are three real numbers α, β, and γ such that:

$$[a_x^0, a_y^0] \quad = \quad \alpha I + \beta a_x^0 + \gamma a_y^0. \tag{4.39}$$

The left–hand side of (4.39) is an antisymmetric operator, due to the fact that:

$$\begin{aligned} [a_x^0, a_y^0]^* &= [a_y^0, a_x^0] \\ &= -[a_x^0, a_y^0], \end{aligned}$$

while the right–hand side is a symmetric operator. Thus:

$$[a_x^0, a_y^0] \quad = \quad 0.$$

3. If $XY = YX$, it follows from (2.6) that $[a_x^-,\, a_y^0] = [a_y^-,\, a_x^0]$, and thus we can take $r' = r$ and $s' = s$. □

Let us not forget that the random variables X and Y are assumed to be symmetric operators. We are presenting now two important lemmas.

Lemma 4.2. *If $(X,\, Y)$ is a centered non–degenerate Meixner random vector of class $\mathcal{M}_L$, then there exists an invertible linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, such that the coefficients b', e', and h' of the Meixner random vector of class $\mathcal{M}_L$, $(X',\, Y') := T(X,\, Y)$, from Lemma 4.1, are $b' = h' = 1$ and $|e'| < 1$.*

Proof. Since both X and Y are centered we have $a_x^0\phi = E[X]\phi = 0$ and $a_y^0\phi = E[Y]\phi = 0$. Thus, we get:

$$\begin{aligned} E[X^2] &= \langle (a_x^+ + a_x^0 + a_x^-)X\phi, \phi\rangle \\ &= \langle X\phi, a_x^-\phi\rangle + \langle X\phi, a_x^0\phi\rangle + \langle a_x^- X\phi, \phi\rangle \\ &= 0 + 0 + \langle a_x^-(a_x^- + a_x^0 + a_x^+)\phi, \phi\rangle \\ &= \langle a_x^- a_x^+\phi, \phi\rangle \\ &= \langle a_x^+ a_x^-\phi, \phi\rangle + \langle [a_x^-, a_x^+]\phi, \phi\rangle \\ &= 0 + \langle (bI + ca_x^0 + da_y^0)\phi, \phi\rangle \\ &= b, \end{aligned}$$

where $\langle\cdot,\,\cdot\rangle$ denotes the inner product of the Hilbert space H supporting the non–commutative probability space $(\mathcal{A},\, \phi)$, in which X and Y are random variables. Similarly, we have:

$$E[Y^2] \quad = \quad h$$

and

$$\begin{aligned} E[XY] &= \langle \left[a_x^-, a_y^+\right] 1, 1\rangle \\ &= e. \end{aligned}$$

Since $(X,\,Y)$ is non–degenerate we have $X\phi \neq 0$ and $Y\phi \neq 0$. Because X is a symmetric operator, we have:

$$\begin{aligned} b &= E[X^2] \\ &= \langle X^2\phi, \phi\rangle \\ &= \langle X\phi, X\phi\rangle \\ &= \parallel X\phi \parallel^2 \\ &> 0, \end{aligned}$$

where $\parallel\cdot\parallel$ denotes the norm of H. Similarly, we have $h > 0$. Let us consider the invertible linear change of variables (in fact a simple re–scaling):

$$(X', Y') \;:=\; \left(\frac{1}{\sqrt{b}}X, \frac{1}{\sqrt{h}}Y\right).$$

It is easy to see now that the coefficients b' and h' of $(X',\,Y')$ are $b' = h' = 1$. Thus by this re–scaling we may assume that $b = h = 1$. It follows now from the Schwarz' inequality, that:

$$\begin{aligned} |e| &= |E[XY]| \\ &= |\langle XY\phi, \phi\rangle| \\ &= |\langle Y\phi, X\phi\rangle| \\ &\leq \parallel X\phi \parallel \cdot \parallel Y\phi \parallel \\ &= \sqrt{b}\cdot\sqrt{h} \\ &= 1. \end{aligned}$$

Thus $|e| \leq 1$. We cannot have $|e| = 1$, since if this were true, then we would have equality is the Schwarz' inequality that we used, which would imply that $X\phi$ and $Y\phi$ are linearly dependent, contradicting the fact that $(X,\,Y)$ is non–degenerate. Hence $|e| < 1$. □

Lemma 4.3. *Let $(X,\,Y)$ be a non–degenerate centered Meixner random vector of class $\mathcal{M}_L$. There exists an invertible 2×2 matrix T, with real entries such that for the non–degenerate centered Meixner random vector $(X_1,\,Y_1) := T(X,\,Y)$ of class $\mathcal{M}_L$, the coefficients q, s, r', and u, from Lemma 4.1 are $q = s = r' = u = 0$. Moreover, $b = h = 1$ and $|e| < 1$.*

Proof. From Lemma 4.2, we may assume that: $b = h = 1$ and $|e| < 1$. Let's find first some relations between the values of the coefficients from Lemma 4.1, that hold for all non–degenerate Meixner random vectors $(X,\,Y)$.

From the Jacobi identity:

$$[a_x^-, [a_x^0, a_y^0]] + [a_x^0, [a_y^0, a_x^-]] + [a_y^0, [a_x^-, a_x^0]] \;=\; 0,$$

using the fact that a_x^- and a_y^- are linearly independent, we get:

$$sr' \;=\; uq \tag{4.40}$$

and

$$s(s' - p) \quad = \quad -q(r - v). \tag{4.41}$$

From the Jacobi identity:

$$[a_y^-, [a_x^0, a_y^0]] + [a_x^0, [a_y^0, a_y^-]] + [a_y^0, [a_y^-, a_x^0]] \quad = \quad 0,$$

we obtain:

$$u(s' - p) \quad = \quad -r'(r - v). \tag{4.42}$$

Let $\gamma := s' - p$ and $\delta := r - v$. Relations (4.41) and (4.42) become now:

$$s\gamma = -q\delta \tag{4.43}$$

and

$$u\gamma = -r'\delta. \tag{4.44}$$

Formulas (4.28), (4.30), (4.32), and (4.34) can now be written as:

$$\begin{aligned}
\left[a_x^-, a_x^0\right] &= pa_x^- + qa_y^- \\
\left[a_x^-, a_y^0\right] &= (v + \delta)a_x^- + sa_y^- \\
\left[a_y^-, a_x^0\right] &= r'a_x^- + (p + \gamma)a_y^- \\
\left[a_y^-, a_y^0\right] &= ua_x^- + va_y^-.
\end{aligned}$$

Case 1. If $q = s = r' = u = 0$, we have nothing to prove and are done.

Case 2. Let us assume that at least one of the numbers q, s, r', and u is not equal to zero.

We try now to find two random variables $Z_w = \alpha_w X + \beta_w Y$, $w \in \{1, 2\}$, where α_w and β_w are real numbers, such that there exist some real constants λ_w and μ_w for which the following relations hold:

$$\left[a_{z_w}^-, a_x^0\right] \quad = \quad \lambda a_{z_w}^-, \tag{4.45}$$

$$\left[a_{z_w}^-, a_y^0\right] \quad = \quad \mu a_{z_w}^-, \tag{4.46}$$

and the matrix:

$$\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix}$$

is invertible. We drop for the moment the index w. It is easy to see that if $Z = \alpha X + \beta Y$, then an (APC) decomposition of Z is given by: $a_z^\epsilon = \alpha a_x^\epsilon + \beta a_y^\epsilon$, for all $\epsilon \in \{-, 0, +\}$. Thus:

$$\begin{aligned}
\left[a_z^-, a_x^0\right] &= \alpha\left[a_x^-, a_x^0\right] + \beta\left[a_y^-, a_x^0\right] \\
&= (\alpha p + \beta r')a_x^- + [\alpha q + \beta(p + \gamma)]a_y^-.
\end{aligned}$$

It follows from here that if we want (4.45) to hold, then the coefficients $\alpha p+\beta r'$ and $\alpha q + \beta(p+\gamma)$ must be proportional to α and β. Since we also want $(\alpha, \beta) \neq (0, 0)$, this fact is equivalent to:

$$\begin{vmatrix} \alpha p + \beta r' & \alpha q + \beta(p+\gamma) \\ \alpha & \beta \end{vmatrix} = 0.$$

This means the following relation must hold:

$$r'\beta^2 - \gamma\beta\alpha - q\alpha^2 = 0. \tag{4.47}$$

A similar calculation shows that if we want relation (4.46) to hold, then we must have:

$$u\beta^2 + \delta\beta\alpha - s\alpha^2 = 0. \tag{4.48}$$

Let us look now at the matrix formed by the coefficient vectors $(r', -\gamma, -q)$ and $(u, \delta, -s)$ of the unknown vector $(\beta^2, \beta\alpha, \alpha^2)$ from equations (4.47) and (4.48). This matrix is:

$$\begin{pmatrix} r' & -\gamma & -q \\ u & \delta & -s \end{pmatrix}.$$

Amazingly, it follows from relations (4.40), (4.43), and (4.44), that the determinant of any 2×2 sub–matrix of this matrix is zero. Thus the two rows of this matrix are linearly dependent and so the equations (4.47) and (4.48) are equivalent, unless one of them is the trivial equation $0 = 0$.

Since at least one of the numbers q, s, r', and u is not zero, we know for sure that at least one of the equations (4.47) and (4.48) is quadratic in either β or α, and the solution(s) of that equation will also be solution(s) of the other one. Let us assume $r' \neq 0$, and focus on equation (4.47). Let us choose $\alpha := 1$. Thus this equation becomes:

$$r'\beta^2 - \gamma\beta - q = 0. \tag{4.49}$$

The solutions of this equation are:

$$\beta_1 = \frac{\gamma - \sqrt{\gamma^2 + 4r'q}}{2r'} \tag{4.50}$$

and

$$\beta_2 = \frac{\gamma + \sqrt{\gamma^2 + 4r'q}}{2r'}. \tag{4.51}$$

We need β_1 and β_2 to be real and distinct. If we show this, then the matrix:

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \tag{4.52}$$

will have a non–zero determinant, and thus the linear transformation $(X, Y) \mapsto (Z_1, Z_2)$ will be invertible.

Thus to achieve our goal we need now to show that the discriminant:

$$\Delta = \gamma^2 + 4r'q$$

of the quadratic equation (4.49) is strictly positive. To show this we use again one of the Jacobi identities, applied to the vacuum vector ϕ, namely:

$$\left[a_x^-, \left[a_x^0, a_y^+\right]\right]\phi + \left[a_x^0, \left[a_y^+, a_x^-\right]\right]\phi + \left[a_y^+, \left[a_x^-, a_x^0\right]\right]\phi = 0.$$

Since $a_x^0\phi = a_y^0\phi = 0$, we get:

$$r' - q = -e\gamma. \tag{4.53}$$

Let us remember, that from the Schwarz' inequality and the fact that (X, Y) is non–degenerate we know that $|e| < 1$. Thus we have:

$$\begin{aligned} -4r'q &\le (r'-q)^2 \quad &(4.54)\\ &= e^2\gamma^2 \\ &\le 1\gamma^2. \quad &(4.55)\end{aligned}$$

It follows now that $\Delta = \gamma^2 + 4r'q \ge 0$. This inequality must be strict, since if $\Delta = 0$, then we must have equality in (4.54), which means $(r' + q)^2 = 0$. Thus we would have $q = -r'$, which would imply: $e\gamma = -2r' \neq 0$. Hence $\gamma \neq 0$, and thus (4.55) is a strict inequality, which shows that $\Delta > 0$.

It follows now easily from (4.45) and (4.46) that, for all $(i, j) \in \{1, 2\}^2$, we have:

$$\left[a_{z_i}^-, a_{z_j}^0\right] \in \mathbb{R}a_{z_i}^-. \tag{4.56}$$

Re–scaling the random variables Z_1 and Z_2, we may assume that their coefficients b and h are both equal to 1. □

Lemma 4.4. *If (X, Y) is a centered non–degenerate Meixner random vector of class $\mathcal{M}_L$, whose constants from Lemma 4.1 satisfy the conditions $b = h = 1$ and $q = s = r' = u = 0$, then:*

$$cs' + dv = 0, \tag{4.57}$$
$$jp + kr = 0, \tag{4.58}$$
$$fp + gr = 0, \tag{4.59}$$
$$fs' + gv = 0. \tag{4.60}$$

Proof. From the Jacobi identity:

$$\left[a_x^-, \left[a_y^-, a_x^+\right]\right] + \left[a_y^-, \left[a_x^+, a_x^-\right]\right] + \left[a_x^+, \left[a_x^-, a_y^-\right]\right] = 0$$

we get:

$$\begin{aligned} 0 &= \left[a_x^-, \left[a_y^-, a_x^+\right]\right] + \left[a_y^-, \left[a_x^+, a_x^-\right]\right] \\ &= \left[a_x^-, I + fa_x^0 + a_y^0\right] - \left[a_y^-, I + ca_x^0 + da_y^0\right] \\ &= (fp + gr)a_x^- - (cs' + dv)a_y^-.\end{aligned}$$

Since $(X,\, Y)$ is non–degenerate, a_x^- and a_y^- are linearly independent. Thus we obtain that relations (4.57) and (4.59)hold.

Similarly, from the Jacobi identity:

$$\left[a_y^-, \left[a_x^-, a_y^+\right]\right] + \left[a_x^-, \left[a_y^+, a_y^-\right]\right] + \left[a_y^+, \left[a_y^-, a_x^-\right]\right] \;=\; 0,$$

we conclude that (4.58) and (4.60) hold. □.

We are ready now to present the main theorem.

Theorem 4.5. *If $(X,\, Y)$ is a non–degenerate centered Meixner random vector, then there exists an invertible linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$, such that the random vector $(X',\, Y') := S(X,\, Y)$ is equivalent (moment equal) to a random vector of two independent Meixner random variables with mixed preservation operators of class $\mathcal{M}_L$. Equivalently, the vector space spanned by the identity operator and the joint (APC) operators of X and Y is isomorphic, as a Lie algebra, to the vector space spanned by the identity operator and the joint (APC) operators of two independent Meixner random variables of class $\mathcal{M}_L$, with mixed preservation operators.*

Proof. From the previous lemmas we may assume that $q = s = r' = u = 0$ and $b = h = 1$.

From the Jacobi identity:

$$\left[a_y^-, \left[a_x^0, a_x^+\right]\right] + \left[a_x^0, \left[a_x^+, a_y^-\right]\right] + \left[a_x^+, \left[a_y^-, a_x^0\right]\right] \;=\; 0,$$

we get:

$$\left[a_y^-, p a_x^+\right] + 0 + \left[a_x^+, s' a_y^-\right] \;=\; 0.$$

This is equivalent to:

$$\gamma\left[a_y^-, a_x^+\right] \;=\; 0. \tag{4.61}$$

Similarly, from

$$\left[a_x^-, \left[a_y^0, a_y^+\right]\right] + \left[a_y^0, \left[a_y^+, a_x^-\right]\right] + \left[a_y^+, \left[a_x^-, a_y^0\right]\right] \;=\; 0,$$

we obtain:

$$\delta\left[a_x^-, a_y^+\right] \;=\; 0. \tag{4.62}$$

Since $\left[a_x^-, a_y^+\right] = \left[a_y^-, a_x^+\right]$, we conclude from (4.61) and (4.62), that if $\gamma \neq 0$ or $\delta \neq 0$, then:

$$\left[a_x^-, a_y^+\right] \;=\; 0, \tag{4.63}$$

and thus we can take $e = f = g = 0$ in (4.25). We analyze now two cases:

Case 1. If $\gamma \neq 0$ or $\delta \neq 0$, then $[a_x^-,\, a_y^+] = [a_y^-,\, a_x^+] = 0$. We know from the previous lemma that relations (4.57) and (4.58) hold. These are exactly the orthogonality relations from Example 2, from the previous section. Moreover the commutators of the joint (APC) operators of X and Y are expressed in terms of I and the joint (APC) operators of X and Y in exactly the same way as in Example 1. So we are done.

Case 2. If $\gamma = \delta = 0$, then $p = s'$ and $v = r$. It follows now from the previous lemma that:

$$\begin{aligned} cp + dv &= 0, && (4.64)\\ jp + dv &= 0, && (4.65)\\ fp + dv &= 0. && (4.66)\end{aligned}$$

Since $p = s'$, $[a_x^-,\, a_x^0] = pa_x^-$, and $[a_y^-,\, a_x^0] = s'a_y^-$, we conclude that, for all $z \in \{x,\, y\}$,

$$\left[a_z^-, a_x^0\right] = pa_z^-.$$

By duality it follows now that:

$$\left[a_x^0, a_z^+\right] = pa_z^+, \qquad (4.67)$$

for all $z \in \{x,\, y\}$. Similarly, we have:

$$\left[a_y^0, a_z^+\right] = va_z^+, \qquad (4.68)$$

for all $z \in \{x,\, y\}$.

Claim. $a_x^0 = p\mathcal{N}$ and $a_y^0 = v\mathcal{N}$, where $\mathcal{N}$ denotes the number operator.

Indeed, for any $n \geq 0$ and any $(z_1,\, z_2,\, \ldots,\, z_n) \in \{x,\, y\}^n$, using the product rule for commutators and the fact that $a_x^0\phi = 0$, we have:

$$\begin{aligned} & a_x^0\left(a_{z_1}^+ a_{z_2}^+ \cdots a_{z_n}^+ \phi\right)\\ =\ & \left(a_{z_1}^+ a_{z_2}^+ \cdots a_{z_n}^+\right) a_x^0\phi + \left[a_x^0, a_{z_1}^+ a_{z_2}^+ \cdots a_{z_n}^+\right]\phi\\ =\ & \sum_{k=1}^{n} a_{z_1}^+ \cdots a_{z_{k-1}}^+ \left[a_x^0, a_{z_k}^+\right] a_{z_{k+1}}^+ \cdots a_{z_n}^+\phi\\ =\ & \sum_{k=1}^{n} a_{z_1}^+ \cdots a_{z_{k-1}}^+ \left(pa_{z_k}^+\right) a_{z_{k+1}}^+ \cdots a_{z_n}^+\phi\\ =\ & p\sum_{k=1}^{n} a_{z_1}^+ a_{z_2}^+ \cdots a_{z_n}^+\phi\\ =\ & pna_{z_1}^+ a_{z_2}^+ \cdots a_{z_n}^+\phi. \end{aligned}$$

Hence, for all $n \geq 0$ and all $\xi \in H_n$, we have $a_x^0\xi = pn\xi$. Since we are interested only on the action of a_x^0 on the space $\mathcal{A}'\phi$, where $\mathcal{A}'$ is the unital algebra generated by X and Y, and $\mathcal{A}'\phi = \cup_{n\geq 0} \oplus_{k=0}^{n} H_k$, we conclude that $a_x^0 = p\mathcal{N}$.

Similarly, we can see that $a_y^0 = v\mathcal{N}$.

It follows from this claim and the relation (4.64) that:

$$\begin{aligned}\left[a_x^-, a_x^0\right] &= I + ca_x^0 + da_y^0\\ &= I + (cp + dv)\mathcal{N}\\ &= I + 0\mathcal{N}\\ &= I.\end{aligned}$$

Similarly, it follows from relations (4.65) and (4.66) that:

$$\left[a_x^-, a_y^0\right] = eI$$

and

$$\left[a_y^-, a_y^0\right] = I.$$

Since $|e| < 1$, we can make the following invertible linear change of variable:

$$\left(X', Y'\right) = \left(\frac{1}{\sqrt{2(1+e)}}(X+Y), \frac{1}{\sqrt{2(1-e)}}(X-Y)\right). \tag{4.69}$$

It easy to see now that:

$$\begin{aligned}\left[a_{x'}^-, a_{x'}^+\right] &= I,\\ \left[a_{x'}^-, a_{y'}^+\right] &= 0,\\ \left[a_{y'}^-, a_{y'}^+\right] &= I,\\ \left[a_{x'}^-, a_{x'}^0\right] &= \frac{p+v}{\sqrt{2(1+e)}}a_{x'}^-,\\ \left[a_{x'}^-, a_{y'}^0\right] &= \frac{p-v}{\sqrt{2(1-e)}}a_{x'}^-,\\ \left[a_{y'}^-, a_{x'}^0\right] &= \frac{p+v}{\sqrt{2(1+e)}}a_{y'}^-,\\ \left[a_{y'}^-, a_{y'}^0\right] &= \frac{p-v}{\sqrt{2(1-e)}}a_{y'}^-.\end{aligned}$$

The proof is now complete for the following reason. Since the Lie algebra generated by the joint (APC) operators of (X', Y') is isomorphic to the Lie algebra of the (APC) operators of two independent Meixner random variables with mixed preservation operators, and because the joint moments can be recovered from the commutators, as shown in [12], in the commutative case, and can easily be extended to the non–commutative case, we conclude that (X', Y') is moment equal to a random vector whose components are independent Meixner random variables of class $\mathcal{M}_L$ with mixed preservation operators. □

Due to the fact that in Example 2, the construction starts with two independent Meixner random vectors T and Z, we can make the following observation.

Observation 4.6. *The Lie algebra generated by the joint (APC) operators of a two dimensional non–degenerate Meixner random vector (X, Y) of class $\mathcal{M}_L$ is isomorphic to a subalgebra of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathbb{R}I$.*

Corollary 4.7. *Let μ be a Meixner probability measure of class $\mathcal{M}_L$ on $\mathbb{R}^2$, such that, μ is not supported by any line of equation $ax+by=c$, with $a^2+b^2>0$. Then, the following statements are true:*

(1) *Up to an invertible affine transformation, μ is a product of two Meixner probability measures of class $\mathcal{M}_L$ on $\mathbb{R}$. That means, there exist an invertible linear transformation $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$, a vector $c = (c_1, c_2) \in \mathbb{R}^2$, and two Meixner probability distributions μ_1 and μ_2 of class $\mathcal{M}_L$ on $\mathbb{R}$, such that the measure:*

$$\nu(B) \quad := \quad \mu(\mathcal{S}B + c), \tag{4.70}$$

can be written as:

$$\nu \quad = \quad \mu_1 \otimes \mu_2, \tag{4.71}$$

where $\mathcal{S}B + c := \{\mathcal{S}(x, y) + c \mid (x, y) \in B\}$, for all Borel subsets B of $\mathbb{R}^2$.

(2) *If μ is not supported by any finite union of lines of equation $ax+by=c$, with $a^2+b^2>0$, then up to an invertible affine transformation μ is a product of two Meixner probability distributions with infinite support, of class $\mathcal{M}_L$ on $\mathbb{R}$.*

Final Comment We left out of our discussion the symmetric two parameter hyperbolic distributions. The Lie Algebra W that we used in this paper, can be enlarged, so that we can include also these distributions and characterize the entire Meixner class using this new Lie Algebra. We hope to do this in another paper. We would like to mention that the recursive relation among the orthogonal polynomials, in d variables, that appears in [11], leads to commutation relations among the annihilation, preservation, and creation operators, that cannot be related to the present work, but to the next paper.

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Department of Mathematics, The Ohio State University at Marion, 1465 Mount Vernon Avenue, Marion, OH 43302, U.S.A., E-mail: *stan.7@osu.edu*