Expander graphs from Curtis Tits groups

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Abstract

Using the construction of a nonorientable Curtis-Tits group of type $\tilde{A}_n$, we obtain new explicit families of expander graphs of valency five for unitary groups over finite fields.

1 Introduction

Expanders are sparse graphs with high connectivity properties. Explicit constructions of expander graphs have potential applications in computer science and is an area of active research. One of the most significant recent results on expanders is that Cayley graphs of finite simple groups are expanders, see [11],[5]. More precisely there is a k and $\epsilon > 0$ such that every non-abelian finite simple group G has a set of k generators for which the Cayley graph $X(G; S)$ is an ϵ expander. The size of k is estimated around 1000.

The present paper is a byproduct of the investigation in [3, 4] of Curtis-Tits structures and the associated groups. A Curtis-Tits (CT) structure over k with (simply laced) Dynkin diagram Γ over a finite set I is an amalgam $\mathcal{A} = \{G_i, G_{i,j} \mid i, j \in I\}$ whose rank-1 groups G_i are isomorphic to $\mathrm{SL}_2(\mathrm{k})$, where $G_{i,j} = \langle G_i, G_j \rangle$, and in which G_i and G_j commute if $\{i, j\}$ is a non-edge in Γ and are embedded naturally in $G_{i,j} \cong \mathrm{SL}_3(\mathrm{k})$ if $\{i, j\}$ is an edge in Γ. It was shown in [3] that such structures are determined up to isomorphisms by group homomorphisms from the fundamental group of the graph Γ and the group $\mathrm{Aut}(\mathrm{k}) \times \mathbb{Z}_2$. Moreover in the case when the diagram is in fact a cycle, all such structure have non collapsing completions described in [4]. It turns out that such groups can be described as fixed subgroups of certain automorphisms of Kac-Moody groups. This is an important point since they will turn out to have Karzdan's property (T) hence they will give rise to expanders. Many of these groups will be Kac-Moody groups themselves but some will not. In particular again in the case of a cycle we obtain a new group which turns out to be a

lattice in $SL_{2n}(K)$ for some local field K and so by a classical theorem it will itself have property (T). Moreover it turns out that the group in question will have finite unitary groups as quotients giving a more concrete result for unitary groups than [11]. In particular we have

Theorem 1 *For any n there exists an $\epsilon > 0$ and a symmetric set $S_{n,q}$ of generators for $SU_{2n}(q)$ so that $S(n,q)$ has size five and the family of Cayley graphs $X(SU_{2n}(q), S_{n,q})$ for $q \geq n$ forms an ϵ-expanding family of unbounded new girth.*

Our methods have been introduced in [6, 8, 9] in a slightly more general setting. The result is weaker than the types of results in [11] and [5] in the fact that the rank of the groups need to be fixed.

2 The groups

Let V be a free $k[t, t^{-1}]$-module of rank $2n$ with basis $\{e_i, f_i \mid i = 1, \ldots, n\}$. In this case $k[t, t^{-1}]$ denotes the ring of commutative Laurent polynomials in the variable t over a field k. Recall that a σ-sesquilinear form β on V is a map $\beta : V \times V \to k$ so that β is linear in the first coordinate and $\beta(u, \lambda v + w) = \sigma(\lambda)\beta(u, v) + \beta(u, w)$. Such a form is determined by its image on a basis. Let β be such that $\beta(e_i, e_j) = \beta(f_i, f_j) = 0, \beta(e_i, f_j) = t\delta_{ij}$ and $\beta(f_i, e_j) = \delta_{ij}$ where $\sigma: k[t, t^{-1}] \to k[t, t^{-1}]$ is the identity on k and interchanges t and t^{-1}. More precisely

$$G^\tau := \{g \in SL_{2n}(k[t, t^{-1}]) | \forall x, y \in V, \beta(gx, gy) = \beta(x, y)\}$$

In [4] it was proved that G^τ is the "nonorientable" Curtis-Tits group.

It turns out that the group G^τ has some very interesting natural quotients and that its action on certain Clifford-like algebras are related to phenomena in quantum physics. see [10] for such constructions.

The aim of this paper is to prove that the group G^τ has Kazhdan's property T. This implies that the finite quotients of this group will form expander families. We will also show, that the Cayley graphs of these quotients have unbounded new girth. Before doing this we record the following lemma.

Lemma 2.1 *The group G^τ can be generated with a symmetric set of size at most 5.*

Proof Consider the element $s \in SL_{2n}(k[t, t^{-1}])$ transforming the basis above as follows. For each $i = 1, \ldots n - 1$, $e_i^s = e_{i+1}$ and $f_i^s = f_{i+1}$, $e_n^s = f_1$ and $f_n^s = t^{-1}e_1$ It is not too hard to see that s is in fact an element of G^τ. Moreover consider the subgroups

$$L_k = \left\{ \left(\begin{array}{ccc|ccc} I_k & & & & & \\ & A & & & & \\ & & I_{n-k-2} & & & \\ \hline & & & I_k & & \\ & & & & {}^t A^{-1} & \\ & & & & & I_{n-k-2} \end{array} \right) \mid A \in SL_2(k) \right\}$$

for each $k = 0, \ldots n - 2$ and and

$$L_{n-1} = \left\{ \left(\begin{array}{cc|cc} a & & & -bt^{-1} \\ & I_{n-2} & & \\ \hline & & a & b \\ & & c & d \\ & & & I_{n-2} \\ -ct & & & & d \end{array} \right) \middle| \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathrm{k}) \right\}.$$

It is immediate that $L_k^s = L_{k+1}$ for $k = 0, \ldots n - 2$ and $L_{n-1}^n = L_0$. Moreover from [4] it follows that G^τ is generated by the L_k's. Finally the groups L_0 can be generated by an involution x and another element y. This means that we can take $S = \{x, y, y^{-1}, s, s^{-1}\}$. $\qquad\qquad\square$

Let $\overline{\mathrm{k}}$ denote the algebraic closure of k. For any $a \in \overline{\mathrm{k}}^*$ consider the specialization map $\epsilon_a : \mathrm{k}[t, t^{-1}] \to \overline{\mathrm{k}}$ given by $\epsilon_a(f) = f(a)$. The map induces a homomorphism $\epsilon_a : \mathrm{SL}_{2n}(\mathrm{k}[t, t^{-1}]) \to \mathrm{SL}_{2n}(\mathrm{k}(a))$. In some instances the map commutes with the automorphism σ and so one can define a map $\epsilon_a : G^\tau \to \mathrm{SL}_{2n}(\overline{\mathrm{k}})$

The most important specialization maps are those given by evaluating t at $a = \pm 1$ or $a = \zeta$, a $(q^m + 1)$-st root of 1 where q is a power of the characteristic.

Consider first $a = -1$. In this case the automorphism σ becomes trivial. Note that for $g \in G^\tau$ we have $\epsilon_{-1}(g) \in \mathrm{Sp}_{2n}(\mathrm{k})$. In this case, the image of the group G^τ is the group $\mathrm{Sp}_{2n}(\mathrm{k})$. Similarly, if $a = 1$, the automorphism σ is trivial and the map ϵ_1 takes G^τ into $\Omega_{2n}^+(\mathrm{k})$.

Finally assume that $\mathrm{k} = \mathbb{F}_q$ and $a \in \overline{\mathbb{F}}_q$ is a primitive $(q^s + 1)$-st root of 1. and define $\tilde{V} = V \otimes_{\mathrm{k}[t, t^{-1}]} \mathrm{k}(a)$ and $\tilde{\beta}$ the respective evaluation of β. We shall also denote by $\overline{\lambda}$ the image of λ under the Galois automorphism given by $a \mapsto a^{-1}$. Define the transvection map $T_v(\lambda) : \tilde{V} \to \tilde{V}$ by $T_v(\lambda)(x) = x + \lambda\tilde{\beta}(x, v)v$. Note that the group $SU_{2n}(q^s)$ is generated by the set

$$\{T_v(\lambda) | \overline{\lambda} + a\lambda = 0, v \in \{\tilde{e}_1, \cdots \tilde{f}_n\}$$

(by Phan theorem for example since T_{e_i}, T_{f_i} generate a weak Phan system, see [2] for details).

Therefore if we can lift each such map to G^τ the theorem will be proved. We propose that for each $v \in \{\tilde{e}_1, \cdots \tilde{f}_n\}$, the lift of $T_v(\lambda)$ would be given by a "transvection" map $\Phi_v(x) = x + F\beta(x, v)v$ where $F \in \mathbb{F}_q[t, t^{-1}]$ would be an appropriate choice of a lift of λ. This map is obviously in $SL_{2n}(\mathbb{F}_q[t, t^{-1}])$ so the only thing one needs to check is the fact that it leaves β invariant. An immediate computation shows that

$$\begin{aligned} \beta(x, y) - \beta(\Phi_v(x), \Phi_v(y)) &= \sigma(F)\beta(x, v)\sigma(\beta(y, v)) + F\beta(x, v)\beta(v, y) \\ &= (\sigma(F) + tF)\beta(x, v)\sigma(\beta(y, v)) \end{aligned}$$

and so the sufficient conditions are $F(a) = \lambda$ and $\sigma(F) + tF = 0$.

Let us assume that $p \in \mathbb{F}_q[t]$ is the minimal (monic) polynomial for a. Note that a and a^{-1} are conjugate. Moreover if b is another root of p then b is a root of $x^{q^s+1} - 1$ so it is a power of a and in particular b^{-1} is also a root of p and of course $b \neq b^{-1}$

since otherwise p will not be irreducible. In conclusion the roots of p come in pairs b, b^{-1}. This means that $p(0) = 1$. Now $\sigma(p(t)) = p(t^{-1}) = t^{-2s}p'(t)$ where p' is a monic irreducible polynomial that has the same roots as p so it must equal p.

We start with a random choice for F so that $F(a) = \lambda$. Since $\overline{\lambda} + a\lambda = 0$, we get that a is a root of $\sigma(F) + tF$ and so $\sigma(F) + tF = pG$ for some $G \in \mathbb{F}_q[t, t^{-1}]$. Applying σ shows that $\sigma(p)\sigma(G) = t^{-1}pG$ and so $\sigma(G) = t^{2s-1}G$.

Assume $G = \sum_{i=-r}^{l} a_i t^i$, the condition above gives that $-l = 2s - 1 - r$ and $a_{-r+i} = a_{l-i}$ for each $i = 1, \ldots l + r$.

We need to find an element $H \in \mathbb{F}_q[t, t^{-1}]$ so that $\sigma(pH) + tpH = pG$. Indeed, $F_1 = F - pH$ will then have the property that $F_1(a) = \lambda$. The condition on H is that $\sigma(H)t^{-2s} + tH = G$. There are many choices for H, one of them will be

$$ H = t^{-l-2s} + t^{-l-2s+1} + \ldots t^{-s-1} + (a_{-s+1} - 1)t^{-s} + \ldots (a_l - 1)t^{l-1} $$

$\square$

For the rest of this section, we deal with unitary representation of topological groups on Hilbert spaces we will follow the notations of [1]. Consider a complex Hilbert space $\mathcal{H}$. We will denote by $\mathcal{U}(\mathcal{H})$ the group of unitary transformations from $\mathcal{H}$ to $\mathcal{H}$, i.e. the group of all invertible bounded linear operators on $\mathcal{H}$ that leave invariant the inner product. A unitary representation of a topological group G is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ so that $g \mapsto \pi(g)\xi$ is continuous for any $\xi \in \mathcal{H}$.

Definition 2.2 Let G be a topological group and $\pi : G \to U(\mathcal{H})$ is a continuous unitary representation of G on a Hilbert space $\mathcal{H}$.

If $Q \subseteq \mathcal{G}$ and $\epsilon > 0$, a vector ξ is called (Q, ϵ) *invariant* if $\sup_{q \in Q} ||\pi(q)\xi - \xi|| < \epsilon$. An *invariant vector* is a vector $\xi \in \mathcal{H}$ such that $\xi = \pi(g)\xi$ for all $g \in G$.

A subset $Q \subseteq G$ is a *Kazhdan set* if there exists $\epsilon > 0$ so that every unitary representation of G that admits a (Q, ϵ) invariant vector admits a nonzero invariant vector.

Finally a group G has *Kazhdan property (T)* if it admits a compact Kazdan set.

In order to prove that our groups have property (T) we will use the following results.

Theorem 2.3 *(Theorem 1.4.15 in [1]) Let K be a local field. The group $SL_n(K)$ has Property (T) for any integer $n \geq 3$.*

Theorem 2.4 *If G is a locally compact group and H is a lattice in G the H has property (T) if and only if G does.*

In order to show that our group G^τ has property (T) it is sufficient to show that G^τ is a lattice in $SL_{2n}(k((t)))$. To do this we use the methods of [6], [9], [8]. In particular the more general argument is briefly described in Remark 7.11 in [8].

For convenience we state Lemma 6.13 and 6.14 from [4]:

Lemma 2.5 *Suppose that $c_\varepsilon \in \Delta$ satisfies $\delta_*(c_\varepsilon, c_\varepsilon^\tau) = w$, let $i \in I$ and suppose that π is the i-panel on c_ε. Then,*

(a) *There exists a word $u \in W$ such that $u(u^{-1})^\tau$ is a reduced expression for w.*

(b) If $l(s_i w) > l(w)$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ except one satisfy $\delta_*(d_\varepsilon, d_\varepsilon^\tau) = w$. The remaining chamber $\check{c}_\varepsilon$ satisfies $\delta_*(\check{c}_\varepsilon, (\check{c}_\varepsilon)^\tau) = s_i w s_{\tau(i)}$.

(c) If $l(s_i w) < l(w)$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ satisfy $\delta_*(d_\varepsilon, d_\varepsilon^\tau) = s_i w s_{\tau(i)}$.

Corollary 2.6 *If $q \geq n$, the group G^τ has property (T).*

Proof Note that the group G^τ acts transitively on the sets $C_w = \{c \in \Delta_+ | \delta_*(c, c^\tau) = w\}$ and these partition Δ_+. For each $u \in W$ pick an element c_u so that $\delta_*(c_u, c_u^\tau) = u(u^{-1})^\tau$ to parametrise the orbits of G^τ on Δ_+. We can therefore apply Lemma 1.4.2 of [6] to conclude that G^τ is a lattice if and only if the series $\sum_{u \in W} \frac{1}{|Stab_{G^\tau}(c_w)|}$ converges. By Lemma 2.5 there are exactly $q^{l(u)}$ elements of C_{1_w} at distance u from c_u and the group $Stab_{G^\tau}(c_w)$ acts transitively on these and so $\frac{1}{|Stab_{G^\tau}(c_w)|} \leq \frac{1}{q^{l(u)}}$. In particular since there are at most $(n-1)^i$ elements of W that have length i,

$$\sum_{u \in W} \frac{1}{q^{l(u)}} \leq \sum_{i=0}^{\infty} \frac{(n-1)^i}{q^i}$$

and so if $q \geq n$ the series converges. $\qquad\square$

3 The expanders

Definition 3.1 Let $X = (V, E)$ be a finite k-regular graph with n vertices. we say that X is an (n, k, c) expander if for any subset $A \subset V$

$$|\partial A| \geq c(1 - \frac{|A|}{N})|A|$$

where $\partial A = \{v \in V \mid d(v, A) = 1\}$.

The following property is due to Margulis [12]

Theorem 3.2 *Let Γ be a finitely generated group that has property (T). Let $\mathcal{L}$ be a family of finite index normal subgroups of Γ and let $S = S^{-1}$ be a finite symmetric set of generators for γ. Then the family $\{X(\Gamma/N, S) \mid N \in \mathcal{L}\}$ of Cayley graphs of the finite quotients of Γ with respect to the image of S is a family of (n, k, c) expanders for $n = |\Gamma/N|, k = |S|$ and some fixed $c > 0$.*

Corollary 3.3 *If S be a symmetric generating set for G^τ then the family of Cayley graphs $X(SU_{2n}(q^k), S)$ is an expanding family.*

An important question about expander families is whether the girth of the graphs in question is bounded or not.

Definition 3.4 Let $\Gamma, \mathcal{L}, S$ as in Theorem 3.2. Consider also the natural map $\phi_N : X(\Gamma, S) \to X(\Gamma/N, S)$. The *new girth* of a graph $X(\Gamma/N, S)$ is the length of the shortest circuit γ in $X(\Gamma/N, S)$ so that γ is not the image of a circuit in $X(\Gamma, S)$ under the map ϕ_N.

Proposition 3.5 *The new girth of the family $X(SU_{2n}(q^k), S)$ is unbounded.*

Proof Suppose that the new girth is bounded by a number M. Since there are finitely many possible words on S of length less than N and infinitely many graphs in the family, it follows that there exists a path γ in $X(G^\tau, S)$ which is not a circuit so that the the images of γ are circuits in $X(SU_{2n}(q^k), S)$ for infinitely many choices of k. Without loss of generality we can assume that γ has the identity of G^τ as its initial point. The statement above means that the end point of γ is the identity in infinitely many $X(SU_{2n}(q^k), S)$. However the end point of γ is an element $g \in SL_{2n}(k[t, t^{-1}])$. However if $g \neq Id_{2n}$ then the set of solutions to the equation and the condition is that $g(t) = Id_{2n}$ in $M_{2n}(k[t, t^{-1}])$ is an algebraic set so it has to be finite, a contradiction. □

Finally Lemma 2.1, Corollary 2.6, Corollary 3.3, and Proposition 3.5 prove Theorem 1.

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