BILINEAR LOCAL SMOOTHING ESTIMATE AND ITS APPLICATION TO THE CRITICAL GKDV EQUATION

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Abstract. We prove an improved version of bilinear local smoothing estimate to Airy solutions. Using this we study a smoothing property of Duhamel part of nonlinear solutions to the mass-critical generalized KdV equation.

1. Introduction

We consider the mass-critical generalized KdV equation:

$$\partial_t u + \partial_x^3 u = \mu \partial_x (u^5) = 0, \qquad u : \mathbb{R}\times\mathbb{R} \to \mathbb{R} \tag{1.1}$$

in the Sobolev space H^s. Here μ is ± 1 which corresponds to focusing or defocusing case respectively. Smooth solutions enjoy the mass and energy conservation laws:

$$M(u) = \int u(t,x)^2 dx = \int u_0(x) dx$$

$$E(u) = \frac{1}{2}\int \partial_x u(t,x)^2 dx - \frac{1}{3}\int u^6 dx.$$

It has scaling invariance, more precisely, if $u(t,x)$ solves (1.1), then so it does $u_\lambda(t,x) := \lambda^{-1/2}u(t/\lambda^3, x/\lambda)$ with initial data $u_{\lambda,0} = u_0(x/\lambda)$. One can check $\|u_\lambda(t,\cdot)\|_{L^2_x} = \|u(t,\cdot)\|_{L^2_x}$ and from this property we refer this scaling property as L^2-critical.

We are interested in the strong solutions $u(t,x)$ to (1.1) on a maximal time interval $[0,T)$ in the sense that $u(t,x) \in C_t H^s([0,T)\times\mathbb{R})$ satisfying the integral equation

$$u(t) = e^{-t\partial_x^3}u_0 - \int_0^t e^{-(t-s)\partial_x^3}\partial_x(u(s)^5) ds$$

where $e^{-t\partial_x^3}u_0$ is a linear solution, i.e.

$$e^{-t\partial_x^3}u_0 = \frac{1}{2\pi}\int e^{it\xi^3 + i(x-y)\xi}u_0(y)\, dy d\xi.$$

It is well known that the initial value problem is locally well-posed at critical regularity $s = 0$ [4]. Indeed, for given initial data $u_0 \in L^2(\mathbb{R})$, there is a unique solution to (1.1) in $u(t,x) \in C_t L^2_x \cap L^5_x L^{10}_t$. For the proof of local well-posedness, the following local smoothing estimate is crucial.

Proposition 1.1. *We have*

$$\|D^\alpha e^{-t\partial_x^3} f\|_{L^q_x L^r_t} \lesssim \|f\|_{L^2_x} \tag{1.2}$$

where $-\alpha + \frac{1}{q} + \frac{3}{r} = \frac{1}{2}$, *and* $\frac{4}{q} + \frac{2}{r} \le 1$, *except at an end point* $(q,r) = (\infty,\infty)$. *Here* D^α *is a homogeneous fractional derivative. See Notations.*
In particular,

$$\|e^{-t\partial_x^3}u_0\|_{L^5_x L^{10}_t} \lesssim \|u_0\|_{L^2_x}. \tag{1.3}$$

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Moreover, we have the inhomogeneous local smoothing estimate.

$$\| \int e^{s\partial_x^3} D^\alpha F(s,x)\, ds \|_{L_x^2} \lesssim \|F\|_{L_x^{q'} L_t^{r'}} \tag{1.4}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $\frac{1}{r} + \frac{1}{r'} = 1$ *where* α, q *and* r *as above.*

In help with (1.3) one can find a nonlinear solution to (1.1) on a time interval $[0, T]$ such that

$$\|e^{-t\partial_x^3} u_0\|_{L_x^5 L^{10}(\mathbb{R}\times[0,T])} \le \epsilon_0$$

for some $\epsilon_0 > 0$. Thus, the forward maximal lifespan T^* does not only depend on $\|u_0\|_{L_x^2}$ but also on the profile of u_0 and so the mass conservation law by itself does not give the global existence. A byproduct of the local theory is a blow up criteria:

$$T^* \Longrightarrow \|u\|_{L^5 L_t^{10}(\mathbb{R}\times[0,T^*))} = \infty.$$

If $u_0 \in H^s(\mathbb{R})$ for $s > 0$, we have *subcritical* local well-posedness, which says that the maximal lifespan depends only on $\|u_0\|_{H_x^s}$. In this case, using scaling symmetry, we obtain the lower bound on the blow up rate

$$\|u(t)\|_{H_x^s} \gtrsim \frac{1}{|T^* - t|^{s/3}}. \tag{1.5}$$

See [2] for detail. In the defocusing case, if $u_0 \in H^1$, then since the energy is finite and positive definite, the energy conservation law immediately implies global existence.

In this note we show an improved version of bilinear local smoothing estimate when the support of two frequencies are separated.

Theorem 1.2. *Let* $M, N > 0$*. Then,*

$$\|D^\alpha e^{-t\partial_x^3} f D^\alpha e^{-t\partial_x^3} g\|_{L_x^{q/2} L_t^{r/2}} \lesssim \left(\frac{M}{N}\right)^{5\theta/12} \|f\|_{L_x^2} \|g\|_{L_x^2} \tag{1.6}$$

where

$$-\alpha + \frac{1}{q} + \frac{3}{r} = \frac{1}{2}, (\frac{1}{q}, \frac{1}{r}) = (\frac{\theta}{6} + \frac{1-\theta}{4}, \frac{\theta}{6}), 0 \le \theta \le 1$$

for all L^2*-functions* f *and* g *with supp* $\widehat{f} \subset \{\xi : |\xi| \le 2M\}$ *and supp* $\widehat{g} \subset \{\xi : N \le |\xi| \le 2N\}, 0 \le M \le N$.

In particular, we have

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_x^{5/2} L_t^5} \lesssim \left(\frac{M}{N}\right)^{1/4} \|f\|_{L_x^2} \|g\|_{L_x^2} \tag{1.7}$$

In the space-time frequency space, the linear wave is supported on the characteristic curve $\tau = \xi^3$. Due to the curvature (or the slope of the tangent line) of the interaction of two linear waves at different frequencies is weaker and so one can have some gain.

Remark 1.3. *This type of estimate is firstly shown by Bourgain for symmetric Strichartz estimate of Schrödinger equation in* $d = 2$ [1]*. Keraani-Vargas* [5] *extended to other dimension for symmetric Strichartz norms and Chae-Cho-Lee* [3] *for non-symmetric norms.*

Remark 1.4. *The exponent in Theorem 1.2 is sharp.*
Define $\widehat{f} = \chi_{M \le \xi \le 2M}$ *and* $\widehat{g} = \chi_{1 \le \xi \le 1 + M^{1/2}}$*. Consider a subset* K *of* $\mathbb{R}\times\mathbb{R}$ *of* (t, x)

$$K = \{(t,x) : |x + 3t| \le \frac{1}{100} M^{-1/2}, |x| \le \frac{1}{100} M^{-1}\}.$$

One can easily observe that for all $(t,x) \in K$,

$$|e^{-t\partial_x^3} f(x)| = |\int_M^{2M} e^{it\xi^3 + ix\xi} d\xi| \sim M$$

and

$$|e^{-t\partial_x^3} g(x)| = |\int_1^{1+M^{1/2}} e^{it\xi^3 + ix\xi}| \sim M^{1/2}.$$

Thus,

$$\begin{aligned} \|D^{e^{-t\partial_x^3}}\| &\geq M^1 M^{\frac{1}{2}} M^{\alpha} \|\chi_K\|_{L_x^{q/2} L_t^{r/2}} \\ &\sim M^{\frac{3}{2}+\alpha} M^{-\frac{1}{2}\cdot\frac{2}{r}} M^{-1\cdot\frac{2}{q}} \\ &\sim M^{1+\frac{2}{r}-\frac{1}{q}} = M^{\frac{5\theta}{12}+\frac{3}{4}} \end{aligned}$$

where used admissible condition of exponents.
Since $\|f\|_{L_x^2} = M^{1/2}$ *and* $\|g\|_{L_x^2} = M^{1/4}$, *we see the estimate* (1.6) *is sharp.*

As an application of the bilinear local smoothing estimate we can obtain the smoothing property of the integral part of the Duhamel formula of nonlinear solutions.

Theorem 1.5. *Suppose* $u_0 \in H^s$ *for* $s > 3/4$. *Let write a solution to* (1.1) *as:*

$$u(t) = e^{-t\partial_x^3} u_0 + w(t) \qquad t \in [0, T^*).$$

Then $w(t) \in H^1$ *as long as the solution exists.*
Furthermore, if $T^* < \infty$, *then we have*

$$\|\partial_x w(t)\|_{L_x^2} \gtrsim \frac{1}{|T^* - t|^{1/3}} \tag{1.8}$$

For H^1 initial data, Martel and Merle [7, 8] studied the existence of blow up solutions and the lower bound of blow up profile when the solution has mass near that of ground state.

Remark 1.6. *It shows that the blow up phenomenon has an* H^1 *mechanism. Despite the fact that* $u(t) \in H^s$ *for* $s < 1$, *all blow up profiles belong to* H^1.

In Section 2 we prove Theorem 1.2 and in section 3 we give the proof of Theorem 1.5.

Notations. We use space-time mixed norm notation:

$$\|u(t,x)\|_{L_x^q L_t^r} := \Big(\int \big(\int |u(t,x)|^r dt \big)^{q/r} dx \Big)^{1/q}.$$

We denote the fractional derivative as $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ and the Sobolev norm as

$$\|f\|_{H^s} = \|\langle D \rangle^s f\|_{L^2}$$

where $\langle \xi \rangle = |\xi| + 1$ and $\widehat{f}$ is the Fourier transform of f. We use $X \lesssim Y$ to denote the estimate $X \leq CY$ where C depends only on the fixed parameters and exponents. We shall need the following Littlewood-Paley projection operators. Let $\phi(\xi)$ be a bump function, $\operatorname{supp} \phi \in \{|\xi| \leq 2\}$ and $\phi(\xi) = 1$

on $\{|\xi| \le 1\}$. For each dyadic number $N = 2^j, j \in \mathbb{N}$,

$$\widehat{P_N f}(\xi) = (\phi(\xi/N) - \phi(2\xi/N))\widehat{f}(\xi)$$
$$\widehat{P_0 f}(\xi) = (\phi(\xi))\widehat{f}(\xi)$$
$$\widehat{P_{\le N} f}(\xi) = \sum_{M \le N} P_M f(\xi)$$
$$\widehat{P_{>N} f}(\xi) = (I - \widehat{P_{\le N}}) f(\xi)$$

and we also use a wider projection operator $\widetilde{P}_N = P_{N/2} + P_N + P_{2N}$.

2. Proof of Theorem 1.2

Since (1.6) is a scaling invariant estimate, by scaling one can assume $N = 1$. In view of (1.2), we may also assume $M \ll 1$. (1.6) follows by interpolating the following two estimates:

$$\|D^{-1/4} e^{-t\partial_x^3} f D^{-1/4} e^{-t\partial_x^3} g\|_{L_x^2 L_t^\infty} \lesssim \|f\|_{L_x^2} \|g\|_{L_x^2} \tag{2.1}$$

$$\|D^{1/6} e^{-t\partial_x^3} f D^{1/6} e^{-t\partial_x^3} g\|_{L_{x,t}^3} \lesssim M^{5/12} \|f\|_{L_x^2} \|g\|_{L_x^2} \tag{2.2}$$

(2.1) is an immediate result of (1.2). Now we prove (2.2). Using Bernstein's inequality and observing frequency support of f and g, we are reduced to show that

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{x,t}^3} \lesssim M^{1/4} \|f\|_{L_x^2} \|g\|_{L_x^2}. \tag{2.3}$$

(2.3) is derived from the following lemma: indeed, it follows from the interpolation of ((2.4))(p=2) and ((2.5))

Lemma 2.1. *Assume that f and g are functions such that supp$|\widehat{f}| \subset [0, 2M]$, supp$|\widehat{g}| \subset [1, 2]$, $M \ll 1$*
(a) Let $p \ge 2$. Then we have

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{x,t}^p} \lesssim \|\widehat{f}\|_{L_x^{p'}} \|\widehat{g}\|_{L_x^{p'}} \tag{2.4}$$

where $p' = \frac{p}{p-1}$.
(b)

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{x,t}^4} \lesssim M^{3/8} \|f\|_{L_x^2} \|g\|_{L_x^2} \tag{2.5}$$

Lemma 2.1(a) follows from a classical argument of Fefferman and Stein [6]. It makes use of Hausdorff-Young inequality. We give a proof in the appendix for the sake of completeness.
In order to show (2.5) we use the example of Remark 1.4. Decompose g into functions whose frequency support are on small intervals of length $M^{1/2}$. Indeed, for integer k, $1 \le k \le M^{-1/2}$, set $I_k = [1 + (k-1)M^{1/2}, 1 + kM^{1/2}]$ and set $\widehat{g_K} = g\chi_{I_k}$. Then $g = \sum_k g_k$. Then we will use the following orthogonality inequality:

Lemma 2.2. *We have*

$$\|\sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{t,x}^4} \lesssim \Big(\sum_k \|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{t,x}^4}^2\Big)^{1/2} \tag{2.6}$$

Assuming that Lemma 2.2 is true for a moment, we are reduced to show

$$\Big(\sum_k \|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|_{L_{t,x}^4}^2\Big)^{1/2} \lesssim M^{3/8} \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

Using (2.4) for $p = 4$, and the size of support of f and g_k, we estimate

$$\begin{aligned}\Big(\sum_k \|e^{-t\partial_x^3} f e^{-t\partial_x^3} g\|^2_{L^4_{t,x}}\Big)^{1/2} &\lesssim \Big(\sum_k \|\widehat{f}\|^2_{L^{4/3}} \|\widehat{g_k}\|^2_{L^{4/3}}\Big)^{1/2} \\ &\lesssim M^{3/8} \|\widehat{f}\|_{L^2} \big(\sum_k \|\widehat{g_k}\|^2_{L^2}\big)^{1/2} \\ &= M^{3/8} \|f\|_{L^2} \|g\|_{L^2}\end{aligned}$$

which concludes (2.5).

It remains to show Lemma 2.2.
We write, using Plancherel theorem,

$$\begin{aligned}\|\sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k\|^2_{L^4_{x,t}} &= \|(\sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k)^2\|_{L^2_{t,x}} \\ &= \|\sum_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k \sum_j e^{-t\partial_x^3} f e^{-t\partial_x^3} g_j\|_{L^2_{t,x}} \\ &= \|\sum_{j,k} \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} g_j} * \widetilde{e^{-t\partial_x^3} g_k}\|_{L^2_{\tau,\xi}}.\end{aligned}$$

where $\widetilde{f(t,x)}(\tau,\xi)$ is the space-time Fourier transform of $f(t,x)$. We denote by $E_{j,k}$ the support of the function $\widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} g_j} * \widetilde{e^{-t\partial_x^3} g_k}$. We claim that the $E_{j,k}$ are essentially disjoint. In other words, there is a constant C, independent of M, so that

$$\sum_{j,k} \chi_{E_{j,k}} \leq C. \tag{2.7}$$

By this claim, we estimate

$$\begin{aligned}&\|\sum_{j,k} \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} g_j} * \widetilde{e^{-t\partial_x^3} g_k}\|_{L^2_{t,x}} \\ &\leq C\Big(\sum_{j,k} \|\widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} f} * \widetilde{e^{-t\partial_x^3} g_j} * \widetilde{e^{-t\partial_x^3} g_k}\|^2_{L^2_{t,x}}\Big)^{1/2} \\ &= C\Big(\sum_{j,k} \|e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_j\|_{L^2_{t,x}}\Big)^{1/2} \\ &= C\Big(\sum_{j,k} \int |e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k e^{-t\partial_x^3} f e^{-t\partial_x^3} g_j|^2\Big)^{1/2} \\ &= C\Big(\int (\sum_k |e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k|^2)^2\Big)^{1/2} \\ &= C\|\sum_k |e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k|^2\|_{L^2} \\ &\leq C\sum_k \||e^{-t\partial_x^3} f e^{-t\partial_x^3} g|^2\|_{L^2} \\ &= C\sum_k \|e^{-t\partial_x^3} f e^{-t\partial_x^3} g_k\|^2_{L^4}.\end{aligned}$$

We are left to show the inequality (2.7). One can easily see that the support of $\widetilde{e^{-t\partial_x^3}g_k}$ is in $E_k = \{(\tau,\xi) : |\xi - kM^{1/2}| \leq M^{1/2}, \tau = \xi^3\}$, and the support of $\widetilde{e^{-t\partial_x^3}f}$ is in $\{(\tau,\xi) : |\xi| \leq 2M, \tau = \xi^3\}$. If $(\rho,\eta) \in E_{j,k}$, then there exists (ξ_1,ξ_2) such that $(\xi_1^3,\xi_1) \in E_k, (\xi_2^3,\xi_2) \in E_j, |\rho - \xi_1^3 - \xi_2^3| \leq 4M$, and $|\eta - \xi_1 - \xi_2| \leq 4M$.
From the identity $4\xi_1^3 + 4\xi_2^3 = (\xi_1+\xi_2)^3 + 3(\xi_1-\xi_2)^2(\xi_1+\xi_2)$, we see that

$$E_{j,k} \subset F_{j,k} = \{(\rho,\eta) : |\eta - (j+k)M^{1/2}| \leq 3M^{1/2}, (3|k-j|^2 - 8)M \leq |4\rho - \eta^3| \leq (6|k-j|^2+8)M\}.$$

It is easy to verify that the $F_{j,k}$'s overlap only a finite number of times and that this number is bounded by a universal constant.

3. Proof of Theorem 1.5

In this section we prove Theorem 1.5. Firstly, we show Proposition 3.1.

Proposition 3.1. *Let $b < 1/4$.*

$$\|\partial_x(e^{-t\partial_x^3}u_0 e^{-t\partial_x^3}v_0)\|_{L_x^{5/2}L_t^5} \lesssim \|u_0\|_{H^b}\|v_0\|_{H^{1-b}} \tag{3.1}$$

Proof. We use the Littlewood-Paley operators to decompose into the paraproduct:

$$\partial_x e^{-t\partial_x^3}u_0 e^{-t\partial_x^3}v_0 = \pi_{lh} + \pi_{hh} + \pi_{hl}$$

where

$$\begin{aligned}
\pi_{lh} &= \sum_{N>M} P_N\partial_x e^{-t\partial_x^3}u_0 P_M e^{-t\partial_x^3}v_0 \\
\pi_{hh} &= \sum_{N\sim M} P_N\partial_x e^{-t\partial_x^3}u_0 P_M e^{-t\partial_x^3}v_0 \\
\pi_{hl} &= \sum_{N<M} P_N\partial_x e^{-t\partial_x^3}u_0 P_M e^{-t\partial_x^3}v_0.
\end{aligned}$$

By the triangle inequality, we have

$$\|\partial_x(e^{-t\partial_x^3}u_0 e^{-t\partial_x^3}v_0)\|_{L_x^{5/2}L_t^5} \leq \|\pi_{lh}\|_{L_x^{5/2}L_t^5} + \|\pi_{hh}\|_{L_x^{5/2}L_t^5} + \|\pi_{hl}\|_{L_x^{5/2}L_t^5}.$$

We estimate term by term. For the first two terms we can use the usual local smoothing estimate since the derivative falls in the low frequency part.

$$
\begin{aligned}
\|\pi_{hh}\|_{L_x^{5/2}L_t^5} &\lesssim \sum_{j=-1}^{\infty} \|P_{2^j}\partial_x e^{-t\partial_x^3}u_0 P_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^{5/2}L_t^5} \\
&\lesssim \sum_{j=-1}^{\infty} \|\widetilde{P}_{2^j}\partial_x e^{-t\partial_x^3}u_0\|_{L_x^5L_t^{10}}\|\widetilde{P}_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^5L_t^{10}} \\
&\lesssim \sum_{j=-1}^{\infty} 2^j\|\widetilde{P}_{2^j}e^{-t\partial_x^3}u_0\|_{L_x^5L_t^{10}}\|\widetilde{P}_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^5L_t^{10}} \\
&= \sum_{j=-1}^{\infty} 2^{bj}\|\widetilde{P}_{2^j}e^{-t\partial_x^3}u_0\|_{L_x^5L_t^{10}}2^{j(1-b)}\|\widetilde{P}_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^5L_t^{10}} \\
&\lesssim \sum_{j=-1}^{\infty} 2^{bj}\|\widetilde{P}_{2^j}u_0\|_{L^2}2^{j(1-b)}\|\widetilde{P}_{2^j}v_0\|_{L_x^2} \\
&\lesssim \|u_0\|_{H^b}\|v_0\|_{H^{1-b}}
\end{aligned}
$$

where $\tilde{P}_{2^j} = P_{2^{j-1}} + P_{2^j} + P_{2^{j+1}}$.

$$
\begin{aligned}
\|\pi_{lh}\|_{L_x^{5/2}L_t^5} &\lesssim \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1} \|P_{2^k}\partial_x e^{-t\partial_x^3}u_0 P_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^{5/2}L_t^5} \\
&\lesssim \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1} \|\widetilde{P}_{2^k}\partial_x e^{-t\partial_x^3}u_0\|_{L_x^5L_t^{10}}\|\widetilde{P}_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^5L_t^{10}} \\
&\lesssim \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1} 2^k\|\widetilde{P}_{2^k}e^{-t\partial_x^3}u_0\|_{L_x^5L_t^{10}}\|\widetilde{P}_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^5L_t^{10}} \\
&= \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1} 2^{b(k-j)}2^{bk}\|\widetilde{P}_{2^k}e^{-t\partial_x^3}u_0\|_{L_x^5L_t^{10}}2^{j(1-b)}\|\widetilde{P}_{2^j}e^{-t\partial_x^3}v_0\|_{L_x^5L_t^{10}} \\
&\lesssim \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1} 2^{b(k-j)}2^{bk}\|\widetilde{P}_{2^k}u_0\|_{L^2}2^{j(1-b)}\|\widetilde{P}_{2^j}v_0\|_{L_x^2} \\
&\lesssim \sum_{i=1}^{\infty} 2^{-bi}\sum_{j\geq i} 2^{b(j-i)}\|\widetilde{P}_{2^{j-i}}u_0\|_{L^2}2^{(1-b)j}\|\widetilde{P}_{2^j}v_0\|_{L^2} \\
&\lesssim \|u_0\|_{H^b}\|v_0\|_{H^{1-b}}.
\end{aligned}
$$

For the last term, the high-low paraproduct, we need to use improved bilinear estimate (1.7).

$$\begin{aligned}\|\pi_{lh}\|_{L_x^{5/2}L_t^5} &\lesssim \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1}\|P_{2^j}e^{-t\partial_x^3}\partial_x u_0 P_{2^k}e^{-t\partial_x^3}v_0\|_{L_x^{5/2}L_t^5}\\ &\lesssim \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1}2^{(k-j)/4}2^j\|P_{2^j}u_0\|_{L^2}\|P_{2^k}v_0\|_{L^2}\\ &= \sum_{j=1}^{\infty}\sum_{k=-1}^{j-1}2^{(k-j)(b-1/4)}2^{j(1-b)}\|P_{2^j}u_0\|_{L^2}2^{kb}\|P_{2^k}v_0\|_{L^2}\\ &\lesssim_b \|u_0\|_{H^b}\|v_0\|_{H^{1-b}}\end{aligned}$$

where we used Berstein's inequality, Cauchy-Schwartz inequality, and (1.7). □

Applying Hölder inequality to (1.7), we easily obtain the following corollary.

Corollary 3.2. *Let $s > 3/4$. Then we have*

$$\|\partial_x(e^{-t\partial_x^3}u_0)^5\|_{L_x^1L_t^2} \lesssim \|u_0\|_{H^s}^5. \tag{3.2}$$

We are now ready to prove Theorem 1.5.
We see from the local well-posedness theory [4] that for a given $\epsilon > 0$, there exists $T = T(\|u_0\|_{H^s}, \epsilon)$ such that

$$\max(\|\langle D\rangle^s u_{lin}\|_{L_x^5L_t^{10}([0,T]\times\mathbb{R})}, \|\langle D\rangle^s w\|_{L_x^5L_t^{10}([0,T]\times\mathbb{R})}) \le \epsilon$$

where u_{lin} is a linear solution, i.e. $u_{lin} = e^{-t\partial_x^3}u_0$.
We claim that

$$\|\partial_x w\|_{L_x^5L_t^{10}([0,T]\times\mathbb{R})} \le C\epsilon \tag{3.3}$$

Indeed, from (3.1) and (3.2), by reducing the value of T if necessary, we have

$$\|\partial_x(e^{-t\partial_x^3}u_0)^5\|_{L_x^1L_t^2([0,T]\times\mathbb{R})} \le \epsilon \tag{3.4}$$

$$\|\partial_x(e^{-t\partial_x^3}u_0 e^{-t\partial_x^3}u_0)\|_{L_x^{5/2}L_t^5([0,T]\times\mathbb{R})} \le \epsilon \tag{3.5}$$

We estimate $\|\partial_x w\|_{L_x^5L_t^{10}([0,T]\times\mathbb{R})}$ using (1.2), its inhomogeneous counter part, and Christ-Kiselev lemma as follows:

$$\begin{aligned}\|\partial_x w\|_{L_x^5L_t^{10}([0,T]\times\mathbb{R})} &= \|\partial_x\int_0^t e^{-(t-s)\partial_x^3}\partial_x(u^5(s))ds\|_{L_x^5L_t^{10}([0,T]\times\mathbb{R})}\\ &\lesssim \|\partial_x\int_0^T e^{s\partial_x^3}\partial_x(u^5)ds\|_{L_x^2}\\ &\lesssim \|\partial_x(u^5)\|_{L_x^1L_t^2([0,T]\times\mathbb{R})}\\ &= \|\partial_x(u_{lin}+w)^5\|_{L_x^1L_t^2([0,T]\times\mathbb{R})}\\ &\lesssim \|\partial_x(w^5)\|_{L_x^1L_t^2([0,T]\times\mathbb{R})} + \|\partial_x(u_{lin}w^4)\|_{L_x^1L_t^2([0,T]\times\mathbb{R})} + \text{the rest}\\ &= I + II + III.\end{aligned}$$

We can estimate

$$I \lesssim \|\partial_x w\|_{L_x^5L_t^{10}}\|w\|_{L_x^5L_t^{10}}^4$$

and I is absorbed to the left hand side by choosing ϵ small.
III is easily estimated by (3.4) and (3.5) since III contains at least two u_{lin}'s. For example, we have

$$\begin{aligned}\|\partial_x(u_{lin}u_{lin})www\|_{L^1_xL^2_t} + \|u^2_{lin}w^2\partial_x w\|_{L^1_xL^2_t} &\lesssim \|D^s u\|^2_{L^5_xL^{10}_t}\|w\|^3_{L^5_xL^{10}_t} + \|\partial_x w\|_{L^5_xL^{10}_t}\|w\|^2_{L^5_xL^{10}_t}\|u_{lin}\|_{L^5_xL^{10}_t} \\ &\lesssim \epsilon^5 + \epsilon^4\|\partial_x w\|_{L^5_xL^{10}_t}.\end{aligned}$$

Choosing ϵ to be small, we can estimate

$$II \lesssim \epsilon.$$

In order to show the bound of II it is enough to show

$$\|\partial_x u_{lin} w\|_{L^{5/2}_x L^5_t([0,T]\times\mathbb{R})} \lesssim \epsilon \tag{3.6}$$

In view of

$$\|\partial_x u_{lin} w\|_{L^{5/2}_x L^5_t([0,T]\times\mathbb{R})} = \|\int_0^t \partial_x e^{-t\partial_x^3}u_0 e^{-(t-s)\partial_x^3}\partial_x(u^5(s))ds\|_{L^{5/2}_x L^5_t([0,T]\times\mathbb{R})}$$

and Christ-Kiselev lemma it suffices to show

$$\|\int_0^T \partial_x e^{-t\partial_x^3}u_0 e^{-(t-s)\partial_x^3}\partial_x(u^5(s))ds\|_{L^{5/2}_x L^5_t([0,T]\times\mathbb{R})} \lesssim \epsilon. \tag{3.7}$$

We estimate, using (3.1) and the inhomogeneous local smoothing estimate (1.4)

$$\begin{aligned}&\|\int_0^T \partial_x e^{-t\partial_x^3}u_0 e^{-(t-s)\partial_x^3}\partial_x(u^5(s))ds\|_{L^{5/2}_x L^5_t([0,T]\times\mathbb{R})} \\ &\quad = \|\partial_x e^{-t\partial_x^3}u_0 e^{-t\partial_x^3}\int_0^T e^{s\partial_x^3}\partial_x(u^5(s))ds\|_{L^{5/2}_x L^5_t([0,T]\times\mathbb{R})} \\ &\quad \lesssim \|u_0\|_{H^s}\|\int_0^T e^{s\partial_x^3}\partial_x(u^5(s))ds\|_{H^s} \\ &\quad \lesssim \|u_0\|_{H^s}\|\langle D\rangle^s(u^5)\|_{L^1_xL^2_t([0,T]\times\mathbb{R})} \\ &\quad \lesssim \|u_0\|_{H^s}\|\langle D\rangle^s u\|^5_{L^5_xL^{10}_t([0,T]\times\mathbb{R})} \\ &\quad \lesssim \|u_0\|_{H^s}\epsilon^5.\end{aligned}$$

Combining altogether, we have

$$\|\langle D\rangle^1 w\|_{L^5_xL^{10}_t([0,T]\times\mathbb{R})} \le C\epsilon.$$

In a similar way one can estimate again using Christ-Kiselev lemma, for $0 \le t \le T$,

$$\begin{aligned}\|D^1 w(t)\|_{L^2} &= \|D^1 e^{-t\partial_x^3}\int_0^t e^{s\partial_x^3}\partial_x(u^5(s))ds\|_{L^2} \\ &\lesssim \|D^1\int_0^T e^{s\partial_x^3}\partial_x(u^5(s))ds\|_{L^2} \\ &\lesssim \|D^1(u^5)\|_{L^1_xL^2_t([0,T]\times\mathbb{R})} \\ &\lesssim \epsilon.\end{aligned}$$

Now we remain to show the lower bound of blow up rate, (1.8).
From (1.5) we have

$$\|w(t)\|_{H^s_x} \gtrsim \frac{1}{|T^* - t|^{s/3}}.$$

Because the H^s norm of the linear part of the solution is conserved, by interpolation,

$$\|w(t)\|_{H^s} \le \|w(t)\|^{1-s}_{L^2}\|w(t)\|^s_{H^1}.$$

Since $\|w(t)\|_{L^2} \le 2\|u_0\|_{L^2}$, we can conclude

$$\|w(t)\|_{\dot{H}^1} \gtrsim \frac{1}{|T^* - t|^{1/3}}.$$

Appendix

Proof of (2.4).
Writing

$$e^{-t\partial_x^3} f e^{-t\partial_x^3} g(t,x) = c \iint e^{ix(\xi_1+\xi_2)+(\xi_1^3+\xi_2^3)} \widehat{f}(\xi_1)\widehat{g}(\xi_2)\, d\xi_1 d\xi_2,$$

we make a change of variables $(u,v) = (\xi_1+\xi_2, \xi_1^3+\xi_2^3)$. Then we obtain

$$e^{-t\partial_x^3} f e^{-t\partial_x^3} g(t,x) = c \iint e^{ixu+itv}\Pi(u,v)\, dudv$$

where $\Pi(u,v) = \widehat{f}(\xi_1)\widehat{g}(\xi_2)|J^{-1}|$ and $J = \det \frac{\partial(u,v)}{\partial(\xi_1,\xi_2)} = \frac{1}{3(\xi_2^2-\xi_1^2)}$.
We can view

$$e^{-t\partial_x^3} f e^{-t\partial_x^3} g(t,x) = \widehat{\Pi}(t,x).$$

Hence, using Hausdorff-Young inequality, for $p \ge 2$,

$$\|e^{-t\partial_x^3} f e^{-t\partial_x^3}\|_{L^p_{t,x}} = \|\widehat{\Pi}\|_{L^p_{t,x}} \le \|\Pi\|_{L^{p'}_{t,x}}$$

where $p' = \frac{p}{p-1}$.
To compute $\|\Pi\|_{L^{p'}}$, we use the fact $|\xi_1 - \xi_2| \ge 1/2$ (i.e. $|J| \sim 1$) and change variables back to ξ_1, ξ_2. Indeed,

$$\begin{aligned}
\|\Pi\|_{L^{p'}}^{p'} &= \iint \left|\widehat{f}(\xi_1)\widehat{g}(\xi_2)J^{-1}\right|^{p'} dudv \\
&= \iint \left|\widehat{f}(\xi_1)\widehat{g}(\xi_2)\right|^{p'} \left|J^{-1}\right|^{p'} \left|J\right| d\xi_1 d\xi_2 \\
&\sim \|\widehat{f}\|_{L^{p'}} \|\widehat{g}\|_{L^{p'}}.
\end{aligned}$$

□

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