Charge conjugation from space-time inversion in QED: discrete and continuous groups

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Abstract

We show that the CPT groups of QED emerge naturally from the $\mathcal{PT}$ and $\mathcal{P}$ (or $\mathcal{T}$) subgroups of the Lorentz group. We also find relationships between these discrete groups and continuous groups, like the connected Lorentz and Poincaré groups and their universal coverings.

Keywords: CPT groups; space-time inversion; Lorentz and Poincaré groups

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1 Introduction

It was shown in [1] that the CPT group, $G_{\hat{\theta}}(\hat{\psi})$ ($\hat{\theta} = \hat{C} * \hat{P} * \hat{T}$), of the Dirac quantum field is a non abelian group with sixteen elements isomorphic to the direct product of the quaternion group, Q, and the cyclic group, $\mathbb{Z}_2$:

$$G_{\hat{\theta}}(\hat{\psi}) \cong Q \times \mathbb{Z}_2. \tag{1}$$

Unlike $G_{\hat{\theta}}(\hat{\psi})$ [1, 2, 3], the CPT group, $G_{\hat{\theta}}(\hat{A}_\mu)$, of the electromagnetic field is an abelian group of eight elements with three generators [2]:

$$G_{\hat{\theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3. \tag{2}$$

As the CPT transformation properties of the interacting $\hat{\psi} - \hat{A}_\mu$ fields are the same as for the free fields [4], the complete CPT group for QED, $G_{\hat{\Theta}}(QED)$, is the direct product of the above mentioned two groups, $G_{\hat{\Theta}}(\hat{\psi})$ and $G_{\hat{\Theta}}(\hat{A}_\mu)$, i.e.,

$$G_{\hat{\Theta}}(QED) = G_{\hat{\Theta}}(\hat{\psi}) \times G_{\hat{\Theta}}(\hat{A}_\mu) \cong (Q \times \mathbb{Z}_2) \times \mathbb{Z}_2^3. \tag{3}$$

2 C from $\mathcal{PT}$

It was shown in [3] that Q becomes isomorphic to a subgroup H of $SU(2)$, being λ the isomorphism:

$$Q \xrightarrow{\lambda} H < SU(2),$$
$$1 \mapsto I, \quad \iota \mapsto -i\sigma_1, \quad \gamma \mapsto -i\sigma_2, \quad \kappa \mapsto -i\sigma_3, \tag{4}$$

where ι, γ, κ are the three imaginary units of the quaternion group and σ_k ($k = 1, 2, 3$) are the Pauli matrices; and taking also into account that $\mathbb{Z}_2$ is isomorphic to the center of $SU(2)$: $\{I, -I\}$, then:

$$G_{\hat{\theta}}(\hat{\psi}) \cong H \times (center\ \ of\ \ SU(2)). \tag{5}$$

Since $SU(2)$ is the universal covering group of $SO(3)$:

$$SU(2) \xrightarrow{\Phi} SO(3), \tag{6}$$

then $\Phi(H)$ has 4 elements and, for that reason, the unique candidates are groups isomorphic to C_4 and $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein group. A simple application of Φ to the elements of H led to:

$$\Phi(H) = \{I,\ R_x(\pi),\ R_y(\pi),\ R_z(\pi)\}, \tag{7}$$

with $R_x(\pi)$, $R_y(\pi)$, $R_z(\pi)$ the rotations in π around the axes x, y and z, respectively, and I, the unit matrix in $SO(3)$. It was then immediately verified that the multiplication table of $\Phi(H) < SO(3)$ is the same as for D_2.

Then, we have:

$$G_{\hat{\theta}}(\hat{\psi}) \cong \Phi^{-1}(D_2) \times \mathbb{Z}_2. \tag{8}$$

Within the Lorentz group $O(3,1)$, the transformations of parity $\mathcal{P}$ and time reversal $\mathcal{T}$, together with their product $\mathcal{PT}$ and the 4×4 unit matrix E, lead to the subgroup of the Lorentz group, called the $\mathcal{PT}$-group, which is also isomorphic to D_2.

On the other hand, $\mathcal{P}$ or $\mathcal{T}$ separately, together with the unit 4×4 matrix E, give rise to the group $\mathbb{Z}_2$. Then, we obtain the desired result for the Dirac field:

$$G_{\hat{\theta}}(\hat{\psi}) \cong \Phi^{-1}(< \{\mathcal{P}, \mathcal{T}\} >)\times < \{\mathcal{P}\} > \tag{9}$$

or

$$G_{\hat{\theta}}(\hat{\psi}) \cong \Phi^{-1}(< \{\mathcal{P}, \mathcal{T}\} >)\times < \{\mathcal{T}\} >; \tag{10}$$

while, for the electromagnetic field, we have:

$$G_{\hat{\Theta}}(\hat{A}_\mu) \cong < \{\mathcal{P}, \mathcal{T}\} > \times < \{\mathcal{P}\} > \tag{11}$$

or

$$G_{\hat{\Theta}}(\hat{A}_\mu) \cong < \{\mathcal{P}, \mathcal{T}\} > \times < \{\mathcal{T}\} > . \tag{12}$$

The above result suggests that the Minkowskian space-time structure of special relativity, in particular the unconnected component of its symmetry group, the real Lorentz group $O(3,1)$, implies the existence of the CPT group as a whole, and therefore the existence of the charge conjugation transformation, and thus the proper existence of antiparticles.

3 Discrete and continuous groups

The relationships between the discrete groups: Q, $G_{PT} =< \{\mathcal{P}, \mathcal{T}\} >$, $G_{\hat{\theta}}(\hat{\psi})$ and $G_{\hat{\Theta}}(\hat{A}_\mu)$ and continuous groups, like the Lorentz group and its universal covering group, can be summarized in the following diagram:

$$\begin{array}{ccccccccc}
\mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_{\hat{\theta}}(\hat{\psi}) \cong Q \times \mathbb{Z}_2 & \stackrel{\alpha}{\leftarrow} & Q & \stackrel{\mu}{\rightarrow} & SU(2) & \stackrel{\beta}{\rightarrow} & SL_2(\mathbb{C}) & \stackrel{\gamma}{\rightarrow} & \mathbb{R}^4 \rtimes SL_2(\mathbb{C}) \\
\downarrow\psi & & \downarrow\rho & & \downarrow\Phi & & \downarrow\widetilde{\Phi} & & \downarrow\widetilde{\widetilde{\Phi}} \\
G_{\hat{\theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3 & \stackrel{\bar{\alpha}}{\leftarrow} & G_{PT} \cong \mathbb{Z}_2^2 & \stackrel{\bar{\mu}}{\rightarrow} & SO(3) & \stackrel{\bar{\beta}}{\rightarrow} & SO^c(3,1) & \stackrel{\bar{\gamma}}{\rightarrow} & \mathbb{R}^4 \rtimes SO^c(3,1).
\end{array} \tag{13}$$

The homomorphism μ is defined by $\mu(q) = \lambda(q)$ (see (4)) and the homomorphism Φ was described in (6); $\widetilde{\Phi}$ and $\widetilde{\widetilde{\Phi}}$ are the homomorphisms between the conected Lorentz ($SO^c(3,1)$) and Poincaré ($\mathbb{R}^4 \rtimes SO^c(3,1) \equiv \mathcal{P}_4^c$) groups, respectively, and their universal coverings ($SL_2(\mathbb{C})$ and $\mathbb{R}^4 \rtimes SL_2(\mathbb{C}) \equiv \bar{\mathcal{P}}_4^c$); while ρ, ψ, $\bar{\mu}$, α, $\bar{\alpha}$, β, $\bar{\beta}$, γ and $\bar{\gamma}$ are given by:

$$Q \stackrel{\rho}{\longrightarrow} \frac{Q}{\mathbb{Z}_2} \cong G_{PT}, \quad q \mapsto [q], \tag{14}$$

$$G_{\hat{\theta}}(\hat{\psi}) \stackrel{\psi}{\longrightarrow} \frac{Q \times \mathbb{Z}_2}{\mathbb{Z}_2} \cong G_{\hat{\theta}}(\hat{A}_\mu), \quad (q,1) \mapsto [(q,1)], \quad (q,-1) \mapsto [(q,1)], \tag{15}$$

$$G_{PT} \stackrel{\bar{\mu}}{\longrightarrow} SO(3), \quad [q] \mapsto \Phi(h), \quad h = \lambda(q), \tag{16}$$

$$Q \stackrel{\alpha}{\longrightarrow} G_{\hat{\theta}}(\hat{\psi}), \quad q \mapsto (q,1), \tag{17}$$

$$G_{PT} \stackrel{\bar{\alpha}}{\longrightarrow} G_{\hat{\theta}}(\hat{A}_\mu), \quad [q] \mapsto [(q,1)], \tag{18}$$

$$SU(2) \stackrel{\beta}{\longrightarrow} SL_2(\mathbb{C}), \quad A \mapsto A, \tag{19}$$

$$SO(3) \stackrel{\bar{\beta}}{\longrightarrow} SO^c(3,1), \quad R \mapsto \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \tag{20}$$

$$SL_2(\mathbb{C}) \xrightarrow{\gamma} \mathbb{R}^4 \rtimes SL_2(\mathbb{C}), \quad B \mapsto (\mathbf{0}, B), \tag{21}$$

$$SO^c(3,1) \xrightarrow{\bar{\gamma}} \mathbb{R}^4 \rtimes SO^c(3,1), \quad \Lambda \mapsto (\mathbf{0}, \Lambda). \tag{22}$$

Let ν the function which goes from $Q \times \mathbb{Z}_2$ to $SU(2)$:

$$Q \times \mathbb{Z}_2 \xrightarrow{\nu} SU(2), \quad (q,g) \mapsto \nu(q,g) := sg(g)\lambda(q), \tag{23}$$

where $sg(g) = 1$ if $g = 1$ and $sg(g) = -1$ if $g = -1$.

Then, it holds:

- ν is an homomorphism.
  Proof:

$$\begin{aligned} \nu((q',g')(q,g)) &= \nu(q'q, g'g) = sg(g'g)\lambda(q'q) = sg(g')sg(g)\lambda(q')\lambda(q) \\ &= (sg(g')\lambda(q'))(sg(g)\lambda(q)) = \nu(q',g')\nu(q,g). \end{aligned} \tag{24}$$

- ν is 2 to 1.
  Proof:

$$\nu(q,-1) = \nu(-q,1). \tag{25}$$

$\bar{\nu}$ is determined by ν due to the commutative diagram:

$$\begin{array}{ccc} G_{\hat{\theta}}(\hat{\psi}) & \xrightarrow{\nu} & SU(2) \\ \downarrow\psi & & \downarrow\Phi \\ G_{\hat{\theta}}(\hat{A}_\mu) & \xrightarrow{\bar{\nu}} & SO(3) \end{array} \tag{26}$$

and is also a 2 to 1 homomorphism. If $x \in \mathbb{Z}_2^3$ then $\psi^{-1}(\{x\}) = \{y_1, y_2\} \subset Q \times \mathbb{Z}_2$. Hence:

$$\begin{aligned} \bar{\nu}(x) &= \bar{\nu}(\psi(y_k)) = \bar{\nu} \circ \psi(y_k) = \Phi \circ \nu(y_k) \\ &= \Phi(\nu(y_k)) = \Phi(\nu(q_k, g_k)), \end{aligned} \tag{27}$$

with $k = 1$ or 2. Then:

- $\bar{\nu}$ is an homomorphism.
  Proof:

$$\begin{aligned} \bar{\nu}(x'x) &= \Phi(\nu((q'_k, g'_k)(q_l, g_l))) = \Phi(\nu(q'_k, g'_k))\Phi(\nu(q_l, g_l)) \\ &= \Phi \circ \nu(q'_k, g'_k)\Phi \circ \nu(q_l, g_l) = \bar{\nu} \circ \ \psi(q'_k, g'_k)\bar{\nu} \circ \psi(q_l, g_l) \\ &= \bar{\nu}(x')\bar{\nu}(x), \end{aligned} \tag{28}$$

  with $l = 1$ or 2.

- $\bar{\nu}$ is 2 to 1.
  Proof: From $\Phi \circ \nu = \bar{\nu} \circ \psi$ and the fact that Φ, ν and ψ are 2 to 1, it follows that $\bar{\nu}$ is also 2 to 1.

Taking into account diagrams (13) and (26), the group homomorphisms:

$$\varphi = \gamma \circ \beta \circ \nu \tag{29}$$

and

$$\bar{\varphi} = \bar{\gamma} \circ \bar{\beta} \circ \bar{\nu}, \tag{30}$$

make commutative the following diagram:

$$\begin{array}{ccc} G_{\hat{\theta}}(\hat{\psi}) & \stackrel{\varphi}{\rightarrow} & \bar{\mathcal{P}}_4^c \\ \downarrow_{\psi} & & \downarrow_{\tilde{\tilde{\Phi}}} \\ G_{\hat{\theta}}(\hat{A}_\mu) & \stackrel{\bar{\varphi}}{\rightarrow} & \mathcal{P}_4^c; \end{array} \tag{31}$$

making explicit the close and possibly deep relationship between these discrete and continuous groups.

4 Discussion

In summary, we have that $G_{\hat{\theta}}(\hat{\psi})$ and $G_{\hat{\Theta}}(\hat{A}_\mu)$, which are groups acting at the quantum field level that include the charge conjugation operator, emerge in a natural way from the $\mathcal{PT}$-group and its $\mathcal{P}$ (or $\mathcal{T}$) subgroups. That is, from matrices acting on Minkowski classical space-time.

It is important to note that G_{PT} generates $G_{\hat{\Theta}}(\hat{A}_\mu)$, the CPT group of the electromagnetic field, without passing through $SU(2)$. That is, without the need of using spinors; while the group $SU(2)$ is needed in order to generate $G_{\hat{\theta}}(\hat{\psi})$, the CPT group of the Dirac field.

Finally, another important thing that we found is the relationship between discrete groups, like $G_{\hat{\Theta}}(\hat{A}_\mu)$ and $G_{\hat{\theta}}(\hat{\psi})$, and continuous groups, like the connected Poincaré group ($\mathcal{P}_4^c$) and its universal covering ($\bar{\mathcal{P}}_4^c$). This is shown in diagram (31).

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References

[1] M. Socolovsky, *The CPT group of the Dirac Field*, Int J Theor Phys, **43** (2004), pp. 1941-1967; arXiv: math-ph/0404038.

[2] B. Carballo Pérez and M. Socolovsky, *Irreducible representations of the CPT groups in QED*, IJPAM (in press) (2010); arXiv: math-ph/0906.2381v3.

[3] B. Carballo Pérez and M. Socolovsky, *Charge conjugation from space-time inversion*, Int J Theor Phys, **48** (2009), pp. 1712-1716; arXiv: hep-th/0811.0842v1.

[4] J. A. de Azcárraga, *P, C, T, θ in Quantum Field Theory*, GIFT 7/75 (1975), 69.