STATIC SKT METRICS ON LIE GROUPS

NICOLA ENRIETTI

Abstract. An SKT metric is an Hermitian metric on a complex manifold whose fundamental 2-form ω satisfies $\partial\overline{\partial}\omega = 0$. Streets and Tian introduced in [STb] a Ricci-type flow that preserves the SKT condition. This flow uses the Ricci form associated to the Bismut connection, the unique Hermitian connection with totally skew-symmetric torsion, instead of the Levi-Civita connection. A SKT metric is static if the (1,1)-part of the Ricci form of the Bismut connection satisfies $(\rho^B)^{(1,1)} = \lambda\omega$ for some real constant λ. We study invariant static metrics on Lie groups, providing in particular a classification in dimension 4.

Introduction

Let (M^{2n}, J, g) be an Hermitian manifold with fundamental 2-form $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$. We say that g is Strong KT (shorten *SKT*) or *pluriclosed* if $\partial\overline{\partial}\omega = 0$. This condition is related to the Bismut connection [Bis89, Gau97], the unique Hermitian connection such that the torsion 3-form

$$c(X, Y, Z) = g(X, T^B(Y, Z))$$

is totally skew-symmetric, and it is well known that $c = -Jd\omega$. So the SKT condition is equivalent to $dc = 0$. SKT metrics were introduced in the context of type II string theory and 2-dimensional supersymmetric σ-models [GHR84, Str86], and they have also relations with generalized Kähler geometry [GHR84, Gua10, Hit06, AG07, CG04, FT09]. Moreover, Gauduchon [Gau84] proved that for compact complex surfaces, in the conformal class of any given Hermitian metric one can find an SKT metric.

In [STa] Streets and Tian introduced a class of parabolic flows on a complex manifold (M, J) for Hermitian metrics with respect to J using the Chern connection, that is the unique Hermitian connection whose torsion has everywhere vanishing (1,1)-part. If the initial condition is Kähler, then the solution of the flow is also a solution of the Kähler-Ricci flow. In [STb], moreover, they studied a particular flow in this class that preserves the SKT condition, with equation

$$\frac{\partial\omega(t)}{\partial t} = -\partial\partial^*\omega - \overline{\partial\partial}^*\omega - \frac{i}{2}\partial\overline{\partial}\log\det g.$$

As noted also in [ST10], this flow is deeply connected with the Bismut connection.

We consider in particular static metrics. We say that an SKT metric g on a complex manifold (M, J) is *static* if

$$-\partial\partial^*\omega - \overline{\partial\partial}^*\omega - \frac{i}{2}\partial\overline{\partial}\log\det g = \lambda\omega$$

for some $\lambda \in \mathbb{R}$. In [STb], it is shown that to any static metrics with $\lambda \neq 0$ we can associate a symplectic form that tames J, what they called an Hermitian-symplectic form. By [EFV10], this condition is equivalent to an SKT metric such that $\partial\omega = \overline{\partial}\beta$ for some ∂-closed (2,0)-form β. So, to find all the static metrics on a complex manifold it is sufficient to study the SKT metrics. In [EFV10]

2000 *Mathematics Subject Classification.* 32J27, 53C55, 53C30, 53D05, 53C25, 53C44.

This work was supported by the Projects MIUR "Riemannian Metrics and Differentiable Manifolds", "Geometric Properties of Real and Complex Manifolds" and by GNSAGA of INdAM.

it was proved that a nilmanifold, i.e. the compact quotient of a nilpotent simply connected Lie group by a discrete subgroup, endowed with an invariant complex structure J cannot admit any symplectic form that tames J.

Our aim is to study static metrics on Lie groups and compact quotiens of Lie groups by discrete subgoups. This is clearly related to the study of SKT metrics on such manifolds, that was developed in [FPS02, MS09, EFV10, Swa08].

We outline the paper. In section 1 we make some preliminar observation about the flow introduced in [STb]. First, we observe that the cohomology class $[\partial\omega] \in H^{2,1}_{\overline{\partial}}(M)$ is preserved by the flow; then, we note that we can write the flow as

$$\frac{\partial\omega(t)}{\partial t} = -(\rho^B)^{(1,1)},$$

where ρ^B is the Ricci tensor of the Bismut connection. So, in the case of a static metric with $\lambda \neq 0$, the symplectic 2-form that tames J is exactly ρ^B.

In section 2 we study static metrics on two classes of Lie groups: semisimple compact Lie groups and nilmanifolds. It is well known that every compact Lie group admits a bi-invariant metric g. Moreover, if we add the semisimple condition, we can find an integrable complex structure J compatible with g. We prove that g is static with respect to J.

By [EFV10], we know that a nilmanifold endowed with an invariant complex structure cannot admit any static metric with $\lambda \neq 0$. Adding the hypothesis that the metric is invariant, we prove that it cannot admit any static metric with $\lambda = 0$, too.

Finally, in section 3, we classify all the static metrics on Lie algebras of dimension 4, obtaining that a Lie algebra of dimension 4 together with an integrable complex structure J and a static metric g is either Kähler-Einstein or the Lie algebra of the Hopf surface.

1. Link with the Bismut connection

Let (M^{2n}, J, g) be an Hermitian manifold with fundamental 2-form $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$. In [STb, ST10] Streets and Tian introduced a new flow

$$\frac{\partial\omega(t)}{\partial t} = \Phi(\omega) \tag{1.1}$$

using the operator

$$\Phi(\omega) = -\partial\partial^*\omega - \overline{\partial}\overline{\partial}^*\omega - \frac{i}{2}\partial\overline{\partial}\log\det g, \tag{1.2}$$

where $\partial + \overline{\partial} = d$ and

$$\partial^* : \Omega(M)^{p,q} \to \Omega(M)^{p-1,q}$$
$$\overline{\partial}^* : \Omega(M)^{p,q} \to \Omega(M)^{p,q-1}$$

are the adjoint operators with respect to the metric g of the operators $\partial, \overline{\partial}$ respectively. They proved that Φ is elliptic on the set of SKT metrics on M and that this condition is preserved by the flow.

Every SKT metric on a complex manifold (M, J) specify a Dolbeaut cohomology class given by $[\partial\omega] \in H^{2,1}_{\overline{\partial}}(M)$. We see that the flow (1.1) preserves this class.

Theorem 1.1.
Let (M, J) be a complex manifold and g_0 an SKT metric with fundamental 2-form ω_0. If $\{\omega(t)\}_{t\in[0,T)}$ is the solution of (1.1) with initial value $\omega(0) = \omega_0$, then for every $t \in [0, T)$

$$[\partial\omega(t)] = [\partial\omega_0] \in H^{2,1}_{\overline{\partial}}(M).$$

Proof.
Using (1.2) we obtain

$$\partial\Phi(\omega) = -\partial\overline{\partial\partial}^*\omega = \overline{\partial}\partial\overline{\partial}^*\omega.$$

So by (1.1)

$$\frac{\partial}{\partial t}[\partial\omega(t)] = \left[\partial\frac{\partial\omega(t)}{\partial t}\right] = [\overline{\partial}\partial\overline{\partial}^*\omega(t)] = [0],$$

thus $[\partial\omega(t)]$ is constant in $H^{2,1}_{\overline{\partial}}(M)$.

□

Let (M^{2n}, J, g) be an Hermitian manifold, and ∇ an Hermitian connection on M; the Ricci form ρ associated to ∇ is defined by

$$\rho(X,Y) = \frac{1}{2}\sum_{k=1}^{2n} g(R(X,Y)\mathbf{e}_k, J\mathbf{e}_k),$$

where $\{\mathbf{e}_i\}$ is a local orthonormal frame of the tangent bundle TM and R is the curvature tensor

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

We can define the Ricci form ρ^B and ρ^C of the Bismut and Chern connection, and they are related by the formula [FG04, AI01]

$$\rho^B = \rho^C + dd^*\omega.$$

Moreover, it is well known that locally

$$\rho^C = \frac{i}{2}\partial\overline{\partial}\log\det g.$$

Hence, recalling that $d^* = \partial^* + \overline{\partial}^*$, we obtain that

$$(\rho^B)^{(1,1)} = \partial\partial^*\omega + \overline{\partial\partial}^*\omega + \frac{i}{2}\partial\overline{\partial}\log\det g,$$

where $(\rho^B)^{(1,1)}$ is the $(1,1)$-part of ρ^B; so, as also noted in [ST10], we can rewrite equation (1.1) as

$$\frac{\partial\omega(t)}{\partial t} = -(\rho^B)^{(1,1)}. \tag{1.3}$$

We recall the following

Definition 1.2 ([STb])**.** An SKT metric g on a complex manifold (M, J) is *static* if

$$\Phi(\omega) = \lambda\omega \tag{1.4}$$

for some $\lambda \in \mathbb{R}$.

Unlike [STb], we don't add a normalization condition, but accept that every multiple of a static metric is still static with the same λ.

In [STb], Proposition 5.10, it is shown that if g is a SKT static metric with $\lambda \neq 0$, then the 2-form

$$\Omega = \omega - \frac{1}{\lambda}(\overline{\partial}\partial^*\omega + \partial\overline{\partial}^*\omega)$$

is symplectic and tames J. Thanks to equation (1.3), a static metric can be also viewed as an SKT metric such that the $(1,1)$-part of ρ^B is a scalar multiple of the fundamental 2-form ω at every point of M, that is

$$-(\rho^B)^{(1,1)} = \lambda\omega, \tag{1.5}$$

so if $\lambda \neq 0$, clearly $\lambda\Omega = -\rho^B$. Therefore, if a metric is static with $\lambda \neq 0$, then $\frac{1}{\lambda}\rho^B$ is a symplectic form and tames J.

Moreover, in [EFV10] it was proved that giving a symplectic form Ω which tames J is equivalent to assign an SKT metric such that $\partial\omega = \overline{\partial}\beta$ for some ∂-closed (2,0)-form β; but this implies that $[\partial\omega] = [0]$ in $H^{2,1}_{\overline{\partial}}(M)$. So, applying Theorem 1.1 we have

Proposition 1.3.
Let (M, J) be a complex manifold, then a solution of the flow (1.1) with initial value ω_0 can reach a static metric with $\lambda \neq 0$ only if $\partial\omega_0$ is $\overline{\partial}$-exact.

2. Static metrics on Lie groups

Let G be a Lie group; we say that a complex structure J on G is *left-invariant* if it is induced by a complex structure $\tilde{J}$ on the Lie algebra $\mathfrak{g}$ of G. In the same way, a left-invariant static metric g on a Lie group G endowed with a left-invariant complex structure J is determined by a $\tilde{J}$-Hermitian metric $\tilde{g}$ on the Lie algebra $\mathfrak{g}$ such that the Ricci tensor of the Bismut connection is proportional to the fundamental 2-form, i.e. $(\rho^B)^{(1,1)} = \lambda\tilde{\omega}$. In this section we consider two significative classes of Lie groups and provide some results about the existence of static metrics.

2.1. Compact Lie groups.

If we choose an SKT metric g on a complex manifold (M, J) such that the Bismut connection ∇^B is identically zero, then

$$\Phi(\omega) = -(\rho^B)^{(1,1)} = 0$$

and g is a static metric with $\lambda = 0$. It is well known that this condition holds in the case of a bi-invariant metric on a Lie group, that is a metric g on G which is both left-invariant and right-invariant. In fact, in view of [DF02] we can write the Bismut connection in terms of Lie brackets as

$$\tilde{g}(\nabla^B_X Y, Z) = \frac{1}{2}\Big\{\tilde{g}([X,Y] - [JX, JY], Z) - \tilde{g}([Y,Z] + [JY, JZ], X) - \tilde{g}([X,Z] - [JX, JZ], Y)\Big\} \quad (2.1)$$

where $\tilde{g}$ is the induced bi-invariant metric on the Lie algebra $\mathfrak{g}$ and satisfy

$$\tilde{g}([X,Y], Z) = -\tilde{g}(Y, [X,Z]); \quad (2.2)$$

using (2.2) and the integrability of J we find $\nabla^B = 0$. To prove that $\tilde{g}$ is SKT we write c in terms of Lie brackets as

$$c(X,Y,Z) = -g([JX, JY], Z) - g([JY, JZ], X) - g([JZ, JX], Y), \quad (2.3)$$

then using the integrability of J we have $c(X,Y,Z) = -\frac{1}{2}g([X,Y], Z)$. Applying (2.2) and the Jacobi identity we obtain $dc = 0$.

Since the work of Samelson and Wang [Sam53],[Wan54], it has been known that every compact even-dimensional Lie group G admits a left-invariant complex structure J_L and a right-invariant one J_R. If morevoer G is semi-simple, the bi-invariant metric g_K induced by the Killing form is compatible with both J_L, J_R. So

Theorem 2.1.
Let G be a compact, even-dimensional semi-simple Lie group. Then it admits a static metric with $\lambda = 0$.

A remarkable example in this class is the Hopf surface $H = S^3 \times S^1$. By Theorem 2.1, H admits a static metric with $\lambda = 0$ as also noticed in [STb]. Moreover, for any SKT metric on H (not necessarily invariant) the cohomology class $[\partial\omega] \in H^{2,1}_{\overline{\partial}}(M)$ is nonzero [Gua10]. Therefore *all the static metrics* (both invariant and non-invariant) *on the Hopf surface have* $\lambda = 0$.

As noted by Gualtieri [Gua10], compact even-dimensional semi-simple Lie groups are also examples of generalized Kähler manifolds. We recall that a generalized Kähler structure on M^{2n} can be seen as a couple of integrable complex structure J_+, J_-, both compatible with a Riemannian metric g, that satisfy the conditions

$$\begin{cases} d^c_+\omega_+ = -d^c_-\omega_- \\ d(d^c_\pm\omega_\pm) = 0, \end{cases}$$

where $\omega_\pm(\cdot,\cdot) = g_k(\cdot, J_\pm\cdot)$ and $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$. If M is a compact even-dimensional semi-simple Lie group, (g, J_L, J_R) defines a generalized Kähler structure.

2.2. **Nilmanifolds.**

We recall that a *nilmanifold* is a compact quotient of a simply connected nilpotent Lie group G by a discrete subgroup Γ. By invariant Riemannian metric (complex structure) on G/Γ we mean one induced by a Riemannian metric (integrable complex structure) on the Lie algebra $\mathfrak{g}$ of G. It is well known that a nilmanifold cannot admit any Kähler metric unless it is a torus (see for example [BG88, Has89]), and results about classification of SKT metrics on nilmanifold have been found in [FPS02, EFV10]. Moreover, in [EFV10] it is proved that a nilmanifold (not a torus) together with an inariant complex structure J cannot admit any symplectic form taming J, so in particular we cannot find any static metric with $\lambda \neq 0$. We will show that if the metric is invariant, then it cannot satisfy the relation $(\rho^B)^{(1,1)} = 0$ unless G/Γ it is a torus, i.e. the torus is the unique nilmanifold that admit invariant static metrics with $\lambda = 0$.

Since we are considering invariant metrics, we can reduce to the study of nilpotent Lie algebras. We recall that a Lie algebra $\mathfrak{g}$ is nilpotent if the descending central series $\{\mathfrak{g}^k\}_{k\geqslant 0}$ defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}] \quad \dots \quad \mathfrak{g}^k = [\mathfrak{g}^{k-1},\mathfrak{g}]$$

vanishes for some $k > 0$. A static metric is in particular SKT, and as found in [EFV10] any SKT-nilpotent Lie algebra $\mathfrak{g}$ is 2-step (i.e. $\mathfrak{g}^2 = \{0\}$) and his center is J-invariant; therefore we can split $\mathfrak{g}$ in $\xi \oplus \xi^\perp$, where ξ is the center, $\xi^\perp$ the orthogonal complement to the center with respect to the SKT metric and $[\xi^\perp, \xi^\perp] \subset \xi$, so for every $X \in \mathfrak{g}$ we have a unique decomposition $X = X^\xi + X^\perp$, where $X^\xi \in \xi$ and $X^\perp \in \xi^\perp$.

In the following lemmas we make some calculations about the Bismut connection and the SKT condition:

Lemma 2.2.
Let $\mathfrak{g}$ be a nilpotent Lie algebra together with an integrable complex structure J and a J-Hermitian SKT metric g, and ∇^B its Bismut connection. Then

(1) $\nabla^B_{X^\xi} Y^\xi = 0$

(2) $\nabla^B_{X^\xi} Y^\perp \in \xi^\perp$ *and*
$$g(\nabla^B_{X^\xi} Y^\perp, Z) = -\frac{1}{2} g([Y^\perp, Z] + [JY^\perp, JZ], X^\xi)$$

(3) $\nabla^B_{X^\perp} Y^\xi \in \xi^\perp$ *and*
$$g(\nabla^B_{X^\perp} Y^\xi, Z) = -\frac{1}{2} g([X^\perp, Z] - [JX^\perp, JZ], Y^\xi);$$
moreover,
$$J\nabla^B_{JX^\perp} Y^\xi = \nabla^B_{X^\perp} Y^\xi \tag{2.4}$$

(4) $\nabla^B_{X^\perp} Y^\perp = \frac{1}{2}([X^\perp, Y^\perp] - [JX^\perp, JY^\perp]) \in \xi.$

Proof.
The relations *(1)*,*(2)*,*(4)* and the first part of *(3)* comes directly applying formula (2.1) and using the definition of ξ. Equation (2.4) is obtained using the first part of *(3)* and the integrability of J.

□

Lemma 2.3.
Let $\mathfrak{g}$ be a nilpotent Lie algebra together with an integrable complex structure J and a J-Hermitian SKT metric g. Then

$$g([X,JX],[Y,JY]) = \frac{1}{2}\big(\|[X,Y]\|^2 + \|[X,JY]\|^2 + \|[JX,Y]\|^2 + \|[JX,JY]\|^2\big)$$

for every $X, Y \in \mathfrak{g}$.

Proof.
If X or Y belongs to the center, then the lemma is obviously true; so we consider the case $X, Y \in \xi^\perp$. Using (2.1) and (2.3) we have

$$\begin{aligned}
0 = dc(X,Y,JX,JY) = & - c([X,Y],JX,JY) + c([X,JX],Y,JY) - c([X,JY],Y,JX) - \\
& - c([Y,JX],X,JY) + c([Y,JY],X,JX) - c([JX,JY],X,Y) \\
= & + g([X,Y],[X,Y]) - g([Y,JY],[X,JX]) + g([X,JY],[X,JY]) \\
& + g([Y,JX],[Y,JX]) - g([Y,JY],[X,JX]) + g([JX,JY],[JX,JY]) \\
= & - 2g([X,JX],[Y,JY]) + \|[X,Y]\|^2 + \|[X,JY]\|^2 + \|[JX,Y]\|^2 + \\
& + \|[JX,JY]\|^2
\end{aligned}$$

as required.

□

Now we are ready to prove the

Theorem 2.4.
Let G/Γ a nilmanifold (not a torus) together with an invariant complex structure J. Then it does not admit any J-Hermitian invariant static metric with $\lambda = 0$.

Proof.
Let $\mathfrak{g}$ the Lie algebra of G, $\tilde{J}$ the induced integrable complex structure and g a $\tilde{J}$-Hermitian SKT metric; we have $\mathfrak{g} = \xi \oplus \xi^\perp$. Choose $\{\mathbf{e}_1, \dots, \mathbf{e}_{2m}\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_{2k}\}$ to be orthonormal basis respectively of $\xi^\perp$ and ξ with $2m + 2k = 2n = \dim \mathfrak{g}$; then $\{\mathbf{e}_1, \dots, \mathbf{e}_{2m}, \mathbf{f}_1, \dots, \mathbf{f}_{2k}\}$ is an orthonormal basis of $\mathfrak{g}$. Note that $(\rho^B)^{(1,1)}(X, \tilde{J}X) = \rho^B(X, \tilde{J}X)$, so in order to prove that $(\rho^B)^{(1,1)} \neq 0$ we will show that $\rho^B(X, \tilde{J}X)$ is not zero for some $X \in \mathfrak{g}$.

Suppose $X \in \xi^\perp$; by definition,

$$\rho^B(X,\tilde{J}X) = \frac{1}{2}\Big(\sum_{i=1}^{2m} g(R^B(X,\tilde{J}X)\mathbf{e}_i, \tilde{J}\mathbf{e}_i) + \sum_{j=1}^{2k} g(R^B(X,\tilde{J}X)\mathbf{f}_j, \tilde{J}\mathbf{f}_j)\Big);$$

we consider the two summation separately.

- By definition of R^B, we obtain

  $$g(R^B(X,\tilde{J}X)\mathbf{e}_i, \tilde{J}\mathbf{e}_i) = g(\nabla^B_X \nabla^B_{\tilde{J}X}\mathbf{e}_i, \tilde{J}\mathbf{e}_i) - g(\nabla^B_{\tilde{J}X}\nabla^B_X \mathbf{e}_i, \tilde{J}\mathbf{e}_i) - g(\nabla^B_{[X,\tilde{J}X]}\mathbf{e}_i, \tilde{J}\mathbf{e}_i).$$

  Applying Lemma 2.2 and using the integrability of $\tilde{J}$ we have

  $$g(\nabla^B_X \nabla^B_{\tilde{J}X}\mathbf{e}_i, \tilde{J}\mathbf{e}_i) = -g(\nabla^B_{\tilde{J}X}\nabla^B_X \mathbf{e}_i, \tilde{J}\mathbf{e}_i) = -\frac{1}{4}\|[X,\mathbf{e}_i] - [\tilde{J}X, \tilde{J}\mathbf{e}_i]\|^2$$

and

$$g(\nabla^B_{[X,\tilde{J}X]}\mathbf{e}_i, \tilde{J}\mathbf{e}_i) = -g([X,\tilde{J}X],[\mathbf{e}_i,\tilde{J}\mathbf{e}_i]),$$

so

$$g(R^B(X,\tilde{J}X)\mathbf{e}_i,\tilde{J}\mathbf{e}_i) = -\frac{1}{2}\|[X,\mathbf{e}_i]-[\tilde{J}X,\tilde{J}\mathbf{e}_i]\|^2 + g([X,\tilde{J}X],[\mathbf{e}_i,\tilde{J}\mathbf{e}_i]). \tag{2.5}$$

- Again by definition of R^B and applying Lemma 2.2 and equation (2.4), we obtain

$$\begin{aligned} g(R^B(X,\tilde{J}X)\mathbf{f}_j,\tilde{J}\mathbf{f}_j) &= g(\nabla^B_X\nabla^B_{\tilde{J}X}\mathbf{f}_j,\tilde{J}\mathbf{f}_j) - g(\nabla^B_{\tilde{J}X}\nabla^B_X\mathbf{f}_j,\tilde{J}\mathbf{f}_j)\\ &= \frac{1}{2}g([X,\nabla^B_{\tilde{J}X}\mathbf{f}_j - \tilde{J}\nabla^B_X\mathbf{f}_j] - [\tilde{J}X,\tilde{J}\nabla^B_{\tilde{J}X}\mathbf{f}_j + \nabla^B_X\mathbf{f}_j],\tilde{J}\mathbf{f}_j)\\ &= g([X,\nabla^B_{\tilde{J}X}\mathbf{f}_j] - [\tilde{J}X,\tilde{J}\nabla^B_{\tilde{J}X}\mathbf{f}_j],\tilde{J}\mathbf{f}_j). \end{aligned}$$

By Lemma 2.2 $\nabla^B_{\tilde{J}X}\mathbf{f}_j \in \xi^\perp$, so we can write

$$\nabla^B_{\tilde{J}X}\mathbf{f}_j = \sum_{i=1}^{2m} g(\nabla^B_{\tilde{J}X}\mathbf{f}_j,\mathbf{e}_i)\,\mathbf{e}_i = -\frac{1}{2}\sum_{i=1}^{2m} g([\tilde{J}X,\mathbf{e}_i]+[X,\tilde{J}\mathbf{e}_i],\mathbf{f}_j)\,\mathbf{e}_i,$$

then

$$[X,\nabla^B_{\tilde{J}X}\mathbf{f}_j] = -\frac{1}{2}\sum_{i=1}^{2m} g([\tilde{J}X,\mathbf{e}_i]+[X,\tilde{J}\mathbf{e}_i],\mathbf{f}_j)[X,\mathbf{e}_i]$$

and

$$[\tilde{J}X,\tilde{J}\nabla^B_{\tilde{J}X}\mathbf{f}_j] = -\frac{1}{2}\sum_{i=1}^{2m} g([\tilde{J}X,\mathbf{e}_i]+[X,\tilde{J}\mathbf{e}_i],\mathbf{f}_j)[\tilde{J}X,\tilde{J}\mathbf{e}_i].$$

Now, using the integrability of $\tilde{J}$ and the $\tilde{J}$-invariance of g

$$g([X,\nabla^B_{\tilde{J}X}\mathbf{f}_j] - [\tilde{J}X,\tilde{J}\nabla^B_{\tilde{J}X}\mathbf{f}_j],\tilde{J}\mathbf{f}_j) = \frac{1}{2}\sum_{i=1}^{2m}\big[g([\tilde{J}X,\mathbf{e}_i]+[X,\tilde{J}\mathbf{e}_i],\mathbf{f}_j)\big]^2;$$

but $\mathbf{f}_j$ is an orthonormal basis of ξ and $[\tilde{J}X,\mathbf{e}_i]+[X,\tilde{J}\mathbf{e}_i]\in\xi$, so

$$\sum_{j=1}^{2k}\big[g([\tilde{J}X,\mathbf{e}_i]+[X,\tilde{J}\mathbf{e}_i],\mathbf{f}_j)\big]^2 = \|[X,\mathbf{e}_i]-[\tilde{J}X,\tilde{J}\mathbf{e}_i]\|^2.$$

Finally, we have

$$\sum_{j=1}^{2k} g(R^B(X,\tilde{J}X)\mathbf{f}_j,\tilde{J}\mathbf{f}_j) = \frac{1}{2}\sum_{i=1}^{2m}\|[X,\mathbf{e}_i]-[\tilde{J}X,\tilde{J}\mathbf{e}_i]\|^2. \tag{2.6}$$

Combining equations (2.5) and (2.6) we obtain

$$\begin{aligned} \rho^B(X,\tilde{J}X) = &+\frac{1}{2}\sum_{i=1}^{2m}\Big(-\frac{1}{2}\|[X,\mathbf{e}_i]-[\tilde{J}X,\tilde{J}\mathbf{e}_i]\|^2 + g([X,\tilde{J}X],[\mathbf{e}_i,\tilde{J}\mathbf{e}_i])\Big)+\\ &+\frac{1}{4}\sum_{i=1}^{2m}\Big(\|[X,\mathbf{e}_i]-[\tilde{J}X,\tilde{J}\mathbf{e}_i]\|^2\Big)\\ =&\frac{1}{2}\sum_{i=1}^{2m} g([X,\tilde{J}X],[\mathbf{e}_i,\tilde{J}\mathbf{e}_i]) \end{aligned}$$

and using Lemma 2.3

$$= \frac{1}{4}\sum_{i=1}^{2m} \left(\|[X,\mathbf{e}_i]\|^2 + \|[X,\tilde{J}\mathbf{e}_i]\|^2 + \|[\tilde{J}X,\mathbf{e}_i]\|^2 + \|[\tilde{J}X,\tilde{J}\mathbf{e}_i]\|^2 \right)$$

$$\geqslant \frac{1}{4}\sum_{i=1}^{2m} \|[X,\mathbf{e}_i]\|^2 > 0$$

since $X \in \xi^{\perp}$; this concludes the proof.

□

Remark 2.5. The results of this section can be summarize as follows: let G/Γ a nilmanifold (not a torus) together with an invariant complex structure J and a J-invariant SKT metric g; then if g is a static metric, it must be non-invariant and with $\lambda = 0$.

Whether such metrics exists or not is still not known to the author, but an approach to the problem could be the following: let G/Γ be a nilmanifold and μ a volume element induced by a bi-invariant one on the Lie group G [Mil76]. After rescaling, we can suppose that G/Γ has volume equal to 1. Suppose that G/Γ is endowed with an invariant complex structure J induced by an integrable complex structure $\tilde{J}$ on the Lie algebra $\mathfrak{g}$ of G; Belgun [Bel00] showed that if we choose a J-Hermitian, non-invariant metric g on G/Γ, then we can define a $\tilde{J}$-Hermitian metric $\tilde{g}$ on $\mathfrak{g}$ by posing

$$\tilde{g}(X,Y) = \int_{m\in M} g_m(X_m, Y_m)\, d\mu$$

for any left-invariant vector fields X, Y. This method is called *symmetrization process*. Moreover, in [Uga07] it was proved that if the metric g is SKT, then $\tilde{g}$ is still SKT. Thus, if a nilmanifold admits a non-invariant static metric with $\lambda = 0$, then it necessarily induces an SKT metric $\tilde{g}$ on $\mathfrak{g}$. In general, however, it is not true that the Ricci form $\tilde{\rho}^B$ of the metric $\tilde{g}$ is obtained by the symmetrization of the Ricci tensor ρ^B of g, so it is an open problem to check if the induced invariant metric $\tilde{g}$ is still static.

3. Static metrics in dimension 4

In this section we classify all the invariant static metrics on Lie groups of dimension 4 endowed with a left-invariant complex structure. Since we are interested in invariant structures on Lie groups, it is sufficient to study the induced structures on the corresponding Lie algebra.

Let $\mathfrak{g}$ a Lie algebra; we can define the *derived series* of $\mathfrak{g}$ as $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}],\ \mathfrak{g}^k = [\mathfrak{g}^{k-1},\mathfrak{g}^{k-1}]$. We say that $\mathfrak{g}$ is *solvable* if there exists an integer s such that $\mathfrak{g}^s = 0$. According to [BB81], a Lie algebra of dimension 4 is either solvable, isomorphic to $\mathfrak{su}(2)\times\mathbb{R}$ or isomorphic to $\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}$.

In the sequel we use a shorten notation to identify Lie algebras: for example, $\mathfrak{g} = (0, -2\cdot 12 - 2\cdot 34, -13, -14)$ means that $\mathfrak{g}^*$ admits a basis $\{\mathbf{f}^i\}$ of real 1-forms such that

$$\begin{cases} d\mathbf{f}^1 = 0 \\ d\mathbf{f}^2 = -2\mathbf{f}^{12} - 2\mathbf{f}^{34} \\ d\mathbf{f}^3 = -\mathbf{f}^{13} \\ d\mathbf{f}^4 = -\mathbf{f}^{14}. \end{cases}$$

Theorem 3.1.
The only 4-dimensional Lie algebras admitting an Hermitian structure (g, J) with g static are isomorphic to

$$\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}} = (0, -12, 0, -34)$$
$$\mathfrak{d}'_{4,|-\frac{1}{t}|} = (0, -2\cdot 12 - 2\cdot 34, -13 - t\cdot 14, t\cdot 13 - 14)$$
$$\mathfrak{d}_{4,\frac{1}{2}} = (0, -2\cdot 12 - 2\cdot 34, -13, -14)$$
$$\mathfrak{r}'_{3,0} \times \mathbb{R} = (0, 0, 14, -13)$$
$$\mathfrak{su}(2) \times \mathbb{R}.$$

Moreover, except for $\mathfrak{su}(2)\times\mathbb{R}$, all such Hermitian structures are Kähler-Einstein.

Proof.
We first consider the case of $\mathfrak{g}$ solvable. A classification of 4-dimensional solvable Lie algebras admitting a left-invariant integrable complex structure can be found in [ABDO05, Ova04, Sno90], and recently Madsen and Swann in [MS09] gave a classification of SKT structures on solvable Lie algebras of dimension four. With the help of a Maple software, we use this classification to compute directly the Ricci tensor of the Bismut connection and then we impose the static condition (1.5) to find all the static metrics. According to [MS09], we can suppose that if $\mathfrak{g}$ is solvable and admits an SKT structure the integrable complex structure J is defined by $Je^1 = e^2,\ Je^3 = e^4$, where $\{e^1, e^2, e^3, e^4\}$ is a basis of $\mathfrak{g}^*$. Moreover, $\mathfrak{g}$ belongs to one of the following cases:

- **Complex case**: $\mathfrak{g}$ has structural equations

$$\begin{cases} d\mathbf{e}^1 = 0\\ d\mathbf{e}^2 = a_1\mathbf{e}^{12}\\ d\mathbf{e}^3 = b_1\mathbf{e}^{12} + b_2\mathbf{e}^{13} + b_3\mathbf{e}^{14} - c_1\mathbf{e}^{23} + c_2\mathbf{e}^{24}\\ d\mathbf{e}^4 = d_1\mathbf{e}^{12} + d_2\mathbf{e}^{13} + d_3\mathbf{e}^{14} - f_1\mathbf{e}^{23} + f_2\mathbf{e}^{24} + h_1\mathbf{e}^{34}\end{cases}$$

  and the real coefficients $a_1, b_i, c_i, d_i, f_i, h_1$ satisfy the following equations:

$$\begin{array}{ll} f_1 - c_2 - d_3 + b_2 = 0 & f_2 + c_1 - d_2 - b_3 = 0\\ a_1c_1 - b_3f_1 - c_2d_2 = 0 & c_2a_1 - c_2b_2 + c_2d_3 - b_3c_1 - b_3f_2 = 0\\ h_1\left({b_2}^2 + {b_3}^2 + {c_1}^2 + {c_2}^2\right) = 0 & f_1a_1 + f_1b_2 - f_1d_3 - d_2c_1 - f_2d_2 + h_1d_1 = 0\\ a_1f_2 + b_1h_1 - b_3f_1 - c_2d_2 = 0 & (a_1 + b_2 + d_3)(b_2 + d_3) + (c_1 - f_2)^2 - h_1d_1 = 0. \end{array}$$

  The fundamental 2-form of the SKT metric is $\omega = \mathbf{e}^{12} + \mathbf{e}^{34}$, and we obtain

$$\begin{aligned}\Phi(X,Y) =& -({a_1}^2 + {b_1}^2 + {d_1}^2 + \frac{1}{2}f_1a_1 - \frac{1}{2}d_3a_1 + \frac{1}{2}c_2a_1 - \frac{1}{2}b_2a_1 + h_1d_1)\cdot\mathbf{e}^{12} -\\ &-(\frac{1}{2}b_1b_2 + \frac{1}{2}d_1d_2 + \frac{1}{2}h_1d_2 + \frac{1}{2}b_1c_2 + \frac{1}{2}d_1f_2 + \frac{1}{2}h_1f_2)\cdot(\mathbf{e}^{13} + \mathbf{e}^{24}) -\\ &-(\frac{1}{2}b_1b_3 + \frac{1}{2}d_1d_3 + \frac{1}{2}d_3h_1 + \frac{1}{2}b_1c_1 + \frac{1}{2}d_1f_1 + \frac{1}{2}h_1f_1)\cdot(\mathbf{e}^{14} - \mathbf{e}^{23}) -\\ &-(\frac{1}{2}f_1c_2 - \frac{1}{4}{d_2}^2 - \frac{1}{2}d_2b_3 - \frac{1}{4}{f_1}^2 + \frac{1}{2}b_3f_2 + h_1d_1 - \frac{1}{2}c_2d_3 - \frac{1}{4}{d_3}^2 +\\ &+\frac{1}{2}f_2d_2 - \frac{1}{4}{f_2}^2 + \frac{1}{2}f_1d_3 + {h_1}^2 - \frac{1}{4}{b_3}^2 - \frac{1}{4}{c_2}^2 + \frac{1}{2}b_2d_3 +\\ &+\frac{1}{2}d_2c_1 - \frac{1}{2}f_2c_1 - \frac{1}{4}{b_2}^2 - \frac{1}{2}f_1b_2 + \frac{1}{2}c_2b_2 - \frac{1}{4}{c_1}^2 + \frac{1}{2}c_1b_3)\cdot\mathbf{e}^{34}.\end{aligned}$$

Imposing the static condition we obtain two possible cases, depending on whether $\lambda = 0$ or $\lambda \neq 0$. In the first case the structure equations are

$$\begin{cases} d\mathbf{e}^1 = d\mathbf{e}^2 = 0 \\ d\mathbf{e}^3 = b_3\mathbf{e}^{14} + c_2\mathbf{e}^{24} \\ d\mathbf{e}^4 = -b_3\mathbf{e}^{13} - c_2\mathbf{e}^{23}, \end{cases} \tag{3.1}$$

and by [MS09] we have that $\mathfrak{g} \cong \mathfrak{r}'_{3,0} \times \mathbb{R}$ and that g is Kähler-Einstein with $\lambda = 0$.

Since g is KE, in particular it is Einstein, so it must be contained in the classification of homogeneous spaces of dimension 4 admitting Einstein metrics given in [Jen69].

With the change of basis given by

$$\mathbf{f}_1 = \frac{b_3\mathbf{e}_1 + c_2\mathbf{e}_2}{b_3^2 + c_2^2}, \qquad \mathbf{f}_2 = \frac{-c_2\mathbf{e}_1 + b_3\mathbf{e}_2}{b_3^2 + c_2^2}, \qquad \mathbf{f}_3 = \frac{\mathbf{e}_3}{\sqrt{b_3^2 + c_2^2}}, \qquad \mathbf{f}_4 = \frac{\mathbf{e}_4}{\sqrt{b_3^2 + c_2^2}}$$

we obtain the structure equations

$$\begin{cases} d\mathbf{f}^1 = d\mathbf{f}^2 = 0 \\ d\mathbf{f}^3 = \mathbf{f}^{14} \\ d\mathbf{f}^4 = -\mathbf{f}^{13}, \end{cases}$$

the metric g becomes

$$g = \frac{1}{b_3^2 + c_2^2}\left(\sum_{i=1}^{4} \mathbf{f}^i \otimes \mathbf{f}^i\right)$$

and J is again defined by $J\mathbf{f}_1 = -\mathbf{f}_2$ and $J\mathbf{f}_3 = -\mathbf{f}_4$; so this corresponds to case 1 of the Theorem of Chapter III in [Jen69].

On the other end, if $\lambda \neq 0$ the structure equations are

$$\begin{cases} d\mathbf{e}^1 = 0 \\ d\mathbf{e}^2 = a_1\mathbf{e}^{12} \\ d\mathbf{e}^3 = 0 \\ d\mathbf{e}^4 = \pm a_1\mathbf{e}^{34} \end{cases} \tag{3.2}$$

with $a_1 \neq 0$; the metric g is Kähler-Einstein with $\lambda = -a_1^2 < 0$, and $\mathfrak{g} \cong \mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$.

With the change of basis given by

$$\mathbf{f}_1 = -\frac{\mathbf{e}_1}{a_1} \qquad \mathbf{f}_2 = -\frac{\mathbf{e}_2}{a_1} \qquad \mathbf{f}_3 = \pm\frac{\mathbf{e}_3}{a_1} \qquad \mathbf{f}_4 = \pm\frac{\mathbf{e}_4}{a_1}$$

we obtain the structure equations

$$\begin{cases} d\mathbf{f}^1 = d\mathbf{f}^3 = 0 \\ d\mathbf{f}^2 = -\mathbf{f}^{12} \\ d\mathbf{f}^4 = -\mathbf{f}^{34}, \end{cases}$$

the metric g becomes

$$g = \frac{1}{a_1^2}\left(\sum_{i=1}^{4} \mathbf{f}^i \otimes \mathbf{f}^i\right)$$

and J is again defined by $J\mathbf{f}_1 = -\mathbf{f}_2$ and $J\mathbf{f}_3 = -\mathbf{f}_4$; so this corresponds to case 4 of the Theorem of Chapter III in [Jen69].

- **Real case I**: $\mathfrak{g}$ has structural equations

$$\begin{cases} d\mathbf{e}^1 = 0 \\ d\mathbf{e}^2 = a_1\mathbf{e}^{12} + a_3(\mathbf{e}^{14} - \mathbf{e}^{23}) + b_2\mathbf{e}^{34} \\ d\mathbf{e}^3 = 0 \\ d\mathbf{e}^4 = d_1\mathbf{e}^{12} + d_3(\mathbf{e}^{14} - \mathbf{e}^{23}) + h_1\mathbf{e}^{34}, \end{cases}$$

where $d\mathbf{e}^2$ and $d\mathbf{e}^4$ are linearly indipendent and the real coefficients satisfy the following equations:

$$\begin{array}{ll} b_2a_1 - b_2d_3 + f_2a_3 - {a_3}^2 = 0 & d_1f_2 - d_1a_3 + d_3a_1 - {d_3}^2 = 0 \\ d_3a_3 - b_2d_1 = 0 & (d_1 - a_3)(f_2 + a_3) - (d_3 + a_1)(d_3 - b_2) = 0. \end{array}$$

In this case we have that $\mathfrak{g} \cong \mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$. The fundamental 2-form of the SKT metric is $\omega = \mathbf{e}^{12} + \mathbf{e}^{34} + t\mathbf{e}^{14} + t\mathbf{e}^{23}$ with $t \in (-1, 1)$, and it is Kähler if and only if $t = 0$. Computing $(\rho^B)^{(1,1)}$ we find

$$\begin{aligned} \Phi(X,Y) = &+ \frac{b_2a_1 + f_2d_1 + {a_1}^2 + {d_1}^2}{t^2 - 1} \cdot \mathbf{e}^{12} + \frac{{b_2}^2 + {f_2}^2 + b_2a_1 + f_2d_1}{t^2 - 1} \cdot \mathbf{e}^{34} + \\ &+ \frac{b_2a_3 + f_2d_3 + a_3a_1 + d_3d_1}{t^2 - 1} \cdot (\mathbf{e}^{14} - \mathbf{e}^{23}), \end{aligned}$$

and imposing the static condition we obtain the structure equations

$$\begin{cases} d\mathbf{e}^1 = d\mathbf{e}^3 = 0 \\ d\mathbf{e}^2 = b_2\,\dfrac{3a_3^2 + b_2^2}{b_2^2 - a_3^2}\,\mathbf{e}^{12} + a_3(\mathbf{e}^{14} - \mathbf{e}^{23}) + b_2\,\mathbf{e}^{34} \\ d\mathbf{e}^4 = a_3\,\mathbf{e}^{12} + b_2(\mathbf{e}^{14} - \mathbf{e}^{23}) + a_3\,\dfrac{a_3^2 + 3b_2^2}{a_3^2 - b_2^2}\,\mathbf{e}^{34}, \end{cases} \tag{3.3}$$

with $a_3 \neq 0$. Moreover $t = 0$, so the metric $g = \sum_{i=1}^4 \mathbf{e}^i \otimes \mathbf{e}^i$ is Kähler-Einstein with $\lambda = -2\,\frac{(a_3^2+b_2^2)^3}{(a_3^2-b_2^2)^2} < 0$.

With the change of basis given by

$$\begin{array}{ll} \mathbf{f}_1 = \dfrac{(a_3^2 - b_2^2)^2}{2(a_3^2 + b_2^2)^2}\left(-\dfrac{\mathbf{e}_1}{a_3 + b_2} - \dfrac{\mathbf{e}_3}{a_3 - b_2}\right) & \mathbf{f}_2 = \dfrac{(a_3^2 - b_2^2)^2}{2(a_3^2 + b_2^2)^2}\left(-\dfrac{\mathbf{e}_2}{a_3 + b_2} - \dfrac{\mathbf{e}_4}{a_3 - b_2}\right) \\ \mathbf{f}_3 = \dfrac{(a_3^2 - b_2^2)^2}{2(a_3^2 + b_2^2)^2}\left(\dfrac{\mathbf{e}_1}{a_3 - b_2} - \dfrac{\mathbf{e}_3}{a_3 + b_2}\right) & \mathbf{f}_4 = \dfrac{(a_3^2 - b_2^2)^2}{2(a_3^2 + b_2^2)^2}\left(\dfrac{\mathbf{e}_2}{a_3 - b_2} - \dfrac{\mathbf{e}_4}{a_3 + b_2}\right) \end{array}$$

we obtain the structure equations

$$\begin{cases} d\mathbf{f}^1 = d\mathbf{f}^3 = 0 \\ d\mathbf{f}^2 = -\mathbf{f}^{12} \\ d\mathbf{f}^4 = -\mathbf{f}^{34}, \end{cases}$$

the metric g becomes

$$g = 2\,\frac{(a_3^2 + b_2^2)^3}{(a_3^2 - b_2^2)^2}\Big(\sum_{i=1}^4 \mathbf{f}^i \otimes \mathbf{f}^i\Big)$$

and J is again defined by $J\mathbf{f}_1 = -\mathbf{f}_2$ and $J\mathbf{f}_3 = -\mathbf{f}_4$; so this corresponds to case 4 of the Theorem of Chapter III in [Jen69].

- **Real case II**: $\mathfrak{g}$ has structural equations

$$\begin{cases} d\mathbf{e}^1 = 0 \\ d\mathbf{e}^2 = -kq^2\,\mathbf{e}^{12} - kqr(\mathbf{e}^{14} - \mathbf{e}^{23}) - kr^2\,\mathbf{e}^{34} \\ d\mathbf{e}^3 = \dfrac{c_3 q}{r}\,\mathbf{e}^{12} + c_3\,\mathbf{e}^{14} \\ d\mathbf{e}^4 = \dfrac{kq^3}{r}\,\mathbf{e}^{12} - c_3\,\mathbf{e}^{13} + kq^2(\mathbf{e}^{14} - \mathbf{e}^{23}) + kqr\,\mathbf{e}^{34}, \end{cases}$$

with $q, r, k \in \mathbb{R}$ such that $q^2+r^2 = 1,\ r > 0$ and $k \neq 0$; in this case $\mathfrak{g} \cong \mathfrak{d}'_{4,0}$. The fundamental 2-form of the SKT metric is $\omega = \mathbf{e}^{12} + \mathbf{e}^{34} + t\mathbf{e}^{14} + t\mathbf{e}^{23}$, with $t \in (-1, 1)$, and it is never Kähler. Computing $(\rho^B)^{(1,1)}$ we find

$$\begin{aligned} \Phi(X,Y) = &+ \frac{q\left(k^2qr^4 - tkr^3c_3 + 2\,k^2q^3r^2 - c_3kq^2tr + {c_3}^2q + k^2q^5\right)}{r^2\,(t^2-1)} \cdot \mathbf{e}^{12} - \\ &- \frac{1}{2}\,\frac{c_3\left(c_3tr + kq^3 + kqr^2\right)}{r\,(t^2-1)} \cdot (\mathbf{e}^{13} + \mathbf{e}^{24}) + \frac{k^2\left(r^4 + 2\,q^2r^2 + q^4\right)}{t^2-1} \cdot \mathbf{e}^{34} + \\ &+ \frac{1}{2}\,\frac{-c_3kq^2tr + 4\,k^2q^3r^2 + 2\,k^2q^5 + {c_3}^2q - tkr^3c_3 + 2\,k^2qr^4}{r\,(t^2-1)} \cdot (\mathbf{e}^{14} - \mathbf{e}^{23}), \end{aligned}$$

and imposing the static condition we obtain that $q = r = 0$, that contradicts the condition $q^2 + r^2 = 1$. So we don't have any static metrics.

- **Real case III**: $\mathfrak{g}$ has structural equations

$$\begin{cases} d\mathbf{e}^1 = 0 \\ d\mathbf{e}^2 = -k(1+q^2)\,\mathbf{e}^{12} - kqr(\mathbf{e}^{14} - \mathbf{e}^{23}) - kr^2\,\mathbf{e}^{34} \\ d\mathbf{e}^3 = \dfrac{c_3 q}{r}\,\mathbf{e}^{12} - \dfrac{k}{2}\,\mathbf{e}^{13} + c_3\,\mathbf{e}^{14} \\ d\mathbf{e}^4 = \dfrac{q}{r}(kq^2 + \dfrac{k}{2})\,\mathbf{e}^{12} - c_3\,\mathbf{e}^{13} + (kq^2 - \dfrac{k}{2})\mathbf{e}^{14} - kq^2\mathbf{e}^{23} + kqr\,\mathbf{e}^{34}, \end{cases}$$

with $q, r, k \in \mathbb{R}$ such that $q^2 + r^2 = 1,\ r > 0$ and $k \neq 0$; if $c_3 = 0$ we have $\mathfrak{g} \cong \mathfrak{d}_{4,\frac{1}{2}}$, otherwise $\mathfrak{g} \cong \mathfrak{d}'_{4,|\frac{k}{2c_3}|}$. The fundamental 2-form of the SKT metric is $\omega = \mathbf{e}^{12} + \mathbf{e}^{34} + t\mathbf{e}^{14} + t\mathbf{e}^{23}$ with $t \in (-1, 1)$, and is Kähler if and only if $q = 0$. Computing $(\rho^B)^{(1,1)}$ we find

$$\begin{aligned} \Phi(X,Y) = &+ \frac{1}{4}\Big(\frac{2\,r^2k^2 - 4\,ktr^3c_3q - 8\,krc_3qt - 12\,krq^3c_3t + 4\,r^4k^2 + 4\,q^2r^2k^2}{r^2\,(-1+t^2)} + \\ &+ \frac{4\,q^4k^2 + 4\,{c_3}^2q^2 + 4\,k^2q^6 + k^2q^2 + 4\,r^4k^2q^2}{r^2\,(-1+t^2)}\Big) \cdot \mathbf{e}^{12} + \\ &+ \frac{1}{8}\,\frac{-4\,{c_3}^2tr + 2\,tk^2rq^2 + k^2tr + 4\,kq^3c_3 + 2\,k^2tr^3 - 4\,r^2kc_3q}{r\,(-1+t^2)} \cdot (\mathbf{e}^{13} + \mathbf{e}^{24}) - \\ &- \frac{1}{8}\Big(\frac{8\,k^2q^5 - k^2q - 16\,k^2q^3r^2 + 4\,rkc_3tq^2 - 8\,k^2r^4q + 4\,rkc_3t - 2\,k^2qr^2 - 4\,{c_3}^2q}{r\,(-1+t^2)} + \\ &+ \frac{2\,q^3k^2 + 4\,c_3tr^3k}{r\,(-1+t^2)}\Big) \cdot (\mathbf{e}^{14} - \mathbf{e}^{23}) - \frac{1}{2}\,\frac{k^2\left(-r^2 - 2\,r^4 - 4\,q^2r^2 + 2\,q^4 + q^2\right)}{-1+t^2} \cdot \mathbf{e}^{34}, \end{aligned}$$

and imposing the static condition we obtain $q = t = 0$, so the structure equations are

$$\begin{cases} d\mathbf{e}^1 = 0 \\ d\mathbf{e}^2 = -k(\mathbf{e}^{12} + \mathbf{e}^{34}) \\ d\mathbf{e}^3 = -\dfrac{k}{2}\,\mathbf{e}^{13} + c_3\,\mathbf{e}^{14} \\ d\mathbf{e}^4 = -c_3\,\mathbf{e}^{13} - \dfrac{k}{2}\mathbf{e}^{14}. \end{cases} \tag{3.4}$$

Therefore $g = \sum_{i=1}^4 \mathbf{e}^i \otimes \mathbf{e}^i$ is a Kähler-Einstein metric with $\lambda = -\frac{3}{2}k^2 < 0$.

With the change of basis given by

$$\mathbf{f}_1 = \frac{2\mathbf{e}_1}{k} \qquad \mathbf{f}_2 = \frac{2\mathbf{e}_2}{k} \qquad \mathbf{f}_3 = \frac{2\mathbf{e}_3}{k} \qquad \mathbf{f}_4 = \frac{2\mathbf{e}_4}{k}$$

and defining $t = -\frac{2c_3}{k}$ we obtain the structure equations

$$\begin{cases} d\mathbf{f}^1 = 0 \\ d\mathbf{f}^2 = -2\mathbf{f}^{12} - 2\mathbf{f}^{34} \\ d\mathbf{f}^3 = -\mathbf{f}^{13} - t\,\mathbf{f}^{14} \\ d\mathbf{f}^4 = t\,\mathbf{f}^{13} - \mathbf{f}^{14}, \end{cases}$$

the metric g becomes

$$g = \frac{4}{k^2}\Big(\sum_{i=1}^4 \mathbf{f}^i \otimes \mathbf{f}^i\Big)$$

and J is again defined by $J\mathbf{f}_1 = -\mathbf{f}_2$ and $J\mathbf{f}_3 = -\mathbf{f}_4$; so this corresponds to case 2 of the Theorem of Chapter III in [Jen69].

Now we consider the other 4-dimensional Lie algebras $\mathfrak{su}(2) \times \mathbb{R}$ and $\mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}$, whose structure equations are

$$\mathfrak{su}(2) \times \mathbb{R} \begin{cases} d\mathbf{e}^1 = -\mathbf{e}^{23} \\ d\mathbf{e}^2 = \mathbf{e}^{13} \\ d\mathbf{e}^3 = -\mathbf{e}^{12} \\ d\mathbf{e}^4 = 0 \end{cases} \qquad \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R} \begin{cases} d\mathbf{e}^1 = -\mathbf{e}^{23} \\ d\mathbf{e}^2 = \mathbf{e}^{13} \\ d\mathbf{e}^3 = \mathbf{e}^{12} \\ d\mathbf{e}^4 = 0 \end{cases}$$

From a more general result in [RAS10] we obtain that the only integrable complex structures on these algebras are defined in both cases by

$$J\mathbf{e}^1 = \mathbf{e}^2, \qquad J\mathbf{e}^3 = -p \cdot \mathbf{e}^3 + (1 + p^2) \cdot \mathbf{e}^4.$$

With a brief calculation, we find that all the metrics compatible with those complex structures are represented by a symmetric real matrix $(m_{ij})_{i,j=1..4}$ whose coefficients satisfy

$$\begin{aligned} &m_{41} = \frac{-m_{32} + p\,m_{31}}{p^2 + 1}, \qquad m_{42} = \frac{m_{31} + p\,m_{32}}{p^2 + 1}, \qquad m_{12} = 0, \\ &m_{43} = p\,\frac{m_{33} + (p^2 + 1)m_{44}}{2(p^2 + 1)}, \qquad m_{33} = (1 + p^2)\,m_{44}. \end{aligned} \tag{3.5}$$

Note that both these Lie algebras are simple, thus by Theorem 8 of [Chu74] they cannot admit any invariant symplectic structure, so no invariant static metric with $\lambda \neq 0$ can be found on these algebras.

We study the two cases separately:

- $\mathfrak{su}(2)\times\mathbb{R}$
  If we consider a metric g represented by a symmetric real matrix $(m_{ij})_{i,j=1..4}$ whose coefficients satisfy (3.5), we find

$$\begin{aligned}\Phi(X,Y) = &-\frac{-2\,{m_{31}}^2-2\,{m_{32}}^2+(1+p^2)\,m_{44}\,m_{11}-{m_{44}}^2(1+p^2)^2}{-{m_{31}}^2-{m_{32}}^2+m_{44}\,m_{11}\,p^2+m_{44}\,m_{11}}\cdot\mathbf{e}^{12}-\\ &-\frac{1}{2}\,\frac{(1+p^2)(m_{44}m_{32}+m_{11}m_{32}+p\cdot m_{44}m_{31})}{-{m_{31}}^2-{m_{32}}^2+m_{44}\,m_{11}\,p^2+m_{44}\,m_{11}}\cdot(\mathbf{e}^{13}+\mathbf{e}^{24})-\\ &-\frac{1}{2}\,\frac{m_{44}\,p^2m_{31}+m_{44}\,m_{31}+p\,m_{11}m_{32}+m_{31}\,m_{11}}{-{m_{31}}^2-{m_{32}}^2+m_{44}\,m_{11}\,p^2+m_{44}\,m_{11}}\cdot(\mathbf{e}^{14}-\mathbf{e}^{23}).\end{aligned}$$

  As said before, this algebra can only admit static metric with $\lambda=0$; imposing that $(\rho^B)^{(1,1)}=0$ we obtain that $m_{11}=(1+p^2)m_{44}$ and $m_{31}=m_{32}=0$, so every metric in the form

$$\begin{pmatrix}(1+p^2)m_{44} & 0 & 0 & 0\\ 0 & (1+p^2)m_{44} & 0 & 0\\ 0 & 0 & (1+p^2)m_{44} & p\,m_{44}\\ 0 & 0 & p\,m_{44} & m_{44}\end{pmatrix}$$

  is a static metric with $\lambda=0$.
  Note that $\mathfrak{su}(2)\times\mathbb{R}$ is the Lie algebra of the Hopf surface H considered before, so one of those metrics is the bi-invariant one. Computing the Bismut connection we find

$$\nabla^B_XY=\frac{p\,x_4y_2}{p^2+1}\,\mathbf{e}_1-\frac{p\,x_4y_1}{p^2+1}\,\mathbf{e}_2,$$

  so for $p=0$ we have the bi-invariant metric, because this implies $\nabla^B_XY=0$.

- $\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}$
  If we consider a metric g represented by a symmetric real matrix $(m_{ij})_{i,j=1..4}$ whose coefficients satisfy (3.5), we find that

$$\begin{aligned}\Phi(X,Y) = &+\frac{-2\,{m_{31}}^2-2\,{m_{32}}^2+(1+p^2)\,m_{44}\,m_{11}+{m_{44}}^2(1+p^2)^2}{-{m_{31}}^2-{m_{32}}^2+m_{44}\,m_{11}\,p^2+m_{44}\,m_{11}}\cdot\mathbf{e}^{12}-\\ &-\frac{1}{2}\,\frac{(1+p^2)(-m_{44}m_{32}+m_{11}m_{32}-p\cdot m_{44}m_{31})}{-{m_{31}}^2-{m_{32}}^2+m_{44}\,m_{11}\,p^2+m_{44}\,m_{11}}\cdot(\mathbf{e}^{13}+\mathbf{e}^{24})-\\ &-\frac{1}{2}\,\frac{-m_{31}\,m_{44}\,p^2-m_{44}\,m_{31}+p\,m_{11}m_{32}+m_{31}\,m_{11}}{-{m_{31}}^2-{m_{32}}^2+m_{44}\,m_{11}\,p^2+m_{44}\,m_{11}}\cdot(\mathbf{e}^{14}-\mathbf{e}^{23}).\end{aligned}$$

  Again, this algebra can only admit static metric with $\lambda=0$, and imposing that $(\rho^B)^{(1,1)}=0$ we obtain that $m_{11}=-(1+p^2)m_{44}$ and $m_{31}=m_{32}=0$; but $m_{11}m_{44}\leqslant 0$, that is a contradiction because g is positive definite. Then $\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}$ does not admit any static metric.

□

Using theorem 3.1 we note that if $(\mathfrak{g},J)$ is a Lie algebra of dimension 4 together with an integrable complex structure J and g is a static metric, then either (g,J) is Kähler-Einstein or $\mathfrak{g}$ is the Lie algebra of the Hopf surface. This reflects a more general situation that was pointed out in Remark 2 of [AI01] using a result of [GI97]:

Proposition 3.2.
Let (M,J) be a compact complex surface and g a static J-Hermitian metric; then (M,g,J) is conformally equivalent either to a Kähler-Einstein manifold or to a Hopf surface.

References

[ABDO05] A. Andrada, M. L. Barberis, I. G. Dotti, and G. P. Ovando, *Product structures on four dimensional solvable Lie algebras*, Homology Homotopy Appl. **7** (2005), no. 1, 9–37.

[AG07] V. Apostolov and M. Gualtieri, *Generalized Kähler manifolds with split tangent bundle*, Comm. Math. Phys. **271** (2007), 561–575.

[AI01] B. Alexandrov and S. Ivanov, *Vanishing theorems on Hermitian manifolds*, Differential Geom. Appl. **14** (2001), no. 3, 251–265.

[BB81] L. Bérard-Bergery, *Les espaces homogènes riemanniens de dimension 4*, Riemannian geometry in dimension 4 (Paris, 1978/1979), Textes Math., vol. 3, CEDIC, 1981, pp. 40–60.

[Bel00] F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), no. 1, 1–40.

[BG88] C. Benson and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), 513–518.

[Bis89] J. M. Bismut, *A local index theorem for non-Kähler manifolds*, Mathematische Annalen **284** (1989), no. 4, 681–699.

[CG04] G. R. Cavalcanti and M. Gualtieri, *Generalized complex structures on nilmanifolds*, J. Symplectic Geom. **2** (2004), 393–410.

[Chu74] B. Y. Chu, *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc. **197** (1974), 145–159.

[DF02] I. G. Dotti and A. Fino, *HyperKähler torsion structures invariant by nilpotent Lie groups*, Classical Quantum Gravity **19** (2002), no. 3, 551–562.

[EFV10] N. Enrietti, A. Fino, and L. Vezzoni, *Hermitian-symplectic structures and SKT metrics*, Preprint arXiv:1002.3099 (2010).

[FG04] A. Fino and G. Grantcharov, *Properties of manifolds with skew-symmetric torsion and special holonomy*, Adv. Math. **189** (2004), no. 2, 439–450.

[FPS02] A. Fino, M. Parton, and S. Salamon, *Families of strong KT structures in six dimensions*, Comment. Math. Helv. **79** (2002), no. 2, 317–340.

[FT09] A. Fino and A. Tomassini, *Non Kähler solvmanifolds with generalized Kähler structure*, J. Symplectic Geom. **7** (2009), no. 2, 1–14.

[Gau84] P. Gauduchon, *La 1-forme de torsione d'une variété Hermitienne compacte*, Math. Ann. **267** (1984), 495–518.

[Gau97] ———, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 257–288.

[GHR84] S. J. Gates, C. M. Hull, and M. Rŏcek, *Twisted multiplets and new supersymmetric nonlinear sigma models*, Nuc. Phys. B **248** (1984), 157–186.

[GI97] P. Gauduchon and S. Ivanov, *Einstein-Hermitian surfaces and Hermitian Einstein-Weyl structures in dimension* 4, Math. Z. **226** (1997), no. 2, 317–326.

[Gua10] M. Gualtieri, *Generalized Kähler geometry*, Preprint arXiv:1007.3485v1 (2010).

[Has89] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), no. 1, 65–71.

[Hit06] N. Hitchin, *Instantons and generalized Kähler geometry*, Comm. Math. Phys. **265** (2006), 131–164.

[Jen69] G. R. Jensen, *Homogeneous Einstein spaces of dimension four*, J. Differential Geometry **3** (1969), 309–349.

[Mil76] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.

[MS09] T. Madsen and A. Swann, *Invariant strong KT geometry on four-dimensional solvable Lie groups*, Preprint arXiv:0911.0535v1 (2009).

[Ova04] G. Ovando, *Complex, symplectic and Kähler structures on four dimensional Lie groups*, Rev. Un. Mat. Argentina **45** (2004), no. 2, 55–67 (2005).

[RAS10] A. Rezaei-Aghdam and M. Sephid, *Complex and biHermitian structures on four dimensional real Lie algebras*, Preprint arXiv:1002.4285v2 (2010).

[Sam53] H. Samelson, *A class of complex-analytic manifolds*, Portugaliae Math. **12** (1953), 129–132.

[Sno90] J. E. Snow, *Invariant complex structures on four-dimensional solvable real Lie groups*, Manuscripta Math. **66** (1990), no. 4, 397–412.

[STa] J. Streets and G. Tian, *Hermitian curvature flow*, Preprint arXiv:0804.4109v3, to appear in J. Eur. Math. Soc.

[STb] ———, *A parabolic flow of pluriclosed metrics*, Preprint arXiv:0903.4418v2, to appear in Int. Math. Res. Not. IMRN.

[ST10] ———, *Regularity results for pluriclosed flow*, Preprint arXiv:1008.2794v1 (2010).

[Str86] A. Strominger, *Superstrings with torsion*, Nuclear Phys. B **274** (1986), 253–284.

[Swa08] A. Swann, *Twisting Hermitian and Hypercomplex geometries*, Preprint arXiv:0812.2780v1, to appear in Duke Math. J. (2008).

[Uga07] L. Ugarte, *Hermitian structures on six-dimensional nilmanifolds*, Transform. Groups **12** (2007), no. 1, 175–202.

[Wan54] H. Wang, *Complex parallisable manifolds*, Proc. Amer. Math. Soc. **5** (1954), 771–776.

Dipartimento di Matematica G. Peano, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: `nicola.enrietti@unito.it`