QUANTUM STATISTICAL MECHANICS, L-SERIES AND ANABELIAN GEOMETRY

GUNTHER CORNELISSEN AND MATILDE MARCOLLI

Abstract. It is known that two number fields with the same Dedekind zeta function are not necessarily isomorphic. The zeta function of a number field can be interpreted as the partition function of an associated quantum statistical mechanical system, which is a C^*-algebra with a one parameter group of automorphisms, built from Artin reciprocity. In the first part of this paper, we prove that isomorphism of number fields is the same as isomorphism of these associated systems. Considering the systems as noncommutative analogues of topological spaces, this result can be seen as another version of Grothendieck's "anabelian" program, much like the Neukirch-Uchida theorem characterizes isomorphism of number fields by topological isomorphism of their associated absolute Galois groups.

In the second part of the paper, we use these systems to prove the following. If there is a continuous bijection $\psi : \widehat{G}_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} \widehat{G}_{\mathbb{L}}^{\mathrm{ab}}$ between the character groups (viz., Pontrjagin duals) of the *abelianized* Galois groups of the two number fields that induces an equality of *all* corresponding L-series $L_{\mathbb{K}}(\chi, s) = L_{\mathbb{L}}(\psi(\chi), s)$ (not just the zeta function), then the number fields are isomorphic.

This is also equivalent to the purely algebraic statement that there exists a topological group isomorphism ψ as a above and a norm-preserving group isomorphism between the ideals of $\mathbb{K}$ and $\mathbb{L}$ that is compatible with the Artin maps via ψ.

Contents

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Introduction

Can one describe isomorphism of two number fields $\mathbb{K}$ and $\mathbb{L}$ from associated analytic or topological objects? Here are some attempts ("no"-answers indexed by **N**; "yes"-answers by **Y**):

(N1) An **equality of their Dedekind zeta functions** (so-called *arithmetic equivalence*) does not imply that $\mathbb{K}$ and $\mathbb{L}$ are isomorphic, as was shown by Gaßmann (**[21]**, cf. also Perlis **[39]**, or **[26]**). An example is provided by

$$\mathbb{K} = \mathbb{Q}(\sqrt[8]{3}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt[8]{3 \cdot 2^4})$$

(**[39]**, **[27]**). However, the implication is true *if* $\mathbb{K}$ and $\mathbb{L}$ are Galois over $\mathbb{Q}$ (Theorem of Bauer **[3] [4]**, nowadays a corollary of Chebotarev's density theorem, see, e.g., Neukirch **[36]** 13.9).

(N2) An **isomorphism of their adele rings** $\mathbf{A}_{\mathbb{K}}$ and $\mathbf{A}_{\mathbb{L}}$ as topological rings (which implies in particular an equality of the zeta functions of $\mathbb{K}$ and $\mathbb{L}$) does not imply that $\mathbb{K}$ and $\mathbb{L}$ are isomorphic, cf. Komatsu (**[28]**). An example is

$$\mathbb{K} = \mathbb{Q}(\sqrt[8]{2 \cdot 17}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt[8]{2^5 \cdot 17}, \sqrt[8]{2 \cdot 3 \cdot 11}).$$

On the other hand, a global function field *is* determined up to isomorphism by its adele ring (Turner **[45]**).

(N3) An **isomorphism of the Galois groups of the maximal abelian extensions** $G_{\mathbb{K}}^{\mathrm{ab}}$ and $G_{\mathbb{L}}^{\mathrm{ab}}$ as topological groups does not imply an isomorphism of the fields $\mathbb{K}$ and $\mathbb{L}$. For example,

$$\mathbb{K} = \mathbb{Q}(\sqrt{-2}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt{-3})$$

have isomorphic abelianized absolute Galois groups (see Onabe **[38]**).

However ...

(Y1) An **isomorphism of their absolute Galois groups** $G_{\mathbb{K}}$ and $G_{\mathbb{L}}$ as topological groups implies isomorphism of the fields $\mathbb{K}$ and $\mathbb{L}$: this is the celebrated theorem of Neukirch and Uchida (In **[35]**, Neukirch proved this for fields that are Galois over $\mathbb{Q}$; in **[46]**, Uchida proved the general case, cf. also **[37]** 12.2, and work of Ikeda and Iwasawa). It can be

considered the first manifestation (zero-dimensional case) of the so-called "anabelian" philosophy of Grothendieck (**[23]**, esp. footnote (3)): the neologism "anabelian" was coined by Grothendieck by contrast with statement **(N3)** above.

(Y2) In an unpublished work, Richard Groenewegen **[22]** proves a **Torelli theorem** for number fields: if two number fields have "strongly monomially equivalent" h^0-function in Arakelov theory (in the sense of van der Geer and Schoof, cf. **[49]**), then they are isomorphic.

The starting point for this work is the observation that the zeta function of a number field $\mathbb{K}$ can be realized as the partition function of a quantum statistical mechanical (QSM) system in the style of Bost and Connes (cf. **[6]** for $\mathbb{K} = \mathbb{Q}$). The QSM-systems for general number fields that we consider are those that were constructed by Ha and Paugam (see section 8 of **[24]**, which is a specialization of their more general class of QSM-systems associated to Shimura varieties), and further studied by Laca, Larsen and Neshveyev in **[31]**. This quantum statistical mechanical system consists of a C^*-algebra $A_{\mathbb{K}}$ (the noncommutative analogue of a topological space) with a time evolution $\sigma_{\mathbb{K}}$ (i.e., a continuous group homomorphism $\mathbb{R} \to \operatorname{Aut} A_{\mathbb{K}}$) — for the exact definition, see Section 2 below, but the structure of the algebra is

$$A_{\mathbb{K}} := C(G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathscr{O}}_{\mathbb{K}}^*} \hat{\mathscr{O}}_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^+,$$

where $\hat{\mathscr{O}}_{\mathbb{K}}$ is the ring of finite integral adeles and $J_{\mathbb{K}}^+$ is the semigroup of ideals, which acts on $G_{\mathbb{K}}^{\mathrm{ab}}$ by Artin reciprocity. The time evolution is only non-trivial on elements $\mathfrak{n}$ of $J_{\mathbb{K}}^+$, where it acts by multiplication with the norm $N(\mathfrak{n})^{it}$. For now, it is important to notice that the structure involves the abelianized Galois group and the adele ring, but not the absolute Galois group. In this sense, it is "not anabelian"; but of course, it is "noncommutative" (in noncommutative topology, the crossed product construction is an analog of taking quotients). In light of the previous discussion, it is now natural to ask whether the QSM-system (which contains simultaneously the zeta function from **(N1)**, the adeles from **(N2)** and the abelianized Galois group from **(N3)**) does characterize the number field.

We call two general QSM-systems *isomorphic* if there is a C^*-algebra isomorphism between the algebras that intertwines the time evolutions. Our main result is that the QSM-system seems to cancel out the defects of **(N1)—(N3)** in exactly the right way:

Theorem 1. *Let $\mathbb{K}$ and $\mathbb{L}$ denote arbitrary number fields. Then the following conditions are equivalent:*

(i) *$\mathbb{K}$ and $\mathbb{L}$ are isomorphic as fields;*
(ii) *the QSM systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ are isomorphic.*

One may now ask whether the "topological" isomorphism from (ii) can somehow be captured by an analytic invariant, such as the Dedekind zeta function, which in itself doesn't suffice. Our second main theorem says that this is indeed the case:

Theorem 2. *Let $\mathbb{K}$ and $\mathbb{L}$ denote arbitrary number fields. Then the following conditions are equivalent:*

(ii) *the QSM systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ are isomorphic;*

(iii) *there is a homeomorphism of (the Pontrjagin duals of) the abelianized Galois groups*

$$\psi \,:\, \widehat{G}_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} \widehat{G}_{\mathbb{L}}^{\mathrm{ab}}$$

such that for every character $\chi \in \widehat{G}_{\mathbb{K}}^{\mathrm{ab}}$, we have an identification of L-series for these generalized Dirichlet characters

$$L_{\mathbb{K}}(s,\chi) = L_{\mathbb{L}}(s,\psi(\chi)).$$

Condition (iii) can be considered as the correct generalization of arithmetic equivalence (which is (iii) for the trivial character only) to an analytic equivalence that *does* capture isomorphism. It should also be observed at this point that (Hecke) L-series occur naturally in the description of generalized equilibrium states (KMS-states) of the QSM-system, and this is how we originally discovered the statement of the theorem.

Finally, there is the following purely algebraic reformulation:

Theorem 3. *Let $\mathbb{K}$ and $\mathbb{L}$ denote arbitrary number fields. Then the following conditions are equivalent:*

(i) *the fields $\mathbb{K}$ and $\mathbb{L}$ are isomorphic;*

(iv) *there is a topological group isomorphism*

$$\hat{\psi} \,:\, G_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} G_{\mathbb{L}}^{\mathrm{ab}}$$

and an isomorphism

$$\Psi \,:\, J_{\mathbb{K}}^{+} \xrightarrow{\sim} J_{\mathbb{L}}^{+}$$

of semigroups of ideals such that the following two compatibility conditions are satisfied:

(a) *compatibility of Ψ with norms: $N_{\mathbb{L}}(\Psi(\mathfrak{n})) = N_{\mathbb{K}}(\mathfrak{n})$ and*

(b) *compatibility with the Artin map: $\vartheta_{\mathbb{L}}(\Psi(\mathfrak{n})) = \hat{\psi}(\vartheta_{\mathbb{K}}(\mathfrak{n}))$*

for all ideals $\mathfrak{n} \in J_{\mathbb{K}}^{+}$.

We first say a few words about the proofs. Of course, (i) implies the other conditions. To prove that (ii) implies (i), we first prove that the fields are arithmetically equivalent (by interpreting the zeta functions as partition functions and studying the relation between the Hamiltonians for the two systems), and then we use some results on isomorphism of crossed product algebras to deduce an identification of the semigroups of integral ideals of $\mathbb{K}$ and $\mathbb{L}$. By studying the endomorphism structure of the QSM-systems, we deduce a homomorphism of $G_{\mathbb{K}}^{\mathrm{ab}}$ with $G_{\mathbb{L}}^{\mathrm{ab}}$, then of unit ideles, and finally, multiplicative groups of the rings of integers. We then deduce an isomorphism of all residue fields (induced by the same map) from a computation in Galois cohomology of the maximal abelian extension.

That (ii) implies (iii) follows from the interpretation of L-series as KMS-states. Conversely, we show that the matching of L-series implies automatically that ψ is a continuous *group* isomorphism. We then get a matching of semigroups of ideals, compatible with the Artin map, by doing some Fourier analysis with the L-series of the number fields. We then extend these maps to the whole algebra. At this point, it is maybe interesting to mention that such an isomorphism is by no means uniquely determined by the matching of L-series (indeed, for example, an automorphism of the system might be applied). In this context, one may try to rewrite the main theorems in a functorial way, as a bijection of certain Hom-sets. It would be interesting to understand the relation to the

functor from number fields to QSM-systems in **[32]**. Finally, it is very easy to see that (iv) implies (iii).

Remark. We make a few remarks about condition (iii) in the theorem. First of all, the equivalence of (i) and (iii) (or (iv)) is a purely number theoretical statement, without reference to QSM-systems. We do not know a direct proof that (iii) implies (i) without passing via (ii) and using basic theory of QSM-systems; so we offer this as a number theoretical challenge (of course, one can clear the current proof of QSM-lingo).

Secondly, one may wonder whether condition (iii) can be replaced by something weaker. As we already observed, requiring (iii) for the trivial character only is not enough, but what about, for example, this condition:

> (iii)$_2$ *All rational quadratic L-series of $\mathbb{K}$ and $\mathbb{L}$ are equal, i.e. for all integers d that are not squares in $\mathbb{K}$ and $\mathbb{L}$, we have $L_{\mathbb{K}}(\chi_d, s) = L_{\mathbb{L}}(\chi_d, s)$.*

By considering only rational characters, one does not need to introduce a bijection of abelianized Galois groups, since there is an automatic matching of conductors. One can also consider a similar statement (iii)$_n$ for all n-th order rational L-series.

We can show that (iii)$_2$ is *not* equivalent to (ii). We prove that as soon as $\mathbb{K}$ and $\mathbb{L}$ have the same zeta functions, condition (iii)$_2$ holds (the proof uses *Gaßmann-equivalence*, and was discovered independently by Lotte van der Zalm in her undergraduate thesis **[50]**.) Another number theoretical challenge is to give a purely analytical proof of this statement (i.e., not using group theory).

Bart de Smit has asked what happens if (iii) is replaced by a matching of zeta functions of ray class fields (under some matching of the conductors).

Finally, we note that condition (iii) is motivic: it gives an identification of L-series of rank one motives over both number fields (in the sense of **[20]**, §8).

Remark (Anabelian vs. noncommutative)**.** The anabelian philosophy is, in the words of Grothendieck (*Esquisse d'un programme*, **[23]**, footnote (3)) "a construction which pretends to ignore [...] the algebraic equations which traditionally serve to describe schemes, [...] to be able to hope to reconstitute a scheme [...] from [...] a purely topological invariant [...]". In the zero-dimensional case, the fundamental group plays no rôle, only the absolute Galois group, and we arrive at the theorem of Neukirch and Uchida (greatly generalized in recent years, notably by Bogomolov-Tschinkel **[5]**, Mochizuki **[34]** and Pop **[40]**, compare **[44]**).

Our main result indicates that QSM-systems for number fields can be considered as some kind of substitute for the absolute Galois group. The link to Grothendieck's proposal arises via a philosophy from noncommutative geometry that "topology = C^*-algebra" and "time evolution = Frobenius". This would become a genuine analogy if one could unearthen a "Galois theory" that describes a categorical equivalence between number fields on the one hand, and their QSM-systems on the other hand. Anyhow, it seems Theorem 1 indicates that one may, in some sense, substitute "noncommutative" for "anabelian"[1] ...

It would be interesting to study the analogue of our results for the case of function fields, and higher dimensional schemes. Jacob **[25]** and Consani-Marcolli **[15]** have constructed function field

[1] Interestingly, the Wikipedia entry for "Anabelian geometry" starts with "Not to be confused with Noncommutative geometry" (retrieved 16 Aug 2010).

analogues of QSM systems that respectively have the Weil and the Goss zeta function as partition function. The paper **[17]** studies arithmetic equivalence of function fields using the Goss zeta function.

Remark (Link with hyperring theory)**.** Connes and Consani have studied the adele class space as a hyperring in the sense of Krasner (**[29]**). They prove in **[9]** (Theorem 3.13) that

> (v) *the two adele class spaces* $\mathbf{A}_{\mathbb{K}}/\mathbb{K}^* \cong \mathbf{A}_{\mathbb{L}}/\mathbb{L}^*$ *are isomorphic as hyperrings over the Krasner hyperfield;*

is equivalent to (i) in our main theorem. The proof is very interesting: it uses classification results from incidence geometry. One may try to prove that (ii) implies (v) directly (thus providing a new proof of (ii) $\Rightarrow$ (i); this is especially tempting, since Krasner developed his theory of hyperrings for applications to class field theory, much of the kind which one sees in our proof of the implication from (ii) to (i)).

Observe that the equivalence of (v) with (i) is rather far from the anabelian philosophy (which would be to describe algebra by topology), since it uses (algebraic) isomorphism of hyperrings to deduce isomorphism of fields. But it might be true that the *topology/geometry* of the hyperring can be used instead. As a hint, we refer to Theorem 7.12 in **[9]**: over a global function field, the groupoid of prime elements of the hyperring of adele classes *is* the abelianized loop groupoid of the curve, cf. also **[8]**, Section 9.

Remark (Analogues in Riemannian geometry)**.** There is a well-known (limited) analogy between the theory of L-series in number theory and the theory of spectral zeta functions in Riemannian geometry. For example, the ideas of Gaßmann were used by Sunada to construct isospectral, non-isometric manifolds (cf. **[43]**): the spectral zeta function does not determine a Riemannian manifold up to isometry (actually, not even up to homeomorphism).

In **[16]**, it was proven that the isometry type of a closed Riemannian manifold is determined by a *family* of Dirichlet series associated to the Laplace-Beltrami operator on the manifold. In **[18]**, it was proven that one can reconstruct a compact hyperbolic Riemann surface from a suitable *family* of Dirichlet series associated to a spectral triple. These can be considered as analogues in manifold theory of the equivalence of (i) and (iii).

One might consider as another analogy of (iii) the matching of all L-series of Riemannian coverings of two Riemannian manifolds, but this appears not to be entirely satisfactory; for example, there exist simply connected isospectral, non-isometric Riemannian manifolds (cf. Schüth **[42]**).

At the other side of the spectrum, one may consider Mostow rigidity (a hyperbolic manifold of dimension at least three is determined by its fundamental group) as an analogue of the anabelian theorem. Again, this is very *an*abelian, since the homology rarely determines a manifold.

There is a further occurence of L-series in geometry (as was remarked to us by Atiyah): the Riemann zeta function is the only Dedekind zeta function that occurs as spectral zeta function of a manifold (namely, the circle); but more general L-series can be found in the geometry of the resolution of the cusps of a Hilbert modular variety (**[2]**, compare **[33]**), a kind of "virtual manifold" that also has a "quotient structure", just like the QSM-system algebra is a noncommutative quotient space.

Disambiguation of notations

There will be one notational sloppiness throughout: we will denote maps that are induced by a given isomorphism φ by the same letter φ.

Since the number theory and QSM literature have conflicting standard notations, we include a table of notations for the convenience of the reader:

R^* ..... invertible elements of a ring R
$R^\times$ ..... non-zero elements of a ring R
K^{ac} ..... algebraic closure of a field K

$\widehat{G}$ ..... Pontrjagin dual: continuous $\mathrm{Hom}(G, S^1)$ of a compact abelian group G
G^0 ..... connected component of identity

$\mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ (blackboard bold capitals) ..... number fields
$L_{\mathbb{K}}(-, \chi) = L_{\mathbb{K}}(\chi, -)$ ..... L-series of field $\mathbb{K}$ for generalized Dirichlet character $\chi \in \widehat{G}_{\mathbb{K}}^{\mathrm{ab}}$

$\mathscr{O}_{\mathbb{K}}$ ..... ring of integers of a number field $\mathbb{K}$
$\hat{\mathscr{O}}_{\mathbb{K}}$ ..... ring of finite integral adeles of a number field $\mathbb{K}$

$J_{\mathbb{K}}^+$ ..... semigroup of integral ideals of a number field $\mathbb{K}$
$N = N_{\mathbb{K}} = N_{\mathbb{Q}}^{\mathbb{K}}$ ..... the norm map on ideals of the number field $\mathbb{K}$
$\mathfrak{n}, \mathfrak{p}, \mathfrak{q}$ (fraktur letters) ..... integral ideals of a number field

$f(\mathfrak{p}\,|p) = f(\mathfrak{p}\,|\mathbb{K})$ ..... inertia degree of $\mathfrak{p}$ over p, in $\mathbb{K}$
$\mathfrak{f}_\chi$ ..... conductor of χ

$\mathbb{K}_{\mathfrak{p}}$ ..... maximal extension of $\mathbb{K}$ in which $\mathfrak{p}$ is unramified
$\mathbb{K}_{\mathfrak{p}}^{\mathrm{ab}}$ ..... maximal abelian extension of $\mathbb{K}$ in which $\mathfrak{p}$ is unramified

$\hat{\mathbb{K}}_{\mathfrak{p}}$ ..... completion of a number field $\mathbb{K}$ at a prime ideal $\mathfrak{p}$
$\hat{\mathscr{O}}_{\mathbb{K},\mathfrak{p}}$ ..... integers in $\hat{\mathbb{K}}_{\mathfrak{p}}$
$\overline{\mathbb{K}}_{\mathfrak{p}}$ ..... residue field of a number field $\mathbb{K}$ at a prime ideal $\mathfrak{p}$
$W(-), F, V, \wp$ ..... Witt functor, Frobenius, Verschiebung, $\wp = F - 1$

$G_{\mathbb{K}}$ ..... absolute Galois group of $\mathbb{K}$
$G_{\mathbb{K}}^{\mathrm{ab}}$ ..... Galois group of maximal abelian extension of $\mathbb{K}$

$\vartheta_{\mathbb{K}}$ ..... Artin reciprocity map $J_{\mathbb{K}}^+ \to G_{\mathbb{K}}^{\mathrm{ab}}$ (or $\mathbf{A}_{\mathbb{K}}^* \to G_{\mathbb{K}}^{\mathrm{ab}}$)
$\mathfrak{n} * \gamma$ ..... Action of ideal $\mathfrak{n}$ on $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$ by the Artin map: $\mathfrak{n} * \gamma = \vartheta_{\mathbb{K}}(\mathfrak{n}) \cdot \gamma$

$\mathbf{A}_{\mathbb{K}}$ ..... adele ring of a number field $\mathbb{K}$
$\mathbf{A}_{\mathbb{K},f}$ ..... finite (non-archimedean) part of the adele ring of a number field $\mathbb{K}$
$A_{\mathbb{K}}$ ..... the C^* algebra of the QSM-system of the number field $\mathbb{K}$

β ..... positive real number representing "inverse temperature"
$X_{\mathbb{K}}$ ..... topological space underlying part of the algebra $A_{\mathbb{K}}$
$\sigma_{\mathbb{K}} = \sigma_t = \sigma_{\mathbb{K},t}$ ..... the time evolution (in time t) of the QSM-system of the number field $\mathbb{K}$
$\rtimes$ ..... crossed product construction of C^*-algebras (not semidirect product of groups)

ω ..... a state of a C^*-algebra
ω_β ..... a KMS_β state of a C^*-algebra
π_ω ..... GNS-representation corresponding to ω
$\mathscr{M}_\omega$ ..... weak closure of algebra in GNS-representation

H ..... Hamiltonian
$\mathscr{H}$ ..... Hilbert space
$H(-,-)$ ..... (group) cohomology

$\mathrm{KMS}_\beta(A,\sigma)$ ..... the set of KMS_β-states of the QSM-system (A,σ)
$\mathrm{KMS}_\beta(\mathbb{K})$ ..... $\mathrm{KMS}_\beta(A_\mathbb{K},\sigma_\mathbb{K})$

$\mu_\mathfrak{n}$ ..... element of the C^*-algebra $A_\mathbb{K}$ corresponding to the ideal $\mathfrak{n} \in J_\mathbb{K}^+$
$\boldsymbol{\mu}_n$ ..... group scheme of n-th roots of unity (n integer)

Part A. QSM-ISOMORPHISM OF NUMBER FIELDS

1. Isomorphism of QSM systems

We recall some definitions and refer to **[7]**, **[11]**, and Chapter 3 of **[12]** for more information and for some physics background. After that, we introduce isomorphism of QSM-systems, and prove they preserve KMS-states (cf. infra).

1.1. Definition. A *quantum statistical mechanical system* (QSM-system) (A,σ) is a (unital) C^*-algebra A together with a so-called *time evolution* σ, which is a continuous group homomorphism

$$\sigma : \mathbb{R} \to \mathrm{Aut}\, A : t \mapsto \sigma_t.$$

A *state* on A is a continuous positive unital linear functional $\omega : A \to \mathbb{C}$. We say ω is a ***KMS**$_\beta$ state* for some $\beta \in \mathbb{R}_{>0}$ if for all $a, b \in A$, there exists a function $F_{a,b}$, holomorphic in the strip $0 < \mathsf{Im}\, z < \beta$ and bounded continuous on its boundary, such that

$$F_{a,b}(t) = \omega(a\sigma_t(b)) \text{ and } F_{a,b}(t+i\beta) = \omega(\sigma_t(b)a) \quad (\forall t \in \mathbb{R}).$$

Equivalently, ω is a σ-invariant state with $\omega(ab) = \omega(b\sigma_{i\beta}(a))$ for a, b in a dense set of σ-analytic elements. The set $\mathrm{KMS}_\beta(A,\sigma)$ of KMS_β states is topologized as a subspace of the convex set of states, a weak* closed subset of the unit ball in the operator norm of bounded linear functionals on the algebra. A KMS_β state is called *extremal* if it is an extremal point in the (compact convex) set of KMS_β states for the weak (i.e., pointwise convergence) topology.

1.2. Remark. This notion of QSM-system is one of the possible physical theories of quantum statistical mechanics; one should think of A as the algebra of observables, represented on some Hilbert space $\mathscr{H}$ with orthonormal basis $\{\Psi_i\}$; the time evolution, in the given representation, is generated by a Hamiltonian H by

$$\sigma_t(a) = e^{itH} a e^{-itH}, \tag{1}$$

and (mixed) states of the system are combinations

$$a \mapsto \sum \lambda_i \langle \Psi_i | a \Psi_i \rangle$$

which will mostly be of the form

$$a \mapsto \mathrm{trace}(\rho a)$$

for some density matrix ρ. A typical equilibrium state (here, this means stable by time evolution) is a Gibbs state

$$a \mapsto \mathrm{trace}(ae^{-\beta H})/\,\mathrm{trace}(e^{-\beta H})$$

at temperature $1/\beta$, where we have normalized by the partition function

$$\mathrm{trace}(e^{-\beta H}).$$

The KMS-condition was introduced by Kubo, Martin and Schwinger in the 1950s as a correct generalization of the notion of equilibrium state to the general case, where the trace class condition

$$\mathrm{trace}(e^{-\beta H}) < \infty$$

needed to define Gibbs states no longer necessarily holds.

1.3. For convenience, we recall the construction of the *(reduced) crossed product algebra* $A := C(X) \rtimes G$, where X is a topological space and G is a semigroup that acts "reasonably" on X. Let $\mathscr{H}$ denote a Hilbert space on which $C(X)$ is represented; then A is the algebra generated by the images of the representation π_1 of $C(X)$ and π_2 of G on $\mathscr{H}_G := L^2(G, \mathscr{H})$ (square summable functions on G with values in $\mathscr{H}$) given by

$$\begin{aligned} \pi_1(f)(\xi)(g) &:= g^{-1}(f)(\xi(g)) \\ \pi_2(g)(\xi)(h) &:= \xi(g^{-1}h) \end{aligned}$$

We now introduce the following equivalence relation for QSM-systems:

1.4. Definition. An *isomorphism* of two QSM-systems (A, σ) and (B, τ) is a C^*-algebra isomorphism $\varphi : A \xrightarrow{\sim} B$ that intertwines time evolutions, i.e., such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow[\sim]{\varphi} & B \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ A & \xrightarrow[\sim]{\varphi} & B \end{array}$$

1.5. Lemma. *Let $\varphi : (A, \sigma) \xrightarrow{\sim} (B, \tau)$ denote an isomorphism of QSM systems. Then for any $\beta > 0$,*

(i) *pullback*

$$\varphi^* : \mathrm{KMS}_\beta(B, \tau) \xrightarrow{\sim} \mathrm{KMS}_\beta(A, \sigma) : \omega \mapsto \omega \circ \varphi$$

is a homeomorphism between the spaces of KMS_β states on B and A;

(ii) *φ^* induces a bijection between extremal KMS_β states on B and A.*

Proof. The map φ obviously induces a bijection between states on B and states on A.

For (i), let $F_{a,b}$ be the holomorphic function that implements the KMS_β-condition for the state ω on (B, τ) at $a, b \in B$, so

$$F_{a,b}(t) = \omega(a\tau_t(b)) \text{ and } F_{a,b}(t + i\beta) = \omega(\tau_t(b)a).$$

The following direct computation then shows that the function $F_{\varphi(c),\varphi(d)}$ implements the KMS_β-condition for the state $\varphi^*\omega$ on (A,σ) at $c,d\in A$:

$$(\omega\circ\varphi)(c\sigma_t(d))=\omega(\varphi(c)\tau_t(\varphi(d))=F_{\varphi(c),\varphi(d)}(t),$$

and similarly at $t+i\beta$. Also, note that pullback is continuous, since C^*-algebra isomorphism is compatible with the topology on the set of KMS-states.

For (ii), if a KMS_β state ω on B is not extremal, then the GNS-representation π_ω of ω is not factorial. As in Prop 3.8 of **[11]**, there exists a positive linear functional, which is dominated by ω, namely $\omega_1\leq\omega$, and which extends from B to the von Neumann algebra given by the weak closure $\mathscr{M}_\omega$ of B in the GNS representation. The functional ω_1 is of the form $\omega_1(b)=\omega(hb)$ for some positive element h in the center of the von Neumann algebra $\mathscr{M}_\omega$. Consider then the pull back

$$\varphi^*(\omega)(a)=\omega(\varphi(a))$$

and

$$\varphi^*(\omega_1)(a)=\omega_1(\varphi(a))=\omega(h\varphi(a))$$

for $a\in A$. The continuous linear functional $\varphi^*(\omega_1)$ has norm $\|\varphi^*(\omega_1)\|\leq 1$. In fact, since we are dealing with unital algebras,

$$\|\varphi^*(\omega_1)\|=\varphi^*(\omega_1)(1)=\omega(h).$$

The linear functional $\omega_2(b)=\omega((1-h)b)$ also satisfies the positivity property $\omega_2(b^*b)\geq 0$, since $\omega_1\leq\omega$. The decomposition

$$\varphi^*(\omega)=\lambda\eta_1+(1-\lambda)\eta_2,$$

with $\lambda=\omega(h)$,

$$\eta_1=\varphi^*(\omega_1)/\omega(h) \text{ and } \eta_2=\varphi^*(\omega_2)/\omega(1-h)$$

shows that the state $\varphi^*(\omega)$ is not extremal. Notice that η_1 and η_2 are both KMS states. To see this, it suffices to check that the state $\omega_1(b)/\omega(h)$ is KMS. In fact, one has for all analytic elements $a,b\in B$:

$$\omega_1(ab)=\omega(hab)=\omega(ahb)=\omega(hb\tau_{i\beta}(a)).$$

□

1.6. Definition. An *automorphism* of a QSM-system (A,σ) is an isomorphism to itself. The group of such automorphisms is denoted by $\mathrm{Aut}((A,\sigma))$.

An *endomorphism* of a QSM-system (A,σ) is a $*$-homomorphism $A\to A$ that commutes with σ_t for all t. We denote them by $\mathrm{End}((A,\sigma))$.

An *inner endomorphism* is defined by $a\mapsto uxu^*$ for some isometry $u\in A$ which is an eigenvector of the time evolution, i.e., $u^*u=1$ and there exists an eigenvalue λ such that $\sigma_t(u)=\lambda^{it}u$ for all t. We denote them by $\mathrm{Inn}((A,\sigma))$. (Inner endomorphisms act trivially on KMS-states, cf. **[12]**, Ch. 3, Section 2.3.)

2. A QSM-system for number fields

Bost and Connes (**[6]**) introduced a QSM-system for the field of rational numbers, and **[13]**, **[14]** did so for imaginary quadratic fields. More general QSM-systems associated to arbitrary number fields were constructed by Ha and Paugam in **[24]** as a special case of their more general class of systems for Shimura varieties, which in turn generalize the $\mathrm{GL}(2)$-system of **[11]**. We recall here briefly the construction of the systems for number fields in the equivalent formulation given in **[31]**.

2.1. We denote by $J_{\mathbb{K}}^+$ the semigroup of integral ideals, with the norm function

$$N \,:\, J_{\mathbb{K}}^+ \to \mathbb{Z} \,:\, \mathfrak{n} \mapsto N(\mathfrak{n}) = N_{\mathbb{Q}}^{\mathbb{K}}(\mathfrak{n}) = N_{\mathbb{K}}(\mathfrak{n}).$$

Denote by $G_{\mathbb{K}}^{\mathrm{ab}}$ the Galois group of the maximal abelian extension of $\mathbb{K}$. The semigroup of ideals maps to the ideles $\mathbf{A}_{\mathbb{K}}^*$, and hence to the idele class group modulo its connected component of the identity. The Artin reciprocity map is an isomorphism of this to $G_{\mathbb{K}}^{\mathrm{ab}}$. By abuse of terminology, we refer to

$$\vartheta_{\mathbb{K}} \,:\, J_{\mathbb{K}}^+ \to G_{\mathbb{K}}^{\mathrm{ab}} \,:\, \mathfrak{n} \to \vartheta_{\mathbb{K}}(\mathfrak{n})$$

as the Artin map. We also have an action on $G_{\mathbb{K}}^{\mathrm{ab}}$ of the group $\mathbf{A}_{\mathbb{K},f}^*$ of finite ideles of $\mathbb{K}$, hence one can consider the fibered product

$$X_{\mathbb{K}} := G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}},$$

where $\hat{\mathcal{O}}_{\mathbb{K}}$ is the ring of finite integral adeles, defined for $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$ and $i \in \hat{\mathcal{O}}_{\mathbb{K}}$ by

$$(\gamma, i) \equiv (\vartheta_{\mathbb{K}}(u^{-1}) \cdot \gamma, ui) \text{ for all } u \in \hat{\mathcal{O}}_{\mathbb{K}}^*.$$

2.2. Definition. The *QSM-system* $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ *associated to a number field* $\mathbb{K}$ is defined by

$$A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^+ = C(G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^+, \tag{2}$$

where the crossed product structure is given by the partially defined action of the group of fractional ideals, seen as $\mathbf{A}_{\mathbb{K},f}^* / \hat{\mathcal{O}}_{\mathbb{K}}^*$ which is the restriction to $G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}}$ of the action on $G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \mathbf{A}_{\mathbb{K},f}$, given by $\mathfrak{n} \in J_{\mathbb{K}}^+$ acting as

$$(\gamma, i) \mapsto (\vartheta_{\mathbb{K}}(\mathfrak{n}^{-1}) \cdot \gamma, \mathfrak{n} \cdot i).$$

The time evolution is given by

$$\sigma_{\mathbb{K},t}(f) = f, \quad \forall f \in C(G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}}), \quad \text{and} \quad \sigma_{\mathbb{K},t}(\mu_{\mathfrak{n}}) = N(\mathfrak{n})^{it} \mu_{\mathfrak{n}}, \quad \forall \mathfrak{n} \in J_{\mathbb{K}}^+. \tag{3}$$

where $\mu_{\mathfrak{n}}$ are the isometries that implement the semigroup action of $J_{\mathbb{K}}^+$.

3. Hilbert space representation, partition function, KMS-states

3.1. A complete classification of the KMS states for the systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ was obtained in **[31]**, Thm. 2.1. In particular, in the low temperature range $\beta > 1$, the extremal KMS_β states are parameterized by elements $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$, and are in Gibbs form, given by normalized L-series

$$\omega_{\beta,\gamma}(f) = \frac{1}{\zeta_{\mathbb{K}}(\beta)} \sum_{\mathfrak{n} \in J_{\mathbb{K}}^+} f(\vartheta_{\mathbb{K}}(\mathfrak{n})\gamma) N(\mathfrak{n})^{-\beta}. \tag{4}$$

In the particular case where $f = \chi$ is a character of $G_{\mathbb{K}}^{\mathrm{ab}}$ (extended as usual by 0 on ideals not coprime to its conductor), we find

$$\omega_{\beta,\gamma}(\chi) = \frac{1}{\zeta_{\mathbb{K}}(\beta)} \cdot \chi(\gamma) \cdot L_{\mathbb{K}}(\chi, s). \tag{5}$$

3.2. Associated to any element $\gamma \in G_{\mathbb{K}}^{ab}$ is a natural representation of the algebra $A_{\mathbb{K}}$ on the Hilbert space $\ell^2(J_{\mathbb{K}}^+)$. Namely, let $\varepsilon_{\mathfrak{m}}$ denote the canonical basis of $\ell^2(J_{\mathbb{K}}^+)$. Then the action on $\ell^2(J_{\mathbb{K}}^+)$ of an element $f_{\mathfrak{n}}\mu_{\mathfrak{n}} \in A_{\mathbb{K}}$ with $\mathfrak{n} \in J_{\mathbb{K}}^+$ and $f_{\mathfrak{n}} \in C(X_{\mathbb{K}})$ is given by

$$\pi_\gamma(f_{\mathfrak{n}}\mu_{\mathfrak{n}})\ \epsilon_{\mathfrak{m}} = f_{\mathfrak{n}}(\mathfrak{n}\,\mathfrak{m} * \gamma)\,\epsilon_{\mathfrak{n}\,\mathfrak{m}}.$$

In this picture, the time evolution is implemented (in the sense of formula (1)) by a Hamiltonian

$$H_{\sigma_{\mathbb{K}}}\epsilon_{\mathfrak{n}} = \log N(\mathfrak{n})\ \epsilon_{\mathfrak{n}}, \tag{6}$$

3.3. In this representation,

$$\mathrm{trace}(\pi_\gamma(f)e^{-\beta H_{\sigma_{\mathbb{K}}}}) = \sum_{\mathfrak{n}\in J_{\mathbb{K}}^+} \frac{f(\mathfrak{n} * \gamma)}{N(\mathfrak{n})^\beta}.$$

Setting $f = 1$, the Dedekind zeta function

$$\zeta_{\mathbb{K}}(\beta) = \sum_{\mathfrak{n}\in J_{\mathbb{K}}^+} N(\mathfrak{n})^{-\beta}$$

appears as the partition function

$$\zeta_{\mathbb{K}}(\beta) = \mathrm{trace}(e^{-\beta H_{\sigma_{\mathbb{K}}}})$$

of the system (convergent for $\beta > 1$).

3.4. Remark (Formulation in terms of $\mathbb{K}$-lattices)**.** As shown in **[12]**, the original Bost–Connes system admits a geometric reformulation in terms of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices, which in Section 3 of **[31]** was generalized to number fields. More specifically, the moduli space of $\mathbb{K}$-lattices up to scaling is the abelian part $C(X_{\mathbb{K}})$ of the algebra (a classical quotient), and the moduli space up to scaling *and* commensurability exhibit the complete algebra (a genuinely noncommutative space). We recall the definitions for convenience.

Denote by $\mathbb{K}_\infty = \prod_{v|\infty} \hat{\mathbb{K}}_v$ the product of the completions at the archimedean places, and by $(\mathbb{K}_\infty^*)^0$ the connected component of the identity in $\mathbb{K}_\infty^*$. An 1-*dimensional* $\mathbb{K}$-*lattice* is a pair (Λ, ϕ), where $\Lambda \subset \mathbb{K}_\infty$ is a lattice with $\mathscr{O}_{\mathbb{K}}\,\Lambda = \Lambda$ and $\phi : \mathbb{K}/\mathscr{O}_{\mathbb{K}} \to \mathbb{K}\,\Lambda/\Lambda$ is an $\mathscr{O}_{\mathbb{K}}$-module homomorphism. The set of 1-dimensional $\mathbb{K}$-lattices can be identified with

$$\mathscr{M}_{\mathbb{K},1} = \mathbb{K}^* \setminus \mathbf{A}_{\mathbb{K}}^* \times_{\hat{\mathscr{O}}_{\mathbb{K}}^*} \hat{\mathscr{O}}_{\mathbb{K}}, \tag{7}$$

as in **[13]** and **[15]**, cf. **[31]** Lemma 3.3. Two $\mathbb{K}$-lattices are *commensurable*, denoted by

$$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2),$$

if $\mathbb{K}\,\Lambda_1 = \mathbb{K}\,\Lambda_2$ and $\phi_1 = \phi_2$ modulo $\Lambda_1 + \Lambda_2$.

The *scaling equivalence* corresponds to identifying 1-dimensional $\mathbb{K}$-lattices (Λ, ϕ) and $(k\Lambda, k\psi)$, where $k \in (\mathbb{K}^*_\infty)^0$ and ψ is a pointwise limit of elements $r\phi$ with $r \in \mathscr{O}^*_\mathbb{K} \cap (\mathbb{K}^*_\infty)^0$. The resulting convolution algebra corresponds to the action of $\mathbf{A}^*_{\mathbb{K},f} / \hat{\mathscr{O}}^*_\mathbb{K} \simeq J_\mathbb{K}$ on the *moduli space of 1-dimensional $\mathbb{K}$-lattices up to scaling*

$$\overline{\mathscr{M}}_{\mathbb{K},1} = \mathbf{A}^*_\mathbb{K} / \overline{\mathbb{K}^*(\mathbb{K}^*_\infty)^0} \times_{\hat{\mathscr{O}}^*_\mathbb{K}} \hat{\mathscr{O}}_\mathbb{K} \simeq G^{ab}_\mathbb{K} \times_{\hat{\mathscr{O}}^*_\mathbb{K}} \hat{\mathscr{O}}_\mathbb{K}.$$

The algebra $A_\mathbb{K}$ can be interpreted as the quotient of the groupoid of the commensurability relation by the scaling action. The Hilbert space construction can be fit into the general framework of groupoid algebra representations.

In the lattice picture, the low temperature KMS states are parameterized by the *invertible* 1-dimensional $\mathbb{K}$-lattices, namely those for which the $\mathscr{O}_\mathbb{K}$-module homomorphism φ is actually an isomorphism, see **[12]**, **[13]**, **[31]**, and Chapter 3 of **[11]**.

4. Hamiltonians and arithmetic equivalence

We first show that the existence of an isomorphism of the quantum statistical mechanical systems implies arithmetic equivalence; this is basically because the zeta functions of $\mathbb{K}$ and $\mathbb{L}$ are the partition functions of the respective systems. Some care has to be taken since the systems are not represented on the same Hilbert space.

4.1. Proposition. *Let $\varphi : (A_\mathbb{K}, \sigma_\mathbb{K}) \to (A_\mathbb{L}, \sigma_\mathbb{L})$ be an isomorphism of QSM-systems of number fields $\mathbb{K}$ and $\mathbb{L}$. Then $\mathbb{K}$ and $\mathbb{L}$ are arithmetically equivalent, i.e., have the same Dedekind zeta function.*

Proof. The isomorphism $\varphi : (A_\mathbb{K}, \sigma_\mathbb{K}) \to (A_\mathbb{L}, \sigma_\mathbb{L})$ induces an identification of the sets of extremal KMS-states of the two systems, via pullback $\varphi^* : \mathrm{KMS}_\beta(\mathbb{L}) \to \mathrm{KMS}_\beta(\mathbb{K})$.

Consider the GNS representations associated to regular low temperature KMS states $\omega = \omega_\beta$ and $\varphi^*(\omega)$. We denote the respective Hilbert spaces by $\mathscr{H}_\omega$ and $\mathscr{H}_{\varphi^*\omega}$. As in Lemma 4.3 of **[10]**, we observe that the factor $\mathscr{M}_\omega$ obtained as the weak closure of $A_\mathbb{L}$ in the GNS representation is of type I_∞, since we are only considering the low temperature KMS states that are of Gibbs form. Thus, the space $\mathscr{H}_\omega$ decomposes as

$$\mathscr{H}_\omega = \mathscr{H}(\omega) \otimes \mathscr{H}',$$

with an irreducible representation π_ω of $A_\mathbb{L}$ on $\mathscr{H}(\omega)$ and

$$\mathscr{M}_\omega = \{T \otimes 1 \,|\, T \in \mathscr{B}(\mathscr{H}(\omega))\}$$

($\mathscr{B}$ indicates the set of bounded operators). Moreover, we have

$$\langle (T \otimes 1)1_\omega, 1_\omega \rangle = \mathrm{Tr}(T\rho)$$

for a density matrix ρ (positive, of trace class, of unit trace).

We know that the low temperature extremal KMS states for the system $(A_\mathbb{L}, \sigma_\mathbb{L})$ are of Gibbs form and given by the explicit expression

$$\omega_{\gamma,\beta}(f) = \frac{1}{\zeta_\mathbb{L}(\beta)} \sum_{\mathfrak{m} \in J^+_\mathbb{L}} \frac{f(\vartheta_\mathbb{L}(\mathfrak{m})\gamma)}{N_\mathbb{L}(\mathfrak{m})^\beta}, \tag{8}$$

for some $\gamma \in G_{\mathbb{L}}^{\mathrm{ab}}/\vartheta_{\mathbb{L}}(\hat{\mathcal{O}}_{\mathbb{L}}^*)$; and similarly for the system $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$. Thus, we can identify $\mathscr{H}(\omega)$ with $\ell^2(J_{\mathbb{L}}^+)$ and the density ρ correspondingly with $e^{-\beta H_{\sigma_{\mathbb{L}}}}/\mathrm{Tr}(e^{-\beta H_{\sigma_{\mathbb{L}}}})$. As in Lemma 4.3 of **[10]**, the evolution group e^{itH_ω} generated by the Hamiltonian H_ω that implements the time evolution $\sigma_{\mathbb{L}}$ in the GNS representation on $\mathscr{H}_\omega$ agrees with $e^{itH_{\sigma_{\mathbb{L}}}}$ on the factor $\mathscr{M}_\omega$. This gives

$$e^{itH_\omega}\pi_\omega(f)e^{-itH_\omega} = \pi_\omega(\sigma_{\mathbb{L}}(f)) = e^{itH_{\sigma_{\mathbb{L}}}}\pi_\omega(f)e^{-itH_{\sigma_{\mathbb{L}}}}.$$

As observed in §4.2 of **[10]**, this gives us that the Hamiltonians differ by a constant,

$$H_\omega = H_{\sigma_{\mathbb{L}}} + \log \lambda_1,$$

for some $\lambda_1 \in \mathbb{R}_+^*$. The argument for the GNS representation for $\pi_{\varphi^*(\omega_\beta)}$ is similar and it gives an identification of the Hamiltonians

$$H_{\varphi^*(\omega)} = H_{\sigma_{\mathbb{K}}} + \log \lambda_2$$

for some constant $\lambda_2 \in \mathbb{R}_+^*$.

The algebra isomorphism φ induces a unitary equivalence Φ of the Hilbert spaces of the GNS representations of the corresponding states, and the Hamiltonians that implement the time evolution in these representations are therefore related by

$$H_{\varphi^*(\omega)} = \Phi H_\omega \Phi^*.$$

In particular the Hamiltonians $H_{\varphi^*(\omega)}$ and H_ω then have the same spectrum.

Thus, we know from the discussion above that

$$H_{\mathbb{K}} = \Phi H_{\mathbb{L}} \Phi^* + \log \lambda,$$

for a unitary operator Φ and a $\lambda \in \mathbb{R}_+^*$. This gives at the level of zeta functions

$$\zeta_{\mathbb{L}}(\beta) = \lambda^{-\beta}\zeta_{\mathbb{K}}(\beta). \tag{9}$$

This identity holds for all $\beta > 1$, and hence by analytic continuation to all $\beta \in \mathbb{C}$. Now consider the left hand side and right hand side as classical Dirichlet series of the form

$$\sum_{n\geq 1} \frac{a_n}{n^\beta} \text{ and } \sum_{n\geq 1} \frac{b_n}{(\lambda n)^\beta},$$

respectively. Since $b(1) = 1 \neq 0$, the identity theorem for Dirichlet series first implies that λ is an integer. Then, since $a(1) = 1 \neq 0$, we actually find $\lambda = 1$. Thus, we obtain $\zeta_{\mathbb{K}}(\beta) = \zeta_{\mathbb{L}}(\beta)$ which gives arithmetic equivalence of the number fields. □

By expanding the zeta functions as Euler products, we deduce

4.2. Corollary. *If the QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ of two number fields $\mathbb{K}$ and $\mathbb{L}$ are isomorphic, then there is a bijection of the primes $\mathfrak{p}$ of $\mathbb{K}$ above p and the primes $\mathfrak{q}$ of $\mathbb{L}$ above p that preserves the inertia degree: $f(\mathfrak{p}\,|\mathbb{K}) = f(\mathfrak{q}\,|\mathbb{L})$.* □

Using some other known consequences of arithmetical equivalence, we get the following **[39]**, Theorem 1) - which will not be used in the sequel:

4.3. Corollary. *If the QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ of two number fields $\mathbb{K}$ and $\mathbb{L}$ are isomorphic, then the number fields have the same degree over $\mathbb{Q}$, the same discriminant, normal closure, isomorphic unit groups, and the same number of real and complex embeddings.* □

However, it does not follow from arithmetical equivalence that $\mathbb{K}$ and $\mathbb{L}$ have the same class group (or even class number), cf. **[19]**.

5. Crossed product structure and QSM-isomorphism

In this section, we study isomorphisms of general algebras obtained as crossed products by endomorphisms, compatible with certain time evolutions. The argument we give here is a modification of the argument of **[41]**. Basically, this setup shows that compatibility with time evolution guarantees that the isomorphism $\varphi : A_{\mathbb{K}} \xrightarrow{\sim} A_{\mathbb{L}}$ induces a separate homeomorphism $X_{\mathbb{K}} \xrightarrow{\sim} X_{\mathbb{L}}$ and a family of semi-group isomorphisms $J^+_{\mathbb{K}} \xrightarrow{\sim} J^+_{\mathbb{L}}$. We first discuss the case of an action of $\mathbb{Z}_+$ and then extend to the higher rank case $\mathbb{Z}^N_+$.

5.1. Let X be a compact Hausdorff topological space and let $\gamma : X \to X$ be a continuous injective map such that $Y = \mathrm{Range}(\gamma) \subset X$ is a clopen set, so that the characteristic function $\chi = \chi_Y \in C(X)$. We then have an endomorphism $\nu : C(X) \to C(X)$, given by

$$\nu(f)(x) = f(\gamma(x)),$$

and another endomorphism $\rho : C(X) \to C(X)$, given by

$$\rho(f)(x) = \chi(x) f(\gamma^{-1}(x)).$$

This is well defined, since, for an injective γ the inverse $\gamma^{-1}(x)$ is well defined for $x \in Y \subset X$. One has $\chi(\gamma(x)) = 1$ for all $x \in X$, so that $\nu(\chi f) = \nu(f)$ for all $f \in C(X)$, and $\nu(\rho(f)) = f$ for all $f \in C(X)$. Thus, the endomorphism ν is surjective on $C(X)$ but not injective. One also has $\rho(\nu(f)) = \chi f$.

5.2. The semigroup crossed product

$$C(X) \rtimes_\rho \mathbb{Z}_+$$

is generated algebraically by elements $f \in C(X)$ and an isometry μ, with the relations

$$\mu^* \mu = 1, \ f\mu = \mu\nu(f), \ \mu f = \rho(f)\mu,$$

for all $f \in C(X)$, with ν and ρ as above. Under the $*$-involution these give also relations of the form $\mu^* f = \nu(f)\mu^*$ and $f\mu^* = \mu^* \rho(f)$. The semigroup action in the crossed product is given by the endomorphism ρ, with $\rho(f) = \mu f \mu^*$.

5.3. Proposition. *Let X and X' be compact Hausdorff spaces, and let*

$$\mathscr{A} = C(X) \rtimes_\rho \mathbb{Z}_+ \ \text{ and } \ \mathscr{A}' = C(X') \rtimes_{\rho'} \mathbb{Z}_+$$

be the semigroup crossed product C^*-algebras associated to fixed-point free injective continuous maps $\gamma : X \to X$ and $\gamma' : X' \to X'$ as above. Suppose given time evolutions on $\mathscr{A}$ and $\mathscr{A}'$ such that

$$\begin{cases} \sigma_t(f) = f & \text{for all } f \in C(X), \\ \sigma_t(\mu) = \lambda^{it}\mu, \end{cases}$$

for some $\lambda \in \mathbb{R}^*_+$, and similarly for σ' on $\mathscr{A}'$ with the same λ. Let

$$\varphi : (\mathscr{A}, \sigma) \xrightarrow{\sim} (\mathscr{A}', \sigma')$$

be an isomorphism of QSM-systems. Then φ induces a homeomorphism

$$\Phi : X \xrightarrow{\sim} X',$$

with

$$\Phi\gamma = \gamma'\Phi.$$

Proof. By the relations in the crossed product algebra, we can decompose elements $a \in \mathscr{A}$ linearly as

$$a = f_0 + \sum_{k>0} (\mu^k f_k + f_{-k}(\mu^*)^k).$$

One writes $E_k(a) = f_k$ for the linear contractive map that gives the "Fourier coefficients" of this decomposition. For a time evolution with $\sigma_t(f) = f$ for $f \in C(X)$ and $\sigma_t(\mu) = \lambda^{it}\mu$, the E_k are in fact the projections onto the eigenspace of the time evolution with eigenvalue λ^{ikt}.

Let $\mathscr{A}_0 \subset \mathscr{A}$ be the closed subalgebra (without involution) generated by $C(X)$ and the isometry μ, but without the adjoint μ^*. Elements in the subalgebra $\mathscr{A}_0$ have $E_k(a) = 0$ for all $k \leq 0$. The isomorphism φ is compatible with the time evolution, hence it maps the eigenspace $\mathscr{E}_k$ in $\mathscr{A}$ with eigenvalue λ^{itk} to the eigenspace $\mathscr{E}'_k$ with the same eigenvalue in $\mathscr{A}'$. Thus, in particular, φ induces an isomorphism $\varphi : \mathscr{A}_0 \xrightarrow{\sim} \mathscr{A}_0$, compatible with the restrictions of the time evolutions to this subalgebra.

Let $\mathscr{C}_0$ denote the closure of the commutator ideal of $\mathscr{A}_0$ and $\mathscr{C}_0^2$ the closure of the span of products of commutators, and let $\mathscr{C}'_0$ and $(\mathscr{C}'_0)^2$ be the same for $\mathscr{A}'_0$. The C^*-isomorphism φ maps $\mathscr{C}_0$ to $\mathscr{C}'_0$ and $\mathscr{C}_0^2$ to $(\mathscr{C}'_0)^2$. Thus, it induces an isomorphism

$$\overline{\varphi} : \mathscr{A}_0 / \mathscr{C}_0 \xrightarrow{\sim} \mathscr{A}'_0 / \mathscr{C}'_0,$$

which gives a bijection of the maximal ideals of $\mathscr{A}_0$ containing $\mathscr{C}_0$ and the maximal ideals of $\mathscr{A}'_0$ containing $\mathscr{C}'_0$.

Given a maximal ideal I_x of $C(X)$, given by all functions vanishing at a point $x \in X$, define

$$\widetilde{I}_{x,0} = \{a \in \mathscr{A}_0 \,:\, E_0(a) \in I_x\}.$$

This is a maximal ideal in $\mathscr{A}_0$, containing $\mathscr{C}_0$. Since $\gamma : X \to X$ has no fixed points, all the maximal ideals of $\mathscr{A}_0$ containing $\mathscr{C}_0$ are of this form.

The bijection between these maximal ideals induced by the isomorphism

$$\overline{\varphi} : \mathscr{A}_0 / \mathscr{C}_0 \xrightarrow{\sim} \mathscr{A}'_0 / \mathscr{C}'_0$$

then gives a bijection Φ between the points of X and the points of X', which induces an algebra isomorphism of $C(X)$ and $C(X')$, hence a homeomorphism

$$\Phi : X \xrightarrow{\sim} X'.$$

One can write commutators

$$[f, \mu] = f\mu - \mu f = \mu(\nu(f) - f) = (f - \rho(f))\mu.$$

Similarly, one has

$$[f,\mu^k] = \mu^k(\nu^k(f) - f) = (f - \rho^k(f))\mu^k$$
$$[f,\mu^*] = (f - \nu(f))\mu^* = \mu^*(\rho(f) - f),$$
$$[f,(\mu^*)^k] = (f - \nu^k(f))(\mu^*)^k = (\mu^*)^k(\rho^k(f) - f).$$

Thus, elements $a \in \mathscr{C}_0$ have Fourier coefficients $E_0(a) = 0$ and $E_k(a)$ in the subspace of $\mathscr{E}_k$ generated by the "coboundaries" $\mu^k h$, with

$$h = (f - \nu^k(f)).$$

Similarly, elements in $\mathscr{C}_0^2$ have $E_0(a) = E_1(a) = 0$ and the $\mathscr{E}_k$ for $k \geq 2$ in the subspace of $\mathscr{E}_k$ spanned by the coboundaries.

Notice then that if $f \in C(X)$ belongs to an ideal I_y, with $y \in Y$, then $\nu(f)$ belongs to the ideal I_x, with $y = \gamma(x)$. Similarly, if $f \in I_x$, then $\rho(f) \in I_{\gamma(x)}$. Moreover, for

$$a = f_0 + \sum_{k>0}(\mu^k f_k + f_{-k}(\mu^*)^k)$$

as above, we have

$$\begin{aligned}
\nu(a) &= \nu(f_0) + \sum_{k>0}\nu(\rho^k(f_k))\mu^k + (\mu^*)^k\nu(\rho^k(f)) \\
&= \nu(f_0) + \sum_{k>0}\mu^k\nu^{k+1}(\rho^k(f_k)) + \nu^{k+1}(\rho^k(f))(\mu^*)^k \\
&= \nu(f_0) + \sum_{k>0}\mu^k\nu(f_k) + \nu(f_{-k})(\mu^*)^k,
\end{aligned}$$

and

$$\rho(a) = \rho(f_0) + \sum_{k>0}(\mu^k\rho(f_k) + \rho(f_{-k})(\mu^*)^k).$$

Thus, we also have that if $a \in \widetilde{I}_{y,0}$, for $y \in Y \subset X$, then $\nu(a) \in \widetilde{I}_{x,0}$, with $y = \gamma(x)$ and if $a \in \widetilde{I}_{x,0}$ then $\rho(a) \in \widetilde{I}_{\gamma(x),0}$. Moreover, for $a \in \widetilde{I}_{y,0}$, we have

$$a\,[f,\mu] = [f,\mu]\,\nu(a) \text{ and } [f,\mu]\,a = \rho(a)\,[f,\mu].$$

This gives

$$\widetilde{I}_{y,0}\,\mathscr{C}_0 + \mathscr{C}_0^2 = \mathscr{C}_0\,\widetilde{I}_{x,0} + \mathscr{C}_0^2\,.$$

Under the isomorphism φ we then have

$$\varphi(\widetilde{I}_{y,0}\,\mathscr{C}_0 + \mathscr{C}_0^2) = \widetilde{I}_{\Phi(y),0}\,\mathscr{C}'_0 + (\mathscr{C}'_0)^2$$

and

$$\varphi(\mathscr{C}_0\,\widetilde{I}_{x,0} + \mathscr{C}_0^2) = \mathscr{C}'_0\,\widetilde{I}_{\Phi(x),0} + (\mathscr{C}'_0)^2,$$

with $y = \gamma(x)$. The same relations applied to the algebra $\mathscr{A}'_0$ then give $\Phi(\gamma(x)) = \gamma'(\Phi(x))$. □

5.4. Remark. The condition that $\gamma : X \to X$ has no fixed points can be left out; the proof gets technically more complicated since one has to include an additional class of maximal ideals of $\mathscr{A}_0$ containing $\mathscr{C}_0$.

We extend the result to the case of a crossed product by an abelian semigroup generated by N commuting isometries μ_i, each corresponding to a pair of endomorphism $\nu_i(f) = f \circ \gamma_i$ and $\rho_i(f) = \chi_i f \circ \gamma_i^{-1}$.

5.5. Proposition. *Let*

$$\mathscr{A} = C(X) \rtimes_\rho \mathbb{Z}_+^N \text{ and } \mathscr{A}' = C(X') \rtimes_{\rho'} \mathbb{Z}_+^N,$$

with

$$\rho_i(f) = \mu_i f \mu_i^* \text{ and } \nu_i(f) = f \circ \gamma_i = \mu_i^* f \mu_i,$$

for fixed-point free maps γ_i. Assume that $\gamma_i(x) \neq \gamma_j(x)$, for all $x \in X$ and all $i \neq j$ and similarly, $\gamma'_i(x') \neq \gamma'_j(x')$ for all $x' \in X'$ and $i \neq j$. Let σ and σ' be time evolutions on $\mathscr{A}$ and $\mathscr{A}'$ with

$$\begin{cases} \sigma_t(f) = f & \text{for all } f \in C(X), \\ \sigma_t(\mu_i) = \lambda^{it}\mu_i, \end{cases}$$

for a $\lambda \in \mathbb{R}_+^*$ and, similarly on $\mathscr{A}'$. Then a QSM-isomorphism

$$\varphi : (\mathscr{A}, \sigma) \xrightarrow{\sim} (\mathscr{A}', \sigma')$$

induces a homeomorphism

$$\Phi : X \xrightarrow{\sim} X'$$

and a locally constant function

$$\alpha : X \to S_N,$$

with S_N the group of permutations of the set $\{1, \dots, N\}$, such that

$$\Phi(\gamma_{\alpha_x(i)}(x)) = \gamma'_i(\Phi(x)),$$

for all $x \in X$.

Proof. The argument proceeds as in the case of a single isometry. One uses the compatibility with the time evolution to induce isomorphisms

$$\varphi : \mathscr{A}_0 \xrightarrow{\sim} \mathscr{A}'_0$$

of the closed subalgebras without involution generated by $C(X)$ (respectively, $C(X')$) and the μ_i (respectively μ'_i). The induced isomorphism

$$\overline{\varphi} : \mathscr{A}_0 / \mathscr{C}_0 \xrightarrow{\sim} \mathscr{A}'_0 / \mathscr{C}'_0$$

again gives an identification between the maximal ideals $\widetilde{I}_{x,0}$ in $\mathscr{A}_0$ containing $\mathscr{C}_0$ and $\widetilde{I}_{x',0}$ in $\mathscr{A}'_0$ containing $\mathscr{C}'_0$, hence a homeomorphism

$$\Phi : X \xrightarrow{\sim} X'.$$

Here one can again describe as before the elements in $\mathscr{C}_0$ and $\mathscr{C}_0^2$ in terms of their projections E_k on the eigenspaces of the time evolution and one sees that $\mathscr{C}_0 / \mathscr{C}_0^2$ is a bimodule for $C(X)$ of the form

$$C(X)\mu_1 + \cdots + C(X)\mu_N,$$

corresponding to the projection onto the eigenspace $\mathscr{E}_1$. As in **[41]**, one then sees that, for a point $x \in X$, the set of points $\{\gamma_i(x)\}_{i=1,\dots,N}$, which are distinct by the assumption, is the set of those $y \in X$ such that the space

$$\widetilde{I}_{x,0}\,\mathscr{C}_0 + \mathscr{C}_0\,\widetilde{I}_{y,0} + \mathscr{C}_0^2$$

has codimension one in $\mathscr{C}_0$. One obtains in this way, for each $x \in X$, an identification between the sets $\{\Phi(\gamma_i(x))\}_{i=1,\dots,N}$ and $\{\gamma'_i(\Phi(x))\}_{i=1,\dots,N}$. This gives for each x a permutation α_x of the set $\{1,\dots,N\}$. By continuity, this gives a locally constant function $\alpha : X \to S_N$. □

5.6. Remark. Thus, in the case of N commuting isometries, we find that a C^*-algebra isomorphism $\varphi : \mathscr{A} \xrightarrow{\sim} \mathscr{A}'$ compatible with the time evolution maps isomorphically

$$\varphi : C(X) \xrightarrow{\sim} C(X'),$$

through a homeomorphism $\Phi : X \xrightarrow{\sim} X'$, and it maps

$$\varphi(\mu_i) = \sum_{j=1}^{N} h_{ij}\mu'_j,$$

where $h_{ij} \in C(X')$ are given by $h_{ij} = \varphi(f_{ij})$, with $f_{ij} \in C(X)$ locally constant functions satisfying

$$f_{ij}(x) = \delta_{j,\alpha_x(i)}.$$

These satisfy $\sum_j f_{ij}(x) \equiv 1$, which is compatible with the relation $\varphi(\mu_i^*\mu_i) = 1$.

6. Application to the QSM-system of a number field

We now return from the general situation to our specific QSM-systems, so

$$A_{\mathbb{K}} = C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^+.$$

We also use the notation $\mathfrak{n} * x$ for the action of $\mathfrak{n} \in J_{\mathbb{K}}^+$ on $X_{\mathbb{K}}$. This action corresponds to the endomorphisms that give the crossed product action in $A_{\mathbb{K}}$, and is clearly fixed-point free. If we factor the algebra into finite pieces as in the following proof, we can deduce from the previous section the following result:

6.1. Proposition. *Let $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \xrightarrow{\sim} (A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ be an isomorphism of the QSM-systems associated to number fields $\mathbb{K}$ and $\mathbb{L}$. Then the isomorphism φ induces a homeomorphism*

$$\varphi : X_{\mathbb{K}} \xrightarrow{\sim} X_{\mathbb{L}}$$

and a family of semigroup isomorphisms

$$\alpha_x : J_{\mathbb{K}}^+ \xrightarrow{\sim} J_{\mathbb{L}}^+,$$

locally constant in $x \in X_{\mathbb{K}}$, with the compatibility condition

$$\varphi(\mathfrak{n} * x) = \alpha_x(\mathfrak{n}) * \varphi(x).$$

Proof. The previous proposition does not apply directly, since $J_{\mathbb{K}}^+$ has infinite rank. However, we can view it as a product over finite rank semigroups, corresponding to sub-semigroups generated by prime ideals of a given norm. The fact that everything is compatible with this splitting follows from the compatibility of φ with the time evolutions: it implies that φ maps the eigenspace of the time evolution $\sigma_{\mathbb{K}}$ with eigenvalue p^{it} to the eigenspace of the time evolution $\sigma_{\mathbb{L}}$ with the same eigenvalue. Thus, φ induces, for each rational prime p, an isomorphism

$$\varphi_p : A_{\mathbb{K},p} \xrightarrow{\sim} A_{\mathbb{L},p}$$

of the subalgebras $A_{\mathbb{K},p} \subset A_{\mathbb{K}}$ and $A_{\mathbb{L},p} \subset A_{\mathbb{L}}$, given by

$$A_{\mathbb{K},p} = C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K},p}^+,$$

with $J_{\mathbb{K},p}^+ \subset J_{\mathbb{K}}^+$ the sub-semigroup generated by the isometries $\mu_{\mathfrak{p}}$ with $\mathfrak{p}$ a prime of $\mathbb{K}$ with $N_{\mathbb{K}}(\mathfrak{p}) = p$, and similarly for $A_{\mathbb{L},p} = C(X_{\mathbb{L}}) \rtimes J_{\mathbb{L},p}^+$. To each of these subalgebras we can apply the result of the previous proposition and obtain an induced homeomorphism

$$\varphi : X_{\mathbb{K}} \xrightarrow{\sim} X_{\mathbb{L}}$$

and a locally constant bijection

$$\alpha_x : \{\mathfrak{p} \in \mathbb{K} \,:\, N_{\mathbb{K}}(\mathfrak{p}) = p\} \xrightarrow{\sim} \{\mathfrak{q} \in \mathbb{L} \,:\, N_{\mathbb{L}}(\mathfrak{q}) = p\}.$$

Notice that we know a priori that $\mathbb{K}$ and $\mathbb{L}$ have the same number of primes over the same rational prime p, because of arithmetic equivalence. Assembling together these identifications for each prime p, one obtains the isomorphism

$$\alpha_x : J_{\mathbb{K}}^+ \xrightarrow{\sim} J_{\mathbb{L}}^+.$$

□

6.2. Proposition. *In the previous proposition 6.1, for $x \in G_{\mathbb{K}}^{\mathrm{ab}}$, the map*

$$\alpha_x : J_{\mathbb{K}}^+ \xrightarrow{\sim} J_{\mathbb{L}}^+,$$

is independent of x. If we denote it by φ, the compatibility condition becomes

$$\varphi(\mathfrak{n} * x) = \varphi(\mathfrak{n}) * \varphi(x)$$

for $x \in G_{\mathbb{K}}^{\mathrm{ab}}$ and $\mathfrak{n} \in J_{\mathbb{K}}^+$.

Proof. The group $G_{\mathbb{K}}^{\mathrm{ab}}$ operates faithfully by endomorphisms on the QSM-system $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$, cf. **[31]**, Remark 2.2(i). We then have a commutative diagram

$$\begin{array}{ccc} X_{\mathbb{K}} & \xrightarrow[\sim]{\varphi} & X_{\mathbb{L}} \\ \Big\downarrow{\scriptstyle \gamma \in G_{\mathbb{K}}^{\mathrm{ab}}} & & \Big\downarrow{\scriptstyle \varphi(\gamma) \in G_{\mathbb{L}}^{\mathrm{ab}}} \\ X_{\mathbb{K}} & \xrightarrow[\sim]{\varphi} & X_{\mathbb{L}} \end{array}$$

Thus,

$$\varphi(\gamma)\varphi(x) = \varphi(\gamma x).$$

We now compute that for $x \in G_{\mathbb{K}}^{\mathrm{ab}}$,

$$\varphi(\mathfrak{n} *\gamma \cdot x) = \varphi(\mathfrak{n} *(\gamma x)) = \alpha_{\gamma x}(\mathfrak{n}) * \varphi(\gamma x).$$

and on the other hand

$$\varphi(\mathfrak{n} *\gamma \cdot x) = \varphi(\gamma)\varphi(\mathfrak{n} *x) = \varphi(\gamma)\alpha_x(\mathfrak{n}) * \varphi(x) = \alpha_x(\mathfrak{n}) * \varphi(\gamma)\varphi(x) = \alpha_x(\mathfrak{n}) * \varphi(\gamma x).$$

So that finally for all $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$,

$$\alpha_{\gamma x} = \alpha_x.$$

Since $G_{\mathbb{K}}^{\mathrm{ab}}$ acts transitively on itself, we do find that α_x for $x \in G_{\mathbb{K}}^{\mathrm{ab}}$ is independent of $x \in G_{\mathbb{K}}^{\mathrm{ab}}$; hence equal to α_1, which we denote by φ. □

6.3. Remark. We cannot conclude that α_x is constant on elements of $\hat{\mathscr{O}}_{\mathbb{K}}$ at this point (but of course it is on the subspace $\hat{\mathscr{O}}_{\mathbb{K}}^*$).

7. From QSM to field isomorphism: multiplicative structure

7.1. We now come to the proof of Theorem 1. Of course, (i) implies (ii). We now show that (ii) implies (i), i.e., that isomorphism of QSM-systems leads to isomorphic fields.

7.2. Remark. The start of the proof of the Neukirch-Uchida theorem is roughly based on the observation that a prime is characterized by its decomposition group in the algebraic closure (a fact apparently going back to F.K. Schmidt), a fact that is totally false in the abelian closure. Hence only based on the correspondence of abelianized Galois groups, we cannot get started with the proof in this way. In our proof, however, by what we have already deduced in previous sections, the isomorphism of QSM-systems induces automatically a bijection between the (prime) ideals of the field.

We first recall the following facts on the symmetries of the QSM-systems of number fields. The statement is analogous to Proposition 2.14 of **[13]** and Proposition 3.124 of **[12]**, where it was formulated for the case of imaginary quadratic fields, and to Theorem 2.14 of **[15]**, formulated in the function field case.

7.3. Lemma. *Let $\mathbb{K}$ denote any number field. The semigroup $\hat{\mathscr{O}}_{\mathbb{K}} \cap \mathbf{A}_{\mathbb{K},f}$ acts by endomorphisms of $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$, with kernel $\mathscr{O}_{\mathbb{K}}^*$. The subset $\hat{\mathscr{O}}_{\mathbb{K}}^*$ acts by automorphisms of the system, and the subset $\mathscr{O}_{\mathbb{K}}^\times = \mathscr{O}_{\mathbb{K}} - \{0\}$ of non-zero elements of the ring of integers acts by inner endomorphisms. This is summarized by following commutative diagram:*

$$\begin{array}{ccccc}
\mathrm{Inn}(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) & \hookrightarrow & \mathrm{End}(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) & \hookleftarrow & \mathrm{Aut}(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathscr{O}_{\mathbb{K}}^\times & \hookrightarrow & \hat{\mathscr{O}}_{\mathbb{K}} \cap \mathbf{A}_{\mathbb{K},f}^* & \hookleftarrow & \hat{\mathscr{O}}_{\mathbb{K}}^* \\
\uparrow & & \uparrow & & \uparrow \\
\mathscr{O}_{\mathbb{K}}^* & = & \mathscr{O}_{\mathbb{K}}^* & = & \mathscr{O}_{\mathbb{K}}^*
\end{array}$$

Proof. Consider a given $s \in \hat{\mathscr{O}}_{\mathbb{K}} \cap \mathbf{A}^*_{\mathbb{K},f}$, and the associated ideal $\mathfrak{n}$ given by

$$\mathfrak{n} = s\hat{\mathscr{O}}_{\mathbb{K}} \cap \mathbb{K}\,.$$

An element $(\gamma, \rho) \in X_{\mathbb{K}}$ is divisible by $\mathfrak{n}$ if $s^{-1}\rho \in \hat{\mathscr{O}}_{\mathbb{K}}$. One obtains in this way an action by endomorphisms of $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ by

$$\varepsilon_s(f)(\gamma, \rho) = f(\gamma, s^{-1}\rho)$$

when (γ, ρ) is divisible by $\mathfrak{n}$ and $\varepsilon_s(f)(\gamma, \rho) = 0$ otherwise. These are compatible by construction with the time evolution,

$$\varepsilon_s \sigma_t = \sigma_t \varepsilon_s, \;\; \forall s \in \hat{\mathscr{O}} \cap \mathbf{A}^*_{\mathbb{K},f}, \;\; \forall t \in \mathbb{R}\,.$$

It is clear from this definition that elements of $\hat{\mathscr{O}}^*_{\mathbb{K}}$ act by automorphisms.

Now consider then the case where $s \in \mathscr{O}^\times_{\mathbb{K}}$. In this case, $\mathfrak{n}$ is the principal ideal generated by s, and the non-zero values of the function $\varepsilon_s(f)$ can be identified as follows:

$$\varepsilon_s(f)(\gamma, \rho) = f(\gamma, s^{-1}\rho) = f(\vartheta_{\mathbb{K}}(\mathfrak{n}) \cdot \gamma, \rho) = (\mu_{\mathfrak{n}} f \mu^*_{\mathfrak{n}})\,(\gamma, \rho),$$

which is an inner endormorphism, since $\mu_{\mathfrak{n}}$ is an eigenvector of time evolution.

□

7.4. Remark. More generally, as we have already observed in the proof of Proposition 6.2, $G^{\mathrm{ab}}_{\mathbb{K}}$ (which contains an image of $\hat{\mathscr{O}}_{\mathbb{K}} \cap \mathbf{A}^*_{\mathbb{K},f}$) acts by endomorphisms of the system, cf. **[31]**, Remark 2.2(i). Also note that automorphisms of the field induce automorphisms of the associated QSM-systems.

7.5. Remark ($\mathbb{K}$-lattices)**.** In terms of $\mathbb{K}$-lattices (Λ, ϕ), the divisibility condition above corresponds to the condition that the homomorphism ϕ factors through

$$\phi : \mathbb{K} \,/\, \mathscr{O}_{\mathbb{K}} \to \mathbb{K}\,\Lambda / \mathfrak{n}\,\Lambda \to \mathbb{K}\,\Lambda/\Lambda.$$

The action of the endomorphisms is then given by

$$\varepsilon_s(f)((\Lambda, \phi), (\Lambda', \phi')) = f((\Lambda, s^{-1}\phi), (\Lambda', s^{-1}\phi'))$$

when both (Λ, ϕ) and (Λ', ϕ') are divisible by s and zero otherwise.

When $s \in \mathscr{O}^\times_{\mathbb{K}}$, we can consider the function

$$\mu_s((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} 1 & \Lambda = s^{-1}\Lambda' \;\text{ and }\; \phi' = \phi; \\ 0 & \text{otherwise.} \end{cases}$$

These are eigenvectors of the time evolution, with $\sigma_t(\mu_s) = N_{\mathbb{K}}(\mathfrak{n})^{it}\mu_s$, and $\varepsilon_s(f) = \mu_s \star f \star \mu^*_s$, for the convolution product of the algebra $A_{\mathbb{K}}$.

7.6. Proposition. *An isomorphism $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \to (A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ of the QSM-systems of two number fields $\mathbb{K}$ and $\mathbb{L}$, induces an isomorphism of topological groups between the Galois groups of their maximal abelian extensions:*

$$\varphi : G^{\mathrm{ab}}_{\mathbb{K}} \xrightarrow{\sim} G^{\mathrm{ab}}_{\mathbb{L}}.$$

Proof. For $\beta > 1$, the set of extremal KMS_β states of $A_\mathbb{K}$ is homeomorphic to $G_\mathbb{K}^{\mathrm{ab}}$ **([31]**, Thm. 2.1(iii)). Hence from the matching of KMS-states from Lemma 1.5, we find that φ induces a *homeomorphism*

$$\varphi \,:\, G_\mathbb{K}^{\mathrm{ab}} \xrightarrow{\sim} G_\mathbb{L}^{\mathrm{ab}}.$$

We now need to prove that this is actually a *group* isomorphism. Again, we use that $G_\mathbb{K}^{\mathrm{ab}}$ is also naturally a faithful symmetry group of the system $(A_\mathbb{K}, \sigma_\mathbb{K})$, so there is a group homomorphism

$$G_\mathbb{K}^{\mathrm{ab}} \hookrightarrow \mathrm{End}(A_\mathbb{K}, \sigma_\mathbb{K})$$

(which even factors modulo inner endomorphisms). Now we also see that the map φ automatically induces (by pullback) an isomorphism

$$\varphi \,:\, \mathrm{End}(A_\mathbb{K}, \sigma_\mathbb{K}) \xrightarrow{\sim} \mathrm{End}(A_\mathbb{L}, \sigma_\mathbb{L}).$$

Although we do not know the complete structure of the endomorphism algebra, by Lemma 1.5, we do know that φ sends the image of $G_\mathbb{K}^{\mathrm{ab}}$ to the image of $G_\mathbb{L}^{\mathrm{ab}}$. Thus, from the commutative diagram

$$\begin{array}{ccc} G_\mathbb{K}^{\mathrm{ab}} & \xrightarrow{\text{grp}} & \mathrm{End}(A_\mathbb{K}, \sigma_\mathbb{K}) \\ \simeq \big\downarrow \varphi\ \text{(top)} & & \simeq \big\downarrow \varphi\ \text{(grp)} \\ G_\mathbb{L}^{\mathrm{ab}} & \xrightarrow{\text{grp}} & \mathrm{End}(A_\mathbb{L}, \sigma_\mathbb{L}) \end{array}$$

we find that φ is indeed a topological group isomorphism $G_\mathbb{K}^{\mathrm{ab}} \xrightarrow{\sim} G_\mathbb{L}^{\mathrm{ab}}$. □

7.7. Remark. All the groups $G_\mathbb{K}^{\mathrm{ab}}$ are homeomorphic to each other (and to the Cantor set). The main point of using the set of KMS-states at the start of the proof is to show that *the map* φ induces such a homeomorphism.

7.8. Remark. The isomorphism type of the infinite abelian group $G_\mathbb{K}^{\mathrm{ab}}$ is determined by its so-called *Ulm invariants*. For $G_\mathbb{K}^{\mathrm{ab}}$, those were computed abstractly by Kubota **([30])**, and Onabe **([38])** computed them explicitly for quadratic imaginary fields. For example, $G_{\mathbb{Q}(i)}^{\mathrm{ab}}$ is never isomorphic to any other group for such a field, but $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$ have isomorphic abelianized absolute Galois groups (and they are not isomorphic as fields).

7.9. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields admitting an isomorphism φ of their QSM-systems $(A_\mathbb{K}, \sigma_\mathbb{K})$ and $(A_\mathbb{L}, \sigma_\mathbb{L})$. Then φ induces a group isomorphism of unit ideles*

$$\varphi \,:\, \hat{\mathscr{O}}_\mathbb{K}^* \xrightarrow{\sim} \hat{\mathscr{O}}_\mathbb{L}^*.$$

Proof. We have already seen that φ induces a homeomorphism

$$\varphi \,:\, X_\mathbb{K} \xrightarrow{\sim} X_\mathbb{L},$$

where $X_\mathbb{K} = G_\mathbb{K}^{\mathrm{ab}} \times_{\hat{\mathscr{O}}_\mathbb{K}^*} \hat{\mathscr{O}}_\mathbb{K}$; the product is balanced over the unit ideles, meaning that $X_\mathbb{K} = G_\mathbb{K}^{\mathrm{ab}} \times \hat{\mathscr{O}}_\mathbb{K} / \sim$ where for $i \in G_\mathbb{K}^{\mathrm{ab}}$ and $m \in \hat{\mathscr{O}}$, we let $(i, m) \sim (u^{-1} i, um)$ for any $u \in \hat{\mathscr{O}}_\mathbb{K}^*$. Also, on $G_\mathbb{K}^{\mathrm{ab}}$, φ is already a group isomorphism. Hence all we need to do is prove that φ maps unit ideles to unit ideles, as subsets of $G_\mathbb{K}^{\mathrm{ab}}$.

Since φ is already a group isomorphism when restricted to the group $G_{\mathbb{K}}^{ab}$, we have $\varphi(1) = 1$. We find $\hat{\mathscr{O}}_{\mathbb{K}}$ as a subspace of $X_{\mathbb{K}}$ by taking a (non-canonical) section

$$\hat{\mathscr{O}}_{\mathbb{K}} \hookrightarrow X_{\mathbb{K}} \,:\, m \mapsto (1, m);$$

indeed, $(1, m) \sim (1, n)$ only for m. We denote equivalence classes for $\sim$ by square brackets. Now

$$\varphi : \hat{\mathscr{O}}_{\mathbb{K}} \to \hat{\mathscr{O}}_{\mathbb{L}}$$

satisfies $\varphi(m) = m'$ if and only if

$$\varphi([(1, m)]) = [(1, m')].$$

Let us now check that this map induced by φ maps unit ideles to unit ideles. For this, we take $m \in \hat{\mathscr{O}}_{\mathbb{K}}^*$ to be a unit idele, and we compute the image of m. First of all, by definition, $\varphi([(1, m)] = [1, \varphi(m)]$. On the other hand, since m is a unit itself, we have $[(1, m)] = [(m, 1)]$. This is mapped by definition to $[(\varphi(m), 1)]$. Hence we find an equivalence

$$(1, \varphi(m)) \sim (\varphi(m), 1),$$

i.e., the existence of a unit $u \in \hat{\mathscr{O}}_{\mathbb{L}}$ with

$$1 = u\varphi(m).$$

This proves the claim that $\varphi(m)$ is also a unit idele. □

7.10. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields admitting an isomorphism φ of their QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$. Then φ induces a semigroup isomorphism:*

$$\varphi \,:\, \mathbf{A}_{\mathbb{K},f}^* \cap \hat{\mathscr{O}}_{\mathbb{K}} \xrightarrow{\sim} \mathbf{A}_{\mathbb{L},f}^* \cap \hat{\mathscr{O}}_{\mathbb{L}}.$$

Proof. We have an exact sequence

$$0 \to \hat{\mathscr{O}}_{\mathbb{K}}^* \to \mathbf{A}_{\mathbb{K},f}^* \cap \hat{\mathscr{O}}_{\mathbb{K}} \to J_{\mathbb{K}}^+, \tag{10}$$

which is (non-canonically) split by choosing a uniformizer $\pi_{\mathfrak{p}}$ at every place $\mathfrak{p}$ of the field. Hence as a semigroup, $\mathbf{A}_{\mathbb{K},f}^* \cap \hat{\mathscr{O}}_{\mathbb{K}} = J_{\mathbb{K}}^+ \times \hat{\mathscr{O}}_{\mathbb{K}}^*$. Now φ induces a bijection

$$J_{\mathbb{K}}^+ \times \hat{\mathscr{O}}_{\mathbb{K}}^* \xrightarrow{\sim} J_{\mathbb{L}}^+ \times \hat{\mathscr{O}}_{\mathbb{L}}^*$$

given by

$$(\mathfrak{n}, i) \mapsto (\alpha_i(\mathfrak{n}), \varphi(i)).$$

This is a group isomorphism precisely if

$$\alpha_{ij}(\mathfrak{m}\,\mathfrak{n}) = \alpha_i(\mathfrak{m})\alpha_j(\mathfrak{n}),$$

which happens exactly if $\alpha_i(\mathfrak{n})$ is independent of i for $i \in \hat{\mathscr{O}}_{\mathbb{K}}^*$. Now $\hat{\mathscr{O}}_{\mathbb{K}}^* \subseteq G_{\mathbb{K}}^{\mathrm{ab}}$, and we have seen in Proposition 6.2 that for such elements, indeed, $\alpha_i = \alpha_1$. □

7.11. Remark. One may prove in a similar way that φ induces a group isomorphism of the finite ideles of $\mathbb{K}$ and $\mathbb{L}$.

7.12. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields admitting an isomorphism φ of their QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$. Then φ induces a semigroup isomorphism between the multiplicative semigroups of non-zero elements of the rings of integers of $\mathbb{K}$ and $\mathbb{L}$:*

$$\varphi \,:\, (\mathscr{O}_{\mathbb{K}}^{\times}, \times) \xrightarrow{\sim} (\mathscr{O}_{\mathbb{L}}^{\times}, \times).$$

Proof. The previous proposition says that φ induces an isomorphism

$$\varphi \,:\, \mathbf{A}_{\mathbb{K},f}^{*} \cap \hat{\mathscr{O}}_{\mathbb{K}} \xrightarrow{\sim} \mathbf{A}_{\mathbb{L},f}^{*} \cap \hat{\mathscr{O}}_{\mathbb{L}}.$$

From Lemma 7.3, we have a natural map

$$\Theta_{\mathbb{K}} \,:\, \mathbf{A}_{\mathbb{K},f}^{*} \cap \hat{\mathscr{O}}_{\mathbb{K}} \to \mathrm{End}(A_{\mathbb{K}}, \sigma_{\mathbb{K}}),$$

and φ induces a map

$$\mathrm{End}(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \xrightarrow{\sim} \mathrm{End}(A_{\mathbb{L}}, \sigma_{\mathbb{L}}).$$

Now φ, as an isomorphism of QSM-systems, also preserves the *inner* endomorphisms:

$$\varphi \,:\, \mathrm{Inn}(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \xrightarrow{\sim} \mathrm{Inn}(A_{\mathbb{L}}, \sigma_{\mathbb{L}}),$$

but we know that

$$\Theta_{\mathbb{K}}^{-1}\left(\mathrm{Inn}(A_{\mathbb{K}}, \sigma_{\mathbb{K}})\right) \cap \left(\mathbf{A}_{\mathbb{K},f}^{*} \cap \hat{\mathscr{O}}_{\mathbb{K}}\right) = \mathscr{O}_{\mathbb{K}}^{\times}$$

(where $\mathscr{O}_{\mathbb{K}}^{\times} = \mathscr{O}_{\mathbb{K}} - \{0\}$), so we also get that φ induces an isomorphism

$$\varphi \,:\, \mathscr{O}_{\mathbb{K}}^{\times} \xrightarrow{\sim} \mathscr{O}_{\mathbb{L}}^{\times}.$$

□

8. From QSM to field isomorphism: additive structure

We have already shown that isomorphism of QSM-systems of two number fields $\mathbb{K}$ and $\mathbb{L}$ implies that the number fields are arithmetically equivalent. It then follows that it gives a *residual equivalence*, i.e., it also induces a bijection of prime ideals that gives an isomorphism between residue fields (**[26]**, Chapter VI, (2.1)). However, this argument is only based on the fact that the cardinalities of these (finite) fields are the same. We now show, using Galois cohomology, that all such residual isomorphisms are in fact naturally induced from the given map φ.

8.1. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields whose QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ are isomorphic. Let $\varphi \,:\, J_{\mathbb{K}}^{+} \xrightarrow{\sim} J_{\mathbb{L}}^{+}$ denote the induced isomorphism of semigroups of ideals. Let $\mathfrak{p}$ denote a prime ideal of $\mathbb{K}$. Set $\mathfrak{p}' := \varphi(\mathfrak{p})$. The map φ induces an isomorphism of additive groups of the corresponding residue fields*

$$\varphi \,:\, (\overline{\mathbb{K}}_{\mathfrak{p}}, +) \xrightarrow{\sim} (\overline{\mathbb{L}}_{\mathfrak{p}'}, +).$$

Proof. Let $\mathbb{N}_{\mathfrak{p}}^{\mathrm{ab}}$ denote the maximal abelian extension of $\mathbb{K}$ in which $\mathfrak{p}$ is unramified, and pick any prime $\mathfrak{P}$ above $\mathfrak{p}$ in $\mathbb{N}_{\mathfrak{p}}^{\mathrm{ab}}$. Observe that $\varphi(\mathbb{N}_{\mathfrak{p}}^{\mathrm{ab}})$ is the maximal abelian extension of $\mathbb{L}$ in which $\varphi(\mathfrak{p})$ is unramified. Also, φ induces a natural isomorphism of the decomposition group of $\mathfrak{P}$ and any prime $\mathfrak{P}'$ above $\mathfrak{p}'$ in $\varphi(\mathbb{N}_{\mathfrak{p}}^{\mathrm{ab}})$. Since $\mathbb{N}_{\mathfrak{p}}^{\mathrm{ab}} \,/\, \mathbb{K}$ and $\varphi(\mathbb{N}_{\mathfrak{p}}^{\mathrm{ab}}) / \,\mathbb{L}$ are abelian, these decomposition groups are independent of the choice of $\mathfrak{P}$ and $\mathfrak{P}'$ and will be denoted by $D_{\mathfrak{p}}^{\mathrm{ab}}$ and $D_{\mathfrak{p}'}^{\mathrm{ab}}$, respectively. Now also observe that since $\mathbb{K}$ has abelian extensions of arbitrary high residue field index at $\mathfrak{p}$, we have

isomorphisms $D_{\mathfrak{p}}^{\mathrm{ab}} \cong \mathrm{Gal}(\overline{\mathbb{K}}_{\mathfrak{p}}^{\mathrm{ac}} / \mathbb{K}_{\mathfrak{p}})$ where $\overline{\mathbb{K}}_{\mathfrak{p}}$ is the residue field of $\mathfrak{p}$ in $\mathbb{K}$ and $\overline{\mathbb{K}}_{\mathfrak{p}}^{\mathrm{ac}}$ is an algebraic closure. As such, we let the groups $D_{\mathfrak{p}}^{\mathrm{ab}}$ and $D_{\mathfrak{p}'}^{\mathrm{ab}}$ act trivially on the module $\mathbb{Z}/p^n$. After taking Galois cohomology, we find that φ induces an isomorphism

$$\varphi \,:\, H^1(D_{\mathfrak{p}}^{\mathrm{ab}}, \mathbb{Z}/p^n) \xrightarrow{\sim} H^1(D_{\mathfrak{p}'}^{\mathrm{ab}}, \mathbb{Z}/p^n).$$

However,

$$H^1(D_{\mathfrak{p}}^{\mathrm{ab}}, \mathbb{Z}/p^n) = \left(W(\overline{\mathbb{K}}_{\mathfrak{p}})/\langle V^n W(\overline{\mathbb{K}}_{\mathfrak{p}}), \wp W(\overline{\mathbb{K}}_{\mathfrak{p}})\rangle, +\right)$$

as abelian groups, where W is the Witt vectors, V is *Verschiebung*, and as usual, $\wp = F - 1$ where F is Frobenius. Taking limits over all n, we hence find that φ induces an isomorphism of abelian groups

$$\varphi \,:\, \left(W(\overline{\mathbb{K}}_{\mathfrak{p}})/\wp W(\overline{\mathbb{K}}_{\mathfrak{p}}), +\right) \xrightarrow{\sim} \left(W(\overline{\mathbb{L}}_{\mathfrak{p}'})/\wp W(\overline{\mathbb{L}}'_{\mathfrak{p}}), +\right).$$

We observe that $\wp W \subseteq (p)$, and by taking the above isomorphism modulo p, we find

$$\varphi \,:\, \left(\overline{\mathbb{K}}_{\mathfrak{p}}, +\right) \xrightarrow{\sim} \left(\overline{\mathbb{L}}_{\mathfrak{p}'}, +\right).$$

□

8.2. Theorem. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields whose QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ are isomorphic. Then $\mathbb{K}$ and $\mathbb{L}$ are isomorphic as fields.*

Proof. Follows immediately from the fact that the map φ induces an isomorphism of multiplicative semigroups of non-zero integers (Proposition 7.12), so it can be extended to a multiplicative isomorphism of $(\mathbb{K}^*, \times)$ with $(\mathbb{L}^*, \times)$. Then, by defining $\varphi(0) = 0$, the result follows, since we have shown that the same map φ induces an isomorphism of additive groups of all residue fields (Proposition 8.1). □

9. Addendum: recovering the multiplicative structure via cohomology

We give an independent cohomological proof of the fact that φ is residually multiplicative, which is not needed in the main argument, and anyhow follows from Proposition 7.12, but which we include since it provides a nice parallel to the additive theory from the previous section. We do remark that, given a "good" matching of ideals, a combination of Propositions 8.1 and 9.2 (below) with a statement that φ induces a natural bijection between the *sets* $\mathbb{K}$ and $\mathbb{L}$ would suffice to prove the main theorem. However, it does not seem to be so easy to find such a bijection (this is essentially done in Proposition 7.12, but immediately in combination with a multiplicative structure).

9.1. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields whose QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ are isomorphic. Let $\varphi \,:\, J_{\mathbb{K}}^+ \xrightarrow{\sim} J_{\mathbb{L}}^+$ denote the induced isomorphism of semigroups of ideals. Let $\mathfrak{p}$ denote a prime ideal of $\mathbb{K}$ above the rational prime p, with ramification index $f = f(\mathfrak{p}\,|p)$ in $\mathbb{K}\,/\,\mathbb{Q}$. Set $\mathfrak{p}' := \varphi(\mathfrak{p})$. The map φ induces an isomorphism of the following quotients of the multiplicative groups of the corresponding completions*

$$\varphi \,:\, \left(\hat{\mathbb{K}}_{\mathfrak{p}}^{\times} / \left(\hat{\mathbb{K}}_{\mathfrak{p}}^{\times}\right)^{p^f-1}, \times\right) \xrightarrow{\sim} \left(\hat{\mathbb{L}}_{\mathfrak{p}'}^{\times} / \left(\hat{\mathbb{L}}_{\mathfrak{p}'}^{\times}\right)^{p^f-1}, \times\right).$$

Proof. Let $D_\mathfrak{p}$ denote the decomposition group of a prime above $\mathfrak{p}$ in $\mathbb{N}_\mathfrak{p} \,/\, \mathbb{K}$, where $\mathbb{N}_\mathfrak{p}$ is the maximal (not necessarily abelian) extension in which $\mathfrak{p}$ is unramified. Then $D_\mathfrak{p} = \mathrm{Gal}(\hat{\mathbb{K}}_\mathfrak{p}^{\mathrm{ac}}/\hat{\mathbb{K}}_\mathfrak{p})$ is the absolute Galois group of the $\mathfrak{p}$-adic completion of K. Recall Kummer theory: for any integer, let $\boldsymbol{\mu}_n$ denote the n-th roots of unity, then

$$H^1(D_\mathfrak{p}, \boldsymbol{\mu}_n) = \hat{\mathbb{K}}_\mathfrak{p}^\times \Big/ \left(\hat{\mathbb{K}}_\mathfrak{p}^\times\right)^n,$$

where we let $D_\mathfrak{p}$ act on $\boldsymbol{\mu}_n$ like the absolute Galois group of $\hat{\mathbb{K}}_\mathfrak{p}$. Recall that if $f = f(\mathfrak{p}\,|p)$ is the ramification index of p in $\mathbb{K}$, then the (p^f-1)-th roots of unity belong to $\hat{\mathbb{K}}_\mathfrak{p}$. Hence the action of $D_\mathfrak{p}$ on $\boldsymbol{\mu}_{p^f-1}$ is trivial, so we find

$$H^1(D_\mathfrak{p}, \boldsymbol{\mu}_{p^f-1}) = \mathrm{Hom}(D_\mathfrak{p}, \boldsymbol{\mu}_{p^f-1}) = \mathrm{Hom}(D_\mathfrak{p}^{\mathrm{ab}}, \boldsymbol{\mu}_{p^f-1}).$$

(The crucial point for us is the last equality, which is obvious since $\boldsymbol{\mu}_n$ is abelian, since it allows us to switch from the absolute local Galois group, about which we have no information, to the abelianized group, which is encoded in our system. But observe that if the action of $D_\mathfrak{p}$ on $\boldsymbol{\mu}_n$ is not trivial, then the group cohomology does not need to factor over the abelianization.) Hence for $n = p^f - 1$, the canonical isomorphism $\varphi : D_\mathfrak{p}^{\mathrm{ab}} \xrightarrow{\sim} D_{\mathfrak{p}'}^{\mathrm{ab}}$ induces the desired isomorphism. □

9.2. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields whose QSM-systems $(A_\mathbb{K}, \sigma_\mathbb{K})$ and $(A_\mathbb{L}, \sigma_\mathbb{L})$ are isomorphic. Let $\varphi : J_\mathbb{K}^+ \xrightarrow{\sim} J_\mathbb{L}^+$ denote the induced isomorphism of semigroups of ideals. Let $\mathfrak{p}$ denote a prime ideal of $\mathbb{K}$. Set $\mathfrak{p}' := \varphi(\mathfrak{p})$. The map φ induces an isomorphism of multiplicative groups of the corresponding residue fields*

$$\varphi : (\overline{\mathbb{K}}_\mathfrak{p}^\times, \times) \xrightarrow{\sim} (\overline{\mathbb{L}}_{\mathfrak{p}'}^\times, \times).$$

Proof. We only have to observe that if $x \in \hat{\mathbb{L}}^\times$ is a (p^f-1)-th power, then $x = 1 \bmod \mathfrak{p}$. Hence the maps from the previous proposition indeed reduces modulo $\mathfrak{p}$ to an isomorphism of multiplicative groups. □

Part B. L-SERIES AND QSM-ISOMORPHISM

Let χ denote a character in the Pontrjagin dual of $G_\mathbb{K}^{\mathrm{ab}}$. We set

$$L_\mathbb{K}(\chi, s) := \sum_{\mathfrak{n} \in J_\mathbb{K}^+} \frac{\chi(\theta_\mathbb{K}(\mathfrak{n}))}{N_\mathbb{K}(\mathfrak{n})^s},$$

where it is always understood that we set $\chi(\theta_\mathbb{K}(\mathfrak{n})) = 0$ if $\mathfrak{n}$ is not coprime to the conductor $\mathfrak{f}_\chi$ of χ. This is also the Artin L-series for χ considered as a representation of the Galois group of the finite extension $\mathbb{K}_\chi \,/\, \mathbb{K}$ through which χ factors injectively (**[36]**, VII.10.6).

10. Matching L-series via QSM-isomorphism

We start by proving that (ii) implies (iii) in Theorem 2.

10.1. Proposition. *An isomorphism $\varphi : (A_\mathbb{K}, \sigma_\mathbb{K}) \to (A_\mathbb{L}, \sigma_\mathbb{L})$ induces an identification of L-series with characters from $\widehat{G}_\mathbb{K}^{\mathrm{ab}}$.*

Proof. By Proposition 7.6, we have an isomorphism $G_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} G_{\mathbb{L}}^{\mathrm{ab}}$, hence by Pontrjagin duality, a identification of characters

$$\psi \,:\, \widehat{G}_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} \widehat{G}_{\mathbb{L}}^{\mathrm{ab}}.$$

Also, the isomorphism of QSM-systems implies the compatibility of this isomorphism with the action of ideals by the respective Artin maps, which translates in the dual group to

$$\chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) = \psi(\chi)(\vartheta_{\mathbb{L}}(\varphi(\mathfrak{n}))$$

for all $\chi \in \widehat{G}_{\mathbb{K}}^{\mathrm{ab}}, \mathfrak{n} \in J_{\mathbb{K}}^{+}$. In particular, the conductors of χ and $\psi(\chi)$ match. By the intertwining of time evolution, we have compatibility with norms

$$N_{\mathbb{K}}(\mathfrak{n}) = N_{\mathbb{L}}(\varphi(\mathfrak{n}))$$

for all $\mathfrak{n} \in J_{\mathbb{K}}^{+}$. Hence we can compute

$$L_{\mathbb{K}}(\chi, s) = \sum_{\mathfrak{n} \in J_{\mathbb{K}}^{+}} \frac{\chi(\vartheta_{\mathbb{K}}(\mathfrak{n}))}{N_{\mathbb{K}}(\mathfrak{n})^{s}} = \sum_{\varphi(\mathfrak{n}) \in J_{\mathbb{L}}^{+}} \frac{\psi(\chi)(\vartheta_{\mathbb{L}}(\varphi(\mathfrak{n}))}{N_{\mathbb{L}}(\varphi(\mathfrak{n}))^{s}} = L_{\mathbb{L}}(\psi(\chi), s).$$

□

10.2. Remark. The above result can also be seen as a manifestation of the matching of KMS_β states. Namely, our isomorphism of QSM-systems gives $\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$ (Proposition 4.1), and an isomorphism of character groups ψ as in the previous proof. Now Lemma 1.5 implies that pullback is an isomorphism of KMS_β-states. Now for $\beta > 1$, such a state $\omega_{\gamma,\beta}^{\mathbb{L}}$ on $A_{\mathbb{L}}$ (corresponding to $\gamma \in G_{\mathbb{L}}^{\mathrm{ab}}$) is pulled back to a similar state

$$\omega_{\gamma,\beta}^{\mathbb{L}}(\varphi(f)) = \omega_{\widetilde{\gamma},\beta}^{\mathbb{K}}(f),$$

for some $\widetilde{\gamma} \in G_{\mathbb{K}}^{\mathrm{ab}}$ and every $f \in A_{\mathbb{K}}$. We can choose in particular $f = \chi \in \widehat{G}_{\mathbb{K}}^{\mathrm{ab}}$ (extended to zero ideals not coprime to the conductor of χ) and then the above identity becomes

$$\frac{1}{\zeta_{\mathbb{L}}(s)} \cdot \psi(\chi)(\gamma) \cdot L_{\mathbb{L}}(\psi(\chi), s) = \frac{1}{\zeta_{\mathbb{K}}(s)} \cdot \chi(\widetilde{\gamma}) \cdot L_{\mathbb{K}}(\chi, s).$$

If we now compare the constant coefficients and use arithmetic equivalence, we find $\psi(\chi)(\gamma) = \chi(\widetilde{\gamma})$, and so finally the identity of these particular KMS-states indeed reads

$$L_{\mathbb{L}}(\psi(\chi), s) = L_{\mathbb{K}}(\chi, s).$$

11. QSM-isomorphism from matching L-series: isomorphism of character groups

Conversely, we now show that (iii) ⇒ (ii) in Theorem 2, namely the identity of the L-functions implies the existence of an isomorphism of the quantum statistical mechanical systems.

We start by proving that if we have a bijection of characters, the matching of L-series automatically implies that this bijection is an isomorphism of groups.

11.1. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields. Suppose ψ is a set-theoretic bijection*

$$\psi \,:\, \widehat{G}_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} \widehat{G}_{\mathbb{L}}^{\mathrm{ab}}$$

that induces an identity of the respective L-functions

$$L_{\mathbb{K}}(\chi,\beta) = L_{\mathbb{L}}(\psi(\chi),\beta).$$

Then ψ is a group *isomorphism.*

Proof. If

$$\sum \frac{a_n}{n^\beta} \text{ and } \sum \frac{b_n}{n^\beta}$$

are two L-series, we mean by their *fusion* the L-series

$$\sum \frac{a_n b_n}{n^\beta}.$$

The first ingredient of our proof is the trivial observation that if two L-series are equal, so are their fusions. The second ingredient is that the only L-series with a pole at $\beta = 1$ is the one with trivial character, i.e., the zeta function (**[36]**, VII.8.5).

If we use the trivial character $\chi = 1$ in the hypothesis, we find that

$$L_{\mathbb{L}}(\psi(1),\beta) = \zeta_{\mathbb{K}}(\beta),$$

which has a pole at $\beta = 1$. Hence $\psi(1) = 1$.

Next, let χ be a character. The fusion of the L-series $L_{\mathbb{L}}(\psi(\chi),\beta)$ and $L_{\mathbb{L}}(\psi(\chi^{-1}),\beta)$ is by hypothesis equal to

$$L_{\mathbb{K}}(\chi\cdot\chi^{-1},\beta) = \zeta_{\mathbb{K}}(\beta).$$

Since this has a pole at $\beta = 1$, we find that $\psi(\chi)\cdot\psi(\chi^{-1})$ is the trivial character, i.e.,

$$\psi(\chi^{-1}) = \psi(\chi)^{-1}.$$

Now let χ and χ' be two characters. We consider the L-series

$$L_{\mathbb{L}}(\psi(\chi\chi')\psi(\chi)^{-1}\psi(\chi')^{-1},\beta),$$

which is the fusion of the L-series corresponding to the three characters

$$\psi(\chi\chi'), \psi(\chi)^{-1} = \psi(\chi^{-1}) \text{ and } \psi(\chi')^{-1} = \psi(\chi'^{-1}).$$

Hence it is equal to the $L_{\mathbb{K}}(1,\beta) = \zeta_{\mathbb{K}}(\beta)$. Since this has a pole at $s = 1$, we conclude that

$$\psi(\chi\chi') = \psi(\chi)\psi(\chi').$$

We conclude from this that ψ is a group isomorphism. □

12. QSM-isomorphism from matching L-series: compatible isomorphism of ideals

12.1. Proposition. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields. Suppose ψ is a* continuous *bijection*

$$\psi : \widehat{G}^{\text{ab}}_{\mathbb{K}} \xrightarrow{\sim} \widehat{G}^{\text{ab}}_{\mathbb{L}}$$

that induces an identity of the respective L-functions

$$L_{\mathbb{K}}(\chi,\beta) = L_{\mathbb{L}}(\psi(\chi),\beta).$$

Then there exists a semigroup isomorphism $\Psi : J_{\mathbb{K}}^+ \to J_{\mathbb{L}}^+$, which is compatible with the Artin reciprocity map under ψ in the sense that

$$\vartheta_{\mathbb{L}} \circ \Psi = (\psi^{-1})^* \circ \vartheta_{\mathbb{K}}.$$

Proof. The previous proposition shows that ψ is a group isomorphism, so $\psi(1) = 1$, which means that the zeta functions (L-series for the trivial character) match on both sides:

$$\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s).$$

This is arithmetic equivalence, and it shows in particular that there is a bijection between the sets of primes of $\mathbb{K}$ and $\mathbb{L}$ above a given rational prime p and with a given inertia degree f. We need to match these primes in such a way that they are compatible with Artin reciprocity. The naive way is to map a prime $\mathfrak{p}$ of $\mathbb{K}$ to a prime $\mathfrak{q}$ above the same p, with the same inertia degree, and such that

$$\vartheta_{\mathbb{L}}(\mathfrak{q}) = (\psi^{-1})^* \vartheta_{\mathbb{K}}(\mathfrak{p}) \tag{11}$$

The main point is to show that it is always possible to find such $\mathfrak{q}$, and to show that one may perform this in a bijective way between primes. We prove this by using a combination of L-series as counting function for the number of such ideals $\mathfrak{q}$. Observe that the identity (11) is equivalent to the dual statement that

$$\chi(\vartheta_{\mathbb{K}}(\mathfrak{p})) = \psi(\chi)(\vartheta_{\mathbb{L}}(\mathfrak{q}))$$

for all characters $\chi \in \widehat{G}_{\mathbb{K}}^{\mathrm{ab}}$, and this provides the link to L-series.

The identification of L-series means that for any character χ, we have

$$\sum_{\mathfrak{n} \in J_{\mathbb{K}}^+} \frac{\chi(\vartheta_{\mathbb{K}}(\mathfrak{n}))}{N_{\mathbb{K}}(\mathfrak{n})^s} = \sum_{\mathfrak{m} \in J_{\mathbb{L}}^+} \frac{\psi(\chi)(\vartheta_{\mathbb{L}}(\mathfrak{m}))}{N_{\mathbb{L}}(\mathfrak{m})^s}. \tag{12}$$

We fix an integer n and consider the norm-n part of this identity:

$$\sum_{\substack{\mathfrak{n} \in J_{\mathbb{K}}^+ \\ N_{\mathbb{K}}(\mathfrak{n}) = n}} \chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) = \sum_{\substack{\mathfrak{m} \in J_{\mathbb{L}}^+ \\ N_{\mathbb{L}}(\mathfrak{m}) = n}} \psi(\chi)(\vartheta_{\mathbb{L}}(\mathfrak{m})). \tag{13}$$

In this notation, remember that we have set χ equal to zero on ideals not coprime to its conductor.

Recall that the character group $\widehat{G}_{\mathbb{K}}^{\mathrm{ab}}$ is a compact topological group, and thus carries a (normalized) Haar measure $d\chi$. We consider the subgroup $\widehat{G}_{\mathbb{K},n}^{\mathrm{ab}}$ of characters whose conductor is coprime to a given integer n (that this is a subgroup follows from $\mathfrak{f}_{\chi\chi'} = \mathfrak{f}_\chi + \mathfrak{f}_{\chi'}$). We integrate the identity (13) against this measure, times the function $\chi(\gamma^{-1})$ for a fixed element $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$ — interchanging the order of integration and summation by absolute convergence, we find

$$\sum_{\substack{\mathfrak{n} \in J_{\mathbb{K}}^+ \\ N_{\mathbb{K}}(\mathfrak{n}) = n}} \left(\int_{\widehat{G}_{\mathbb{K},n}^{\mathrm{ab}}} \chi(\gamma^{-1}) \chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) d\chi \right) = \sum_{\substack{\mathfrak{m} \in J_{\mathbb{L}}^+ \\ N_{\mathbb{L}}(\mathfrak{m}) = n}} \left(\int_{\widehat{G}_{\mathbb{K},n}^{\mathrm{ab}}} \chi(\gamma^{-1}) \psi(\chi)(\vartheta_{\mathbb{L}}(\mathfrak{m})) d\chi \right). \tag{14}$$

Recall the following fact:

12.2. Lemma. *On any compact topological group $\mathscr{G}$ with normalized Haar measure $d\mu$, and normalized Haar measure $d\alpha$ on its dual $\widehat{G}$, it holds true for any $g \in \mathscr{G}$ that*

$$\int_{\widehat{G}} \alpha(g^{-1})\alpha d\alpha = \delta(-, g)$$

is the characteristic function of g. □

Let us introduce the following set of ideals for $n \in \mathbb{Z}_{\geq 1}$ and $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$:

$$B_n(\gamma) = \{\mathfrak{n} \in J_{\mathbb{K}}^{+} \,:\, N_{\mathbb{K}}(\mathfrak{n}) = n \text{ and } \vartheta_{\mathbb{K}}(\mathfrak{n}) = \gamma\}$$

and denote the cardinality of this set by

$$b_n(\gamma) := \#B_n(\gamma),$$

(or $b_{\mathbb{K},n}(\gamma)$ if we want to indicate the dependence on the ground field $\mathbb{K}$). By the Lemma, the value of the left hand side of Equation (14) is

$$\mathrm{LHS}(14) = \frac{\mathrm{vol}(G_{\mathbb{K},n}^{\mathrm{ab}})}{\mathrm{vol}(G_{\mathbb{K}}^{\mathrm{ab}})} \cdot b_{\mathbb{K},n}(\gamma).$$

We now perform a base change in the bracketed integral on the right hand side of (14), using the homeomorphism

$$\psi \,:\, \widehat{G}_{\mathbb{K}}^{\mathrm{ab}} \to \widehat{G}_{\mathbb{L}}^{\mathrm{ab}},$$

which we can do since ψ preserves the subgroups indexed by n:

$$\psi(G_{\mathbb{K},n}^{\mathrm{ab}}) = G_{\mathbb{L},n}^{\mathrm{ab}}.$$

Indeed, if $\mathfrak{f}_\chi$ is not coprime to n, $L_{\mathbb{K}}(\chi, s)$ has a missing Euler factor at a prime number p dividing n. Hence, by the equality of L-series, also $L_{\mathbb{L}}(\psi(\chi), s)$ has such a missing Euler factor, so $f_{\psi(\chi)}$ is not coprime to p (hence n).

Also recall that a continuous group homomorphism of compact topological groups is automatically an isometry for normalized Haar measure (this follows from the uniqueness of Haar measure up to scaling). This applies in our case, since from Proposition 11.1 above, we know that ψ is a group isomorphism. Observe that ψ is also an isometry when restricted to the n-th level group $G_{\mathbb{K},n}^{\mathrm{ab}}$.

Hence the integral at the right hand side of (14) becomes

$$\int_{\widehat{G}_{\mathbb{L},n}^{\mathrm{ab}}} \psi^{-1}(\eta)(\gamma^{-1})\eta(\vartheta_{\mathbb{L}}(\mathfrak{m}))d\eta \tag{15}$$

We now use Lemma 12.2, but on the dual group $\widehat{G}_{\mathbb{L},n}^{\mathrm{ab}}$. For this, we observe that for fixed $\mathfrak{m}$ coprime to $\mathfrak{f}_\eta$,

$$\Xi_{\mathfrak{m}} \,:\, \eta \mapsto \psi^{-1}(\eta)(\gamma^{-1})\eta(\vartheta_{\mathbb{L}}(\mathfrak{m}))$$

is a character on $\widehat{G}_{\mathbb{L},n}^{\mathrm{ab}}$. The lemma implies that

$$\int_{\widehat{G}_{\mathbb{L},n}^{\mathrm{ab}}} \psi^{-1}(\eta)(\gamma^{-1})\eta(b)d\eta = \begin{cases} \mathrm{vol}(G_{\mathbb{L},n}^{\mathrm{ab}})/\mathrm{vol}(G_{\mathbb{L}}^{\mathrm{ab}}) & \text{if } \Xi_{\mathfrak{m}} \equiv 1; \\ 0 & \text{otherwise.} \end{cases}$$

Now $\Xi_{\mathfrak{m}} \equiv 1$ means that

$$\eta(\vartheta_{\mathbb{L}}(\mathfrak{m})) = \psi^{-1}(\eta)(\gamma) \text{ for all } \eta \in G^{\mathrm{ab}}_{\mathbb{L},n}.$$

Plugging everything back in, we find that the right hand side of Equation (14) becomes

$$\begin{aligned} \text{RHS}(14) &= \frac{\mathrm{vol}(G^{\mathrm{ab}}_{\mathbb{L},n})}{\mathrm{vol}(G^{\mathrm{ab}}_{\mathbb{L}})} \cdot \#\{\mathfrak{m} \in J^+_{\mathbb{L}} \text{ with } N_{\mathbb{L}}(\mathfrak{m}) = n \text{ and } \vartheta_{\mathbb{L}}(\mathfrak{m}) = (\psi^{-1})^*(\gamma)\} \\ &= \frac{\mathrm{vol}(G^{\mathrm{ab}}_{\mathbb{L},n})}{\mathrm{vol}(G^{\mathrm{ab}}_{\mathbb{L}})} \cdot b_{\mathbb{L},n}((\psi^{-1})^*(\gamma)). \end{aligned}$$

Since ψ is an isometry, it preserves volume, so we conclude that for all $\gamma \in G^{\mathrm{ab}}_{\mathbb{K}}$ and for all integers n,

$$b_{\mathbb{K},n}(\gamma) = b_{\mathbb{L},n}((\psi^{-1})^*(\gamma)). \tag{16}$$

We now switch to the subsets consisting of prime ideals. For this purpose, we introduce the subset $C_n(\gamma)$ of $B_n(\gamma)$:

$$C_n(\gamma) = \{\mathfrak{n} \in J^+_{\mathbb{K}} \,:\, \mathfrak{n} \text{ prime ideal}, N_{\mathbb{K}}(\mathfrak{n}) = n \text{ and } \vartheta_{\mathbb{K}}(\mathfrak{n}) = \gamma\},$$

and denote the cardinality of this set by $c_n(\gamma) = \#C_n(\gamma)$.

Then we have a decomposition

$$B_n(\gamma) = C_n(\gamma) \coprod_{\substack{n_1 | n \\ n_1 \neq 1,n}} \coprod_{\substack{\mathfrak{n}_1 \\ N(\mathfrak{n}_1) = n_1}} B_{\frac{n}{n_1}}\left(\frac{\gamma}{\vartheta_{\mathbb{K}}(\mathfrak{n}_1)}\right) / S_2.$$

Let us explain this notation. The set $B_n(\gamma)$ is the disjoint union of $C_n(\gamma)$ together with all "old" sets that correspond to choosing a non-trivial factorisation $n = n_1 \cdot n_2$, choosing $\mathfrak{n}_1$ with norm n_1 (a non-trivial divisor of n) and such that $\mathfrak{n}_1\,\mathfrak{n}_2$ has image by reciprocity γ — or, what is the same, $\mathfrak{n}_2$ has norm n/n_1 and image by reciprocity $\gamma/\vartheta_{\mathbb{K}}(\mathfrak{n}_1)$. Now all these choices are disjoint, up to permuting the factors in the factorisation $n = n_1 \cdot n_2$, which we indicate in the notation by dividing by the symmetric group S_2. The upshot is that the functions $c_n(\gamma)$ are a univeral expression in terms of $b_n(\gamma)$ and an enumeration of ideals of given norm.

Hence we conclude also that

$$c_{\mathbb{K},n}(\gamma) = c_{\mathbb{L},n}((\psi^{-1})^*(\gamma)).$$

This says exactly that the number of prime ideals with a given image under reciprocity and given norm in $\mathbb{K}$ is the same as the number of prime ideals of $\mathbb{L}$ with the same norm and compatible Artin action. We define a map of semigroups of ideals

$$\Psi \,:\, J^+_{\mathbb{K}} \to J^+_{\mathbb{L}}$$

on generators, by sending a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ to any prime ideal $\mathfrak{q}$ of $\mathbb{L}$ with the same norm $N_{\mathbb{K}}(\mathfrak{q}) = N_{\mathbb{K}}(\mathfrak{p}) = p^{f(\mathfrak{p}\,|\,\mathbb{K})}$ and with $\vartheta_{\mathbb{L}}(\mathfrak{q}) = (\psi^{-1})^*\vartheta_{\mathbb{K}}(\mathfrak{p})$. The above count shows exactly that the ambiguity in choosing such $\mathfrak{q}$ is the same as the ambiguity in choosing a prime ideal $\mathfrak{p}'$ of $\mathbb{K}$ with the same norm (recall that by arithmetic equivalence, the inertia degrees match) and the same image by the Artin map as $\mathfrak{p}$. This shows that the map can be made bijective on prime ideals, hence on all ideals, and compatible with norms (inertia degrees) and the Artin map. □

12.3. Remark. It seems there are many possible choices for the map Ψ to be compatible with Artin reciprocity and the given isomorphism of character groups. This is not so strange in the light of the fact that we cannot expect to construct a *unique* isomorphism of QSM-systems from the matching of L-series; for example, there are automorphisms of the QSM-system that induce the identity on L-series, just like there are automorphisms of a number field that induce equalities of L-series, cf. the discussion already in Artin (**[1]**, after *Satz 5*) about relations between L-series on number fields different from $\mathbb{Q}$.

13. QSM-isomorphism from matching L-series: homeomorphism on $X_{\mathbb{K}}$

We now proceed to show that ψ also induced a natural map from the whole abelian part $C(X_{\mathbb{K}})$, not just on the part $\psi : C(G_{\mathbb{K}}^{ab}) \xrightarrow{\sim} C(G_{\mathbb{L}}^{ab})$ where it is automatically defined (by continuity of ψ). We check this on "finite" parts of these algebras that exhaust the whole algebra, cf. also **[31]**, proof of Thm. 2.1 (or section 3 of **[13]** for a description in terms of $\mathbb{K}$-lattices).

13.1. Lemma. *The map ψ extends to an algebra isomorphism*

$$\psi : C(G_{\mathbb{K}}^{ab} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}}) \to C(G_{\mathbb{L}}^{ab} \times_{\hat{\mathcal{O}}_{\mathbb{L}}^*} \hat{\mathcal{O}}_{\mathbb{L}}).$$

Proof. Recall that the map $\psi : \widehat{G}_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} \widehat{G}_{\mathbb{L}}^{\mathrm{ab}}$ induces by duality a group isomorphism

$$(\psi^{-1})^* : G_{\mathbb{K}}^{\mathrm{ab}} \xrightarrow{\sim} G_{\mathbb{L}}^{\mathrm{ab}},$$

and let $\Psi : J_{\mathbb{K}}^+ \xrightarrow{\sim} J_{\mathbb{L}}^+$ denote the compatible isomorphism of semigroups of ideals introduced in the previous section.

We use the terminology from **[31]**. Let $\mu_{\mathbb{K}}$ denote the measure on

$$X_{\mathbb{K}} = G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}}$$

given as the products of normalized Haar measures on $G_{\mathbb{K}}^{\mathrm{ab}}$ and on every factor $\hat{\mathcal{O}}_{\mathbb{K},\mathfrak{p}}$ of $\hat{\mathcal{O}}_{\mathbb{K}}$ (so that $\hat{\mathcal{O}}_{\mathbb{K},\mathfrak{p}}^*$ has measure $1 - 1/N_{\mathbb{K}}(\mathfrak{p})$). Take a *finite* set of primes $B \subseteq J_{\mathbb{K}}^+$ and consider the space

$$X_{\mathbb{K},B} := G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K},B},$$

where $\hat{\mathcal{O}}_{\mathbb{K},B} = \prod_{\mathfrak{p}\in B} \hat{\mathcal{O}}_{\mathbb{K},\mathfrak{p}}$. Then

$$X_{\mathbb{K}} = \varinjlim_{B} X_{\mathbb{K},B}.$$

Let $J_{\mathbb{K},B}^+$ denote the subsemigroup of $J_{\mathbb{K}}^+$ generated by B. Let us also introduce the non-standard notation

$$X_{\mathbb{K},B}^* := G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K},B}^*.$$

As a group, it is isomorphic to

$$X_{\mathbb{K},B}^* \xrightarrow{\sim} G_{\mathbb{K}}^{\mathrm{ab}}/\vartheta_{\mathbb{K}}(\hat{\mathcal{O}}_{\mathbb{K},B^c}^*),$$

where $\vartheta_{\mathbb{K}} : \mathbf{A}_{\mathbb{K}}^* \to G_{\mathbb{K}}^{\mathrm{ab}}$ is the Artin map at the level of ideles, and B^c is the complement of B in the set of prime ideals of $\mathbb{K}$.

We can decompose

$$X_{\mathbb{K},B} = X_{\mathbb{K},B}^1 \coprod X_{\mathbb{K},B}^2$$

with

$$X^1_{\mathbb{K},B} := \bigcup_{\mathfrak{n}\in J^+_{\mathbb{K},B}} \mathfrak{n} * X^*_{\mathbb{K},B} \text{ and } X^2_{\mathbb{K},B} := \bigcup_{\mathfrak{p}\in B} Y_{\mathbb{K},\mathfrak{p}},$$

where

$$Y_{\mathbb{K},\mathfrak{p}} = \{(\gamma,\rho) \in X_{\mathbb{K},B} \,:\, \rho_{\mathfrak{p}} = 0\}.$$

As usual, $\mathfrak{n}*-$ is the action by Artin reciprocity. Observe that $X^2_{\mathbb{K},B}$ is a set of $\mu_{\mathbb{K}}$-measure zero. By total disconnectedness, the algebra $C(X_{\mathbb{K},B})$ is generated by the characteristic functions of clopen sets. Now $X_{\mathbb{K},B}$ has no open sets of Haar measure zero. Indeed, a $\mathfrak{p}$-adic ring of integers $\hat{\mathscr{O}}_{\mathfrak{p}}$ does not have non-empty open sets U of measure zero, since U contains a ball of sufficiently small radius around any point in it, and this will have Haar measure the $\mathfrak{p}$-adic absolute value of the radius; the same argument applies to $G^{\mathrm{ab}}_{\mathbb{K}}$, by considering it as the idele class group modulo connected component of the identity and using the idele norm. It follows that $X^1_{\mathbb{K},B}$ is dense in $X_{\mathbb{K},B}$, as the complement cannot contain any open set. Define the functions $f_{\mathfrak{n},\chi}$ by

$$f_{\mathfrak{n},\chi} \,:\, X_{\mathbb{K},B} \to \mathbb{C} \,:\, x \mapsto \begin{cases} \chi(\vartheta_{\mathbb{K}}(\mathfrak{n})^{-1}x) & \text{if } x \in \mathfrak{n} * X^*_{\mathbb{K},B}; \\ 0 & \text{otherwise,} \end{cases}$$

for χ running through the characters of conductor $\mathfrak{f}_\chi \in J^+_{\mathbb{K},B}$, and $\mathfrak{n}$ any ideal. In Section 2 of **[31]**, it is proven that the span of the functions $f_{\mathfrak{n},\chi}$ contains the characteristic functions of the clopen subsets of $X^1_{\mathbb{K},B}$, hence by density they span $C(X_{\mathbb{K},B})$.

We can then define a map

$$\psi_B \,:\, C(X_{\mathbb{K},B}) \to C(X_{\mathbb{L},\Psi(B)})$$

as the closure of the map given by

$$f_{\mathfrak{n},\chi} \mapsto f_{\Psi(\mathfrak{n}),\psi(\chi)}.$$

The map is a vector space isomorphism by construction, since both ψ and Ψ are bijective.

To see that the map is well-defined, we need to check that the conductors match, i.e., that $\mathfrak{f}_{\psi(\chi)}$ and $\Psi(\mathfrak{f}_\chi)$ have the same prime divisors; but the prime divisors $\mathfrak{q} = \Psi(\mathfrak{p})$ of $\mathfrak{f}_{\psi(\chi)}$ are exactly those $\mathfrak{q}$ for which $\psi(\chi)(\vartheta_{\mathbb{L}}(\mathfrak{q})) = 0$. Since we have by construction that $\psi(\chi)(\vartheta_{\mathbb{L}}(\Psi(\mathfrak{p}))) = \chi(\vartheta_{\mathbb{K}}(\mathfrak{p}))$, this set of $\mathfrak{q}$ is the image under Ψ of the $\mathfrak{p}$ with $\chi(\vartheta_{\mathbb{K}}(\mathfrak{p})) = 0$, viz., dividing the conductor of χ.

By taking direct limits, we arrive at a topological vector space isomorphism

$$\psi = \varinjlim_B \psi_B \,:\, C(X_{\mathbb{K}}) \xrightarrow{\sim} C(X_{\mathbb{L}}).$$

To see that the map ψ is an algebra homomorphism, we need to check it is compatible with multiplication. For this, we observe that where the function $f_{\mathfrak{n},\chi}$ is nonzero, it is given by a pullback. Indeed, for $y \in \Psi(\mathfrak{n}) * X^*_{\mathbb{L},\Psi(B)}$ we can see y as an element of $G^{\mathrm{ab}}_{\mathbb{L}}$, so we can find $x \in \mathfrak{n} * X^*_{\mathbb{L},B}$ with $y = (\psi)^* x$, and we have that

$$\psi_B(f_{\mathfrak{n},\chi})(y) = f_{\Psi(\mathfrak{n}),\psi(\chi)}(y) = \psi(\chi)(\vartheta_{\mathbb{L}}(\Psi(\mathfrak{n}))^{-1}y) = \chi(\vartheta_{\mathbb{K}}(\mathfrak{n})^{-1}x) = (\psi^{-1})^* f_{\mathfrak{n},\chi}(y).$$

By compatibility with conductors, the map is also a pullback on elements where the function is zero. Hence if χ and χ' are two characters in $\widehat{G}^{\mathrm{ab}}_{\mathbb{K}}$, and $\mathfrak{n}, \mathfrak{n}'$ are two ideals in $J^+_{\mathbb{K},B}$ for B sufficiently

large, we find

$$\psi(f_{\mathfrak{n},\chi}\cdot f_{\mathfrak{n}',\chi'})=(\psi^{-1})^*\left(f_{\mathfrak{n},\chi}\cdot f_{\mathfrak{n}',\chi'}\right)=(\psi^{-1})^*\left(f_{\mathfrak{n},\chi}\right)\cdot(\psi^{-1})^*\left(f_{\mathfrak{n}',\chi'}\right)=\psi(f_{\mathfrak{n},\chi})\cdot\psi(f_{\mathfrak{n}',\chi'}),$$

which, after taking linear combinations and closures, implies that ψ is multiplicative.

□

13.2. Remark. Note that we have proven that $X_{\mathbb{K}}$ and $X_{\mathbb{L}}$ are homeomorphic; we didn't prove a this point that the rings of integral adeles $\hat{\mathcal{O}}_{\mathbb{K}}$ and $\hat{\mathcal{O}}_{\mathbb{L}}$ are isomorphic (as rings).

13.3. Remark ($\mathbb{K}$-lattices)**.** Let $\overline{\mathscr{M}}_{\mathbb{K},1}$ denote the space of 1-dimensional $\mathbb{K}$-lattices up to scaling; recall that $C(X_{\mathbb{K}})=C(\overline{\mathscr{M}}_{\mathbb{K},1})$. The preceding proof organizes this space into an inductive system of the spaces $C(\overline{\mathscr{M}}_{\mathbb{K},1,B})$ of functions that depend on the datum ϕ of a $\mathbb{K}$-lattice (Λ,ϕ) only through its projection to $\hat{\mathcal{O}}_B$.

14. QSM-isomorphism from matching L-series: end of proof

14.1. Theorem. *Let $\mathbb{K}$ and $\mathbb{L}$ denote two number fields. Suppose ψ is a homeomorphism*

$$\psi\,:\,\widehat{G}_{\mathbb{K}}^{\mathrm{ab}}\xrightarrow{\sim}\widehat{G}_{\mathbb{L}}^{\mathrm{ab}}$$

that induces an identity of the respective L-functions

$$L_{\mathbb{K}}(\chi,\beta)=L_{\mathbb{L}}(\psi(\chi),\beta).$$

Then there is an isomorphism of QSM-systems $\varphi:(A_{\mathbb{K}},\sigma_{\mathbb{K}})\to(A_{\mathbb{L}},\sigma_{\mathbb{L}})$.

Proof. We assemble all our maps into the C^*-algebra isomorphism

$$\varphi\,:\,A_{\mathbb{K}}=C(X_{\mathbb{K}})\rtimes J_{\mathbb{K}}^{+}\to A_{\mathbb{L}}=C(X_{\mathbb{L}})\rtimes J_{\mathbb{L}}^{+}\,:\,(f,\mathfrak{n})\mapsto(\psi(f),\Psi(\mathfrak{n})).$$

Looking at the construction of the reduced crossed product, we indeed get a C^*-algebra isomorphism, since by construction, the map is compatible with the action of the semigroup on the abelian part.

It remains to verify that this map is indeed a QSM-isomorphism, i.e., that it commutes with time evolution. On the abelian part, there is nothing to verify, since it is stable by time evolution. On the semigroup part, it is a simple consequence of the fact that Ψ preserves norms:

$$N_{\mathbb{L}}(\Psi(\mathfrak{n}))=N_{\mathbb{K}}(\mathfrak{n}).$$

This only needs to be verified on prime ideals $\mathfrak{n}=\mathfrak{p}$, where it is equivalent to

$$f(\mathfrak{p}\,|\,\mathbb{K})=f(\Psi(\mathfrak{p})|\mathbb{L}),$$

which holds true by the construction of Ψ. So finally, on the one hand

$$\sigma_{\mathbb{L},t}(\varphi(\mu_{\mathfrak{n}}))=N_{\mathbb{L}}(\Psi(\mathfrak{n}))^{it}\mu_{\Psi(\mathfrak{n})},$$

and on the other hand,

$$\varphi(\sigma_{\mathbb{K},t}(\mu_{\mathfrak{n}}))=\varphi(N_{\mathbb{K}}(\mathfrak{n})^{it}\mu_n)=N_{\mathbb{K}}(\mathfrak{n})^{it}\mu_{\Psi(\mathfrak{n})}.$$

This finishes the proof that $\sigma_{\mathbb{L},t}\circ\varphi=\varphi\circ\sigma_{\mathbb{K},t}$. □

14.2. Remark. In the particular case of QSM-systems of number fields, we find that an equality of KMS_β states at all inverse temperatures $\beta > 1$ (manifesting themselves here as L-series with characters) implies that the systems are isomorphic. One may wonder in how far a QSM-system is characterized by its generalized equilibrium states in some sense.

14.3. Remark. As quoted in the introduction, in **[16]**, it was shown that an equality of infinitely many Dirichlet series associated to a map between closed Riemannian manifolds is equivalent to this map being an isometry. In the same reference, it is then shown how to use this theorem to define a distance between closed Riemannian manifolds, as infimum over a usual distance between complex functions. With number fields, we are now in a very analogous situation, in that we characterize number fields by an equality of Dirichlet series. One might use this to define a distance on the set of all number fields up to isomorphism. It then remains to investigate whether this (forcedly discrete) distance on a countable set has an interesting completion (much like passing from $\mathbb{Q}$ to $\mathbb{R}$): are there interesting 'limits' of number fields?

15. Proof of Theorem 3

It suffices to show that (iv) implies (iii). Let $\chi \in \widehat{G}^{\mathrm{ab}}_{\mathbb{K}}$, then, by performing a change in summation $\mathfrak{m} = \Psi(\mathfrak{n})$ in the L-series as follows (using that norms are preserved, and Artin maps intertwined):

$$L_{\mathbb{K}}(\chi, s) = \sum_{\mathfrak{n} \in J^+_{\mathbb{K}}} \frac{\chi(\vartheta_{\mathbb{K}}(\mathfrak{n}))}{N_{\mathbb{K}}(\mathfrak{n})^s} = \sum_{\mathfrak{m} \in J^+_{\mathbb{L}}} \frac{(\hat{\psi}^{-1})^*(\chi)(\vartheta_{\mathbb{L}}(\mathfrak{m}))}{N_{\mathbb{L}}(\mathfrak{m})^s} = L_{\mathbb{L}}((\hat{\psi}^{-1})^*(\chi), s). \quad \square$$

15.1. Remark. In Uchida's proof of the function field case of the Neukirch-Uchida theorem (**[48]**), the construction of a multiplicative map of global function fields $(\mathbb{K}^*, \times) \stackrel{\sim}{\to} (\mathbb{L}^*, \times)$ is based on the existence of topological group isomorphisms of the ideles $\Psi : \mathbf{A}^*_{\mathbb{K}} \stackrel{\sim}{\to} \mathbf{A}^*_{\mathbb{L}}$ and of the abelianized Galois groups $\hat{\psi} : G^{\mathrm{ab}}_{\mathbb{K}} \stackrel{\sim}{\to} G^{\mathrm{ab}}_{\mathbb{L}}$ which are compatible with the Artin maps, using that in a function field $\mathbb{K}$, the group $\mathbb{K}^*$ is the kernel of the Artin map (which is not surjective in this case). The conditions that go into this proof are a bit similar to the ones in Theorem 3. Our theorem shows that similar conditions imply the same result for number fields as for function fields, albeit with a rather different proof.

16. Relaxing the conditions on L-series

16.1. One may now wonder whether condition (iii) of the main Theorem 2, can be weakened. For example, is it possible to restrict to characters of fixed type? At least for rational characters of order two (i.e., arising from quadratic extensions by the square root of a rational number), this is not the case, as the following proposition shows.

16.2. Proposition. *Suppose $\mathbb{K}$ and $\mathbb{L}$ are number fields with the same Dedekind zeta function. Then for any quadratic character χ whose conductor is a rational non-square in $\mathbb{K}$ nor $\mathbb{L}$, we have an equality of L-series $L_{\mathbb{K}}(\chi, s) = L_{\mathbb{L}}(\chi, s)$.*

Proof. We have

$$\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s) \tag{17}$$

This says that $\mathbb{K}$ and $\mathbb{L}$ are arithmetically equivalent, which we can express in group theoretical terms by Gaßmann's criterion (**[39]**) as follows: let $\mathbb{N}$ be Galois over $\mathbb{Q}$ containing $\mathbb{K}$ and $\mathbb{L}$; then $\mathrm{Gal}(\mathbb{N}\,/\,\mathbb{K})$ and $\mathrm{Gal}(\mathbb{N}\,/\,\mathbb{L})$ intersect all conjugacy classes in $\mathrm{Gal}(\mathbb{N}\,/\,\mathbb{Q})$ in the same number of elements.

Let $\mathbb{M} = \mathbb{Q}(\sqrt{d})$ for a rational non-square d. It is easy to see from Gaßmann's criterion for arithmetic equivalence that then, the composita $\mathbb{K}\,\mathbb{M}$ and $\mathbb{L}\,\mathbb{M}$ are also arithmetically equivalent (cf. e.g. Uchida **[47]**, Lemma 1): choose $\mathbb{N}$ so it also contains $\mathbb{M}$, and verify that $\mathrm{Gal}(\mathbb{N}\,/\,\mathbb{K}\,\mathbb{M})$ and $\mathrm{Gal}(\mathbb{N}\,/\,\mathbb{L}\,\mathbb{M})$ intersect all conjugacy classes in $\mathrm{Gal}(\mathbb{N}\,/\,\mathbb{Q})$ in the same number of elements. We conclude that the zeta functions of $\mathbb{K}\,\mathbb{M} = \mathbb{K}(\sqrt{d})$ and $\mathbb{L} = \mathbb{L}(\sqrt{d})$ are equal:

$$\zeta_{\mathbb{K}\,\mathbb{M}}(s) = \zeta_{\mathbb{L}\,\mathbb{M}}(s) \tag{18}$$

Let χ be the quadratic character that belongs to d. By Artin factorization, we can write

$$\zeta_{\mathbb{K}\,\mathbb{M}}(s) = \zeta_{\mathbb{K}}(s)\cdot L_{\mathbb{K}}(\chi,s) \text{ and } \zeta_{\mathbb{L}\,\mathbb{M}}(s) = \zeta_{\mathbb{L}}(s)\cdot L_{\mathbb{L}}(\chi,s). \tag{19}$$

We find the conclusion by combining (17), (18) and (19). □

16.3. Remark. We do not know a direct "analytic" proof that equality of zeta functions implies equality of all quadratic twist L-series. As a matter of fact, looked at in a purely analytic way, the result does not appear to be so obvious at all.

16.4. A matching of all L-series implies in particular a matching of all zeta functions of ray class fields. Bart de Smit suggests to investigate whether condition (iii) can be weakened to

(iii)' *There is a bijection of ideals* $\Psi : J_{\mathbb{K}}^{+} \to J_{\mathbb{L}}^{+}$ *such that*

$$\zeta_{\mathbb{K}(\mathfrak{n})}(s) = \zeta_{\mathbb{L}(\Psi(\mathfrak{n}))}(s)$$

for all ideals $\mathfrak{n} \in J_{\mathbb{K}}^{+}$.

Here, $\mathbb{K}(\mathfrak{n})$ is the ray class field corresponding to $\mathfrak{n}$ (the maximal abelian extension of $\mathbb{K}$ of conductor $\mathfrak{n}$). The connection arises from the fact that the zeta function of the ray class field is (by Artin factorisation) the product of the L-series corresponding to the characters of the ray class group modulo $\mathfrak{n}$.

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Mathematisch Instituut, Universiteit Utrecht, Postbus 80.010, 3508 TA Utrecht, Nederland
E-mail address: g.cornelissen@uu.nl

Mathematics Department, Mail Code 253-37, Caltech, 1200 E. California Blvd. Pasadena, CA 91125, USA
E-mail address: matilde@caltech.edu