

Unbreakable loops

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Abstract

We say that a loop is unbreakable when it does not have nontrivial subloops. While the cyclic groups of prime order are the only unbreakable finite groups, we show that nonassociative unbreakable loops exist for every order $n \geq 5$. We describe two families of commutative unbreakable loops of odd order, $n \geq 7$, one where the loop's multiplication group is isomorphic to the alternating group \mathcal{A}_n and another where the multiplication group is isomorphic to the symmetric group \mathcal{S}_n . We also prove for each even $n \geq 6$ that there exist unbreakable loops of order n whose multiplication group is isomorphic to \mathcal{S}_n .

Keywords: Loops, multiplicative monoid, alternating group, symmetric group

1 Introduction

We say that a finite loop is *unbreakable* whenever it doesn't have proper subloops, that is, other than itself and the trivial one-element loop. While it is easy to see that the finite associative unbreakable loops are exactly the cyclic groups of prime order, it turns out that finite, nonassociative unbreakable loops are numerous and diverse. Our interest for these loops arose in the context of a research effort on the classes of word languages defined in terms of finite loops; the proof of the main theorem of [2] requires the existence for infinitely many integers $n \geq 5$ of a group-free loop of order n . A loop is group-free if none of its nontrivial subloops or quotients is a group; an unbreakable loop is just a special case of these loops. We prove in this article that nonassociative unbreakable loops exist for every order $n \geq 5$. More precisely, we prove existence theorems for unbreakable loops of orders $n \geq 5$, with constraints on their multiplication group and, for odd $n \geq 7$, the additional condition that the loop is commutative. Moreover, when n is odd we are able to give fully constructive proofs. Our results are summarized as follows.

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Theorem 1.1 *There exists a nonassociative unbreakable loop for every order $n \geq 5$. Furthermore:*

- i. for every odd $n \geq 7$, there exists a commutative unbreakable loop of order n whose multiplication group is the symmetric group \mathcal{S}_n , and another one whose multiplication group is the alternating group \mathcal{A}_n ;*
- ii. for every even $n \geq 6$, there exists an unbreakable loop of order n whose multiplication group is the symmetric group \mathcal{S}_n .*

We refer the reader to [14, 4] for detailed background on loops. In this article, all loops are finite. Let G be a loop of order n ; its operation is denoted by an asterisk, e.g. $a * b = c$. To each loop element a we associate its *right and left actions*, R_a and L_a respectively, defined by $R_a(b) = b * a$ and $L_a(b) = a * b$. Both actions are permutations of G . The actions generate $\mathcal{M}(G) = \langle \{ L_a, R_a \mid a \in G \} \rangle$, the *multiplication group* of G . In the literature, these objects are also called the left and right translations and the translation group, respectively. Note that in a commutative loop, we have $L_a = R_a$ for every a ; we then speak of the *action of a* and use the notation L_a .

Our descriptions and proofs use only basic notions and facts on groups and permutations; they can be found in fundamental texts such as [11] and we assume that they are familiar to the reader. The only exceptions are Propositions 3.3 and 4.1, taken from Piccard's work on generating sets for the symmetric and the alternating groups [15].

We denote by $G = \{0, 1, \dots, n - 1\}$ the underlying set of a loop G of order n . To make our descriptions simpler, we write them as if G were a subset of \mathbb{N} and use relations and operations usually encountered in these contexts, such as " \leq " and "+". The *symmetric group* over G is the set of all $n!$ permutations of $\{0, 1, \dots, n - 1\}$; its subgroup the *alternating group* \mathcal{A}_n is the set of all even permutations of G ; this group is simple and unsolvable for every order $n \geq 5$. An *even permutation* can be identified in several ways; in this article we use the following.

- i. A permutation τ is even iff it contains an even number of inversions; an inversion is a pair i, j such that $i < j$ and $\tau(i) > \tau(j)$.*
- ii. A permutation is even iff its cyclic representation contains an even number of cycles of even length.*

We regard the multiplication group $\mathcal{M}(G)$ as a subset of \mathcal{S}_n ; we therefore write statements like " $\mathcal{M}(G) = \mathcal{S}_n$ " instead of " $\mathcal{M}(G)$ is isomorphic to \mathcal{S}_n ".

For a given loop, most of our work is done on the table of its operation (the *Cayley table*), where rows and columns are labelled with the loop's elements, and where entry $[a, b]$ contains the value $a * b$. It is well known that a finite groupoid is a quasigroup iff its Cayley table is a latin square; it is commutative iff the table is symmetric.

The notion of multiplication group of a loop was introduced by Albert [1]. The properties of this group have been the object of extensive study, see e.g. [3, 4, 13, 9]. Certain results have some relationship to the topic of our paper: the multiplication group of a loop is solvable only if the loop itself is solvable [17], which implies that the multiplication loop of a nonassociative unbreakable loop

Order	Number, total	Number, unbreakable	Proportion unbrk. vs. total
5	6	2	1/3
6	109	28	25.7 %
7	23 746	9 906	41.7 %
8	106 228 849	43 803 136	41.2 %
9	9 365 022 303 540	?	?

Figure 1: Unbreakable loops of size 5 to 9

Order n	Number of loops	Multiplication group \mathcal{S}_n	\mathcal{A}_n	\mathbb{Z}_n	Other
5	2	1	0	1	0
6	28	28	0	0	0
7	9 906	9 904	1	1	0
8	43 803 136	43 799 370	3 765	0	1

Figure 2: Multiplication group of unbreakable loops, sizes 5 to 8

is always unsolvable; certain groups cannot be the multiplicative group of any nonassociative loop [16]; and for every $n \neq 2, 4, 5$, the alternating group \mathcal{A}_n can be the multiplication group of a loop of order n [7].

In the next section, we report on the exhaustive analysis we made on loops of orders 5 to 8. Section 3 contains the proof of our main theorem, and is followed by a short conclusion. Our paper contains a large number of constructions and examples; we inserted in the main text only those we deemed absolutely necessary for understanding, and gathered the rest in the Appendix.

2 Small loops

Scrutiny of the exhaustive lists of small latin squares (see for example [12]) shows that nonassociative loops exist with orders 5 and 6. Among them, there are one unbreakable loop of order 5 and 28 of order 6; their multiplication groups are equal to the symmetric group of the same order. Samples loops of orders 5 and 6 are displayed in the Appendix.

The number of loops (always counted up to isomorphism) increases rapidly with the order; an exhaustive study of all loops of size $n = 7$ and $n = 8$ is possible, but it is already unthinkable for $n = 9$, see Figure 1 (the number for $n = 9$ is quoted from [12]). The first step in our work consisted in analyzing every loop of size 6 to 8; the problem of generating a list of these loops has already been addressed [8]. For each loop we computed its multiplication group and for $n = 7$, verified whether the loop is commutative. The following fact allowed us skip this test for $n = 8$.

Proposition 2.1 *Unbreakable loops of even order cannot be commutative.*

Proof. In a symmetric latin square, the number of occurrences of a given element on the diagonal has the same parity as the size of the square [6]. Thus, in a loop of even size, since $0 * 0 = 0$ there must be some $a \neq 0$ such that $a * a = 0$, which means that $\{0, a\}$ is a subloop isomorphic to the group \mathbb{Z}_2 . \square

The results of our exhaustive search are summarized on Figure 2. We notice a number of interesting facts.

- i.* $\mathcal{M}(G) = \mathcal{S}_n$ for the vast majority of unbreakable loops; however there are loops of sizes 7 and 8 for which $\mathcal{M}(G) = \mathcal{A}_n$.
- ii.* There are eight commutative unbreakable loops of order 7. One of them is the only unbreakable loop of order 7 whose multiplication group is \mathcal{A}_7 ; its Cayley table is displayed in the Appendix.
- iii.* There is also a lone loop of size 8 for which $\mathcal{M}(G)$ is neither \mathcal{S}_n nor \mathcal{A}_n ; we determined with the GAP software that this multiplication group has order 1344 and is isomorphic to the semidirect product $\mathbb{Z}_2^3 \rtimes PSL(2, 7)$; its Cayley table is displayed in the Appendix.

Building on these observations, we undertook to verify that for every odd n , there exists an unbreakable, commutative loop of size n such that $\mathcal{M}(G) = \mathcal{S}_n$ and another one such that $\mathcal{M}(G) = \mathcal{A}_n$. For loops of even order we cannot use commutativity to make our work easier; we nevertheless prove the existence for every even $n \geq 10$ of an unbreakable loop which satisfies $\mathcal{M}(G) = \mathcal{S}_n$; examples for $n = 6$ and $n = 8$ are given in the Appendix.

3 Unbreakable loops of odd size

In this section, we prove Theorem 1.1 for the odd values of n . We do so by building two families of loops, one with $\mathcal{M}(G) = \mathcal{S}_n$ for each $n \geq 21$, and the other family with $\mathcal{M}(G) = \mathcal{A}_n$ for each $n \geq 43$. For those odd values of n not covered by our proofs, we give in the Appendix a set of sample loops of order n . The rest of this section is structured as follows. First, we build a $n \times n$ symmetric partial latin square, which we call the *template*, and we show that it can be completed to yield a commutative unbreakable loop whose multiplication group is either \mathcal{S}_n or \mathcal{A}_n , provided that an additional constraint is respected. Next, we prove how to fill the template in order to ensure that $\mathcal{M}(G) = \mathcal{S}_n$ or $\mathcal{M}(G) = \mathcal{A}_n$.

3.1 A template for the Cayley table

From now on, let $n = 2p + 1$. We denote by $[i, j]$ the content of the table at line i and column j . Since we build a symmetric table, it is enough to specify $[i, j]$ for $i \leq j$. The partial latin square resulting from the forthcoming specifications is called the *template*.

- i.* For all i, j with $0 \leq i \leq n - 1$ and $0 \leq j \leq n - i$, let $[i, j] = i + j \pmod{n}$.
- ii.* For all i, j with $i \geq 7$ and $n - i + 6 \leq j \leq n - 1$, let $[i, j] = i + j \pmod{n}$.

iii. Modify lines 1, 2, and column $n - 1$ as follows: $[1, 2] = 0$; $[1, p + 2] = 3$; $[p + 4, n - 1] = 5$.

iv. Complete lines 1 through 5 as follows:

$$[1, n - 1] = p + 3; [2, n - 2] = 1; [2, n - 1] = 3;$$

$$[3, n - 3] = 1; [3, n - 2] = 2; [3, n - 1] = 0.$$

$$[4, n - 4] = 1; [4, n - 3] = 0; [4, n - 2] = 3; [4, n - 1] = 2;$$

$$[5, n - 5] = 2; [5, n - 4] = 0; [5, n - 3] = 3; [5, n - 2] = 4; [5, n - 1] = 1.$$

These positions define the *top right region*. By symmetry, this also defines a *bottom left region*.

v. Finally, let

$$[p + 1, p + 1] = 3; [p + 1, p + 2] = 0; [p + 1, p + 3] = 5; [p + 1, p + 4] = 4;$$

$$[p + 1, p + 5] = 2;$$

$$[p + 2, p + 2] = 5; [p + 2, p + 3] = 4; [p + 2, p + 4] = p + 3;$$

$$[p + 3, p + 3] = 1.$$

These positions define the *central triangle*.

The template for $n = 21$ is represented on Figure 3. In this figure, the cells whose content is not specified are identified with a question mark "?". Also, entries $[i, j]$ where the template differs from the table of \mathbb{Z}_n , i.e. those where $[i, j] \not\equiv i + j \pmod{n}$, are printed in boldface. Borders are drawn around the central triangle and the top right and bottom left regions. Observe that $[5, 16] = [11, 15] = 2$: therefore it is not possible to build a smaller template consistent with the above specifications.

Those cells whose content is not specified in the template are at positions $[i, j]$ such that $n \leq i + j \leq n + 5$; they must eventually be filled with an element of $\{0, 1, 2, 3, 4, 5\}$. Located on either side of the central triangle, they constitute a region which we call the *undefined zone*.

This template was obtained through experiments where we built a partially defined latin square in which all positions $[i, j]$ such that $i + j < n$ or $i + j > n + 5$ were filled as above and then let a computer try to fill the remaining positions in order to yield a suitable latin square. We observed that a small number of combinations of central triangle and top right region occur in our results for every n above a reasonable threshold. Among these combinations we chose for this proof the one for which the threshold is minimal.

Loops defined out of this template have a number of useful properties; we state and prove them in the rest of this subsection.

Lemma 3.1 *If a loop has a Cayley table consistent with the template and if it also satisfies the constraint that $[i, j] \neq 0$ in every position where $i + j = n$, then it is unbreakable.*

Proof. Let $\langle k \rangle$ denote the subloop generated by $k \in G$; we show that $\langle k \rangle = G$ for every element $k \neq 0$. We first consider $k = 2$: it is readily seen from the above specifications that $[2, j] = j + 2$ for every $2 \leq j \leq n - 3$, which implies that 2 generates all even values between 2 and $n - 1$. Next, $[2, n - 1] = 3$, and from this all odd values between 5 and $n - 2$ can be generated. Finally, $[2, n - 2] = 1$ and $[2, 1] = 0$ yield $\langle 2 \rangle = G$. Since $[1, 1] = 2$, it follows immediately that $\langle 1 \rangle = G$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	0	4	5	6	7	8	9	10	11	12	3	14	15	16	17	18	19	20	13
2	2	0	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	3
3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	2	0
4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	0	3	2
5	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	2	0	3	4	1
6	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	?	?	?	?	?	?
7	7	8	9	10	11	12	13	14	15	16	17	18	19	20	?	?	?	?	?	?	6
8	8	9	10	11	12	13	14	15	16	17	18	19	20	?	?	?	?	?	?	6	7
9	9	10	11	12	13	14	15	16	17	18	19	20	?	?	?	?	?	?	6	7	8
10	10	11	12	13	14	15	16	17	18	19	20	?	?	?	?	?	?	6	7	8	9
11	11	12	13	14	15	16	17	18	19	20	?	3	0	5	4	2	6	7	8	9	10
12	12	3	14	15	16	17	18	19	20	?	?	0	5	4	13	6	7	8	9	10	11
13	13	14	15	16	17	18	19	20	?	?	?	5	4	1	6	7	8	9	10	11	12
14	14	15	16	17	18	19	20	?	?	?	?	4	13	6	7	8	9	10	11	12	5
15	15	16	17	18	19	20	?	?	?	?	?	2	6	7	8	9	10	11	12	13	14
16	16	17	18	19	20	2	?	?	?	?	?	6	7	8	9	10	11	12	13	14	15
17	17	18	19	20	1	0	?	?	?	?	6	7	8	9	10	11	12	13	14	15	16
18	18	19	20	1	0	3	?	?	?	6	7	8	9	10	11	12	13	14	15	16	17
19	19	20	1	2	3	4	?	?	6	7	8	9	10	11	12	13	14	15	16	17	18
20	20	13	3	0	2	1	?	6	7	8	9	10	11	12	5	14	15	16	17	18	19

Figure 3: The template; $n = 21$.

Reasoning as in the case $k = 2$, it is easily verified that $\langle 3 \rangle = \langle 4 \rangle = \langle 5 \rangle = G$.

In the central triangle we observe $[p+1, p+1] = 3$, $[p+2, p+2] = 5$, $[p+3, p+3] = 1$; therefore, $\langle p+1 \rangle = \langle p+2 \rangle = \langle p+3 \rangle = G$.

Next, we show $\langle n-1 \rangle = G$. This follows from the observation that $[n, j] = j-1$ for every $p+5 \leq j \leq n-1$, that $[n-1, p+4] = 5$ and $[n-1, 5] = 1$. Since $[p, p] = n-1$, we also have $\langle p \rangle = G$.

We deal with the other $k \in G$ by induction. Since $[k, k] < k$ for every $k \geq p+4$, we only have to consider the case $6 \leq k \leq p-1$. We have $[k, j] = k+j$ for all $1 \leq j \leq n-k-1$, which means that every $tk \leq n-1$ is generated by k ; let sk denote the largest such multiple of k . Also, observe that the content of cells $[k, n-k]$ to $[k, n-1]$ is a permutation of the set $\{0, \dots, k-1\}$. Therefore, $[k, sk] \in \{0, \dots, k-1\}$. If n is a multiple of k , which means $sk = n-k$, then position $[k, n-k]$ is in the undefined zone, and is subject to the condition $[k, n-k] \neq 0$ of the lemma's statement: this yields $[k, n-k] \in \{1, 2, 3, 4, 5\}$. Otherwise k does not divide n , i.e. $n = (s+1)k - t$ with $0 < t < k$, and $[k, sk]$ is either nonzero, in which case we are done by induction hypothesis and our reasoning on $k \leq 5$, or $[k, sk] = 0$ and we move on to consider the value $[sk, sk]$: since $2sk > n$, we have $[sk, sk] = 2sk \pmod{n} = r$ for some r not a multiple of k . Then $[k, (j-1)k+r] = jk+r$ belongs to $\langle k \rangle$ for every $j \geq 1$ such that $jk+r < n$; let $\ell k+r$ be the largest such value: we have $[k, \ell k+r] \in \{1, \dots, k-1\}$. \square

The problem of generating the symmetric or the alternating group with a pair of permutations was studied exhaustively by Piccard. We quote from her work the following definition and result (Proposition 5, page 20 in [15]); then we proceed to show that in every commutative loop built from the template we can find in $\mathcal{M}(G)$ two permutations which satisfy the conditions of Proposition 3.3.

$$L_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n-1 \\ 2 & 0 & 4 & 5 & 6 & \cdots & n-2 & n-1 & 1 & 3 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 \\ 3 & 4 & 5 & 6 & \cdots & n-2 & n-1 & 1 & 2 & 0 \end{pmatrix}$$

Figure 4: Permutations L_2 and L_3

Definition 3.2 Let $n \geq 2$ be an integer and $a, b \in \{0, 1, \dots, n-1\}$ with $a \neq b$. The distance between a and b , denoted \overline{ab} , is the unique solution of $a + \overline{ab} \equiv b \pmod{n}$ which lies in $\{0, 1, \dots, n-1\}$.

Proposition 3.3 Let $n \geq 5$ be an odd integer and $a, b, c \in \{0, 1, \dots, n-1\}$ be pairwise distinct. Let $\varphi, \psi \in \mathcal{S}_n$ with $\varphi = (0 \ 1 \ \cdots \ n-1)$ and $\psi = (a \ b \ c)$. The pair $\{\varphi, \psi\}$ generates \mathcal{A}_n if, and only if the largest common divisor of a , b and c (i.e., $\gcd(a, b, c)$) is 1.

Lemma 3.4 If the Cayley table of an order- n loop G is consistent with the template, then \mathcal{A}_n is a subgroup of $\mathcal{M}(G)$.

Proof. Consider the left actions L_2 and L_3 of 2 and 3, respectively, in a loop consistent with the template; they are totally defined by the template and are represented, in matrix notation, on Figure 4. The reader can verify that both permutations consist of a unique cycle of length n , that $L_2(x) = x + 2$ for all $x \notin \{1, n-2, n-1\}$, and that $L_3(x) = x + 3$ for all $x \notin \{n-3, n-2, n-1\}$. The compositions $\alpha = L_2 \circ L_3$ and $\beta = L_3 \circ L_2$ differ only on elements 2, 3 and 6, and $\gamma = \alpha^{-1} \circ \beta = (2 \ 3 \ 6)$. Let f be the automorphism of \mathcal{S}_n which satisfies $f(L_2) = \varphi$ and verify that $f(\gamma) = (1 \ p+1 \ 3)$ can play the role of ψ in Proposition 3.3. \square

We prove finally that for most loop elements, testing whether their actions are even permutations is actually quite simple.

Lemma 3.5 For every $i \in \{6, \dots, n-2\}$ other than $p+2$ and $p+4$, the action L_i is an even permutation iff the table entries $[i, n-i]$ to $[i, n-i+5]$ constitute an even permutation of $\{0, 1, 2, 3, 4, 5\}$.

Proof. Consider the permutation L_i , $i \in \{6, \dots, n-2\} \setminus \{p+2, p+4\}$. To count the inversions in L_i , we distinguish three regions in row i of the Cayley table:

- the leftmost $n-i$ positions contain $L_i(x) = x+i \pmod{n}$ for $0 \leq x \leq n-i-1$; this is the increasing sequence $i, i+1, \dots, n-1$;
- the six positions $[i, n-i]$ to $[i, n-i+5]$ constitute the intersection of line i with the undefined zone; they contain a permutation of $\{0, 1, 2, 3, 4, 5\}$;
- the remaining $i-6$ positions contain $L_i(x) = x+i \pmod{n}$ for $n-i+6 \leq x \leq n-1$, that is, the increasing sequence $6, \dots, i-1$.

$p \geq 10$ and $p \equiv 2 \pmod{4}$

6				3	1	2	5	0	4
7				2	1	0	3	4	5
8			3	1	0	4	5	2	
9		1	2	0	5	3	4		
10	1	2	0	3	4	5			

$p \geq 11$ and $p \equiv 3 \pmod{4}$

6					3	1	2	5	0	4
7					1	2	0	3	4	5
8				1	5	0	3	4	2	
9			3	2	0	1	4	5		
10		1	2	0	3	4	5			
11	1	2	0	3	4	5				

$p \geq 12$ and $p \equiv 0 \pmod{4}$

6						3	1	2	5	0	4
7						2	1	0	3	4	5
8					1	3	0	4	5	2	
9				1	5	0	2	3	4		
10			3	2	0	1	4	5			
11		1	2	0	3	4	5				
12	1	2	0	3	4	5					

$p \geq 13$ and $p \equiv 1 \pmod{4}$

6							3	1	2	5	0	4
7							3	1	0	4	2	5
8						1	2	0	5	3	4	
9					1	2	0	4	3	5		
10				2	0	3	1	5	4			
11			3	1	5	0	4	2				
12		1	2	0	3	4	5					
13	1	2	0	3	4	5						

Figure 5: Final patterns for the proof of Lemma 3.6.

Eventually, the proximity of the upper right block makes it impossible to use this pattern, and the remaining rows must be completed using another method. This can be done with one of the four "final patterns" represented on Figure 5; the appropriate pattern is selected depending on the value of $p \bmod 4$.

Observe that the action L_6 is the same in every final pattern and that the content of positions $[6, n-6]$ to $[6, n-1]$ is the odd permutation $(0\ 3\ 5\ 4)$ of $\{0, 1, 2, 3, 4, 5\}$; therefore, by Lemma 3.5 L_6 is an odd permutation of $\{0, \dots, n-1\}$. Notice also that none of the patterns locates 0 at a position $[i, n-i]$; therefore Lemma 3.1 applies on the loops built with this set of patterns. \square

The smallest loop constructible by this method has size 21; its Cayley table is displayed in the Appendix.

3.3 Loops with $\mathcal{M}(G) = \mathcal{A}_n$

In this section, we show how to build from the template a loop in which the action of every element is an even permutation of $\{0, \dots, n-1\}$.

Lemma 3.7 *For every odd $n \geq 43$, there exists a commutative unbreakable loop G which satisfies $\mathcal{M}(G) = \mathcal{A}_n$.*

Proof. We fix the content of a further set of positions in the template in order to obtain what we call the *augmented template*; the top right part of the resulting table is displayed on Figure 6 for $n = 43$.

i. In the top right region, working downwards, let

$$\begin{aligned} [6, n-6] &= 3; [6, n-5] = 1; [6, n-4] = 5; [6, n-3] = 2; [6, n-2] = 0; \\ [6, n-1] &= 4; \\ [7, n-7] &= 1; [7, n-6] = 2; [7, n-5] = 0; [7, n-4] = 3; [7, n-3] = 4; \\ [7, n-2] &= 5; \\ [8, n-4] &= 4; [8, n-3] = 5; [9, n-4] = 2. \end{aligned}$$

ii. Immediately above the central triangle, working upwards, let

$$\begin{aligned} [p, p+1] &= 1; [p, p+2] = 2; [p, p+3] = 0; [p, p+4] = 3; [p, p+5] = 4; \\ [p, p+6] &= 5; \\ [p-1, p+2] &= 1; [p-1, p+3] = 2; [p-1, p+4] = 0; [p-1, p+5] = 3; \\ [p-1, p+6] &= 4; [p-1, p+7] = 5; \\ [p-2, p+3] &= 3; [p-2, p+4] = 1; [p-2, p+5] = 5; \\ [p-3, p+4] &= 2; [p-3, p+5] = 0; [p-4, p+5] = 1. \end{aligned}$$

In the augmented template, each row and column which intersects the central triangle is completely specified. Furthermore, on Figure 6 we highlight two regions by surrounding them with a borderline; they consist of 15 positions each, and their shape and content are identical. We call them *butterflies*. Observe that both ends of the undefined zone are delimited with a butterfly.

We define a special type of patterns which we call *blocks*. A block of index m is an array of $6(m+1) + 9$ cells located on six consecutive antidiagonals; there are $m+1$ complete rows (six cells each) and 9 cells placed on 5 incomplete rows.

	...	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
0	...	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
1	...	23	3	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	24
2	...	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	1	3
3	...	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	1	2	0
4	...	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	1	0	3	2
5	...	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	2	0	3	4	1
6	...	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	3	1	5	2	0	4
7	...	29	30	31	32	33	34	35	36	37	38	39	40	41	42	1	2	0	3	4	5	6
8	...	30	31	32	33	34	35	36	37	38	39	40	41	42	?	?	?	?	4	5	6	7
9	...	31	32	33	34	35	36	37	38	39	40	41	42	?	?	?	?	2	6	7	8	
10	...	32	33	34	35	36	37	38	39	40	41	42	?	?	?	?	?	6	7	8	9	
11	...	33	34	35	36	37	38	39	40	41	42	?	?	?	?	?	?	6	7	8	9	10
12	...	34	35	36	37	38	39	40	41	42	?	?	?	?	?	?	6	7	8	9	10	11
13	...	35	36	37	38	39	40	41	42	?	?	?	?	?	?	6	7	8	9	10	11	12
14	...	36	37	38	39	40	41	42	?	?	?	?	?	?	6	7	8	9	10	11	12	13
15	...	37	38	39	40	41	42	?	?	?	?	?	?	6	7	8	9	10	11	12	13	14
16	...	38	39	40	41	42	?	?	?	?	?	?	6	7	8	9	10	11	12	13	14	15
17	...	39	40	41	42	1	?	?	?	?	?	6	7	8	9	10	11	12	13	14	15	16
18	...	40	41	42	2	0	?	?	?	?	6	7	8	9	10	11	12	13	14	15	16	17
19	...	41	42	3	1	5	?	?	?	6	7	8	9	10	11	12	13	14	15	16	17	18
20	...	42	1	2	0	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
21	...	1	2	0	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
22	...	3	0	5	4	2	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
23	...	0	5	4	24	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
24	...	5	4	1	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
25	...	4	24	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	5
26	...	2	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
:	...	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:

Figure 6: Augmented template for the alternating group; $n = 43$

The content of every cell is defined, every complete row and column is an even permutation of $\{0, 1, 2, 3, 4, 5\}$, and the ends of this array constitute two disjoint copies of the butterfly. Two blocks can be combined to build a larger block, by making the top right butterfly of one block overlap with the bottom left butterfly of the other, as illustrated in Figure 7. Combining two blocks of orders m and q , respectively, creates a block of order $m + q$.

Thus, we can turn the augmented template into the Cayley table of a loop with $\mathcal{M}(G) = \mathcal{A}_n$ simply by inserting a block which fits the undefined zone. Rows 7 to $p - 1$ in the table coincide with the $m + 1$ fully defined rows in the block, so that its order is $m = p - 8$, or conversely $n = 2m + 17$.

Experimentally, we found that the collection of blocks

$$B_{10}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}, B_{18}, B_{19}, B_{21}, B_{22},$$

depicted in the Appendix, enables us to define a loop with $\mathcal{M}(G) = \mathcal{A}_n$ for $n = 37$ (built from B_{10}) and for every odd $n \geq 43$. Each full row and column in these blocks is an even permutation of $\{0, 1, 2, 3, 4, 5\}$. Also, since 0 never occurs at a position $[i, n - i]$, the loops built from these blocks satisfy the condition of Lemma 3.1. In other words, a loop built from the augmented template and our list of blocks is unbreakable, commutative, and such that $\mathcal{M}(G) = \mathcal{A}_n$. \square

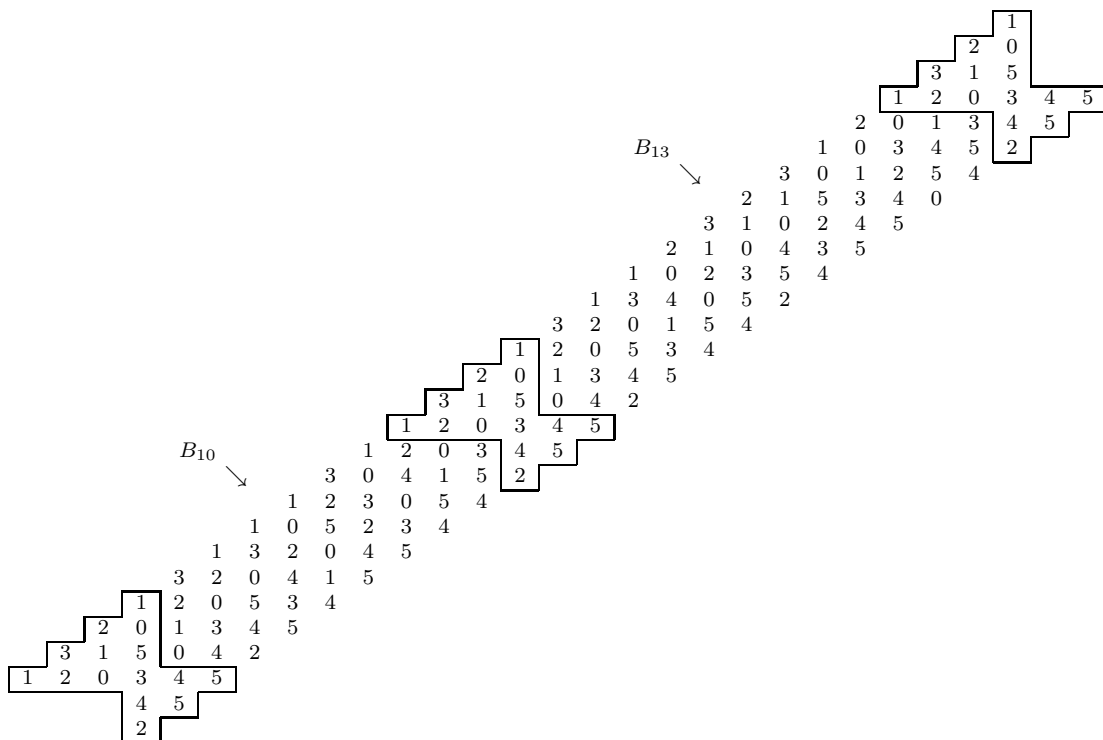


Figure 7: Concatenation of blocks B_{10} and B_{13}

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	5	6	7	8	0	9	4	3	2
2	2	9	5	6	7	8	1	0	4	3
3	3	8	9	5	6	7	2	1	0	4
4	4	7	8	9	5	6	3	2	1	0
5	5	6	7	8	9	4	0	3	2	1
6										
7										
8										
9										

Figure 8: Partially filled Cayley table for $n = 10$

4 Unbreakable loops of even size

In this section we prove the part of Theorem 1.1 which concerns the loops of even order.

Let $n = 2p$, with $p \geq 5$; let $G = \{0, 1, \dots, n-1\}$. We often use the notation $2p$ instead of n . We specify the first $p+1$ rows of a $n \times n$ table, as follows.

- i.* Row 0: for every $j \in G$, $[0, j] = j$.
- ii.* Row 1: besides $[1, 0] = 1$, we have
 - for every j , $1 \leq j \leq p-1$, $[1, j] = p + j - 1$;
 - $[1, p] = 0$ and $[1, p+1] = 2p-1$;
 - for every k , $2 \leq k \leq p-1$, $[1, p+k] = p+1-k$.
- iii.* Row i , for $2 \leq i \leq p-1$: besides $[i, 0] = i$, we have
 - for every j , $1 \leq j \leq i-1$, $[i, j] = 2p - (i-j)$;
 - for every j , $i \leq j \leq p$, $[i, j] = p + (j-i)$;
 - for every k , $1 \leq k \leq i$, $[i, p+k] = i-k$;
 - for every k , $i+1 \leq k \leq p-1$, $[i, p+k] = i + (p-k)$.
- iv.* Row p : besides $[p, 0] = p$, we have
 - for every j , $1 \leq j \leq p-1$, $[p, j] = p+j$;
 - $[p, p] = p-1$ and $[p, p+1] = 0$;
 - for every k , $2 \leq k \leq p-1$, $[p, p+k] = p-k$.

The resulting partially filled table is represented on Figure 8 for $n = 10$. The reader can verify that the above specifies a $(p+1) \times 2p$ latin rectangle on $2p$ objects; there always exists a way to extend it into a $2p \times 2p$ latin square [10], and the bottom $p-1$ lines can be permuted in order to have $[j, 0] = j$ for every j , and thus obtain the Cayley table of a loop.

We now show that this loop is unbreakable. First, we verify that $\langle p \rangle = G$ by observing $p * p = p-1$, $p * (p-1) = 2p-1$, and $p * (2p-1) = 1$; then for every i , $1 \leq i \leq p-2$, $i * (2p-1) = i+1$; meanwhile $1 * j = p+j-1$ for every j , $2 \leq j \leq p-1$. Next, we have $i * i = p$ for every i , $1 \leq i \leq p-1$. There remains the case $j \geq p+1$: for every such j , its left inverse (k such that $k * j = 0$) belongs to the set $\{1, \dots, p\}$, and is a generator of G .

Finally, we prove that the actions L_1 and L_p generate \mathcal{S}_n . In the permutation L_1 , two cycles are of length three, namely $(0 \ 1 \ p)$ and $(2 \ p+1 \ 2p-1)$. If p is odd ($p = 2q - 1$ for some integer q), then the remaining $n - 6$ elements are evenly divided into 4-cycles of the form

$$(k \ p+k-1 \ p-k+2 \ 2p-k+1)$$

for $3 \leq k \leq p$; if p is even, however, there is a cycle $(q+1 \ 3q)$ and the remaining $n - 8$ elements belong to 4-cycles of the above form. Meanwhile, in permutation L_p there is a unique 6-cycle,

$$(0 \ p \ p-1 \ 2p-1 \ 1 \ p+1);$$

if p is even, there is also a 2-cycle $(q \ 3q)$. The other elements of G belong to 4-cycles of the form

$$(k \ p+k \ p-k \ 2p-k),$$

$2 \leq k \leq p-2$. Globally, L_1 and L_p contain the same number of 2-cycles and 4-cycles; the $n - 6$ elements not located in these cycles build up either one (in L_p) or zero (in L_1) cycle of even length; therefore, L_1 and L_p are always of opposite parity.

We again refer to a result by Piccard ([15], Proposition 23, page 53) and proceed to show that L_1 and L_p generate permutations which have exactly the same form as stipulated in the following.

Proposition 4.1 *Let $n \geq 10$ be an even number and $a, b, c, d, e \in \{0, \dots, n-1\}$. Permutations $\varphi = (0 \ 1 \ 2 \ \dots \ n-1)$ and $\psi = (a \ b \ c \ d \ e)$ generate the group \mathcal{S}_n if, and only if, the largest common divisor of \overline{ab} , \overline{ac} , \overline{ad} , \overline{ae} and n is 1.*

Consider $R = L_1^3 \circ L_p$:

- $R(0) = p$; $R(1) = p+1$;
- for every k , $2 \leq k \leq p-2$, $R(k) = k+1$;
- $R(p-1) = 2p-1$; $R(p) = p+2$; $R(p+1) = 0$;
- for every k , $2 \leq k \leq p-3$, $R(p+k) = p+k+1$;
- $R(2p-2) = 2$; $R(2p-1) = 1$.

This permutation consists in a unique n -cycle. Next, let

- $P = L_1^4$, whose cyclic representation is $(0 \ 1 \ p) (2 \ p+1 \ 2p-1)$,
- $Q = L_p^4$, whose cyclic representation is $(0 \ 1 \ p-1) (p \ p+1 \ 2p-1)$, and
- $S = P \circ Q^2$, whose cyclic representation is $(0 \ p-1 \ p \ 2 \ p+1)$.

The automorphism of \mathcal{S}_n which maps R to φ is defined by:

- $0 \mapsto 0$; $1 \mapsto 2p-2$; $p-1 \mapsto 2p-4$; $p \mapsto 1$; $p+1 \mapsto 2p-1$ and
- for every k , $2 \leq k \leq p-2$, $k \mapsto p+k-3$ and $p+k \mapsto k$.

This automorphism maps permutation S to $(0 \ 2p-4 \ 1 \ p-1 \ 2p-1)$, which satisfies the conditions of Proposition 4.1. \square

5 Conclusion

In this article, we proved that unbreakable loops exist for every order $n \geq 5$; we did this with a combination of careful experiments on a computer and of fairly

simple mathematical techniques. We also gathered evidence that these loops are abundant and that certain of them can have interesting or useful additional properties. An obvious and tantalizing extension for our work would be to look for unbreakable loops whose multiplication group is neither of \mathcal{S}_n or \mathcal{A}_n ; coming up with examples of such loops is likely to be a challenging problem, however. It would also be interesting to look for loops with combinatorial properties other than commutativity, or to evaluate how the proportion of unbreakable loops versus the total evolves as the order n increases. The algebraic and combinatorial properties of the variety generated by the unbreakable loops (i.e., their closure under homomorphism, quotient and finite direct product) also deserve to be investigated.

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Appendix

A Small unbreakable loops

In this section, we display the Cayley tables of unbreakable loops of orders between 5 and 13. Starting at order 9, we restrict ourselves to loops such that $\mathcal{M}(G) = \mathcal{A}_n$.

Loop of order 5; $\mathcal{M}(G) = \mathcal{S}_5$.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	0	4	3
2	2	3	4	0	1
3	3	4	1	2	0
4	4	0	3	1	2

Loop of order 6; $\mathcal{M}(G) = \mathcal{S}_6$.

	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	0	4	5	3
2	2	0	1	5	3	4
3	3	5	4	1	0	2
4	4	3	5	2	1	0
5	5	4	3	0	2	1

Commutative loop of order 7; $\mathcal{M}(G) = \mathcal{S}_7$.

	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	1	5	6	0	4
3	3	4	5	6	0	1	2
4	4	5	6	0	3	2	1
5	5	6	0	1	2	4	3
6	6	0	4	2	1	3	5

Commutative loop of order 7; $\mathcal{M}(G) = \mathcal{A}_7$.

	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	0	4	3	6	5
2	2	0	3	5	6	4	1
3	3	4	5	6	1	2	0
4	4	3	6	1	5	0	2
5	5	6	4	2	0	1	3
6	6	5	1	0	2	3	4

Loop of order 8; $\mathcal{M}(G) = \mathcal{S}_8$.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	0	7	6
2	2	3	5	6	7	1	4	0
3	3	5	0	7	6	2	1	4
4	4	6	1	2	3	7	0	5
5	5	7	6	0	1	4	2	3
6	6	4	7	1	0	3	5	2
7	7	0	4	5	2	6	3	1

Loop of order 8; $\mathcal{M}(G) = \mathcal{A}_8$.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	0	7	6
2	2	3	5	6	7	1	0	4
3	3	5	0	7	6	2	4	1
4	4	6	7	0	1	3	5	2
5	5	7	6	1	3	4	2	0
6	6	4	1	2	0	7	3	5
7	7	0	4	5	2	6	1	3

Loop of order 8 with $\mathcal{M}(G) \neq \mathcal{S}_8$ and $\mathcal{M}(G) \neq \mathcal{A}_8$.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	4	7	6	1	3	0	5
3	3	6	1	5	2	7	4	0
4	4	0	5	7	6	2	3	1
5	5	7	0	4	3	1	2	6
6	6	5	4	2	7	0	1	3
7	7	3	6	1	0	4	5	2

Commutative loop of order 9; $\mathcal{M}(G) = \mathcal{A}_9$.

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	6	1	3	7	8	4	5
3	3	4	1	5	7	8	0	6	2
4	4	5	3	7	8	6	1	2	0
5	5	3	7	8	6	1	2	0	4
6	6	7	8	0	1	2	4	5	3
7	7	8	4	6	2	0	5	3	1
8	8	6	5	2	0	4	3	1	7

Commutative loop of order 11; $\mathcal{M}(G) = \mathcal{A}_{11}$.

	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	0	4	5	6	3	8	9	10	7
2	2	0	3	7	8	1	5	9	10	4	6
3	3	4	7	8	1	2	9	10	5	6	0
4	4	5	8	1	7	9	10	2	6	0	3
5	5	6	1	2	9	10	8	3	0	7	4
6	6	3	5	9	10	8	4	0	7	2	1
7	7	8	9	10	2	3	0	6	4	1	5
8	8	9	10	5	6	0	7	4	1	3	2
9	9	10	4	6	0	7	2	1	3	5	8
10	10	7	6	0	3	4	1	5	2	8	9

Commutative loop of order 13; $\mathcal{M}(G) = \mathcal{A}_{13}$.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	0	4	5	6	7	3	9	10	11	12	8
2	2	0	3	5	6	7	1	9	10	11	12	8	4
3	3	4	5	1	7	9	8	10	11	12	6	2	0
4	4	5	6	7	8	1	10	11	12	3	2	0	9
5	5	6	7	9	1	10	11	12	4	8	0	3	2
6	6	7	1	8	10	11	12	2	5	0	4	9	3
7	7	3	9	10	11	12	2	6	0	4	8	5	1
8	8	9	10	11	12	4	5	0	7	2	3	1	6
9	9	10	11	12	3	8	0	4	2	5	1	6	7
10	10	11	12	6	2	0	4	8	3	1	9	7	5
11	11	12	8	2	0	3	9	5	1	6	7	4	10
12	12	8	4	0	9	2	3	1	6	7	5	10	11

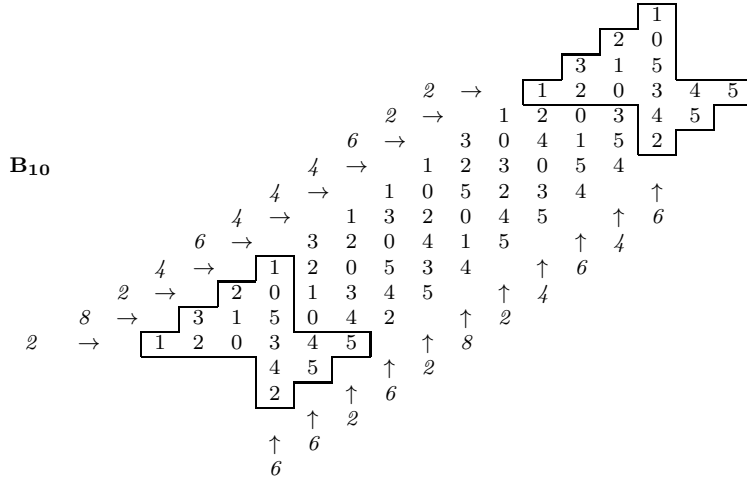
B Example of a loop built from the template

To illustrate the method of Lemma 3.6, we display in this section the full Cayley table of a commutative loop of order 21 built from the template, such that $\mathcal{M}(G) = \mathcal{A}_{21}$.

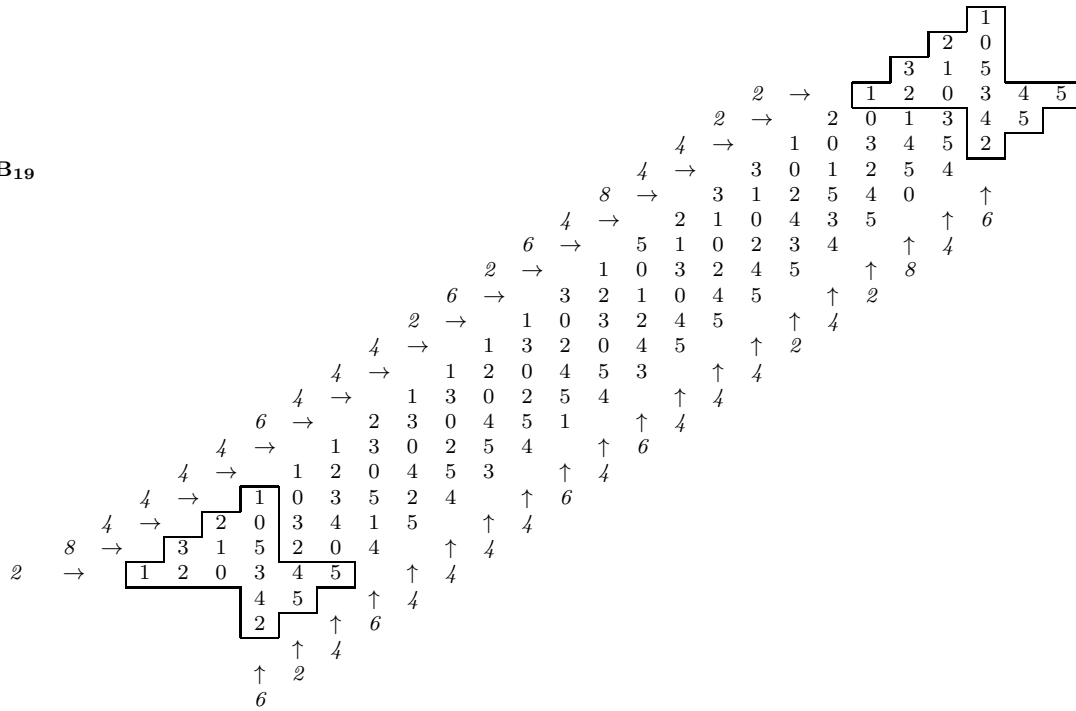
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	0	4	5	6	7	8	9	10	11	12	3	14	15	16	17	18	19	20	13
2	2	0	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	3
3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	2	0
4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	0	3	2
5	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	2	0	3	4	1
6	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	3	1	2	5	0	4
7	7	8	9	10	11	12	13	14	15	16	17	18	19	20	2	1	0	3	4	5	6
8	8	9	10	11	12	13	14	15	16	17	18	19	20	3	1	0	4	5	2	6	7
9	9	10	11	12	13	14	15	16	17	18	19	20	1	2	0	5	3	4	6	7	8
10	10	11	12	13	14	15	16	17	18	19	20	1	2	0	3	4	5	6	7	8	9
11	11	12	13	14	15	16	17	18	19	20	1	3	0	5	4	2	6	7	8	9	10
12	12	3	14	15	16	17	18	19	20	1	2	0	5	4	13	6	7	8	9	10	11
13	13	14	15	16	17	18	19	20	1	2	0	5	4	1	6	7	8	9	10	11	12
14	14	15	16	17	18	19	20	2	3	0	3	4	13	6	7	8	9	10	11	12	5
15	15	16	17	18	19	20	3	1	0	5	4	2	6	7	8	9	10	11	12	13	14
16	16	17	18	19	20	2	1	0	4	3	5	6	7	8	9	10	11	12	13	14	15
17	17	18	19	20	1	0	2	3	5	4	6	7	8	9	10	11	12	13	14	15	16
18	18	19	20	1	0	3	5	4	2	6	7	8	9	10	11	12	13	14	15	16	17
19	19	20	1	2	3	4	0	5	6	7	8	9	10	11	12	13	14	15	16	17	18
20	20	13	3	0	2	1	4	6	7	8	9	10	11	12	5	14	15	16	17	18	19

C Blocks

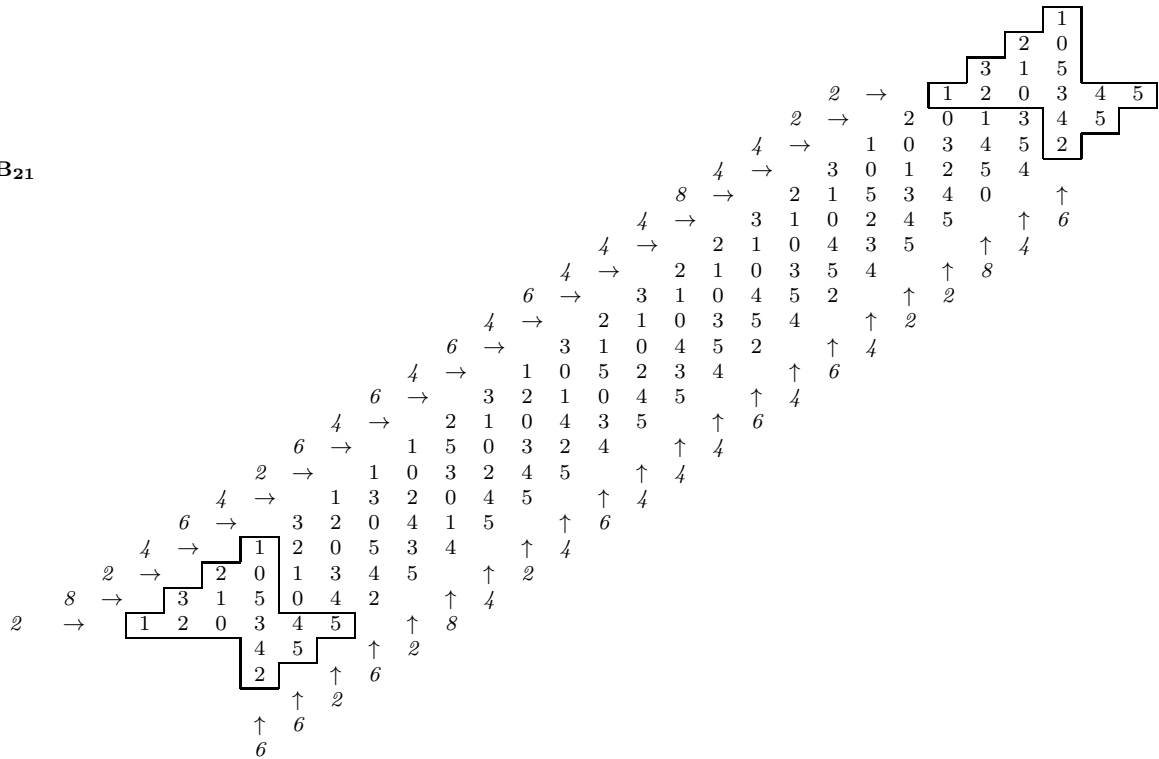
We display in this section Blocks B_{10} to B_{22} , which can be combined with the augmented template to construct a commutative unbreakable loop with $\mathcal{M}(G) = \mathcal{A}_n$ for any order $n \geq 43$; see the proof of Lemma 3.7. In the figures, the numbers in italics are the numbers of inversions in the corresponding row or column; they confirm that all permutations of $\{0, 1, 2, 3, 4, 5\}$ are even.



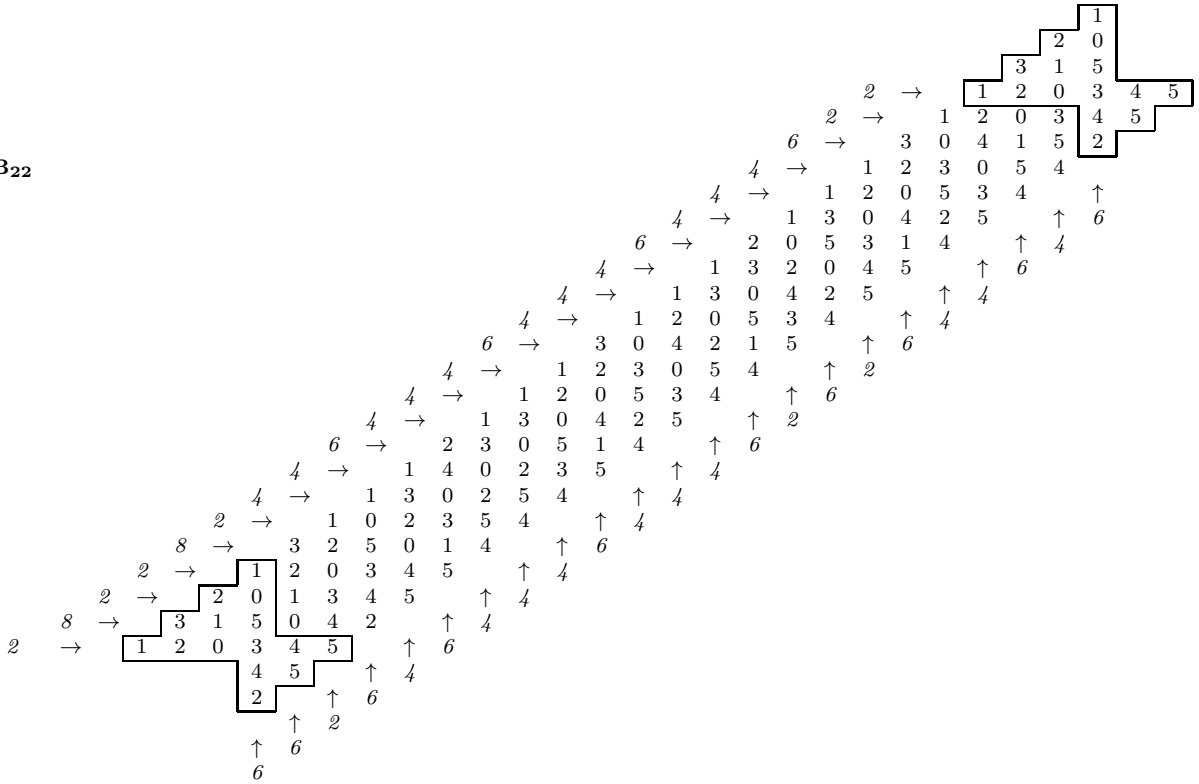
B₁₉



B₂₁



B₂₂



D Blocks for loops of intermediate order

In this section, we display examples of commutative unbreakable loops of odd order n , $15 \leq n \leq 41$ and $n \neq 37$, which satisfy $\mathcal{M}(G) = \mathcal{A}_n$. For loops of these orders, the method of Lemma 3.7 cannot be applied; it is nevertheless possible to build a Cayley table in an analogous manner, starting with a simplified template and completing its undefined zone with a suitable array of entries.

In the template used for orders 15 to 23, all positions $[i, j]$ except those for which $n \leq i + j \leq n + 5$ are filled exactly as in the original template. Since the specification for rows 1 and 2 leaves and the constraint that L_2 be an even permutation leave no flexibility for $[1, n - 1]$, $[2, n - 2]$ and $[2, n - 1]$, all positions $[i, j]$ such that $i \geq 3$, $j \geq 3$ and $n \leq i + j \leq n + 5$ remain undefined. A suitable loop can be built from the simplified template with the data displayed below, which specify the intersection of rows 3 to $p + 3$ and columns p to $n - 1$ with the undefined zone.

Starting with $n = 25$, it is possible to add the further constraint that the central triangle (i.e. positions $[i, j]$ such that $p + 1 \leq i, j \leq p + 5$ and $n \leq i + j \leq n + 5$) is specified exactly as in the original template. Therefore, for orders $n \geq 25$, it is enough to display the intersection of rows 3 to p and columns p to $n - 1$ with the undefined zone.

n = 15					2	0	1
				1	3	2	0
			3	0	1	4	2
		0	1	2	5	3	4
	1	2	0	3	4	5	
	3	5	2	4	0		
	5	1	4	10			
	2	4	5				

n = 17					1	2	0
				1	0	3	2
				2	0	3	4
			3	1	5	2	0
		1	2	0	3	4	5
	1	2	0	3	4	5	
	3	0	5	4	2		
	0	5	4	11			
	5	4	1				

n = 19					1	2	0
				2	3	0	1
				0	1	2	3
			1	3	0	5	4
			2	3	1	4	0
		5	1	0	2	3	4
	1	0	3	2	4	5	
	3	2	0	4	5		
	2	1	4	12			
	0	4	5				

n = 21					1	2	0
				0	2	3	1
				2	1	3	0
				1	3	4	0
			3	0	1	2	5
			0	1	2	5	3
		1	2	0	3	4	5
	3	2	1	4	5	0	
	1	0	5	2	4		
	0	5	4	13			
	5	4	3				

n = 23					1	2	0
				2	3	0	1
				0	1	2	3
				0	1	3	5
				3	1	2	4
			1	0	2	3	5
			2	3	1	5	4
			0	1	4	2	3
		1	2	3	0	5	4
	3	5	0	2	4		
	5	1	4	14			
	0	4	3				

n = 25

```

1 2 0
  1 0 3 2
    2 0 3 4 1
      1 3 2 5 0 4
        1 3 0 4 2 5
          5 0 2 1 3 4
            2 1 3 0 4 5
              3 1 0 2 4 5
                1 2 0 3 4 5
                  1 2 0 3 4 5
                    1 3 0 5 4 2

```

n = 27

```

1 2 0
  1 0 3 2
    2 0 3 4 1
      2 1 3 5 0 4
        0 1 3 4 2 5
          1 2 3 0 5 4
            1 5 3 0 4 2
              3 0 2 1 4 5
                2 1 3 0 4 5
                  1 0 2 4 3 5
                    1 2 3 0 5 4
                      1 3 0 5 4 2

```

n = 29

```

1 2 0
  1 0 3 2
    2 0 3 4 1
      1 5 3 2 0 4
        1 0 3 2 4 5
          1 3 2 0 4 5
            3 2 0 4 1 5
              1 2 0 5 3 4
                2 0 1 3 4 5
                  3 1 5 0 4 2
                    1 2 0 3 4 5
                      1 2 0 3 4 5
                        1 3 0 5 4 2

```

n = 31

```

1 2 0
  0 2 3 1
    2 1 3 0 4
      4 1 3 0 5 2
        1 3 0 2 5 4
          1 0 2 3 5 4
            1 2 3 0 5 4
              1 3 0 2 5 4
                5 0 2 3 4 1
                  2 1 3 0 4 5
                    3 1 0 2 4 5
                      1 2 0 3 4 5
                        1 2 0 3 4 5
                          1 3 0 5 4 2

```

n = 33

```

1 2 0
  1 0 3 2
    2 0 3 4 1
      1 5 3 2 0 4
        1 0 3 2 4 5
          1 3 2 0 4 5
            2 4 0 3 1 5
              1 3 0 2 5 4
                1 3 0 2 5 4
                  5 0 2 1 3 4
                    2 1 3 0 4 5
                      3 1 0 2 4 5
                        1 2 0 3 4 5
                          1 2 0 3 4 5
                            1 3 0 5 4 2

```

n = 35

```

1 2 0
  1 0 3 2
    2 0 3 4 1
      3 1 5 2 0 4
        1 2 0 3 4 5
          1 2 0 3 4 5
            0 4 3 1 5 2
              3 1 2 0 5 4
                3 1 2 0 5 4
                  2 1 0 3 5 4
                    1 0 2 5 4 3
                      3 0 1 4 2 5
                        2 1 3 5 0 4
                          1 0 2 4 3 5
                            1 2 3 0 5 4
                              1 3 0 5 4 2

```

n = 39

```

1 2 0
  2 3 0 1
    0 1 2 3 4
      3 1 5 0 4 2
        2 1 3 0 4 5
          1 0 2 4 3 5
            1 0 3 5 2 4
              3 0 2 1 4 5
                1 2 4 3 5 0
                  1 3 0 2 5 4
                    3 2 0 1 5 4
                      1 2 0 4 5 3
                        5 0 1 3 2 4
                          2 1 3 0 4 5
                            3 1 0 2 4 5
                              1 2 0 3 4 5
                                1 2 0 3 4 5
                                  1 3 0 5 4 2

```

