ON THE CLASSIFICATION OF QUASIHOMOGENEOUS SINGULARITIES

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Abstract. The motivation for this paper are computer calculations of complete lists of weight systems of quasihomogeneous polynomials with isolated singularity at 0 up to rather large Milnor numbers. We review combinatorial characterizations of such weight systems for any number of variables. This leads to certain types and graphs of such weight systems. Using them, we prove an upper bound for the common denominator (and the order of the monodromy) by the Milnor number, and we show surprising consequences if the Milnor number is a prime number.

Contents

1. Introduction

Several people have (re)discovered characterizations of those weight systems which admit quasihomogeneous polynomials with isolated singularity at 0. Section 2 collects and compares these characterizations and gives all references which we found. The results of this section are not new. But the references are not well known and for several reasons,

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it is not so easy to extract the results from them. Also, we will need part of the characterizations for a good control of such weight systems in the later sections.

In section 3 a part of the conditions is used to associate after a choice a type and a graph to a quasihomogeneous singularity. The idea for this is contained in [Ar][AGV], and there it is carried out for 2 and 3 variables. The general case is carried out in [OR1], but that part of [OR1] was never published. As we will need the graphs in the sections 4 and 6, we rewrite the general case. Section 3 also makes the classification in the case of 4 variables in [YS] more precise, showing how necessary and sufficient conditions are obtained.

Section 4 gives an estimate $d \leq \text{const}(n) \cdot \mu$ for the weighted degree d of a reduced weight system of a quasihomogeneous singularity from above by the Milnor number μ. The calculations start with the well known formula for μ in terms of the weights, but refine this formula using a graph and a type of the singularity. The estimate is useful for a computer calculation of all reduced weight systems of quasihomogeneous singularities up to a given Milnor number. We carried out such computer calculations for 2, 3 and 4 variables and $\mu \leq 1500$, 1000 and 500. The long tables will be available on a homepage. Some observations from them are formulated in section 5.

Section 6 proves a surprising fact which we found looking at these tables. If the Milnor number of a quasihomogeneous singularity is a prime number, then the only type which one can associate to it is the chain type (up to adding or removing squares from the singularity), and furthermore, all eigenvalues of the monodromy have multiplicity one. The proof further refines the formula for the Milnor number from section 4.

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2. Combinatorial characterizations of weight systems of quasihomogeneous singularities

We note $\mathbb{N}_0 = \{0, 1, 2, ...\} \supset \mathbb{N} = \{1, 2, ...\}$. The support of a polynomial

$$f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \cdot x^\alpha \in \mathbb{C}[x_1, ..., x_n] \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$$

is $\operatorname{supp} f = \{\alpha \in \mathbb{N}_0^n \mid a_\alpha \neq 0\}$. The polynomial is called *quasihomogeneous* with weight system $(w_1, ..., w_n, d) \in \mathbb{R}_{>0}^{n+1}$ if

$$\sum_{i=1}^{n} \alpha_i \cdot w_i = d \quad \text{for any } \alpha \in \operatorname{supp} f.$$

Here $w_1, ..., w_n$ are the weights and d is the weighted degree. If a polynomial is quasihomogeneous with some weight system it is also quasihomogeneous with a weight system $(w_1, ..., w_n, d) \in \mathbb{Q}_{>0}^{n+1}$. If a quasihomogeneous polynomial has an isolated singularity at 0, that is, if the $\frac{\partial f}{\partial x_i}$ vanish simultaneously precisely at 0, then $w_i < d$ for all i. Therefore, from now on throughout the whole paper we consider only weight systems with

$$(w_1, ..., w_n, d) \in \mathbb{Q}_{>0}^{n+1} \qquad \text{and} \qquad w_i < d \text{ for all } i.$$

Furthermore, from now on we reserve the letters $v_1, ..., v_n$ for weights of weight systems $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$, and the letters $w_1, ..., w_n$ for weights of normalized weight systems $(w_1, ..., w_n, 1) \in \mathbb{Q}_{>0}^{n+1}$, that is, with weighted degree 1.

A weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ is called *reduced* if $\gcd(v_1, ..., v_n, d) = 1$. In later chapters, but not in this one, we will also use a result in [Sa1] and restrict to weight systems with $v_i \leq \frac{d}{2}$ and $w_i \leq \frac{1}{2}$.

Fix $n \in \mathbb{N}$ and denote $N := \{1, ..., n\}$ and $e_i := (\delta_{ij})_{j=1,...,n} \in \mathbb{N}_0^n$. For $J \subset N$ and a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ (with $v_i < d$) and $k \in \mathbb{N}_0$ denote

$$\begin{aligned} \mathbb{N}_0^J &:= \{\alpha \in \mathbb{N}_0^n \mid \alpha_i = 0 \text{ for } i \notin J\}, \\ (\mathbb{N}_0^n)_k &:= \{\alpha \in \mathbb{N}_0^n \mid \sum_i \alpha_i \cdot v_i = k\}, \\ (\mathbb{N}_0^J)_k &:= \mathbb{N}_0^J \cap (\mathbb{N}_0^n)_k. \end{aligned}$$

The following combinatorial lemma will help to compare in theorem 2.2 several characterizations of weight systems which admit quasihomogeneous polynomials with isolated singularities. A discussion of the history and references will be given after theorem 2.2.

Lemma 2.1. *Fix a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ and a subset $R \subset (\mathbb{N}_0^n)_d$. For any $k \in N$ define*

$$R_k := \{\alpha \in (\mathbb{N}_0^n)_{d-v_k} \mid \alpha + e_k \in R\}.$$

The following five conditions (C1), (C1)', (C2), (C2)' and (C3) are equivalent.

$$\begin{aligned}
\textit{(C1):}\quad & \forall\ J \subset N \text{ with } J \neq \emptyset \\
& \qquad \exists\ \alpha \in R \cap \mathbb{N}_0^J \\
& \quad \textit{or}\ \exists\ K \subset N - J \text{ with } |K| = |J| \\
& \qquad\qquad\qquad\qquad \text{and } \forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{N}_0^J.
\end{aligned}$$

(C1)': *As (C1), but only J with* $|J| \leq \frac{n+1}{2}$.

$$\begin{aligned}
\textit{(C2):}\quad & \forall\ J \subset N \text{ with } J \neq \emptyset \\
& \qquad \exists\ K \subset N \text{ with } |K| = |J| \\
& \qquad\qquad\qquad \text{and } \forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{N}_0^J.
\end{aligned}$$

(C2)': *As (C2), but only J with* $|J| \leq \frac{n+1}{2}$.

$$\begin{aligned}
\textit{(C3):}\quad & \forall\ I,\ J \subset N \text{ with } |I| < |J| \\
& \qquad \exists\ k \in N - I \text{ and } \exists\ \alpha \in R_k \cap \mathbb{N}_0^J.
\end{aligned}$$

Proof: (C1)$\Rightarrow$(C1)' and (C2)$\Rightarrow$(C2)' are trivial.

(C1)'$\Rightarrow$(C1): Consider $J \subset N$ with $|J| > \frac{n+1}{2}$ and $I \subset J$ with $|I| = \left[\frac{n+1}{2}\right]$. If there exists $\alpha \in R \cap \mathbb{N}_0^I$, then also $\alpha \in R \cap \mathbb{N}_0^J$. If not, then there exists $K \subset N - I$ with $|K| = |I|$ and $\forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{N}_0^I$. Then n is even and $K = N - I$, and $K \cap J \neq \emptyset$, and for $k \in K \cap J$ and $\alpha \in R_k \cap \mathbb{N}_0^I$ one finds $\alpha + e_k \in R \cap \mathbb{N}_0^J$.

(C2)'$\Rightarrow$(C1)': Consider $J \subset N$ with $0 < |J| \leq \frac{n+1}{2}$ and $K \subset N$ such that J and K satisfy (C2)'.

1st case: $K \subset N - J$. Then J and K satisfy (C1)'.

2nd case: $K \cap J \neq \emptyset$. Then for $k \in K \cap J$ and $\alpha \in R_k \cap \mathbb{N}_0^J$ one obtains $\alpha + e_k \in R \cap \mathbb{N}_0^J$, so J satisfies (C1)'.

(C3)$\Rightarrow$(C2): Consider $J \subset N$ with $J \neq \emptyset$. Construct elements $k_1, ..., k_{|J|} \in N$ and subsets $I_j = \{k_1, ..., k_j\}$ for $0 \leq j \leq |J| - 1$ and $K := \{k_1, ..., k_{|J|}\}$ as follows. (C3) gives for J and I_j an element $k_{j+1} \in N - I_j$ with $R_{k_{j+1}} \cap \mathbb{N}_0^J \neq \emptyset$. Obviously $|K| = |J|$, and J and K satisfy (C2).

(C1)$\Rightarrow$(C3): Consider $I, J \subset N$ with $|I| < |J|$. Then $J \neq \emptyset$.

1st case, J and some set K satisfy (C1): Because of $|I| < |J| = |K|$ there is a $k \in (N - I) \cap K$ with $R_k \cap \mathbb{N}_0^J \neq \emptyset$.

2nd case, $R \cap \mathbb{N}_0^J \neq \emptyset$: If $R \cap \mathbb{N}_0^J \neq R \cap \mathbb{N}_0^{J \cap I}$ then there exist an $\alpha \in R \cap \mathbb{N}_0^J - R \cap \mathbb{N}_0^{J \cap I}$ and a $k \in J - J \cap I$ with $\alpha_k > 0$. Then $k \in N - I$ and $\alpha - e_k \in R_k \cap \mathbb{N}_0^J$.

So suppose $R \cap \mathbb{N}_0^J = R \cap \mathbb{N}_0^{J \cap I}$. Then $J_1 := J - J \cap I \neq \emptyset$ because of $|I| < |J|$, and $R \cap \mathbb{N}_0^{J_1} = \emptyset$, so there exists a $K_1 \subset N - J_1$ such that J_1 and K_1 satisfy (C1). If $K_1 \cap J \neq \emptyset$ then for $k \in K_1 \cap J$ and

$\alpha \in R_k \cap \mathbb{N}_0^{J_1}$ one has $\alpha + e_k \in R \cap \mathbb{N}_0^J - R \cap \mathbb{N}_0^{J\cap I}$, a contradiction. Thus $K_1 \cap J = \emptyset$.

This and $|K_1| = |J_1| > |I-(J-I)|$ give $K_1 - I = K_1 - (I-(J-I)) \neq \emptyset$. Any $k \in K_1 - I$ satisfies $\emptyset \neq R_k \cap \mathbb{N}_0^{J_1} \subset R_k \cap \mathbb{N}_0^J$. $\square$

Theorem 2.2. *Let $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ be a weight system.*

(a) Let $f \in \mathbb{C}[x_0, ..., x_n]$ be a quasihomogeneous polynomial. The condition

(IS1): f has an isolated singularity at 0,

implies that $R := \operatorname{supp} f \subset (\mathbb{N}_0^n)_d$ satisfies (C1) to (C3).

(b) Let R be a subset of $(\mathbb{N}_0^n)_d$. The following conditions are equivalent.

(IS2): There exists a quasihomogeneous polynomial f with $\operatorname{supp} f \subset R$ and an isolated singularity at 0.
(IS2)': A generic quasihomogeneous polynomial with $\operatorname{supp} f \subset R$ has an isolated singularity at 0.
(C1) to (C3): R satisfies (C1) to (C3).

(c) In the case $R = (\mathbb{N}_0^n)_d$ obviously $R_k = (\mathbb{N}_0^n)_{d-v_k}$. The following conditions are equivalent.

(IS3): There exists a quasihomogeneous polynomial f with an isolated singularity at 0.
(IS3)': A generic quasihomogeneous polynomial has an isolated singularity at 0.
(C1) to (C3): $R = (\mathbb{N}_0^n)_d$ satisfies (C1) to (C3).

Remarks 2.3. Several people (re)discovered parts of this theorem. We will not reprove it here, but comment on the history and the references.

(i) Of course, (IS2) $\iff$ (IS2)' and (IS3) $\iff$ (IS3)' and (b)$\Rightarrow$(c) and (a)$\Rightarrow$ $\big($(IS2) $\Rightarrow$ (C1) to (C3)$\big)$.

(ii) Part (a) is quite elementary, for example (IS1)$\Rightarrow$(C1) is contained in K. Saito's paper [Sa1, Lemma 1.5], and it can also be extracted from [Sh, Remark 3].

(iii) (IS2) $\iff$ (C2) is part of an equivalence for more general functions in [Ko1, Remarque 1.13 (ii)], but there Kouchnirenko did not carry out the proof in detail. He gave a short proof of the refined version (IS2) $\iff$ (C2)' in [Ko2, Theorem 1]. This reference [Ko2] seems to have been cited up to now only in [Sh], it seems to have been almost completely ignored.

(iv) Around the same time as Kouchnirenko, Orlik and Randell proved (IS3) $\iff$ (C3) in the preprint [OR1, Theorem 2.12], but the published part [OR2] of it does not contain this result. It seems that they have not published this result.

(v) O.P. Shcherbak stated a more general result [Sh, Theorem 1] from which one can extract (IS2) $\iff$ (C1), but he did not provide a proof. That was done by Wall [Wa, Ch. 5], who also stated explicitly (IS2) $\iff$ (C1) and (IS3) $\iff$ (C1), they are Theorem 5-1 and Theorem 5-3 in [Wa] for the hypersurface case (explicit in (5-7)). But as he covers a much more general case, his proof is long.

(vi) A short proof of (IS3) $\iff$ (C1) is given by Kreuzer and Skarke [KS, proof of Theorem 1], though it requires some work to see that the condition stated in [KS, Theorem 1] is equivalent to (C1).

(vii) In theorem 2.2 (c) conditions $(\mathbb{N}_0^J)_k \neq \emptyset$ for some $k \in \mathbb{N}_0$ arise. For $k \in \mathbb{Z}$ denote $\mathbb{Z}^J$, $(\mathbb{Z}^n)_k$ and $(\mathbb{Z}^J)_k$ analogously to $\mathbb{N}_0^J$, $(\mathbb{N}_0^n)_k$ and $(\mathbb{N}_0^J)_k$. Then $(\mathbb{Z}^J)_k \neq \emptyset$ is equivalent to $\gcd(v_j \,|\, j \in J) \,|\, k$. But $(\mathbb{N}_0^J)_k \neq \emptyset$ (for $k \geq 0$) is more delicate. In the case $J = \{1,2\}$ sufficient conditions are $\gcd(v_1, v_2) \,|\, k$ and $\operatorname{lcm}(v_1, v_2) - v_1 - v_2 + 1 \leq k$, because then

$$\left(\frac{\operatorname{lcm}(v_1, v_2)}{v_1} - 1\right) \cdot v_1 + (-1) \cdot v_2 = (-1) \cdot v_1 + \left(\frac{\operatorname{lcm}(v_1, v_2)}{v_2} - 1\right) \cdot v_2$$

is the largest integer missing in $\mathbb{N}_0 \cdot v_1 + \mathbb{N}_0 \cdot v_2$.

For any weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ define the rational function

$$\rho_{(v,d)}(t) := \prod_{i=1}^{n} (t^d - t^{v_i})(t^{v_i} - 1)^{-1}.$$

It is well known that $\rho_{(v,d)}(t) \in \mathbb{N}_0[t]$ if a quasihomogeneous polynomial with isolated singularity at 0 exists.

The conditions $\rho_{(v,d)}(t) \in \mathbb{Z}[t]$ and $\rho_{(v,d)}(t) \in \mathbb{N}_0[t]$ are in general weaker than (C1) to (C3), but $\rho_{(v,d)}(t) \in \mathbb{Z}[t]$ is equivalent to a surprisingly similar statement. Denote by $\overline{(C1)}$ and $\overline{(C2)}$ the conditions obtained from (C1) and (C2) in lemma 2.1 with $\mathbb{N}_0^J$ replaced by $\mathbb{Z}^J$.

Lemma 2.4. *Fix a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$. The following conditions are equivalent.*

$\overline{(IS3)}$: $\rho_{(v,d)}(t) \in \mathbb{Z}[t]$.
(GCD): $\forall\ J \subset N$ *the* $\gcd(v_j \,|\, j \in J)$ *divides at least* $|J|$ *of the numbers* $d - v_k$.
$\overline{(C2)}$: *for* $R = (\mathbb{N}_0^n)_d$.
$\overline{(C1)}$: *for* $R = (\mathbb{N}_0^n)_d$.

Proof: $\overline{(IS3)}$ means that all zeros of $\prod_{i=1}^n(t^{v_i}-1)$ are zeros of $\prod_{i=1}^n(t^{d-v_i}-1)$ with at least the same multiplicity. This shows $\overline{(IS3)} \iff$ (GCD). The equivalence (GCD) $\iff \overline{(C2)}$ is trivial. The equivalence $\overline{(C2)} \iff \overline{(C1)}$ follows as (C2)'⇒(C1)' and (C1)⇒(C2) in the proof of lemma 2.1. □

Lemma 2.5. *Fix a weight system $(v_1,...,v_n,d)\in\mathbb{N}^{n+1}$ with $v_i<d$. If $n\le 3$ then (IS3) $\iff \overline{(IS3)}$.*

Proof: We restrict to the case $n=3$. It is sufficient to show $\overline{(C1)}$ ⇒(C1) for $R=(\mathbb{N}_0^n)_d$. The ($|J|=1$)-parts of $\overline{(C1)}$ and (C1) coincide.

Consider $J=\{1,2\}$, $J_1=\{1\}$, $J_2=\{2\}$. Then J satisfies (C1) if and only if $(\mathbb{N}_0^J)_d\neq\emptyset$. Now consider the different possibilities how J_1 and J_2 can satisfy (C1). The only case where $(\mathbb{N}_0^J)_d\neq\emptyset$ is not obvious is the case when $J_1\&K_1=\{3\}$ and $J_2\&K_2=\{3\}$ satisfy (C1), that is, when $v_1|(d-v_3)$ and $v_2|(d-v_3)$. Of course, then also $\mathrm{lcm}(v_1,v_2)|(d-v_3)$ and $\mathrm{lcm}(v_1,v_2)\le d-v_3$.

$\overline{(C1)}$ for J gives $(\mathbb{Z}^J)_d\neq\emptyset$, that is, $\gcd(v_1,v_2)|d$.

The conditions $\gcd(v_1,v_2)|d$ and $\mathrm{lcm}(v_1,v_2)\le d-v_3$ imply $(\mathbb{N}_0^J)_d\neq\emptyset$ by remark 2.3 (vii), so J satisfies (C1).

The ($|J|=3$)-part of (C1) follows from the ($|J|=2$)-part. □

Remarks 2.6. (i) Lemma 2.5 is Theorem 3 in [Sa2]. It is also stated in [Ar, remark after cor. 4.13] and [AGV, 2nd remark in 12.3].

(ii) For $n\ge 4$ $\overline{(IS3)}$ is weaker than (IS3). [AGV, 12.3] contains the example $(v_1,v_2,v_3,v_4,d)=(1,33,58,24,265)$ of Ivlev. Here $\rho_{(v,d)}(t)\in\mathbb{N}_0[t]$, but (C1) fails for $J=\{2,4\}$.

(iii) The equivalence $\overline{(IS3)} \iff \overline{(C1)}$ in lemma 2.4 is (up to rewriting their condition as $\overline{(C1)}$) Lemma 1 in [KS].

(iv) Chapter 3 in [Wa] contains results and short proofs for 0-dimensional quasihomogeneous complete intersections which are very close to theorem 2.2 (b)+(c), lemma 2.4 and lemma 2.5.

3. Types and graphs of quasihomogeneous singularities

Here a classification of quasihomogeneous polynomials with isolated singularity at 0 by certain *types*, which are encoded in certain *graphs*, will be given. For $n\in\{2,3\}$ this is treated in [Ar][AGV], the general case is carried out in a part of [OR1] which is not published in [OR2].

The type will come from some choice. Often several choices are possible, and they may lead to different types or the same type, so, often there are several types for one quasihomogeneous polynomial.

Now consider $n \in \mathbb{N}$, $N = \{1, ..., n\}$, a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ and a quasihomogeneous polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ with an isolated singularity at 0. Then $\operatorname{supp} f \subset (\mathbb{N}_0^n)_d$ satisfies (C2) by theorem 2.2 (a).

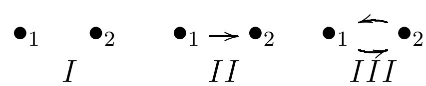
The choice is a map $\kappa : N \to N$ such that for any $j \in N$ the sets $J = \{j\}$ and $K = \{\kappa(j)\}$ satisfy (C2) with $R = \operatorname{supp} f$, that is, f contains a summand $b \cdot x_j^a \cdot x_{\kappa(j)}$ for some $b \in \mathbb{C}^*$, $a \in \mathbb{N}$. The *type* is the conjugacy class of this map κ with respect to the symmetric group S_n. The graph which encodes the map κ is the ordered graph with n vertices with numbers $1, ..., n$ and an arrow from j to $\kappa(j)$ for any $j \in N$ with $j \neq \kappa(j)$. The ordered graph without the numbering of the vertices obviously encodes the type.

In order to describe the graphs, an oriented tree is called *globally oriented* if each vertex except one has exactly one outgoing arrow. Then the exceptional vertex has only incoming arrows and is called *root*. Starting at any vertex and following the arrows one arrives at the root.

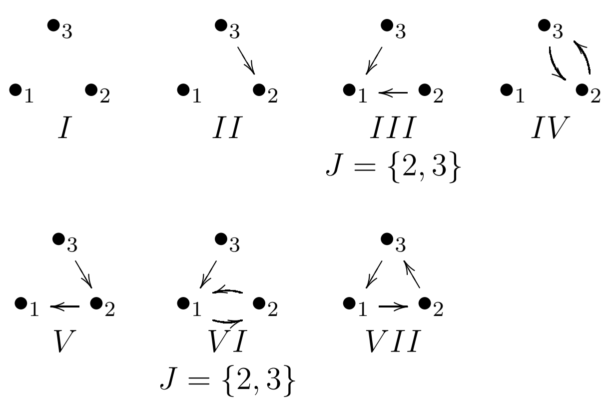
An oriented cycle is called *globally oriented* if each vertex has one incoming and one outgoing arrow. Following the arrows one runs around the cycle. The following lemma is obvious.

Lemma 3.1. *Exactly those graphs occur as graphs of maps $\kappa : N \to N$ whose components either are globally oriented trees or consist of one globally oriented cycle and finitely many globally oriented trees whose roots are on the cycle.*

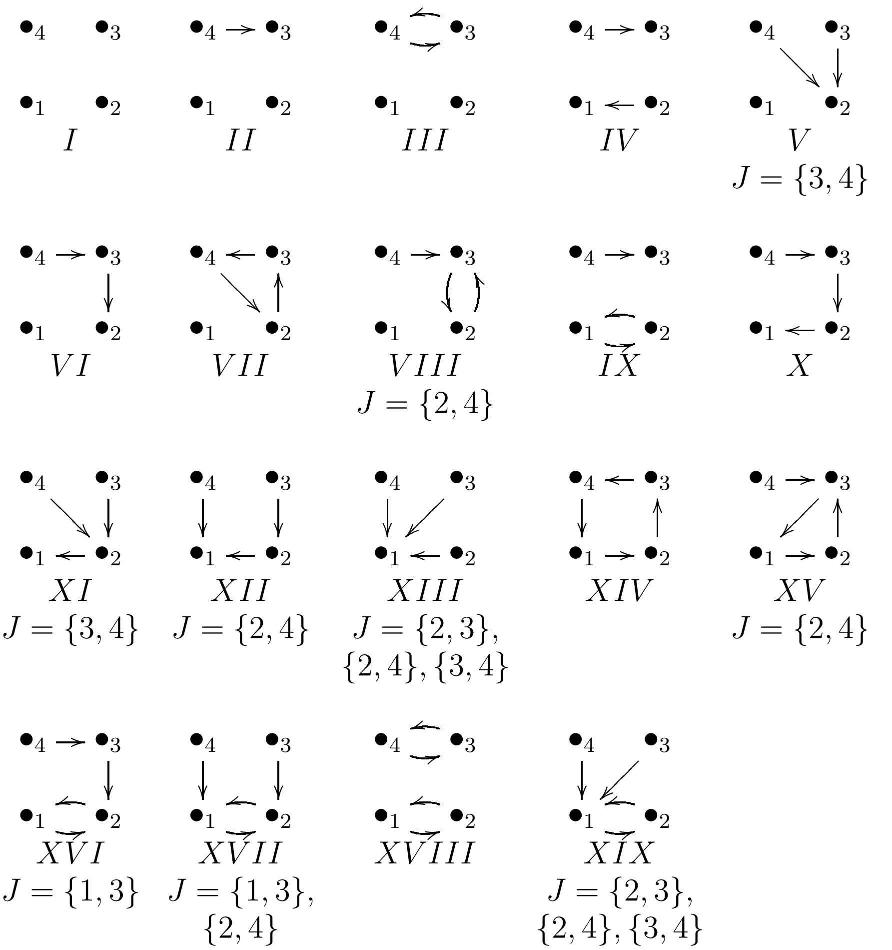
Examples 3.2. (i) $n = 2$: [Ar][AGV] 3 types,



(ii) $n = 3$: [Ar][AGV] 7 types. The sets J under the graphs III and VI are explained in example 3.6.



(iii) $n = 4$: [OR1] and [YS] 19 types. We follow the numbering in [YS, Proposition 3.5]. The sets J under 9 of the 19 graphs are explained in example 3.6.



(iv) $n = 5$: 47 types.
(v) $n = 6$: 128 types.

Remark 3.3. Fix a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$, a quasihomogeneous polynomial f and a map $\kappa : N \to N$ as above. Then for any $j \in N$ the sets $J = \{j\}$ and $K = \{\kappa(j)\}$ satisfy (C2) with $R = \operatorname{supp} f$ *in a unique way*: There is a unique $a_j \in \mathbb{N}$ with $\alpha := a_j e_j \in R_{\kappa(j)} \cap \mathbb{N}_0^J$, that is, there is a unique monomial $x_j^{a_j} x_{\kappa(j)}$ with exponent $a_j e_j + e_{\kappa(j)}$ in the support of f.

Now we forget $(v_1, ..., v_n, d)$ and f and start anew with such a tuple of monomials. We fix $n \in \mathbb{N}$, $N = \{1, ..., n\}$, a map $\kappa : N \to N$, numbers $a_1, ..., a_n \in \mathbb{N}$ and the set $R := \{a_j e_j + e_{\kappa(j)} \mid j \in N\} \subset \mathbb{N}_0^n$ of exponents of the monomials $x_j^{a_j} x_{\kappa(j)}$.

Always $|R| \leq n$, and most often $|R| = n$. The difference $n - |R|$ is the number of 2-cycles in the graph of κ with vertices j_1 and j_2 and numbers $a_{j_1} = a_{j_2} = 1$.

Lemma 3.4. *A weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ and $R \subset (\mathbb{N}_0^n)_d$ exists if and only if any even cycle with vertices $j_1, ..., j_l$ (l even) satisfies either*

(EC1) neither $a_{j_1} = a_{j_3} = ... = a_{j_{l-1}} = 1$
nor $a_{j_2} = a_{j_4} = ... = a_{j_l} = 1$,

or

(EC2) $a_{j_1} = a_{j_2} = ... = a_{j_l} = 1$

(here EC stands for Even Cycle). If such a weight system exists it is unique up to rescaling if and only if all even cycles satisfy (EC1).

Proof: We work with a normalized weight system $(w_1, ..., w_n, 1) \in (\mathbb{Q} \cap (0, 1))^n \times \{1\}$. It is a solution of the system of linear equations $a_j w_j + w_{\kappa(j)} = 1$, $j \in N$. We discuss in this order (1) roots of trees not on a cycle, (2) vertices on a cycle, (3) vertices on trees different from the roots.

(1) If j is the root of a tree and is not on a cycle then $\kappa(j) = j$ and $w_j = \frac{1}{a_j+1} \in \mathbb{Q} \cap (0, 1)$.

(2) The restriction of the equations $a_j w_j + w_{\kappa(j)} = 1$, $j \in N$, to the vertices $j_1, ..., j_l$ of a cycle with $\kappa(j_l) = j_1$ and $\kappa(j_i) = j_{i+1}$ for $1 \leq i \leq l - 1$ has a unique solution $(w_{j_1}, ..., w_{j_l}) \in \mathbb{Q}^l$ if and only if

$$0 \neq \det \begin{pmatrix} a_{j_1} & 1 & & \\ & a_{j_2} & & \\ & & \ddots & 1 \\ 1 & & & a_{j_l} \end{pmatrix} = a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l,$$

that is, if the cycle is odd or does not satisfy (EC2).

In that case one calculates easily that the solution is

$$w_{j_i} = \frac{\rho(a_{j_{i+1}}, a_{j_{i+2}}, ..., a_{j_l}, a_{j_1}, ..., a_{j_{i-1}})}{a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l} \tag{3.1}$$

where

$$\rho : \bigcup_{k=0}^{\infty} \mathbb{Z}^k \rightarrow \mathbb{Z}, \tag{3.2}$$

$$\rho(x_1, ..., x_k) = x_1 ... x_k - x_2 ... x_k + ... + (-1)^{k-1} x_k + (-1)^k. \tag{3.3}$$

If all $x_i \geq 1$ then $\rho(x_1, ..., x_k) \geq 0$, and then $\rho(x_1, ..., x_k) = 0$ if and only if k is odd and $x_1 = x_3 = ... = x_k = 1$. Therefore in the case

$a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l \neq 0$ all $w_{j_i} > 0$ if and only if the cycle is odd or is even and satisfies (EC1). In that case the inequalities $w_{j_i} > 0$ and $w_{j_{i+1}} > 0$ and the equation $a_{j_i} w_{j_i} + w_{j_{i+1}} = 1$ show also $0 < w_{j_i}$ and $0 < w_{j_{i+1}}$.

In the case $a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l = 0$ the cycle is even and satisfies (EC2), and the equations $a_{j_i} w_{j_i} + w_{j_{i+1}} = 1$ give only

$$w_{j_1} = w_{j_3} = ... = w_{j_{l-1}} \quad \text{and} \quad w_{j_2} = w_{j_4} = ... = w_{j_l} = 1 - w_{j_1}.$$

Any choice $w_{j_1} \in \mathbb{Q} \cap (0,1)$ works.

(3) The weights of vertices on the trees different from the roots are successively determined by

$$w_j = \frac{1 - w_{\kappa(j)}}{a_j}$$

and automatically satisfy $0 < w_j < 1$. □

The following lemma 3.5 is related to the notion of *invertible polynomial* [ET] and is known to some specialists. We keep the situation after remark 3.3. We need some notations.

The map $\kappa : N \to N$ is of *Fermat type* if $\kappa = \mathrm{id}$, that is, if its graph has no arrows. It is of *cycle type* if its graph is a cycle. It is of *chain type* if it has the vertices $j_1, ..., j_n$ and the $n-1$ arrows from j_i to j_{i+1} for $1 \leq i \leq n-1$. The type of κ is a *sum of Fermat type, cycle types and chain types* if its graph is a union of the corresponding graphs.

Lemma 3.5. *Let $n \in \mathbb{N}$, $N = \{1, ..., n\}$, $\kappa : N \to N$, $a_1, ..., a_n \in \mathbb{N}$ and $R = \{a_j e_j + e_{\kappa(j)} \,|\, j \in N\}$ be as above such that any even cycle in the graph of κ satisfies (EC1) or (EC2) (in lemma 3.4). Let $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ be a weight system with $v_i < d$ and $R \subset (\mathbb{N}_0^n)_d$ (it exists by lemma 3.4). Then the following 2 conditions are equivalent:*

(IS4) A generic linear combination of the (at most n) monomials $x_j^{a_j} x_{\kappa(j)}$, $j \in N$, is a quasihomogeneous polynomial with an isolated singularity at 0.

(FCC) The type of κ is a sum of Fermat type, cycle types and chain types.

Proof: By theorem 2.2 (b), (IS4) is equivalent to (C2) for R as above. The implication (FCC)⇒(IS4) is well known, also a direct proof of (FCC)⇒(C2) is easy.

The other implication (C2)⇒(FCC) will be proved indirectly: Suppose that (FCC) does not hold. Then there are two indices $j_1, j_2 \in N$ with $j_1 \neq j_2$ and $\kappa(j_1) = \kappa(j_2)$. The set $J := \{j_1, j_2\}$ does not satisfy (C2) for R as above. □

Examples 3.6. We return to the examples 3.2.

(i) For a fixed map $\kappa : N \to N$ and numbers $a_1, ..., a_n \in \mathbb{N}$, the conditions (EC1) and (EC2) in lemma 3.4 are not empty if the graph of κ contains an even cycle, that is type III for $n = 2$, the types IV and VI for $n = 3$ and the types III, VIII, IX, XIV, XVI, XVII, XVIII and XIX for $n = 4$.

(ii) $n = 2$: Type I is Fermat type, type II is chain type, type III is cycle type. In type III one must avoid $a_1 = 1,\ a_2 > 1$ and $a_1 > 1,\ a_2 = 1$. Apart from that (IS4) holds for arbitrary $a_1, a_2 \in \mathbb{N}$.

(iii) $n = 3$ and $n = 4$: 5 of the 7 types with $n = 3$ and 10 of the 19 types with $n = 4$ are sums of Fermat type, cycle types and chain types. There (IS4) and (IS3) (in theorem 2.2 (c)) hold for almost arbitrary $a_1, ..., a_n \in \mathbb{N}$, with the only constraints from (EC1) or (EC2) in lemma 3.4.

For the other types, the sets J which fail to satisfy (C1)' for $R = \{a_j e_j + e_{\kappa(j)} \mid j \in N\}$ are indicated under the graphs in example 3.2 (ii) and (iii). For these types one needs more monomials than those with exponents in R in order to satisfy (C1)'. This leads to further constraints on the numbers $a_1, ..., a_n$.

(iv) $n = 3$: In both cases, III and VI, the failing set is $J = \{2, 3\}$. Suppose that a weight system $(v_1, ..., v_n, d)$ as in lemma 3.4 is determined from $a_1, a_2, a_3 \in \mathbb{N}$ (uniquely except for $a_1 = a_2 = 1$ in type VI). For (IS3) to hold one needs $(\mathbb{N}_0^J)_d \neq \emptyset$. By lemma 2.5 this is equivalent to $(\mathbb{Z}^J)_d \neq \emptyset$ and to $\gcd(v_1, v_2) \mid d$. This condition is made explicit in [Ar][AGV, 13.2].

(v) $n = 4$: Suppose that a weight system (v_1, v_2, v_3, v_4, d) as in lemma 3.4 is determined from $a_1, a_2, a_3, a_4 \in \mathbb{N}$. Consider in each of the 9 cases which are not sums of Fermat type, cycle types and chain types a failing set $J = \{j_1, j_2\}$, that is, with $\kappa(j_1) = \kappa(j_2) = j_3$ and $\{1, 2, 3, 4\} = \{j_1, j_2, j_3, j_4\}$. For (IS3) to hold one needs $(\mathbb{N}_0^J)_d \neq \emptyset$ or $(\mathbb{N}_0^J)_{d-v_{j_4}} \neq \emptyset$. As in lemma 2.5, the condition $(\mathbb{N}_0^J)_d \neq \emptyset$ is equivalent to $(\mathbb{Z}^j)_d \neq \emptyset$ and to $\gcd(v_{j_1}, v_{j_2}) \mid d$. But the condition $(\mathbb{N}_0^J)_{d-v_{j_4}} \neq \emptyset$ may be stronger than $(\mathbb{Z}^J)_{d-v_{j_4}} \neq \emptyset$ and $\gcd(v_{j_1}, v_{j_2}) \mid d - v_{j_4}$.

(vi) We consider the case XII with $n = 4$ in detail. There one starts with arbitrary $a_1, a_2, a_3, a_4 \in \mathbb{N}$ and with the monomials $x_1^{a_1+1}$, $x_2^{a_2}x_1$, $x_3^{a_3}x_2$, $x_4^{a_4}x_1$. The weight system

$$
\begin{aligned}
&(v_1, v_2, v_3, v_4, d) \\
&= (a_2a_3a_4, a_1a_3a_4, ((a_1+1)(a_2-1)+1)a_4, a_1a_2a_3, (a_1+1)a_2a_3a_4)
\end{aligned}
$$

is unique up to rescaling. The only failing set is $J = \{2, 4\}$, and $\kappa(2) = \kappa(4) = 1$, so $(\mathbb{N}_0^J)_{d-v_1} \neq \emptyset$. One needs $(\mathbb{N}_0^J)_d \neq \emptyset$ or $(\mathbb{N}_0^J)_{d-v_3} \neq \emptyset$

for (IS3) to hold. Now

$$\begin{aligned} (\mathbb{N}_0^J)_d \neq \emptyset &\iff (\mathbb{Z}^J)_d \neq \emptyset \\ &\iff \gcd(v_2, v_4) \,|\, d \iff a_1 \,|\, \mathrm{lcm}(a_2, a_4). \end{aligned}$$

And

$$\begin{aligned} (\mathbb{N}_0^J)_{d-v_3} \neq \emptyset &\implies (\mathbb{Z}^J)_{d-v_3} \neq \emptyset \\ &\iff \gcd(v_2, v_4) \,|\, d - v_3 \\ &\iff a_1 a_3 \,|\, \frac{a_4}{\gcd(a_2, a_4)}(((a_1+1)(a_2a_3 - a_2 + 1) - 1). \end{aligned}$$

(vii) Ivlev's example (remark 2.6 (ii), [AGV, 12.3]) $(v_1, v_2, v_3, v_4, d) = (1, 33, 58, 24, 265)$ is of type XII with the monomials x_1^{265}, $x_2^8 x_1$, $x_3^4 x_2$, $x_4^{11} x_1$, so $(a_1, a_2, a_3, a_4) = (264, 8, 4, 11)$. Here $(\mathbb{N}_0^J)_d = \emptyset$ and $(\mathbb{N}_0^J)_{d-v_3} = \emptyset$, so (IS3) does not hold, but $(\mathbb{Z}^J)_{d-v_3} \neq \emptyset$, so $\overline{(IS3)}$ holds and $\rho_{\mathbf{v},d}(t) \in \mathbb{Z}[t]$, even $\in \mathbb{N}_0[t]$.

Two function germs $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n,0}$ are right equivalent if there is a local coordinate change $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f_1 \circ \varphi = f_2$. Often in one right equivalence class of functions with an isolated singularity at 0, there are several quasihomogeneous functions with different weight systems. For example $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3 x_1$ with weight system $(v_1, v_2, v_3, d) = (a_2, 1, a_1a_2 - a_2 + 1, a_1a_2 + 1)$ and $x_1^{a_1a_2+1} + x_2^2 + x_3^2$ with weight system $(v_1', v_2', v_3', v_3', d') = (2, a_1a_2 + 1, a_1a_2 + 1, 2a_1a_2 + 2)$ are in the same right equivalence class of $A_{a_1a_2}$-singularities [ET]. The ambiguity was analysed in [Sa1].

Theorem 3.7. [Sa1] *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a function germ with an isolated singularity at 0.*

(a) f is right equivalent to a quasihomogeneous polynomial if and only if

$$f \in J_f := \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right) \subset \mathcal{O}_{\mathbb{C}^n,0}.$$

(b) If f is quasihomogeneous with normalized weight system $(w_1, ..., w_n, 1)$ with $0 < w_1 \leq ... \leq w_n < 1$ and if $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$, then the weight system is unique and $0 < w_1 \leq ... \leq w_n < \frac{1}{2}$.

(c) If $f \in J_f$ then f is right equivalent to a quasihomogeneous polynomial $g(x_1, ..., x_k) + x_{k+1}^2 + ... + x_n^2$ with $g \in \mathbf{m}^3_{\mathbb{C}^k,0}$. Especially, its normalized weight system satisfies $0 < w_1 \leq ... \leq w_k < w_{k+1} = ... = w_n = \frac{1}{2}$.

(d) If f and $\widetilde{f} \in \mathcal{O}_{\mathbb{C}^n,0}$ are right equivalent and quasihomogeneous with normalized weight systems $(w_1, ..., w_n, 1)$ and $(\widetilde{w}_1, ..., \widetilde{w}_n, 1)$ with $w_1 \leq ... \leq w_n \leq \frac{1}{2}$ and $\widetilde{w}_1 \leq ... \leq \widetilde{w}_n \leq \frac{1}{2}$ then $w_i = \widetilde{w}_i$.

Remarks 3.8. (i) Part (b) can be proved with the arguments in the proof of lemma 3.4. The condition $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$ is equivalent to all $a_2 \geq 2$. The implication $w_j < \frac{1}{2}$ is nontrivial only in case (2) in the proof of lemma 3.4.

(ii) Part (c) follows from (a) and the splitting lemma and (b).

(iii) An argument for part (d) different from the proof in [Sa1] is as follows. If f is quasihomogeneous with some weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ then $\rho_{(\mathbf{v},d)}(t) \in \mathbb{N}_0[t]$, so

$$\rho_{(\mathbf{v},d)}(t^{1/d}) = \sum_{j=1}^{\mu} t^{\alpha_j}$$

for certain numbers $\alpha_1, ..., \alpha_\mu \in \frac{1}{d}\mathbb{N}$. These numbers and $\rho_{(\mathbf{v},d)}(t^{1/d})$ are invariants of the right equivalence class of f. This is well known and follows essentially from calculations in [Br]. The numbers $\alpha_1, ..., \alpha_\mu$ are the *exponents* of the right equivalence class of f. By part (c) there exists a weight system $(\widetilde{v}_1, ...\widetilde{v}_n, \widetilde{d})$ with $\widetilde{v}_i \leq \frac{\widetilde{d}}{2}$ and

$$\sum_{j=1}^{\mu} t^{\alpha_j} = \rho_{(\widetilde{\mathbf{v}},\widetilde{d})}(t^{1/\widetilde{d}}).$$

It is easy to see that one can recover the normalized weight system $\frac{1}{\widetilde{d}}(\widetilde{v}_1, ..., \widetilde{v}_n, \widetilde{d})$ from the exponents and this equation. Therefore this normalized weight system is unique.

4. Milnor number versus weighted degree

Let p_i, $i \in \mathbb{N}$, be the i-th prime number, so $(p_1, p_2) = (2, 3)$. Define

$$l(n) := \prod_{i=1}^{n} \frac{p_i}{p_i - 1},$$

so $(l(1), l(2), l(3), l(4), l(5)) = (2, 3, \frac{15}{4}, \frac{35}{8}, \frac{77}{16})$. The prime number theorem in the form $p_n = n \log n \cdot (1 + o(1))$ [HW, Theorem 8] and Mertens' theorem

$$\prod_{prime\ numbers\ p \leq x} \frac{p}{p-1} = e^\gamma \cdot \log x \cdot (1 + o(1))$$

with $\gamma =$ Euler's constant [HW, Theorem 429] imply

$$l(n) = e^\gamma \cdot \log n \cdot (1 + o(1)).$$

Theorem 4.1. *(a) Let* $f \in \mathbb{C}[x_1, ..., x_n]$ *be a quasihomogeneous polynomial with an isolated singularity at* 0 *and reduced weight system*

$(v_1,...,v_n,d)\in\mathbb{N}^{n+1}$ with $v_i\le\frac{d}{2}$ for all i (reduced: $\gcd(v_1,...,v_n,d)=1$*). Then*

$$d\le l(n)\cdot\mu.$$

(b) If $v_i<\frac{d}{2}$ for all i and $n\ge 2$ then

$$d\le l(n-1)\cdot\mu.$$

These estimates rely only on the conditions for J with $|J|=1$ in (C1)-(C3) for $R=(\mathbb{N}_0^n)_d$, the conditions for $|J|\ge 2$ are not needed. Theorem 4.3 formulates this more general case. Both theorems are proved after stating theorem 4.3.

Remarks 4.2. (i) These estimates are useful for a classification of such weight systems using computer, for a fixed number of variables and with Milnor numbers up to a chosen bound. See section 5.

(ii) Calculations in [Br] show that for a quasihomogeneous polynomial f as in theorem 4.1 the monodromy on the Milnor lattice is semisimple with eigenvalues $e^{-2\pi i\alpha_1}$, ..., $e^{-2\pi i\alpha_\mu}$, where α_1,..., α_μ are the exponents considered in remark 3.8 (iii). For $f\in\mathbf{m}^3_{\mathbb{C}^n,0}$ the procedure mentioned in remark 3.8 (iii), which recovers the normalized weights $(w_1,...,w_n)$ from the exponents, shows that the tuples $(w_1,...,w_n)$ and $(\alpha_1,...,\alpha_\mu)$ have the same common denominator d. Therefore in the case $f\in\mathbf{m}^3_{\mathbb{C}^n,0}$ the order of the monodromy is d. Adding squares $x^2_{n+1}+...x^2_{n+m}$ changes the eigenvalues by the factor $(-1)^m$ and replaces d by $\widetilde{d}$ with $\widetilde{d}=2d$ for odd d and $\widetilde{d}=d$ for even d. Then the order of the monodromy is $\widetilde{d}$ or $\frac{\widetilde{d}}{2}$.

Theorem 4.3. *Fix $n\in\mathbb{N}$, $N=\{1,...,n\}$, a map $\kappa:N\to N$, numbers $a_1,...,a_n\in\mathbb{N}$ and the set $R=\{a_je_j+e_{\kappa(j)}\,|\,j\in N\}$ of exponents of the monomials $x_j^{a_j}x_{\kappa(j)}$. Suppose that $a_j\ge 2$ for all $j\in N$ which lie in components C of the graph of κ with $|C|\ge 2$.*

(a) By lemma 3.4 there is a unique reduced weight system $(v_1,...,v_n,d)\in\mathbb{N}^{n+1}$ with $R\subset(\mathbb{N}_0^n)_d$. It satisfies $v_j<\frac{d}{2}$ for $a_j\ge 2$ and $v_j=\frac{d}{2}$ for $a_j=1$.

(b)

$$d\le l(n)\cdot\mu.$$

(c) If all $a_j\ge 2$ and $n\ge 2$ then

$$d\le l(n-1)\cdot\mu.$$

(d) If $n=1$ then $d=a_1+1$ and $\mu=a_1$.

Proof of theorem 4.1: Suppose $v_1\le...\le v_k<v_{k+1}=...=v_n=\frac{1}{2}$ for some k with $0\le k\le n$. By theorem 3.7 f is right equivalent to a quasihomogeneous polynomial $g(x_1,...,x_k)+x^2_{k+1}+...+x^2_n$ with

$g \in \mathbf{m}^3_{\mathbb{C}^k,0}$ with an isolated singularity at 0 and the same weight system $(v_1, ..., v_n, d)$.

Choose a map $\kappa : N \to N$ for $g + x_{k+1}^2 + ... + x_n^2$ as in section 3. By remark 3.3 there are unique numbers $a_1, ..., a_n \in \mathbb{N}$ such that $a_j e_j + e_{\kappa(j)}$ are in $\mathrm{supp}(g + x_{k+1}^2 + ... + x_n^2)$. The hypotheses in theorem 4.3 are satisfied. Theorem 4.3 (b) and (c) give theorem 4.1 (a) and (b). □

Proof of theorem 4.3: (a) The first part follows from lemma 3.4. If $a_j = 1$ then j is itself a component of the graph of κ, so $(a_j+1)v_j = d$, so $v_j = \frac{d}{2}$. If $a_j \geq 2$ then j lies in a component C of the graph of κ with $a_i \geq 2$ for all $i \in C$. Then $a_j \geq 2$ follows as in remark 3.8 (i) with the arguments in the proof of lemma 3.4.

(b) and (c) Write $\frac{v_j}{d} = w_j = \frac{s_j}{t_j}$ with $w_j \in \mathbb{Q} \cap (0, \frac{1}{2}]$ and $s_j, t_j \in \mathbb{N}$, $\gcd(s_j, t_j) = 1$. An elementary, but important observation is

$$j \neq \kappa(j) \Longrightarrow t_j = t_{\kappa(j)} \cdot \beta_j \text{ for some } \beta_j \in \mathbb{N} \text{ with } \beta_j \,|\, a_j. \tag{4.1}$$

This follows from

$$\frac{s_j}{t_j} = w_j = \frac{1 - w_{\kappa(j)}}{a_j} = \frac{t_j - s_j}{t_j \cdot a_j} \quad \text{and} \quad \gcd(t_j, t_j - s_j) = 1.$$

For any subset $C \subset N$ define

$$\begin{aligned} \mu(C) &:= \prod_{j \in C} (\frac{1}{w_j} - 1), \quad \text{especially } \mu(\emptyset) = 1,\ \mu(N) = \mu, \\ d(C) &:= \mathrm{lcm}(t_j \,|\, j \in C), \quad \text{especially } d(\emptyset) = 1,\ d(N) = d. \end{aligned}$$

Let C_{Fermat} be the union of all components C of the graph of κ with $|C| = 1$. For $j \in C_{Fermat}$ $w_j = \frac{1}{a_j+1}$, so

$$\mu(C_{Fermat}) = \prod_{j \in C_{Fermat}} a_j, \tag{4.2}$$

$$d(C_{Fermat}) = \mathrm{lcm}(a_j + 1 \,|\, j \in C_{Fermat}). \tag{4.3}$$

Now we will study $\mu(C)$ and $d(C)$ for a component C of the graph of κ with $|C| \geq 2$. By hypothesis $a_j \geq 2$ for $j \in C$.

Case 1, C is a cycle: Suppose $C = \{1, ..., m\}$ with $\kappa(j) = j - 1$ for $2 \leq j \leq m$ and $\kappa(1) = m$. (4.1) gives immediately $t_1 = t_2 = ... = t_m = d(C)$. (3.1) shows (with ρ as in (3.2))

$$d(C) = t_1 = ... = t_m = \frac{1}{\gamma} \cdot (a_1...a_m - (-1)^m) \tag{4.4}$$

$$\text{where } \gamma = \gcd(a_1...a_m - (-1)^m, \rho(a_{j-1}, ..., a_1, a_m, ..., a_{j+1})) \tag{4.5}$$

for any $j \in \{1, ..., m\}$. Define here $\widetilde{d}(C) := \gamma \cdot d(C) = a_1...a_m - (-1)^m$.

One calculates

$$\begin{aligned}\mu(C) &= \prod_{j=1}^{m}\frac{d-v_j}{v_j} = \prod_{j=1}^{m}\frac{a_1...a_m-(-1)^m-\rho(a_{j-1},...,a_1,a_m,...a_{j+1})}{\rho(a_{j-1},...,a_1,a_m,...,a_{j+1})}\\ &= \prod_{j=1}^{m}\frac{a_{j+1}\cdot\rho(a_j,...,a_1,a_m,...,a_{j+2})}{\rho(a_{j-1},...,a_1,a_m,...,a_{j+1})} = a_1\cdot...\cdot a_m. \qquad (4.6)\end{aligned}$$

Case 2, C is not a cycle: Then C is either a tree or a cycle with one or several attached trees. If C is a tree suppose $C_1=\{1\}\subset C$ is the root, and define $m:=1$. If C is a cycle with attached trees suppose $C_1=\{1,...,m\}$ is the cycle, and $\kappa(j)=j-1$ for $2\leq j\leq m$, $\kappa(1)=m$. In both cases the *set of leaves* is the subset $C_2\subset C-C_1$ of vertices with no incoming arrows. For any leaf $j\in C_2$ denote by $C(j)$ the set of vertices on the path from j to C_1, excluding the vertex in C_1, so

$$C(j)=\{j,\kappa(j),...,\kappa^{l(j)}(j)\}\subset C-C_1 \text{ with } \kappa^{l(j)+1}(j)\in C_1.$$

Then with $\gamma:=1$ if $m=1$ and γ as in (4.5) if $m\geq 2$ one has

$$d(C_1) = \frac{1}{\gamma}\cdot(a_1...a_m-(-1)^m).$$

With (4.1) and β_i as defined in (4.1) one finds

$$\begin{aligned}t_j &= d(C_1)\cdot\prod_{i\in C(j)}\beta_i \qquad \text{for } j\in C_2, \qquad (4.7)\\ d(C) &= \mathrm{lcm}(t_j\,|\,t_j\in C_2)\\ &= d(C_1)\cdot\mathrm{lcm}(\prod_{i\in C(j)}\beta_i\,|\,j\in C_2). \qquad (4.8)\end{aligned}$$

We will estimate $d(C)$ by $\widetilde{d}(C)$ with $d(C)\,|\,\widetilde{d}(C)$ and

$$\widetilde{d}(C) := (a_1...a_m-(-1)^m)\cdot\left(\prod_{j\in C-(C_1\cup C_2)}a_j\right)\cdot\mathrm{lcm}(a_j\,|\,j\in C_2). \quad (4.9)$$

In order to estimate $\mu(C)$ from above, we choose a decomposition of $C-C_1$ into a disjoint union

$$C-C_1=\dot{\bigcup_{j\in C_2}}\widetilde{C}(j)$$

with $\widetilde{C}(j)\subset C(j)$ being a suitable sub-chain of $C(j)$,

$$\widetilde{C}(j)=\{j,\kappa(j),...,\kappa^{\widetilde{l}(j)}(j)\} \quad \text{for some } \widetilde{l}(j)\leq l(j).$$

To simplify notations suppose for a moment that one such sub-chain $\widetilde{C}(j)$ takes the form $\widetilde{C}(j) = \{j, j-1, ..., k\}$ with $\kappa(i) = i-1$ for $k \leq i \leq j$. Using $w_l = \frac{1-w_{\kappa(l)}}{a_l}$ repeatedly one finds by an easy induction for $k \leq i \leq j$

$$w_i = \frac{\rho(a_{i-1}, ..., a_{k+1}, a_k) + (-1)^{i-1-k} w_{k-1}}{a_k a_{k+1} ... a_{i-1} a_i}. \tag{4.10}$$

Therefore

$$\begin{aligned} \mu(\widetilde{C}(j)) &= \prod_{i \in \widetilde{C}(j)} \frac{1-w_j}{w_j} = \prod_{i \in \widetilde{C}(j)} \frac{\rho(a_i, ..., a_{k+1}, a_k) + (-1)^{i-k} w_{k-1}}{\rho(a_{i-1}, ..., a_{k+1}, a_k) + (-1)^{i-1-k} w_{k-1}} \\ &= \frac{\rho(a_j, ..., a_{k+1}, a_k) + (-1)^{j-k} w_{k-1}}{1 - w_{k-1}}. \end{aligned} \tag{4.11}$$

Because all $a_i \geq 2$ for $i \in C$, one can estimate

$$\begin{aligned} \rho(a_j, ..., a_{k+1}, a_k) + (-1)^{j-k} w_{k-1} &> a_k ... a_{j-1} \cdot (a_j - 1), \\ \mu(\widetilde{C}(j)) > \frac{a_k ... a_{j-1} \cdot (a_j - 1)}{1 - w_{k-1}} &> a_k ... a_{j-1} \cdot (a_j - 1). \end{aligned} \tag{4.12}$$

The following additional estimate is relevant only for odd m. But it holds for all m, and it will be smoother to treat even and odd m simultaneously. For $k-1 \in C_1$

$$\begin{aligned} & \mu(C_1) \cdot \frac{1}{1 - w_{k-1}} \\ = \ & a_1 ... a_m \cdot \frac{a_1 ... a_m - (-1)^m}{a_1 ... a_m - (-1)^m - \rho(a_{k-2}, ..., a_1, a_m, ..., a_k)} \\ \geq \ & a_1 ... a_m - (-1)^m. \end{aligned} \tag{4.13}$$

Now we put together the pieces and estimate $\mu(C)$ from above. There is (at least) one leaf $j_0 \in C_2$ with $\widetilde{C}(j_0) = C(j_0)$, so $k-1 := \kappa^{\widetilde{l}(j_0)+1}(j) \in C_1$. For this leaf j_0 we use the finer estimate in (4.12)

$$\mu(\widetilde{C}(j_0)) > \frac{1}{1 - w_{k-1}} \cdot (a_{j_0} - 1) \cdot \prod_{i \in C(j_0) - \{j_0\}} a_i.$$

Together with (4.12) for all other leaves $j \in C_2$ and (4.13) we obtain

$$\begin{aligned} & \mu(C) = \mu(C_1) \cdot \prod_{j \in C_2} \mu(\widetilde{C}(j)) \\ \geq \ & (a_1 ... a_m - (-1)^m) \cdot \left(\prod_{j \in C - (C_1 \cup C_2)} a_j \right) \cdot \left(\prod_{j \in C_2} (a_j - 1) \right) \end{aligned} \tag{4.14}$$

Now case 2 is finished. We can estimate d and μ and their quotient. $C_{Leaf} \subset N - C_{Fermat}$ denotes the union of the leaves of all components C with $|C| \geq 2$. For any such C the notations of case 2 are preserved, C_1 is the root or the cycle in it, and C_2 is the set of leaves in it. If C is a cycle then $C = C_1$.

$$\begin{aligned}
d &= \operatorname{lcm}\left(d(C_{Fermat});\ d(C) \text{ for all components } C \text{ with } |C| \geq 2\right) \\
&\leq \operatorname{lcm}\Big(d(C_{Fermat});\ \widetilde{d}(C) \text{ for } C \text{ with } |C| \geq 2\Big) \\
&\leq \prod_{C \text{ with } |C|\geq 2, C \text{ not an odd cycle}} \left(\left(\prod_{j\in C_1} a_j - (-1)^{|C_1|}\right) \cdot \prod_{j\in C-(C_1\cup C_2)} a_j\right) \cdot \\
&\quad \operatorname{lcm}\Big(a_j + 1 \text{ for } j \in C_{Fermat}; \\
&\quad \prod_{j\in C} a_j + 1 \text{ for } C \text{ an odd cycle };\ a_j \text{ for } j \in C_{Leaf}\Big). \qquad (4.15)
\end{aligned}$$

$$\begin{aligned}
\mu &= \mu(C_{Fermat}) \cdot \prod_{C \text{ a cycle}} \mu(C) \cdot \prod_{C \text{ not a cycle}, |C|\geq 2} \mu(C) \\
&\geq \prod_{j\in C_{Fermat}} a_j \cdot \prod_{C \text{ a cycle}} \left(\prod_{j\in C} a_j\right) \cdot \qquad (4.16) \\
&\quad \prod_{C \text{ not a cycle}, |C|\geq 2} \left(\left(\prod_{j\in C_1} a_j - (-1)^{|C_1|}\right) \cdot \prod_{j\in C-(C_1\cup C_2)} a_j \cdot \prod_{j\in C_2} (a_j - 1)\right).
\end{aligned}$$

$$\frac{d}{\mu} \leq \frac{\operatorname{lcm}\left(\begin{array}{c} a_j + 1 \text{ for } j \in C_{Fermat}; \\ \prod_{j\in C} a_j + 1 \text{ for } C \text{ an odd cycle};\ a_j \text{ for } j \in C_{Leaf} \end{array}\right)}{\prod_{j\in C_{Fermat}} a_j \cdot \prod_{C \text{ an odd cycle}} \left(\prod_{j\in C} a_j\right) \cdot \prod_{j\in C_{Leaf}} (a_j - 1).} \qquad (4.17)$$

In lemma 4.4 (a) two numbers $l_1(n)$ and $l_2(n) \in \mathbb{Q}_{>0}$ are defined. Obviously $\frac{d}{\mu} \leq l_1(n)$, and if all $a_j \geq 2$ and $n \geq 2$ then $\frac{d}{\mu} \leq \max(l_2(n), l_1(n-1))$. The parts (b) and (c) of theorem 4.3 follow now with lemma 4.4. Part (d) is trivial. $\square$

Lemma 4.4. *For $n \in \mathbb{N}$ define*

$$\begin{aligned}
l_1(n) &= \max\left(\frac{\operatorname{lcm}(b_1, ..., b_n)}{(b_1 - 1) \cdot ... \cdot (b_n - 1)} \mid b_1, ..., b_n \in \mathbb{N} - \{1\}\right), \\
l_2(n) &= \max\left(\frac{\operatorname{lcm}(b_1, ..., b_n)}{(b_1 - 1) \cdot ... \cdot (b_n - 1)} \mid b_1, ..., b_n \in \mathbb{N} - \{1, 2\}\right).
\end{aligned}$$

Then

$$l_1(n) = l(n) := \prod_{i=1}^{n} \frac{p_i}{p_i - 1} \geq l_2(n+1),$$

here p_i is the i-th prime number.

Proof: First, $l_1(n) = l(n)$ will be proved. Choose $b_1, ..., b_n \in \mathbb{N}$ arbitrarily. Write $\mathrm{lcm}(b_1, ..., b_n) = \prod_{i \in I} p_i^{r_i}$ with $I \subset \mathbb{N}$ finite, $r_i \geq 1$ for $i \in I$. For any $i \in I$ choose $\beta(i) \in N$ with $p_i^{r_i} \mid b_{\beta(i)}$. Define

$$\widetilde{b}_j := \prod_{i \text{ with } \beta(i) = j} p_i^{r_i}.$$

For any j with $\widetilde{b}_j > 1$ let $i(j)$ be the minimal i with $\beta(i) = j$. Then

$$\begin{aligned} \mathrm{lcm}(b_1, ..., b_n) &= \mathrm{lcm}(\widetilde{b}_j \mid \widetilde{b}_j > 1) = \prod_{j \text{ with } \widetilde{b}_j > 1} \widetilde{b}_j, \\ \frac{\mathrm{lcm}(b_1, ..., b_n)}{(b_1 - 1) \cdot ... \cdot (b_n - 1)} &\leq \prod_{j \text{ with } \widetilde{b}_j > 1} \frac{\widetilde{b}_j}{\widetilde{b}_j - 1} \\ &\leq \prod_{j \text{ with } \widetilde{b}_j > 1} \frac{p_{i(j)}}{p_{i(j)} - 1} \leq \prod_{i=1}^{n} \frac{p_i}{p_i - 1}. \end{aligned}$$

This proves $l_1(n) \leq l(n)$. The choice $b_i = p_i$ proves $l_1(n) \geq l(n)$.

Analogously one shows for $n \geq 2$

$$l_2(n) = \frac{3}{3-1} \cdot \frac{4}{4-1} \cdot \prod_{i=3}^{n} \frac{p_i}{p_i - 1}.$$

$l_2(2) = 2 = l(1)$. For $n \geq 2$ the estimate $l_2(n+1) \leq l(n)$ follows from

$$\frac{4}{4-1} \cdot \frac{p_{n+1}}{p_{n+1} - 1} \leq \frac{4}{3} \cdot \frac{p_3}{p_3 - 1} = \frac{5}{3} < \frac{2}{2-1}.$$

□

5. Computer calculations

Theorem 2.2 (c) gives combinatorial characterizations (C1)-(C3) of those reduced weight systems $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ for which quasihomogeneous polynomials with an isolated singularity at 0 exist. These characterizations can be used in computer programs to find all such weight systems with Milnor number up to some chosen bound. Because of theorem 3.7 for most purposes it is sufficient to restrict to weight systems with $v_i < \frac{d}{2}$. Theorem 4.1 (b) gives then the bound $d \leq l(n-1) \cdot \mu$ for d if $n \geq 2$.

The second author carried out such computer calculations for $n = 2,3,4$. The following table lists for $n = 2,3,4$ the number of reduced weight systems $(v_1,...,v_n,d)$ (up to reordering of $v_1,...,v_n$) with $v_i < \frac{d}{2}$ which satisfy (C1)-(C3) for $R = (\mathbb{N}_0^n)_d$ and whose Milnor number is less or equal than the number μ in the left column.

μ	$n=1$	$n=2$	$n=3$	$n=4$
50	50	187	217	100
100	100	493	806	570
150	150	847	1627	1442
200	200	1242	2623	2678
300	300	2083	5027	6059
400	400	2998	7832	10459
500	500	3957	10931	15634
1000	1000	9246	30241	?
1500	1500	15058	?	?

On a homepage tables with all these weight systems and the characteristic polynomials of the monodromy will be made available. Of course for $n = 1$ one has just the A_μ-singularities $x_1^{\mu+1}$ with $(v_1,d) = (1,\mu+1)$ for $\mu \geq 1$. The A_1-singularity is taken into account in the column for $n = 1$ despite $v_1 = \frac{d}{2}$ in that case.

For example, the total number of reduced weight systems for $n = 4$ with $v_i \leq \frac{d}{2}$ and (C1)-(C3) and $\mu \leq 50$ is $50 + 187 + 217 + 100$.

The weight system $(\frac{v_1}{d}, \frac{v_2}{d}, \frac{v_3}{d}, \frac{v_4}{d})$ with $\frac{v_i}{d} < \frac{1}{2}$ and the largest d within $\mu \leq 500$ is $(\frac{1}{58}, \frac{1}{5}, \frac{1}{3}, \frac{57}{116})$ with $\mu = 473$, $d = 1740$, $l(3) \cdot \mu = 1773,75$. This indicates that the estimate in theorem 4.1 (b) cannot be improved much.

For any n the weight system with $v_i < \frac{d}{2}$ with the smallest Milnor number is $(1,...,1,3)$ with $d = 3$ and $\mu = 2^n$. This follows from [KS, Lemma 2]. This lemma says that there is an injective map

$$\nu : \{i \mid v_i > \frac{1}{3}\} \to \{i \mid v_i < \frac{1}{3}\} \quad \text{with} \quad v_{\nu(i)} = d - 2v_i.$$

Then

$$(\frac{d}{v_i} - 1)(\frac{d}{v_{\nu(i)}} - 1) > 4.$$

For $n = 2$ weight systems with $v_i < \frac{d}{2}$ exist for any $\mu \geq 4$, because of the D_μ-singularities $x_1^{\mu-1} + x_2^2 x_1$. But for $n = 3$ and $n = 4$ there are some gaps, some numbers $> 2^n$ which are not Milnor numbers of any quasihomogeneous singularities $f \in \mathbf{m}_{\mathbb{C}^n,0}^3$. We list all gaps up to

$\mu = 1000$ for $n = 3$ and up to $\mu = 500$ for $n = 4$.

$$\begin{array}{rrl} n=3: & \mu = & 9,13,37,61,73,157,193,277,313,397,421, \\ & & 457,541,613,661,673,733,757,877,997. \\ n=4: & \mu = & 17,18,19,23,27,47,59,74,83,107,167,179, \\ & & 219,227,263,314,347,359,383,467,479. \end{array}$$

Corollary 6.3 will give an explanation of the majority of these gaps in terms of Sophie Germain prime numbers and similar prime numbers.

Yonemura [Yo] had classified all reduced weight systems (v_1, v_2, v_3, v_4, d) with $\sum_i v_i = d$ and (C1)-(C3) for $R = (\mathbb{N}_0^n)_d$. Using our lists, we recovered his 95 weight systems. 48 are in our list for $n = 3$ with $\sum_{i=1}^3 v_i = \frac{d}{2}$, with Milnor numbers ranging between 125 ($(1,1,1,6)$ and 492 ($(1,6,14,42)$). 47 are in our list for $n = 4$, with Milnor numbers ranging between 81 ($(1,1,1,1,4)$) and 264 ($(1,3,7,10,21)$).

6. The case Milnor number = prime number

The computer calculations mentioned in section 5 led us to expect the following result. This section is devoted to its proof.

Theorem 6.1. *Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0 and normalized weight system $(w_1, ..., w_n, 1) \in (\mathbb{Q} \cap (0, \frac{1}{2}))^n \times \{1\}$ such that its Milnor number μ is a prime number.*

(a) There are numbers $a_1, ..., a_n \in \mathbb{N} - \{1\}$ and $c_1, ..., c_n \in \mathbb{C}^*$ such that

$$f = c_1 x_1^{a_1+1} + c_2 x_2^{a_2} x_1 + ... + c_n x_n^{a_n} x_{n-1}.$$

Therefore f is of chain type by the map $\kappa : N \to N$ with $\kappa(1) = 1$, $\kappa(j) = j - 1$ for $2 \le j \le n$. And this is the only possible map κ as in section 3. Also, by rescaling of $x_1, ..., x_n$ one can arrange $c_1 = ... = c_n = 1$. So, f is unique up to right equivalence.

(b) Write $w_i = \frac{s_i}{t_i}$ with $s_i, t_i \in \mathbb{N}$, $\gcd(s_i, t_i) = 1$. Then

$$\begin{aligned} t_i &= a_i ... a_2 \cdot (a_1 + 1), \quad d = t_n, \\ s_i &= \rho(a_{i-1}, ..., a_2, a_1 + 1) \quad (\text{with } \rho \text{ as in (3.2)}), \\ s_1 &= 1, \quad s_{i+1} = t_i - s_i = t_i - t_{i-1} + t_{i-2} - ... + (-1)^i, \\ \mu &= \rho(a_n, ..., a_2, a_1 + 1). \end{aligned}$$

(c) The characteristic polynomial of the monodromy on the Milnor lattice of f is $\prod_{m:(6.1)} \Phi_m$, here Φ_m is the cyclotomic polynomial of the

m-th primitive unit roots, and (6.1) *is the condition*

$$m \mid a_n ... a_2(a_1+1), \quad \min(i \mid m \mid a_i ... a_2(a_1+1)) \equiv n \mod 2. \quad (6.1)$$

Especially, all eigenvalues have multiplicity 1.

Examples 6.2. For $n = 2, 3$ all tuples $(a_1, ..., a_n)$ as in theorem 6.1 with $\mu \leq 23$ are listed below, for $n = 4$ all tuples with $\mu \leq 31$.

μ	$n = 2$	$n = 3$	$n = 4$
5	(3,2)	-	-
7	(5,2), (2,3)	-	-
11	(9,2), (4,3)	(3,2,2), (2,3,2)	-
13	(11,2), (5,3), (3,4), (2,5)	-	-
17	(15,2), (7,3), (3,5)	(5,2,2), (2,5,2)	-
19	(17,2), (8,3), (5,4), (2,7)	(4,3,2), (3,4,2), (3,2,3)	-
23	(21,2), (10,3)	(7,2,2), (5,3,2), (3,5,2), (2,7,2)	-
29	4 types	6 types	(3,2,3,2)
31	6 types	2 types	(5,2,2,2)

Proof of theorem 6.1: Let $\kappa : N \to N$ be a map as in section 3, so for any $j \in N$ the sets $J = \{j\}$ and $K = \{\kappa(j)\}$ satisfy (C2) for $R = \text{supp}\, f$.

The proof proceeds in 4 steps: Step 1 extends some notations and formulas from the proof of theorem 4.3. Step 2 shows that κ is of chain type. Step 3 shows all remaining statements in (a) and (b). Step 4 proves part (c).

Step 1. We consider the graph of κ. The union of components C with $|C| = 1$ is called C_{Fermat}. For a component C let $C_1 \subset C$ be the root of C if C is a tree, the cycle in C if C contains a cycle, and $C_1 = C$ if $|C| = 1$.

For a component C with $|C| \geq 2$ let $C_2 \subset C - C_1$ be the set of leaves, that is, the vertices without incoming arrows, and let $C_3 \subset C - C_2$ be the set of branch points, that is, the vertices with ≥ 2 incoming arrows. The multiplicity $r(j) \in \mathbb{N}$ of a branch point $j \in C_3$ is the number of incoming arrows minus 1. If C is not a cycle then $C_3 \neq \emptyset$, $C_3 \cap C_1 \neq \emptyset$ and $\sum_{c \in C_3} r(c) = |C_2|$.

The union of all leaves is called C_{Leaf}, the union of all branch points is called C_{Branch}.

For a component C with $|C| \geq 2$ and for $j \in C$ let

$$\widehat{C}(j) = (j, \kappa(j), ..., \kappa^{\widehat{l}(j)}(j))$$

be the longest tuple witout repetition: If C is a tree then $\kappa^{\widehat{l}(j)}(j)$ is the root and $\kappa^{\widehat{l}(j)-1}(j)$ is not the root. If C contains a cycle, $\widehat{C}(j)$ hits the cycle and runs around it almost once, so it hits the cycle in $\kappa^{\widehat{l}(j)+1}(j)$. If $k\in\widehat{C}(j)$ let $C(j,k)$ be the tuple from j to k, $C(j,k)=(j,\kappa(j),...,k)$.

The definition of $C(j)$ in the proof of theorem 4.3 is slightly changed here: For $j\in C-C_1$ (not only $j\in C_2$), let $C(j)=(j,\kappa(j),...,\kappa^{l(j)}(j))$ be the sub-tuple of $\widehat{C}(j)$ which stops just before reaching C_1, so $\kappa^{l(j)}(j)\notin C_1$, $\kappa^{l(j)+1}(j)\in C_1$. For $j\in C_1$ define $C(j):=\emptyset$.

For any $C(j,k)$ define with ρ as in (3.2)

$$\widehat{\rho}(C(j,k)):=\rho(j,\kappa(j),...,k)$$

and define $\widehat{\rho}(\emptyset):=1$.

Now formula (3.1) for the weight w_j of a vertex $j\in C_1$ on a cycle can be rephrased as

$$w_j=\frac{\widehat{\rho}(\widehat{C}(j)-\{j\})}{\prod_{k\in C_1}a_k-(-1)^{|C_1|}}. \tag{6.2}$$

And formula (4.11) generalizes to

$$\mu(C(j,k))=\prod_{i\in C(j,k)}\left(\frac{1}{w_i}-1\right)=\frac{\widehat{\rho}(C(j,k))+(-1)^{|C(j,k)|+1}w_{\kappa(k)}}{1-w_{\kappa(k)}}. \tag{6.3}$$

For $j,k\in C-C_1$ and $k\in C(j)$ the tuple $\widehat{C}(j)$ contains the tuple $\widehat{C}(\kappa(k))$, they hit the cycle or root C_1 at the same vertex $l_1\in C_1$ and end at the same vertex $l_2\in C_1$, with $\kappa(l_2)=l_1$. For such j and k one calculates with (6.2) and (6.3)

$$\begin{aligned}
\mu(C(j,k)&=\frac{\mu(\widehat{C}(j))}{\mu(\widehat{C}(\kappa(k)))}\\
&=\frac{\widehat{\rho}(\widehat{C}(j))+(-1)^{|\widehat{C}(j)|+1}w_{l_1}}{\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|\widehat{C}(\kappa(k))|+1}w_{l_1}}\\
&=\frac{\left(\prod_{l\in C_1}a_l-(-1)^{|C_1|}\right)\widehat{\rho}(\widehat{C}(j))+(-1)^{|\widehat{C}(j)|+1}\widehat{\rho}(\widehat{C}(l_1)-\{l_1\})}{\left(\prod_{l\in C_1}a_l-(-1)^{|C_1|}\right)\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|\widehat{C}(\kappa(k))|+1}\widehat{\rho}(\widehat{C}(l_1)-\{l_1\})}\\
&=\frac{\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(\widehat{C}(j))+(-1)^{|C_1|+1}\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(C(j))}{\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|C_1|+1}\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(C(\kappa(k)))}\\
&=\frac{\widehat{\rho}(\widehat{C}(j))+(-1)^{|C_1|+1}\widehat{\rho}(C(j))}{\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|C_1|+1}\widehat{\rho}(C(\kappa(k))}
\end{aligned} \tag{6.4}$$

A component C with $|C|\geq 2$ which is not a cycle is a tree or a cycle with attached trees. One can choose a map $\beta:C_2\to C_3$ from the leaves to the branch points such that $k\in C_3$ is the image of $r(k)$

leaves and $\beta(j) \in \widehat{C}(j)$ for any leaf j. Then $C - C_1$ is the disjoint union $\bigcup_{j\in C_2}(C(j) - C(\beta(j)))$, here the sets underlying the tuples are meant. Therefore

$$\begin{aligned}\mu(C) &= \prod_{j\in C_1} a_j \cdot \prod_{j\in C_2} \mu(C(j)) \cdot \prod_{j\in C_3} \mu(C(j))^{-r(j)} \\ &= \prod_{j\in C_1} a_j \cdot \prod_{j\in C_2} \Big(\widehat{\rho}(\widehat{C}(j)) + (-1)^{|C_1|+1}\widehat{\rho}(C(j))\Big) \\ &\quad \cdot \prod_{j\in C_3} \Big(\widehat{\rho}(\widehat{C}(j)) + (-1)^{|C_1|+1}\widehat{\rho}(C(j))\Big)^{-r(j)} \qquad (6.5)\end{aligned}$$

Step 2. If $C_{Leaf} = \emptyset$ then the graph of κ is a union of points and cycles, and

$$\mu = \prod_{j\in C_{Fermat}} (a_j + 1) \cdot \prod_{C\ cycle} \prod_{j\in C} a_j.$$

Then $\mu =$ prime number and all $a_j \geq 2$ imply $n = 1$.

So suppose $C_{Leaf} \neq \emptyset$. Then there is a leaf $j_0 \in C_{Leaf}$ such that compared to all leaves $j \in C_{Leaf}$ the number $\widehat{\rho}(\widehat{C}(j)) + (-1)^{|C_1|+1}\widehat{\rho}(C(j))$ is maximal for $j = j_0$. Here and later by a slight abuse of notation we denote for any $j \in C - C_{Fermat}$ the cycle or root in the component of j by C_1. Now choose a map $\beta : C_{Leaf} \to C_{Branch}$ as at the end of step 1 and with the additional property $\beta(j_0) \in C_1$, so $\widehat{C}(j_0)$ hits C_1 in $\beta(j_0)$. This is possible. Define the following natural numbers

$$\begin{aligned}A_0 &:= \widehat{\rho}(\widehat{C}(j_0)) + (-1)^{|C_1|+1}\widehat{\rho}(C(j_0)), \\ A_j &:= a_j + 1 \qquad \text{for } j \in C_{Fermat}, \\ A_j &:= a_j \qquad \text{for } j \in \bigcup_{cycles\ C} C, \\ B_{j_0} &:= \mu(C_1) = \prod_{j\in C_1} a_j \qquad (C_1 \text{ for } j_0), \\ B_j &:= \widehat{\rho}(\widehat{C}(j)) + (-1)^{|C_1|+1}\widehat{\rho}(C(j)) \qquad \text{for } j \in C_{Leaf} - \{j_0\}, \\ D_j &:= \widehat{\rho}(\widehat{C}(\beta(j))) + (-1)^{|C_1|+1}\widehat{\rho}(C(\beta(j))) \qquad \text{for } j \in C_{Leaf}.\end{aligned}$$

Then

$$\mu = A_0 \cdot \prod_{j\in C_{Fermat}\cup(\text{all cycles})} A_j \cdot \prod_{j\in C_{Leaf}} \frac{B_j}{D_j}, \qquad (6.6)$$

and

$$\begin{aligned} & \text{all } A_j \geq 2, \\ & A_0 \geq B_j \qquad \text{for } j \in C_{Leaf} - \{j_0\}, \\ \frac{B_j}{D_j} &= \mu(C(j,\beta(j)) - \{\beta(j)\}) > 1 \qquad \text{for } j \in C_{Leaf} - \{j_0\}. \end{aligned}$$

For $j = j_0$ the map β was chosen with $\beta(j_0) \in C_1$, so $C(\beta(j_0)) = \emptyset$, $\widehat{\rho}(C(\beta_0)) = 1$, and

$$\frac{B_{j_0}}{D_{j_0}} = \frac{\prod_{j\in C_1} a_j}{\widehat{\rho}(\widehat{C}(\beta(j_0))) + (-1)^{|C_1|+1}\widehat{\rho}(C(\beta(j_0)))} \begin{cases} = 1 \text{ if } |C_1| = 1 \\ > 1 \text{ if } |C_1| > 1 \end{cases}$$

And

$$\begin{aligned} A_0 &= \big(B_{j_0} - (-1)^{|C_1|}\big)\,\widehat{\rho}(C(j_0)) + (-1)^{|C(j_0)|+1}\widehat{\rho}(\widehat{C}(\beta(j_0) - \{\beta(j_0)\})), \\ 1 &\leq \widehat{\rho}(\widehat{C}(\beta(j_0)) - \{\beta(j_0)\}) < B_{j_0}, \\ 1 &\leq \widehat{\rho}(C(j_0)) \quad \text{ and} \\ 3 &\leq \widehat{\rho}(C(j_0)) \text{ if } |C(j_0)| \geq 2, \end{aligned}$$

so always

$$A_0 \geq B_{j_0}.$$

Summarizing, we obtain

$$\begin{aligned} & A_j < \mu \text{ for } j \neq 0, \quad B_j \leq A_0, \quad A_0 \leq \mu, \qquad (6.7) \\ A_0 = \mu \iff & C_{Fermat} \cup (cycles) = \emptyset, \quad C_{Leaf} = \{j_0\}, \quad |C_1| = 1 \\ \iff & \kappa \text{ is of chain type with the chain } \widehat{C}(j_0). \qquad (6.8) \end{aligned}$$

μ is a prime number by assumption. It must divide one of the factors A_j or B_j in (6.6). Because of (6.7) this forces $A_0 = \mu$. Because of (6.8) κ is of chain type with the chain $\widehat{C}(j_0)$.

Step 3. After renumbering of the vertices of its graph, $\kappa : N \to N$ is the map with $\kappa(1) = 1$, $\kappa(i) = i - 1$ for $2 \leq i \leq n$. Then f contains the monomials $x_1^{a_1+1}, x_2^{a_2}x_1, ..., x_n^{a_n}x_{n-1}$. The Milnor number is

$$\mu = A_0 = \widehat{\rho}(\widehat{C}(n)) + \widehat{\rho}(C(n)) = \rho(a_n, ..., a_2, a_1 + 1).$$

The weights w_i and the numbers $s_i, t_i \in \mathbb{N}$ with $gcd(s_i, t_i) = 1$, $w_i = \frac{s_i}{t_i}$ are determined recursively by $w_1 = \frac{1}{a_1+1}$, $s_1 = 1$, $t_1 = a_1 + 1$,

$$\begin{aligned}
\frac{s_{i+1}}{t_{i+1}} &= w_{i+1} = \frac{1-w_i}{a_{i+1}} = \frac{t_i - s_i}{t_i \cdot a_{i+1}}, \\
s_{i+1} &= \frac{t_i - s_i}{\gamma_i},\ t_{i+1} = \beta_i t_i, \\
\text{where} \qquad & a_{i+1} = \beta_i \gamma_i,\ \gamma_i = \gcd(a_{i+1}, t_i - s_i).
\end{aligned}$$

Thus

$$\mu = \prod_{i=1}^{n} \left(\frac{1}{w_i} - 1 \right) = \prod_{i=1}^{n} \frac{t_i - s_i}{s_i} = \gamma_1 \cdot ... \cdot \gamma_{n-1} \cdot (t_n - s_n).$$

μ being a prime number forces $\gamma_i = 1$, $\beta_i = a_{i+1}$, $s_{i+1} = t_i - s_i$ and

$$\begin{aligned}
t_i &= a_i t_{i-1} = a_i ... a_2 \cdot (a_1 + 1), \\
s_i &= \rho(a_{i-1}, ..., a_2, a_1 + 1) = t_{i-1} - t_{i-2} + ... + (-1)^{i-1}.
\end{aligned}$$

Finally we show that the only monomials of weighted degree d are $x_1^{a_1+1}, x_2^{a_2} x_1, ..., x_n^{a_n} x_{n-1}$. Then f is as claimed in (a). Let $\sum_{i=1}^n \delta_i e_i \in (\mathbb{N}_0^n)_d$. Let j be maximal with $\delta_j > 0$. Then

$$\delta_j \cdot \frac{s_j}{t_j} = 1 - \sum_{i<j} \delta_i \cdot \frac{s_i}{t_i}.$$

The denominator of the rational number on the right hand side is a divisor of t_{j-1}, and $t_j = a_j t_{j-1}$. Therefore $\delta_j = a_j \varepsilon$ for some $\varepsilon \in \mathbb{N}$. But

$$a_j w_j + w_{j-1} = 1, \quad \text{so } 2a_j w_j > 1, \quad \text{so } \varepsilon = 1, \quad \text{so } \sum_{i<j} \delta_i w_i = w_{j-1}.$$

Then $\delta_{j-1} = 1$, $\delta_i = 0$ for $i < j-1$, so $\sum_i \delta_i e_i = a_j e_j + e_{j-1}$.

Step 4. Following [MO], we define the divisor $\operatorname{div} p(t)$ of a unitary polynomial $p(t) = \prod_{i=1}^k (t - \lambda_i)$ with zeros $\lambda_i \in S^1$ as the element

$$\operatorname{div} p(t) := \sum_{i=1}^{k} \langle \lambda_j \rangle \in \mathbb{Q}[S^1]$$

in the group ring $\mathbb{Q}(S^1)$. Denote $\Lambda_k := \operatorname{div}(t^k - 1)$. Then $1 = \Lambda_1$ is a unit element and $\Lambda_a \cdot \Lambda_b = \gcd(a,b) \cdot \Lambda_{lcm(a,b)}$.

By [MO, Theorem 4] the divisor of the characteristic polynomial $\Delta(t)$ of the monodromy of f is

$$\operatorname{div} \Delta(t) = \prod_{i=1}^{n} \left(\frac{1}{s_i} \Lambda_{t_i} - 1 \right).$$

Using $s_{i+1} = t_i - t_{i-1} + ... + (-1)^i$ and $\Lambda_{t_i} \cdot \Lambda_{t_j} = t_i \cdot \Lambda_{t_j}$ for $i \leq j$, we calculate

$$\begin{aligned} \operatorname{div} \Delta(t) &= (\Lambda_{t_1} - 1)(\frac{1}{s_2}\Lambda_{t_2} - 1) \cdot ... \\ &= (\frac{t_1 - 1}{s_2}\Lambda_{t_2} - \Lambda_{t_1} + 1) \cdot ... = (\Lambda_2 - \Lambda_1 + 1) \cdot ... \\ &= ... = \Lambda_{t_n} - \Lambda_{t_{n-1}} + ... + (-1)^{n-1}\Lambda_{t_1} + (-1)^n. \end{aligned}$$

This shows part (c) of theorem 6.1 □

For a fixed $n \in \mathbb{N}$ a natural number $\mu > 2^n$ is called an *n-gap* if there does not exist a quasihomogeneous polynomial $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$ with an isolated singularity at 0 and Milnor number μ.

Corollary 6.3. *For $n \geq 3$ the set of n-gaps contains the set*

$$\{2p + (-1)^n \,|\, p \text{ and } 2p + (-1)^n \text{ are prime numbers}, 2p + (-1)^n > 2^n\}.$$

Proof: Consider a $p \in \mathbb{N}$ such that $\mu = 2p + (-1)^n$ is bigger than 2^n and is a prime number, but not an n-gap. Then by theorem 6.1 there exist $a_1, ..., a_n \in \mathbb{N} - \{1\}$ with

$$\begin{aligned} 2p + (-1)^n &= \rho(a_n, ..., a_2, a_1 + 1) \\ \text{thus} \quad 2p &= (a_1 + 1)(\rho(a_n, ..., a_2) + (-1)^{n-1}). \end{aligned}$$

But $a_1 + 1 \geq 3$ and $\rho(a_n, ..., a_2) + (-1)^{n-1} \geq 3$ if $n \geq 3$, thus p cannot be a prime number. □

Remarks 6.4. (i) [Ri] A natural number p such that p and $2p+1$ are prime numbers is called a *Sophie Germain prime number*. There are conjectures of Dickson (1904) (and a generalization called *hypothesis H* of Schinzel (1956)) and of Hardy and Littlewood (1923) which would imply that the set of Sophie Germain prime numbers as well as the set $\{p \,|\, p \text{ and } 2p - 1 \text{ are prime numbers}\}$ are infinite. But the infinity of both sets seems to be unknown.

(ii) It is also interesting to ask how many other n-gaps exist for $n \geq 3$. There are 20 3-gaps with $8 < \mu \leq 1000$, 19 of them are of the type $2p-1$ with p and $2p-1$ being prime numbers, 9 is the only other gap. There are 21 4-gaps with $16 < \mu \leq 500$, 14 of them are of the type $2p+1$ with p a Sophie Germain prime number, the other ones are 17, 18, 19, 27, 74, 219, 314.

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