

# ON TENSOR PRODUCTS OF WEAK MIXING VECTOR SEQUENCES AND THEIR APPLICATIONS TO $C^*$ - DYNAMICAL SYSTEMS

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**ABSTRACT.** We prove that, under certain conditions, uniform weak mixing (to zero) of the bounded sequences in Banach space implies uniform weak mixing of its tensor product. Moreover, we prove that ergodicity of tensor product of the sequences in Banach space implies its weak mixing. Applications of the obtained results to  $C^*$ -dynamical systems are given.

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## 1. INTRODUCTION

Let  $X$  be a Banach spaces with dual space  $X^*$ . In what follows  $B_X$  denotes the unit ball in  $X$ , i.e.  $B_X = \{x \in X : \|x\| \leq 1\}$ .

Recall that a sequence  $\{x_k\}$  in  $X$  is said to be

(i) *weakly mixing to zero* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |f(x_k)| = 0, \quad \text{for all } f \in X^*;$$

(ii) *uniformly weakly mixing to zero* if

$$\lim_{n \rightarrow \infty} \sup \left\{ \frac{1}{n} \sum_{k=1}^n |f(x_k)| : f \in B_{X^*} \right\} = 0;$$

(iii) *weakly ergodic* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} f(x_k) \right| = 0 \quad \text{for all } f \in X^*;$$

(iii) *ergodic* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} x_k \right\| = 0.$$

From the definitions one can see that uniform weakly mixing implies weakly mixing, as well as ergodicity implies weak ergodicity. But, the converse is not true at all.

*Example 1.1.* [20] Let  $X = L^2([0, 1])$  and  $1 < n_1 < n_2 \cdots$  be a sequence in  $\mathbb{N}$  such that

$$\frac{n_j - 1}{n_{j+1} - 1} \leq \frac{1}{2}, \quad j \in \mathbb{N}$$

(for example,  $n_1 = 1$ ,  $n_2 = 2$  and  $n_{j+1} = 2n_j - 1$  for  $j \in \mathbb{N}$ ). Let

$$1 > t_1 > t_2 > \cdots > 0, \quad t_j \rightarrow 0$$

be real numbers and  $g_i : [0, 1] \rightarrow [0, \infty)$ ,  $j \in \mathbb{N}$  be continuous functions such that

$$\text{supp}(g_i) \subset [t_{j+1}, t_j] \quad \text{and} \quad \|g_j\|_2 = 1$$

for all  $j \in \mathbb{N}$ .

Put

$$f_k = g_i \quad \text{for } n_j \leq k \leq n_{j+1},$$

then  $(f_k)_{k \in \mathbb{1}}$  is a bounded sequence in  $L^2([0, 1])$ , which is weakly convergent to zero, and so is weakly mixing to zero, but which is not uniformly weakly mixing to zero.

Recall [20] that a sequence  $\{x_k\}$  in a Banach space  $X$  is called *convex shift-bounded* if there exists a constant  $c > 0$  such that

$$\left\| \sum_{j=1}^p \lambda_j x_{j+k} \right\| \leq c \left\| \sum_{j=1}^p \lambda_j x_j \right\|, \quad k \geq 1$$

holds for any  $p \in \mathbb{N}$  and  $\lambda_1, \dots, \lambda_p \geq 0$ .

One can see that every convex shift-bounded sequence is bounded.

*Example 1.2.* Let  $U : X \rightarrow X$  be a power bounded linear operator (i.e. the sequence  $\{\|U^k(x)\|\}$  is bounded for every  $x \in X$ ). Take  $x \in X$  then the sequence  $\{\|U^k(x)\|\}$  is convex shift-bounded.

The following theorem (see [20]) characterizes weak mixing to zero which is a counter part of the Blum-Hanson theorem [5],[11].

**Theorem 1.1.** *For a convex shift-bounded sequence  $\{x_k\}$  in a Banach space  $X$  the following conditions are equivalent:*

- (i)  $\{x_k\}$  is weakly mixing to zero;
- (ii)  $\{x_k\}$  is uniformly weakly mixing to zero;

There is also a characterization of uniformly weak mixing to zero by mean ergodic convergence.

**Theorem 1.2.** *For a bounded sequence  $\{x_k\}$  in a Banach space  $X$  the following conditions are equivalent:*

- (i)  $\{x_k\}$  is uniformly weakly mixing (resp. weakly mixing) to zero;
- (ii) For every sequence  $k_1 < k_2, \dots$  in  $\mathbb{N}$  with  $\sup_{n \in \mathbb{N}} \frac{k_n}{n} < +\infty$  the sequence  $\{x_{k_n}\}$  is ergodic (resp. weakly ergodic).

From this theorem we conclude that weakly ergodicity does not imply ergodicity too.

In the mentioned and others related papers (see [4, 11, 12]) tensor product of sequences which obey mixing and ergodicity were not considered. Section 2 of this note is devoted to the extension of the well-known classical results, stating that a transformation is weakly mixing if and only if its Cartesian square is ergodic [1], for the tensor product of sequences in Banach spaces. In next section 3, we provide some applications of the obtained results to uniquely  $E$ -ergodic, uniquely  $E$ -weak mixing  $C^*$ -dynamical systems. Note that such dynamical systems were investigated in [2, 9, 10, 15, 16].

## 2. MAIN RESULTS

Let  $X$  be a Banach spaces with dual spaces  $X^*$ . For given  $r > 0$  and  $a \in X$  denote

$$B_{r,X}(a) = \{x \in X : \|x - a\| \leq r\}.$$

Let us prove the following useful theorem.

**Theorem 2.1.** *For a bounded sequence  $\{x_k\}$  in a Banach space  $X$  the following conditions are equivalent:*

- (i)  $\{x_k\}$  is uniformly weakly mixing to zero;
- (ii) For every  $r > 0$  and  $y \in X^*$  one has

$$\lim_{n \rightarrow \infty} \sup \left\{ \frac{1}{n} \sum_{k=1}^n |f(x_k)| : f \in B_{r,X^*}(y) \right\} = 0;$$

*Proof.* (i)  $\Rightarrow$  (ii). Let  $r > 0$  and  $y \in X^*$ . Then from (i) one gets

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |y(x_k)| = \lim_{n \rightarrow \infty} \frac{\|y\|}{n} \sum_{k=1}^n \left| \frac{y}{\|y\|}(x_k) \right| = 0 \quad (1)$$

Now take an arbitrary  $g \in B_{r, X^*}(y)$  and denote  $f = \frac{1}{r}(g - y)$ . One can see that  $f \in B_{X^*}$ . Therefore, from (1), (i) and

$$\sup \left\{ \frac{1}{n} \sum_{k=1}^n |g(x_k)| : g \in B_{r, X^*}(y) \right\} \leq \frac{r}{n} \sum_{k=1}^n |y(x_k)| + \sup \left\{ \frac{1}{n} \sum_{k=1}^n |f(x_k)| : f \in B_{X^*} \right\}$$

we get the required assertion.

The implication (ii) $\Rightarrow$ (i) is obvious.  $\square$

Let  $X, Y$  be two Banach spaces with dual spaces  $X^*$  and  $Y^*$ , respectively. Completion of the algebraic tensor product  $X \odot Y$  with respect to a cross norm  $\alpha$  is denoted by  $X \otimes_\alpha Y$ . By  $\alpha^*$  we denote conjugate cross norm to  $\alpha$  defined on  $X^* \odot Y^*$ .

For the dual Banach spaces  $X^*$  and  $Y^*$  denote

$$B_{X^*} \odot B_{Y^*} = \left\{ \sum_{k=1}^n \lambda_k x_k \otimes y_k \mid \begin{array}{l} \{x_k\}_{k=1}^n \subset B_{X^*}, \{y_k\}_{k=1}^n \subset B_{Y^*}, \\ \lambda_k \geq 0, \sum_{k=1}^n \lambda_k \leq 1, n \in \mathbb{N} \end{array} \right\}.$$

By  $B_{X^*} \otimes_{\alpha^*} B_{Y^*}$  denote the closure of  $B_{X^*} \odot B_{Y^*}$  with respect to conjugate cross-norm  $\alpha^*$ . One can see that  $B_{X^*} \otimes_{\alpha^*} B_{Y^*} \subset B_{(X \otimes_\alpha Y)^*}$ .

In what follows we consider the following two conditions:

- (I) There is a number  $r > 0$  ( $r \leq 1$ ) and an element  $y \in (X \otimes_\alpha Y)^*$  such that

$$B_{r, (X \otimes_\alpha Y)^*}(y) \subset B_{X^*} \otimes_{\alpha^*} B_{Y^*};$$

- (II)  $X^* \otimes_{\alpha^*} Y^* = (X \otimes_\alpha Y)^*$ .

One has the following

**Proposition 2.2.** *Let (I) be satisfied, then (II) holds.*

*Proof.* Assume that (I) is satisfied. Then one immediately sees that  $y \in B_{X^*} \otimes_{\alpha^*} B_{Y^*}$ . Now let us take an arbitrary  $f \in (X \otimes_\alpha Y)^*$ , and show that it can be approximated by elements of  $X^* \otimes_{\alpha^*} Y^*$ . Indeed, denote  $g = y + r \frac{f}{\|f\|}$ . Then  $g \in B_{r, (X \otimes_\alpha Y)^*}(y)$ . Due to (I) we conclude that there is a sequence  $\{g_n\} \subset B_{X^*} \otimes_{\alpha^*} B_{Y^*}$  such that  $g_n$  converges to  $g$  in  $\alpha^*$ -norm. Since for all  $g_n$  one has  $g_n \in X^* \otimes_{\alpha^*} Y^*$ , therefore, we have  $g \in X^* \otimes_{\alpha^*} Y^*$ . Hence,  $f = \frac{\|f\|}{r}(g - y)$  belongs to  $X^* \otimes_{\alpha^*} Y^*$ .  $\square$

**Example 2.1.** Let us give some more example which satisfy (I) and (II) conditions.

- (i) Let  $1 < p, q < \infty$ , with conjugate indices  $p', q'$  (i.e.  $p' = \frac{p}{p-1}$ ). Consider  $\ell_p, \ell_q$ . Then for the projective norm  $\pi$  one has  $(\ell_p \otimes_\pi \ell_q)^* = \ell_{p'} \otimes_{\pi^*} \ell_{q'}$  if and only if  $p > q'$  (see Corollary 4.24, Theorem 4.21 [18]).
- (ii) We give here a sufficient condition to satisfy (II). The proof can be found in (see Theorem 5.33 [18]).

Let  $X$  and  $Y$  be Banach spaces such that  $X^*$  has the Radon-Nikodym property and either  $X^*$  or  $Y^*$  has the approximation property. Then

$$(X \otimes_\epsilon Y)^* = X^* \otimes_\pi Y^*$$

here  $\epsilon$  and  $\pi$  are the injective and the projective norms, respectively.

Note that more examples are given in [18].

**Theorem 2.3.** *Let  $X$  and  $Y$  be two Banach spaces with a cross-norm  $\alpha$  such that condition (I) is satisfied. Let  $\{x_k\}$  be a bounded sequence in  $X$ . Then the following assertions are equivalent*

- (i) *for any bounded sequence  $\{y_k\}$  in  $Y$ , the sequence  $\{x_k \otimes y_k\}$  in  $X \otimes_\alpha Y$  is uniformly weakly mixing to zero;*
- (ii)  *$\{x_k\}$  is uniformly weakly mixing to zero.*

*Proof.* (i) $\Rightarrow$ (ii). Let us take any nonzero element  $y \in Y$ . Define a sequence  $\{y_k\}$  by  $y_k = y$  for all  $k \in \mathbb{N}$ . For the defined sequence due to condition (i) we have

$$\lim_{n \rightarrow \infty} \sup \left\{ \frac{1}{n} \sum_{k=1}^n |f(x_k \otimes y)| : f \in B_{(X \otimes_{\alpha} Y)^*} \right\} = 0. \quad (2)$$

Now take  $f = g \otimes h$  with  $g \in B_{X^*}$  and  $h \in B_{Y^*}$ ,  $h(y) \neq 0$ . Then from (2) one gets

$$\lim_{n \rightarrow \infty} \left( \sup_{g \in B_{X^*}} \left\{ \frac{1}{n} \sum_{k=1}^n |g(x_k)| \right\} \right) |h(y)| = 0$$

which implies the assertion.

(ii) $\Rightarrow$ (i). Let  $\{y_k\}$  be an arbitrary bounded sequence in  $Y$ , and  $f \in B_{X^*}$ ,  $g \in B_{Y^*}$  be any functionals. Then the Schwarz inequality yields

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n |f(x_k)g(y_k)| &\leq \sqrt{\frac{1}{n} \sum_{k=1}^n |f(x_k)|^2} \sqrt{\frac{1}{n} \sum_{k=1}^n |g(y_k)|^2} \\ &\leq \max_k \{\|y_k\|\} \|g\| \sqrt{\frac{1}{n} \sum_{k=1}^n |f(x_k)|^2}. \end{aligned} \quad (3)$$

Moreover,

$$\sup_{f \in B_{X^*}} \left\{ \frac{1}{n} \sum_{k=1}^n |f(x_k)|^2 \right\} \leq \max_k \{\|x_k\|\} \sup_{f \in B_{X^*}} \left\{ \frac{1}{n} \sum_{k=1}^n |f(x_k)| \right\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, (3) implies that

$$\lim_{n \rightarrow \infty} \sup_{\substack{f \in B_{X^*}, \\ g \in B_{Y^*}}} \left\{ \frac{1}{n} \sum_{k=1}^n |f \otimes g(x_k \otimes y_k)| \right\} = 0. \quad (4)$$

Hence, using the norm-denseness of the elements  $\sum_{k=1}^m \lambda_k f_k \otimes g_k$ ,  $\{f_k\} \subset B_{X^*}$ ,  $\{g_k\} \subset B_{Y^*}$  (where  $\lambda_k \geq 0$ ,  $\sum_{k=1}^m \lambda_k \leq 1$ ) in  $B_{X^*} \otimes_{\alpha}^* B_{Y^*}$  from (4) one gets

$$\lim_{n \rightarrow \infty} \sup_{\varphi \in B_{X^*} \otimes_{\alpha}^* B_{Y^*}} \left\{ \frac{1}{n} \sum_{k=1}^n |\varphi(x_k \otimes y_k)| \right\} = 0. \quad (5)$$

Thanks to condition (I) one has

$$\sup_{f \in B_{r, (X \otimes_{\alpha} Y)^*}(y)} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} |f(x_k \otimes y_k)| \right\} \leq \sup_{w \in B_{X^*} \otimes_{\alpha}^* B_{Y^*}} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} |w(x_k \otimes y_k)| \right\},$$

consequently (5) with Theorem 2.1 yields the required statement.  $\square$

**Remark.** From the proof of Theorem 2.3 one can see that the implication (i) $\Rightarrow$ (ii) is still valid without condition (I).

Using the same argument as above given the proof we get the following

**Corollary 2.4.** *Let  $X$  and  $Y$  be two Banach spaces with a cross-norm  $\alpha$  such that condition (II) is satisfied. Let  $\{x_k\}$  be a bounded sequence in  $X$ . Then for the following assertions are equivalent*

- (i) *for any bounded sequence  $\{y_k\}$  in  $Y$ , the sequence  $\{x_k \otimes y_k\}$  in  $X \otimes_{\alpha} Y$  is weakly mixing to zero;*
- (ii)  *$\{x_k\}$  is weakly mixing to zero.*

**Proposition 2.5.** *Let  $X$  be a Banach space and  $\{x_k\}$  be a bounded sequence in  $X$  such that the sequence  $\{x_k \otimes x_k\}$  is ergodic in  $X \otimes_{\alpha} X$ . Then  $\{x_k\}$  is uniformly weakly mixing to zero.*

*Proof.* Ergodicity of the the sequence  $\{x_k \otimes x_k\}$  means that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=1}^n x_k \otimes x_k \right\| = 0,$$

whence we find

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{f \in B_{X^*}} \left\{ \frac{1}{n} \left| f \otimes f \left( \sum_{k=0}^{n-1} x_k \otimes x_k \right) \right| \right\} &= \lim_{n \rightarrow \infty} \sup_{f \in B_{X^*}} \left\{ \frac{1}{n} \left| \sum_{k=0}^{n-1} f \otimes f(x_k \otimes x_k) \right| \right\} \\ &= \lim_{n \rightarrow \infty} \sup_{f \in B_{X^*}} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} |f(x_k)|^2 \right\} = 0. \end{aligned} \quad (6)$$

The Schwartz inequality implies that

$$\sup_{f \in B_{X^*}} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} |f(x_k)| \right\} \leq \sqrt{\sup_{f \in B_{X^*}} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} |f(x_k)|^2 \right\}}$$

Therefore, due to (6) we find that  $\{x_k\}$  is uniformly weakly mixing to zero.  $\square$

Similarly, one can prove

**Corollary 2.6.** *Let  $X$  be a Banach space and  $\{x_k\}$  be a bounded sequence in  $X$  such that the sequence  $\{x_k \otimes x_k\}$  is weakly ergodic in  $X \otimes_{\alpha} X$ . Then  $\{x_k\}$  is weakly mixing to zero.*

**Theorem 2.7.** *Let  $X$  be a Banach spaces with a cross-norm  $\alpha$  on  $X \odot X$  such that condition (I) is satisfied with  $Y = X$ . Let  $\{x_k\}$  be a be bounded sequence in  $X$ . Then the following assertions are equivalent*

- (i) *the sequence  $\{x_k \otimes x_k\}$  is ergodic in  $X \otimes_{\alpha} X$ ;*
- (ii) *the sequence  $\{x_k \otimes x_k\}$  is uniformly weakly mixing to zero in  $X \otimes_{\alpha} X$ ;*
- (iii)  *$\{x_k\}$  is uniformly weakly mixing to zero.*

*Proof.* The implication (i) $\Rightarrow$ (iii) immediately follows from Proposition 2.5. The implication (iii) $\Rightarrow$ (ii) follows from Theorem 2.3. The implication (ii) $\Rightarrow$ (i) is evident.  $\square$

**Corollary 2.8.** *Let  $X$  be a Banach spaces with a cross-norm  $\alpha$  on  $X \odot X$  such that condition (II) is satisfied with  $Y = X$ . Let  $\{x_k\}$  be a be bounded sequence in  $X$ . Then the following assertions are equivalent*

- (i) *the sequence  $\{x_k \otimes x_k\}$  is weakly ergodic in  $X \otimes_{\alpha} X$ ;*
- (ii) *the sequence  $\{x_k \otimes x_k\}$  is weakly mixing to zero in  $X \otimes_{\alpha} X$ ;*
- (iii)  *$\{x_k\}$  is weakly mixing to zero.*

**Theorem 2.9.** *Let  $X$  and  $Y$  be two Banach spaces with a cross-norm  $\alpha$  on  $X \odot Y$  such that condition (I) (resp. (II)) is satisfied. Let  $\{x_k\}$  be a bounded sequence in  $X$ . The following assertions are equivalent*

- (i) *for any bounded sequence  $\{y_k\}$  in  $Y$ , the sequence  $\{x_k \otimes y_k\}$  in  $X \otimes_{\alpha} Y$  is ergodic (resp. weakly ergodic);*
- (ii)  *$\{x_k\}$  is uniformly weakly mixing (resp. weakly mixing) to zero.*

*Proof.* (i) $\Rightarrow$ (ii). Let us take any nonzero element  $y \in Y$ . Define a sequence  $\{y_k\}$  by  $y_k = y$  for all  $k \in \mathbb{N}$ . For the defined sequence due to condition (i) we have

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n x_k \otimes y \right\| = \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\| \|y\| = 0 \quad (7)$$

which means  $\{x_k\}$  is ergodic. The condition yields that  $\{x_k \otimes x_k\}$  is ergodic, hence Theorem 2.7 implies that that  $\{x_k\}$  is uniformly weakly mixing to zero.

(ii) $\Rightarrow$ (i). According to Theorem 2.3 we find that  $\{x_k \otimes y_k\}$  is uniformly weakly mixing to zero, for every bounded sequence  $\{y_k\}$  in  $Y$ . Hence, it is ergodic.  $\square$

3. APPLICATIONS TO  $C^*$ -DYNAMICAL SYSTEMS

In this section  $\mathfrak{A}$  will be a  $C^*$ -algebra with the unity  $\mathbf{1}$ . Recall a linear functional  $\varphi \in \mathfrak{A}^*$  is called *positive* if  $\varphi(x^*x) \geq 0$  for every  $x \in \mathfrak{A}$ . A positive functional  $\varphi$  is said to be a *state* if  $\varphi(\mathbf{1}) = 1$ . By  $\mathcal{S}(\mathfrak{A})$  we denote the set of all states on  $\mathfrak{A}$ . A linear operator  $T : \mathfrak{A} \mapsto \mathfrak{A}$  is called *positive* if  $Tx \geq 0$  whenever  $x \geq 0$ . By  $M_n(\mathfrak{A})$  we denote the set of all  $n \times n$ -matrices  $a = (a_{ij})$  with entries  $a_{ij}$  in  $\mathfrak{A}$ . A linear mapping  $T : \mathfrak{A} \mapsto \mathfrak{A}$  is called *completely positive* if the linear operator  $T_n : M_n(\mathfrak{A}) \mapsto M_n(\mathfrak{A})$  given by  $T_n(a_{ij}) = (T(a_{ij}))$  is positive for all  $n \in \mathbb{N}$ . A completely positive map  $T : \mathfrak{A} \mapsto \mathfrak{A}$  with  $T\mathbf{1} = \mathbf{1}$  is called a *unital completely positive (ucp)* map. A pair  $(\mathfrak{A}, T)$  consisting of a  $C^*$ -algebra  $\mathfrak{A}$  and a ucp map  $T : \mathfrak{A} \mapsto \mathfrak{A}$  is called a  *$C^*$ -dynamical system*. Let  $\mathfrak{B}$  be another  $C^*$ -algebra with unit. A completion of the algebraic tensor product  $\mathfrak{A} \odot \mathfrak{B}$  with respect to the minimal  $C^*$ -tensor norm on  $\mathfrak{A} \odot \mathfrak{B}$  is denoted by  $\mathfrak{A} \otimes \mathfrak{B}$ , and it would be also a  $C^*$ -algebra with a unit (see, [19]). It is known [19] that if  $(\mathfrak{A}, T)$  and  $(\mathfrak{B}, H)$  are two  $C^*$ -dynamical systems, then  $(\mathfrak{A} \otimes \mathfrak{B}, T \otimes H)$  is also  $C^*$ -dynamical system. Since a mapping  $T \otimes H : \mathfrak{A} \otimes \mathfrak{B} \mapsto \mathfrak{A} \otimes \mathfrak{B}$  given by  $(T \otimes H)(x \otimes y) = Tx \otimes Hy$  is a ucp map.

Let  $(\mathfrak{A}, T)$  be a  $C^*$ -dynamical system, and  $\mathfrak{B}$  be a subspace of  $\mathfrak{A}$ . Let  $E : \mathfrak{A} \rightarrow \mathfrak{B}$  be a norm-one projection, i.e.  $E^2 = E$ . In [8] (see also [9, 16]) it has been introduced the following notations

**Definition 3.1.**

(i)  $(\mathfrak{A}, T)$  is said to be *unique  $E$ -ergodic* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi(T^k(x)) = \varphi(E(x)), \quad x \in \mathfrak{A}, \varphi \in \mathcal{S}(\mathfrak{A}). \quad (8)$$

(ii)  $(\mathfrak{A}, T)$  is said to be *unique  $E$ -weakly mixing* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\varphi(T^k(x)) - \varphi(E(x))| = 0, \quad x \in \mathfrak{A}, \varphi \in \mathcal{S}(\mathfrak{A}). \quad (9)$$

It can readily seen (cf. [9]) that the map  $E$  below is a norm one projection onto the fixed point subspace  $\mathfrak{A}^T = \{x \in \mathfrak{A} : Tx = x\}$ . Therefore, in what follows we denote it by  $E_T$ . In [2], (i) is called unique ergodicity w.r.t. the fixed point subalgebra, whereas (ii) is called in [9]  $E$ -strictly weak mixing. In addition, when  $E = \omega(\cdot)\mathbf{1}$  (i.e. when there is a unique invariant state for  $T$ ), (i) is the well-known unique ergodicity, and (ii) is called strict (unique) weak mixing in [16]. Note that in [3] relations between unique ergodicity, minimality and weak mixing was studied.

By using the Jordan decomposition of bounded linear functionals (cf. [19]), one can replace  $\mathcal{S}(\mathfrak{A})$  with  $\mathfrak{A}^*$  in Definition 3.1.

Note that in [9, 15] it has been shown that the free shift on the reduced amalgamated free product  $C^*$ -algebra, and length-preserving automorphisms of the reduced  $C^*$ -algebra of  $RD$ -group for the length-function, including the free shift on the free group on infinitely many generators are enjoy unique  $E$ -mixing property. Such class of dynamical systems first time was defined and studied in [2]. Note that in [10] more other complicated unique  $E$ -ergodic and unique mixing  $C^*$ -dynamical systems arising from free probability have been studied. Note that in [7] sufficient and necessary conditions for ergodicity in terms of joinings are studied.

In this section we are going to apply the results of the previous section to the given notions.

**Theorem 3.2.** *Let  $(\mathfrak{A}, T)$ ,  $(\mathfrak{B}, H)$  be two  $C^*$ -dynamical systems with  $(\mathfrak{A} \otimes \mathfrak{B})^{T \otimes H} = \mathfrak{A}^T \otimes \mathfrak{B}^H$ , and assume that  $(\mathfrak{A} \otimes \mathfrak{B})^* = \mathfrak{A}^* \otimes \mathfrak{B}^*$  is satisfied. Then the following assertions are equivalent:*

- (i) *The  $C^*$ -dynamical system  $(\mathfrak{A} \otimes \mathfrak{B}, T \otimes H)$  is unique  $E_T \otimes E_H$ -weak mixing;*
- (ii)  *$(\mathfrak{A}, \varphi, T)$  and  $(\mathfrak{B}, \varphi_1, H)$  are unique  $E_T$  and  $E_H$  weak mixing, respectively.*

*Proof.* First note that  $(\mathfrak{A} \otimes \mathfrak{B})^{T \otimes H} = \mathfrak{A}^T \otimes \mathfrak{B}^H$  implies that  $E_{T \otimes H} = E_T \otimes E_H$ , therefore, in the proof we will deal with  $E_T \otimes E_H$ . Consequently, the implication (i) $\Rightarrow$ (ii) immediately follows from the definition.

Let us consider the implication (ii) $\Rightarrow$ (i). Let  $x \in \mathfrak{A}$  and  $y \in \mathfrak{B}$ . Define two sequences as follows

$$x_k = T^k(x) - E_T(x), \quad y_k = H^k(y) - E_H(y), \quad k \in \mathbb{N}. \quad (10)$$

Then one can see that the sequences are weakly mixing. Hence, Corollary 2.4 implies that the sequence  $\{x_k \otimes y_k\}$  is weakly mixing as well. This means that for every  $\omega \in (\mathfrak{A} \otimes \mathfrak{B})^*$  one has

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n & \left| \omega(T^k(x) \otimes H^k(y)) - \omega(T^k(x) \otimes E_H(y)) \right. \\ & \left. - \omega(E_T(x) \otimes H^k(y)) + \omega(E_T(x) \otimes E_H(y)) \right| = 0 \end{aligned} \quad (11)$$

Now define two functionals  $\omega_1$  and  $\omega_2$  on  $\mathfrak{A}$  and  $\mathfrak{B}$ , respectively, as follows:

$$\omega_1(\cdot) = \omega(\cdot \otimes E_H(y)) \quad \omega_2(\cdot) = \omega(E_T(x) \otimes \cdot), \quad (12)$$

here  $E_T(x)$  and  $E_H(y)$  are fixed. Then according weak mixing condition (see (ii)) one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\omega_1(T^k(x)) - \omega_1(E_T(x))| = 0, \quad (13)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\omega_2(H^k(y)) - \omega_2(E_H(y))| = 0. \quad (14)$$

The last relations (13),(14) with (12) mean that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\omega(T^k(x) \otimes E_H(y)) - \omega(E_T(x) \otimes E_H(y))| = 0, \quad (15)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\omega(E_T(x) \otimes H^k(y)) - \omega(E_T(x) \otimes E_H(y))| = 0. \quad (16)$$

The inequality

$$\begin{aligned} |\omega(T^k \otimes H^k(x \otimes y)) - \omega(E_T(x) \otimes E_H(y))| & \leq \left| \omega(T^k(x) \otimes H^k(y)) - \omega(T^k(x) \otimes E_H(y)) \right. \\ & \left. - \omega(E_T(x) \otimes H^k(y)) + \omega(E_T(x) \otimes E_H(y)) \right| \\ & + |\omega(T^k(x) \otimes E_H(y)) - \omega(E_T(x) \otimes E_H(y))| \\ & + |\omega(E_T(x) \otimes H^k(y)) - \omega(E_T(x) \otimes E_H(y))| \end{aligned}$$

with (11),(15) and (16) imply that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\omega(T^k \otimes H^k(x \otimes y)) - \omega(E_T \otimes E_H(x \otimes y))| = 0. \quad (17)$$

The norm-denseness of the elements  $\sum_{i=1}^m x_i \otimes y_i$  in  $\mathfrak{A} \otimes \mathfrak{B}$  with (17) yields

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\omega(T^k \otimes H^k(\mathbf{z})) - \omega(E_T \otimes E_H(\mathbf{z}))| = 0.$$

for arbitrary  $\mathbf{z} \in \mathfrak{A} \otimes \mathfrak{B}$ . So,  $(\mathfrak{A} \otimes \mathfrak{B}, T \otimes H)$  is unique  $E_T \otimes E_H$ -weak mixing.  $\square$

**Remark.** The proved theorem extends some results of [14, 15]. We note that in [3, 13] similar results were proved for non-commutative dynamical systems defined over von Neumann algebras.

*Example 3.1.* Let us consider some more examples of  $C^*$ -algebras which satisfy the condition  $(\mathfrak{A} \otimes \mathfrak{B})^* = \mathfrak{A}^* \otimes \mathfrak{B}^*$ .

- (i) Let  $K$  be a compact metric space. By  $C(K)$  we denote the set of all continuous functions. Then one can see that  $C(K)$  and its dual  $C(K)^*$  are separable. Then due to Corollary 5.42 [18] the space  $C(K)^*$  has the Radon-Nikodým property (see [18] for more details). Moreover, it also has the approximation property (see [18], p.74). Therefore, according to Theorem 5.33 [18] we have  $(C(K) \otimes \mathfrak{A})^* = C(K)^* \otimes \mathfrak{A}^*$  for every  $C^*$ -algebra  $\mathfrak{A}$ .

- (ii) Consider the space  $c_0$ , which dual  $\ell_1$  has the Radon-Nikodým and the approximation properties (see [18]). Hence, again Theorem 5.33 [18] implies that  $(c_0 \otimes \mathfrak{A})^* = (c_0)^* \otimes \mathfrak{A}^*$  for every  $C^*$ -algebra  $\mathfrak{A}$ .
- (iii) Let  $M_n(\mathbb{C})$  be the algebra of  $n \times n$  matrices over  $\mathbb{C}$ . Then owing to Theorem 5.33 [18] one has  $(M_n(\mathbb{C}) \otimes \mathfrak{A})^* = (M_n(\mathbb{C}))^* \otimes \mathfrak{A}^*$  for every  $C^*$ -algebra  $\mathfrak{A}$ .

**Theorem 3.3.** *Let  $(\mathfrak{A}, T)$  be a  $C^*$ -dynamical systems. Then for the following assertions*

- (i)  $(\mathfrak{A}, T)$  is unique  $E_T$ -weak mixing;
- (ii) for every  $(\mathfrak{B}, H)$  - unique  $E_H$ -ergodic  $C^*$ -dynamical system with  $E_{T \otimes H} = E_T \otimes E_H$  and  $\mathfrak{A}^* \otimes \mathfrak{B}^* = (\mathfrak{A} \otimes \mathfrak{B})^*$ , the  $C^*$ -dynamical system  $(\mathfrak{A} \otimes \mathfrak{B}, T \otimes H)$  is unique  $E_T \otimes E_H$ -ergodic;

the implication (i) $\Rightarrow$ (ii) holds true.

*Proof.* Let  $(\mathfrak{B}, H)$  be a  $C^*$ -dynamical system as in (ii). Now take arbitrary elements  $x \in \mathfrak{A}$  and  $y \in \mathfrak{B}$ , and consider the corresponding sequences  $\{x_k\}$  and  $\{y_k\}$  given by (10). Then due to the condition  $\{x_k\}$  is weak mixing and  $\{y_k\}$  is weak ergodic. Hence, Theorem 2.9 yields that  $\{x_k \otimes y_k\}$  is weak ergodic, which means for every  $\omega \in (\mathfrak{A} \otimes \mathfrak{B})^*$  one has

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\omega(T^k(x) \otimes H^k(y)) - \omega(T^k(x) \otimes E_H(y)) \\ - \omega(E_T(x) \otimes H^k(y)) + \omega(E_T(x) \otimes E_H(y))) = 0 \end{aligned} \quad (18)$$

Using similar arguments as in the proof of Theorem 3.2 we find

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\omega(T^k(x) \otimes E_H(y)) - \omega(E_T(x) \otimes E_H(y))| = 0, \quad (19)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\omega(E_T(x) \otimes H^k(y)) - \omega(E_T(x) \otimes E_H(y))) = 0. \quad (20)$$

From

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=1}^n (\omega(T^k \otimes H^k(x \otimes y)) - \omega(E_T(x) \otimes E_H(y))) \right| \\ \leq \left| \frac{1}{n} \sum_{k=1}^n (\omega(T^k(x) \otimes H^k(y)) - \omega(T^k(x) \otimes E_H(y)) \right. \\ \left. - \omega(E_T(x) \otimes H^k(y)) + \omega(E_T(x) \otimes E_H(y))) \right| \\ + \frac{1}{n} \sum_{k=1}^n |\omega(T^k(x) \otimes E_H(y)) - \omega(E_T(x) \otimes E_H(y))| \\ + \left| \frac{1}{n} \sum_{k=1}^n (\omega(E_T(x) \otimes H^k(y)) - \omega(E_T(x) \otimes E_H(y))) \right| \end{aligned}$$

and (18)-(20) we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\omega(T^k \otimes H^k(x \otimes y)) - \omega(E_T \otimes E_H(x \otimes y))) = 0.$$

Finally, the density argument shows that  $(\mathfrak{A} \otimes \mathfrak{B}, T \otimes H)$  is unique  $E_T \otimes E_H$ -ergodic.  $\square$

**Remark.** We note that all the results of this section extends the results of [14, 15] to uniquely  $E$ -ergodic and uniquely  $E$ -weak mixing.

**Remark.** We have to stress that the unique ergodicity  $T \otimes H$  does not imply unique weak mixing of  $T$ . Indeed, let us consider the following examples.



*Example 3.2.* Let  $\mathfrak{A} = \mathbb{C}^2$  and

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that  $\mathfrak{A}^T = \mathbb{C}\mathbf{1}$ , so  $T$  is ergodic, i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n T^k(x, y) = \frac{x+y}{2}(1, 1). \quad x, y \in \mathbb{C}$$

From the equality

$$\left| T^k(x, y) - \frac{x+y}{2}(1, 1) \right| = \left| \frac{x-y}{2} \right|$$

we infer that  $T$  is not unique weak mixing.

On the other hand, the equality

$$(\mathfrak{A} \otimes \mathfrak{A})^{T \otimes T} = \{(x, y, y, x) : x, y \in \mathbb{C}\},$$

implies unique  $E_{T \otimes T}$ -ergodicity of  $T \otimes T$ .

*Example 3.3.* Let  $\mathfrak{A} = \mathbb{C}^3$  and  $\mathfrak{B} = \mathbb{C}^2$ . Consider the a mapping  $P : \mathfrak{A} \rightarrow \mathfrak{A}$  given by

$$P(x, y, z) = (y, x, uy + vz), \quad (21)$$

where  $u, v > 0$  and  $u + v = 1$ . It is clear that  $P$  is positive and unital. Direct calculations show that  $\mathfrak{A}^P = \mathbb{C}\mathbf{1}$ , which means  $P$  is uniquely ergodic.

Now consider the mapping  $P \otimes T$ , where  $T$  is defined as above. One can see that such a mapping acts as follows

$$P \otimes T(\mathbf{x}, \mathbf{y}) = (P\mathbf{y}, P\mathbf{x})$$

where  $\mathbf{x}, \mathbf{y} \in \mathfrak{A}$ . Hence, we find

$$(\mathfrak{A} \otimes \mathfrak{B})^{P \otimes T} = \{(\mathbf{x}, P\mathbf{x}) : \mathbf{x} \in \mathfrak{A}^{P^2}\}.$$

Therefore, from (21) one immediately gets

$$P^2(x, y, z) = (x, y, ux + uv y + v^2 z). \quad (22)$$

Thus, we find

$$\mathfrak{A}^{P^2} = \left\{ \left( x, y, \frac{x + vy}{1 + v} \right) : x, y \in \mathbb{C} \right\}.$$

On the other hand, we have  $\mathfrak{A}^P \otimes \mathfrak{B}^T = \mathbb{C}\mathbf{1}$ , which means  $(\mathfrak{A} \otimes \mathfrak{B})^{P \otimes T} \neq \mathfrak{A}^P \otimes \mathfrak{B}^T$ .

Similarly reasoning as in Example 3.2 we can show that  $P \otimes T$  is uniquely  $E_{P \otimes T}$ -ergodic.

Note that, from the provided examples we infer the importance of condition  $E_{T \otimes H} = E_T \otimes E_H$ .

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