

# Geometric renormalisation and Hausdorff dimension for loop-approximable geodesics escaping to infinity

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## Abstract

The main result of this paper is to show that if  $N$  is a normal subgroup of a Kleinian group  $G$  such that  $G/N$  contains a coset which is represented by some loxodromic element, then the Hausdorff dimension of the transient limit set of  $N$  coincides with the Hausdorff dimension of the limit set of  $G$ . This observation extends previous results by Fernández and Melián for Riemann surfaces.

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## 1 Introduction and statement of results

In this paper we study fractal geometric aspects of the limit set  $L(N)$  of a normal subgroup  $N$  of some given non-elementary Kleinian group  $G$  acting on  $(m + 1)$ -dimensional hyperbolic space  $\mathbb{D}^{m+1}$ . We always assume that  $G/N$  contains a coset which is represented by some loxodromic isometry  $\gamma \in G$ . It is well known that in this situation  $L(N)$  coincides with the limit set  $L(G)$  of the larger group  $G$ . However, a comparison of finer aspects of these two limit sets usually turns out to be far more involved, as can be seen, for instance, in the work of Brooks [3] and Rees [5, 6].

In this paper we investigate the set  $L_t(N)$  of directions at some arbitrary point  $z$  on the manifold  $\mathcal{M}_N$  associated with  $N$  for which the resulting geodesic movement on  $\mathcal{M}_N$  eventually escapes from every compact region on  $\mathcal{M}_N$ , but which is nevertheless contained in the  $\epsilon$ -neighbourhood of some sequence of closed loops starting and ending at  $z$  on  $\mathcal{M}_N$ , for each  $\epsilon > 0$ . That is, we consider the transient limit set  $L_t(N)$  of  $N$ , given by

$$L_t(N) := \{\xi \in L(N) : \lim_{r \rightarrow \infty} d(\xi(r), N(0)) = \infty\}.$$

Here,  $d$  refers to the hyperbolic metric in  $\mathbb{D}^{m+1}$  and  $\xi(r)$  denotes a  $\mathbb{R}_+$ -parametrisation of the geodesic ray from the origin to  $\xi$ .

The following theorem gives the main result of this paper.

**Main Theorem.** *Let  $G$  and  $N$  be given as above. We then have*

$$\dim_H(L_t(N)) = \dim_H(L(G)).$$

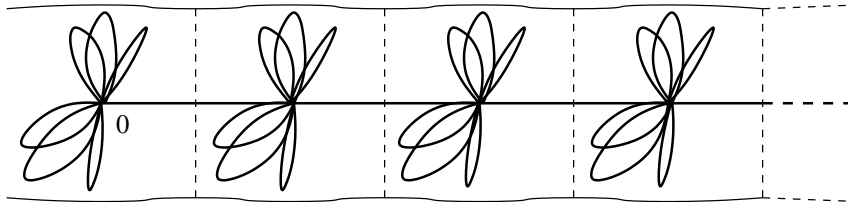
This theorem gives an extension of results by Fernández and Melián [4], who studied the set  $\mathcal{E}$  of escaping, not necessarily loop-approximable directions on a complete oriented non-compact Riemann surface  $\mathcal{R}$  with fundamental group  $\Gamma$ . That is,  $\mathcal{E} := \{\xi \in \mathbb{S}^1 : \lim_{r \rightarrow \infty} \rho(\xi(r), \Gamma(0)) = \infty\}$ , where  $\mathbb{S}^1$  refers to the boundary at infinity of  $\mathbb{D}^2$  and  $\rho$  denotes the hyperbolic metric in  $\mathbb{D}^2$ . The main result of [4, Theorem 1] was to establish the following trichotomy.

- (i) If  $\mathcal{R}$  has finite area, then  $\mathcal{E}$  is countable.
- (ii) If Brownian motion on  $\mathcal{R}$  is transient, then  $\mathcal{E}$  has full Lebesgue measure.
- (iii) If  $\mathcal{R}$  has infinite area and Brownian motion is recurrent, then  $\mathcal{E}$  has zero Lebesgue measure, but its Hausdorff dimension is equal to 1.

Therefore, for hyperbolic manifolds which are normal coverings of some hyperbolic manifold and which possess a loxodromic representative  $\gamma \in G/N$ , our Main Theorem extends the results by Fernández and Melián to arbitrary dimensions and to the situation where the boundary at infinity of hyperbolic space is replaced by the limit set of the fundamental group. However, let us emphasize that our proof does require the existence of a loxodromic  $\gamma \in G/N$ , and hence our extension is restricted to normal coverings with this property.

Our proof hinges on two renormalisation procedures. These are used to locate a certain family of subsystems within the loop-approximable, non-recurrent dynamics on the manifold  $\mathcal{M}_N$ . At the boundary of the universal covering space these subsystems are described by a family of Cantor sets contained in  $L(N)$ , and here the key observation is that this family contains Cantor sets whose Hausdorff dimension is arbitrarily close to the Hausdorff dimension of  $L(G)$ . These Cantor sets are constructed inductively using the following two renormalisation procedures. The first of these employs a well-known construction by Bishop and Jones for computing the Hausdorff dimension of bounded dynamics (see [2], [7]). This construction gives rise to a certain weighted scaling law, which we then apply a sufficient number of times in order to prepare for the second renormalisation step. On the manifold  $\mathcal{M}_N$  this first step of the overall construction corresponds to a family of well separated quasi-geodesics within a bounded region of  $\mathcal{M}_N$ , each starting at the same point. The second renormalisation procedure consists in prolonging each of these quasi-geodesics by a long geodesic segment which is contained in the projection of the axis

Figure 1: Dynamics in  $\mathcal{M}_N$ .



$A_\gamma$  of  $\gamma$  onto  $\mathcal{M}_N$ , and which is chosen such that it leads out of the bounded region which contained the original quasi-geodesics. The resulting dynamical behaviour on  $\mathcal{M}_N$  is sketched in Figure 1.

We assume that the reader is familiar with the proof of Bishop and Jones' result on the relationship between the exponent of convergence  $\delta(G)$  of a non-elementary Kleinian group  $G$  and the Hausdorff dimension of the radial limit set  $L_r(G)$  of  $G$  (see [2] and for a more detailed proof [7]). Here, the reader might like to recall that the radial limit set  $L_r(G)$  represents those limit points  $\xi$  for which the projection of a hyperbolic ray towards  $\xi$  returns infinitely often to some compact part of the manifold associated to  $G$ .

## 2 Preliminaries

Throughout, let  $G$  be a non-elementary Kleinian group acting on  $(m+1)$ -dimensional hyperbolic space  $\mathbb{D}^{m+1}$ . Also, let  $N$  be a normal subgroup of  $G$  such that  $G/N$  contains a coset  $[\gamma]$ , for some loxodromic  $\gamma \in G$ . Moreover, we always assume without loss of generality that  $0 \in \mathbb{D}^{m+1}$  is an element of the axis  $A_\gamma$  of  $\gamma$ , and we let  $\eta_-$  ( $\eta_+$  resp.) refer to the repulsive (attractive resp.) fixed point of  $\gamma$ . Next, recall that to any arbitrary Kleinian group  $\Gamma$  we can associate its truncated Poincaré series  $\mathcal{P}_t(\Gamma, s, w)$ , as well as its Poincaré series  $\mathcal{P}(\Gamma, s, w)$ . These series are given, for  $s, t \in \mathbb{R}_+$  and  $w \in \mathbb{D}^{m+1}$ , by

$$\mathcal{P}_t(\Gamma, s, w) := \sum_{\substack{h \in \Gamma \\ d(w, h(w)) \leq t}} e^{-sd(w, h(w))}, \quad \text{and} \quad \mathcal{P}(\Gamma, s, w) := \lim_{t \rightarrow \infty} \mathcal{P}_t(\Gamma, s, w).$$

The abscissa of convergence of the infinite series  $\mathcal{P}(\Gamma, s, w)$  is called the exponent of convergence of  $\Gamma$ , and it will be denoted by  $\delta(\Gamma)$ . Here, note that Bishop and Jones [2] (see also [7]) showed that if  $\Gamma$  is non-elementary, then we always have that

$$\delta(\Gamma) = \dim_H(L(\Gamma)).$$

Also, we will make use of the following standard facts and notations for the Poincaré model  $(\mathbb{D}^{m+1}, d)$  of the  $(m+1)$ -dimensional hyperbolic space. Let  $B(w, r)$  refer to

the hyperbolic ball centred at  $w$  of radius  $r$ , and let  $\Pi : \mathbb{D}^{m+1} \rightarrow \mathbb{S}^m$  denote the radial projection from the origin to the boundary  $\mathbb{S}^m$  of hyperbolic space. That is, for  $E \subset \mathbb{D}^{m+1}$  we have  $\Pi(E) := \{\xi \in \mathbb{S}^m : s_\xi \cap E \neq \emptyset\}$ , where  $s_\xi$  refers to the Euclidean straight line between the origin and  $\xi$ . Also, we require the following analogue of Pythagoras's Theorem for hyperbolic triangles. For this, consider a hyperbolic triangle with sides of finite lengths  $a$ ,  $b$  and  $c$ , and with  $\alpha_0 \in (0, \pi)$  denoting the angle opposite to the side of length  $a$ . A straightforward application of the hyperbolic cosine rule (see e.g. [1]) then gives that there exists a constant  $K > 0$ , depending only on  $\alpha_0$ , such that

$$b + c - K \leq a \leq b + c.$$

Moreover, we require the following additional observation from elementary hyperbolic geometry. For this, consider some arbitrary hyperbolic geodesic  $A \subset \mathbb{D}^{m+1}$  which does not contain the origin, and let  $s_\eta$  denote the geodesic ray connecting the origin with one of the endpoints  $\eta \in \mathbb{S}^m$  of  $A$ . Also, let  $\hat{z}_A$  refer to the summit of the geodesic  $A$ . That is,  $\hat{z}_A$  is uniquely determined by  $d(0, \hat{z}_A) = \min\{d(0, w) : w \in A\}$ . A straightforward exercise in hyperbolic geometry then shows that there exists a universal constant  $\tau > 0$  such that

$$\min\{d(w, \hat{z}_A) : w \in s_\eta\} < \tau. \tag{1}$$

In fact, an elementary calculation shows that  $\tau$  is equal to  $\log(1 + \sqrt{2})$ , which is often referred to as Schweikart's constant.

For a sequence  $(w_n)_{n \in \mathbb{N}_0}$  of distinct points in  $\mathbb{D}^{m+1}$ , let

$$[w_0, w_1, w_2, \dots]$$

denote the quasi-geodesic path obtained by connecting  $w_n$  and  $w_{n+1}$  with the unique geodesic arc between them, for each  $n \in \mathbb{N}$ . A standard observation from hyperbolic geometry then shows that if the lengths of these geodesic segments are uniformly bounded away from 0, and if each of the angles between adjacent geodesic segments is uniformly bounded from below by some  $\alpha_0 > 0$ , then  $[w_0, w_1, w_2, \dots]$  is a quasi-geodesic ray towards a unique point at infinity. That is, each  $w_n$  is, with respect to the hyperbolic metric, uniformly bounded (depending on  $\alpha_0$ ) away from the geodesic ray from  $w_0$  towards the uniquely determined limit at the boundary at infinity of the sequence  $(w_n)$ .

Finally, note that we use the common notation  $a_n \asymp b_n$  if two sequences of positive real numbers  $a_n$  and  $b_n$  are comparable, that is, if the ratio  $a_n/b_n$  is uniformly bounded from below by  $1/c$  and from above by  $c$ , for some  $c > 1$  and for all  $n \in \mathbb{N}$ .

### 3 The two renormalisation procedures

Let us begin with by giving our first renormalisation procedure. Here, a *geodesic  $N$ -tree*  $\mathcal{T}(z)$  rooted at  $z$  refers to an infinite tree whose set of vertices  $V(\mathcal{T}(z))$  is

contained in  $N(z)$  and whose edges are finite geodesic segments between the vertices, such that each vertex  $u \in \mathcal{T}(z)$  has a finite set  $\mathcal{S}(u)$  of successors of cardinality at least 2 and such that each element in  $V(\mathcal{T}(z)) \setminus \{z\}$  has a unique predecessor.

In the following, let  $z_n$  be defined by  $z_n := \gamma^n(0)$ , for each  $n \in \mathbb{Z}$ . Note that our first renormalisation procedure is well known for the special case in which each  $z_n$  lies in the orbit  $N(0)$ . In this situation, its outcome has already been obtained in [7, Proposition 3.5]. The novelty here is that for the normal subgroup  $N$  the result of [7, Proposition 3.5] continues to hold for each element of the orbit  $\{z_n : n \in \mathbb{Z}\}$  of the origin under  $\langle \gamma \rangle$ .

**Recurrent Renormalisation Procedure ([RRP]).**

For each  $0 < s < \delta(N)$ , there exist  $\kappa > 0$ ,  $\ell_s > 0$  and  $K_s > 1$  such that for each  $h \in N$  and  $n \in \mathbb{Z}$  there exists a geodesic  $N$ -tree  $\mathcal{T} = \mathcal{T}_s(h_0(z_n))$  rooted at  $h(z_n)$  with the following properties.

- (i) If  $u \in V(\mathcal{T})$ , then  $\Pi(B(v, \kappa)) \subset \Pi(B(u, \kappa))$  for each  $v \in \mathcal{S}(u)$ .
- (ii) If  $v \in \mathcal{S}(u)$  for some  $u \in V(\mathcal{T})$ , then  $d(u, v) \leq \ell_s$ .
- (iii) If  $v, w \in \mathcal{S}(u)$  for some  $u \in V(\mathcal{T})$ , then  $\exp(d(0, v)) \asymp \exp(d(0, w))$  and  $\Pi(B(v, \kappa)) \cap \Pi(B(w, \kappa)) = \emptyset$ .
- (iv) For each  $u \in V(\mathcal{T})$  we have

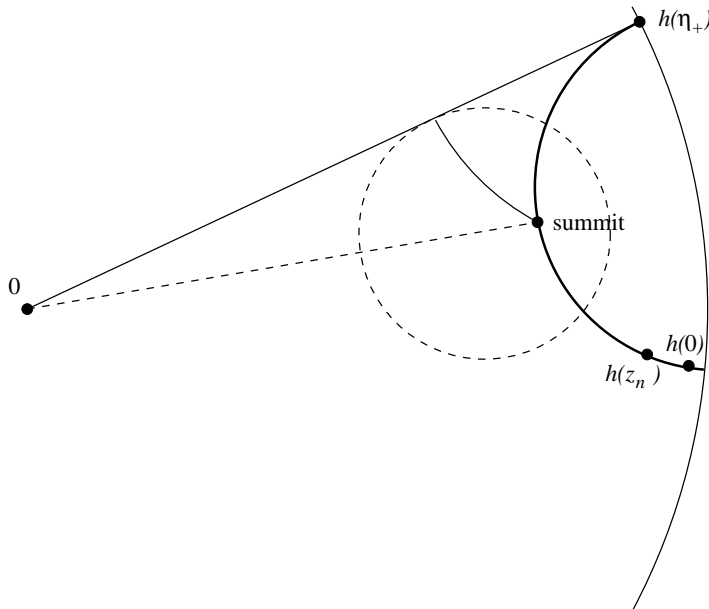
$$\sum_{v \in \mathcal{S}(u)} (\text{diam}(\Pi(B(v, \kappa))))^s \geq K_s (\text{diam}(\Pi(B(u, \kappa))))^s.$$

We will say that the so derived family  $\{B(v, \kappa) : v \in \mathcal{S}(u)\}$  is obtained by applying the recurrent renormalisation procedure to  $u \in V(\mathcal{T})$ .

*Proof.* As already mentioned before, for  $n = 0$  the assertion in this procedure has been obtained in [7, Proposition 3.5] (see also [2]), and we refer to these papers for the proof in this case. In fact, note that the main idea of the proof of [7, Proposition 3.5] consists of a geometrization of the rate of increase, for  $t$  tending to infinity, of the truncated Poincaré series  $\mathcal{P}_t(H, s, 0)$  for  $s < \delta(H)$ . For the proof of the general situation, that is, for some arbitrary  $n \in \mathbb{Z}$ , note that the value of the truncated Poincaré series associated with  $N$  does not change if we exchange the observation point  $z_0 = 0$  by some arbitrary point in  $\{z_n : n \in \mathbb{Z}\}$ . More precisely, since  $N$  is normal in  $G$ , we have for each  $n \in \mathbb{Z}$  and  $s, t \in \mathbb{R}_+$ ,

$$\begin{aligned} \mathcal{P}_t(N, s, z_n) &= \sum_{\substack{h \in N \\ d(z_n, h(z_n)) \leq t}} e^{-sd(z_n, h(z_n))} = \sum_{\substack{h \in N \\ d(\gamma^n(0), h\gamma^n(0)) \leq t}} e^{-sd(\gamma^n(0), h\gamma^n(0))} \\ &= \sum_{\substack{\gamma^{-n}h\gamma^n \in N \\ d(0, \gamma^{-n}h\gamma^n(0)) \leq t}} e^{-sd(0, \gamma^{-n}h\gamma^n(0))} = \sum_{\substack{h \in N \\ d(0, h(0)) \leq t}} e^{-sd(0, h(0))} \\ &= \mathcal{P}_t(N, s, 0). \end{aligned}$$

Figure 2: The location of  $h(\eta_+)$ .



Using this observation, the assertion now follows from a straightforward adaptation of the arguments in the proof of [7, Proposition 3.5].  $\square$

**Transient Renormalisation Procedure ([TRP]).**

For  $0 < s < \delta(N)$ ,  $n \in \mathbb{Z}$  and  $h_0 \in N$ , let  $\mathcal{T} = \mathcal{T}_s(h_0(z_n))$  denote the geodesic  $N$ -tree obtained in the recurrent renormalisation procedure [RRP]. Then there exists a constant  $0 < k_\gamma < 1$  such that for each  $q \in \mathbb{N}_0$  sufficiently large and for each  $h \in N$  with  $h(z_n) \in V(\mathcal{T}) \setminus \{h_0(z_n)\}$ , we have that

$$\text{diam}(\Pi(B(h(z_{n+q}), \tau))) \geq k_\gamma^q \text{diam}(\Pi(B(h(z_n), \tau))),$$

where  $\tau := \log(1 + \sqrt{2})$ . We will say that the ball  $B(h(z_{n+q}), \tau)$  is obtained by starting at  $h(z_n)$  and applying the transient renormalisation procedure  $q$  times.

*Proof.* Let  $\mathcal{T} = \mathcal{T}_s(h_0(z_n))$  and  $h \in N$  be given as stated in the renormalisation procedure. Let us first show that  $h(z_n)$  lies always close to the summit  $\hat{z}_{h(A_\gamma)}$  of the geodesic  $h(A_\gamma)$ , the image of the axis  $A_\gamma$  under  $h$ . Indeed, since  $h(z_n) \in V(\mathcal{T}) \setminus \{h_0(z_n)\}$ , the statement in (1) of [RRP] implies that there exists  $u \in V(\mathcal{T})$  such that  $\Pi(B(h(z_n), \kappa)) \subset \Pi(B(u, \kappa))$  (see also Figure 2).

The elementary observation in (1) shows that the distance from the summit of a geodesic to each of the two rays from the origin to the endpoints of the geodesic is

less than  $\tau$ . Thus, by construction, we have that

$$h(\eta_+) \in \Pi(B(\hat{z}_{h(A_\gamma)}, \tau)).$$

Recall that in the recurrent renormalisation procedure [RRP] we have already derived the existence of the parameter  $\ell_s$ , which is the upper bound of the lengths of the edges in the tree  $\mathcal{T}$ .

Next, consider the geodesic  $h(A_\gamma)$  containing the points  $h(0)$  and  $h(z_n)$ ; one of its endpoints will be  $h(\eta_+)$ . Assume, by way of contradiction, that the distance between  $h(z_n)$  and the summit  $\hat{z}_{h(A_\gamma)}$  of  $h(A_\gamma)$  is larger than  $2\ell_s + \tau$ . Projecting onto the manifold  $\mathcal{M}_N$  associated to  $N$  and using the hyperbolic triangle inequality, we obtain a contradiction to the fact that  $h(z_n) \in V(\mathcal{T}) \setminus \{h_0(z_n)\}$ . It immediately follows that

$$\text{diam}(\Pi(B(\hat{z}_{h(A_\gamma)}, \tau))) \asymp \text{diam}(\Pi(B(h(z_n), \tau))),$$

where the comparability constant depends only on the distance  $2\ell_s + \tau$ , and therefore, only on  $N$  and  $s$ . The statement now follows by applying  $h\gamma^q h^{-1}$  to the ball  $B(h(z_n), \tau)$ , which immediately gives that

$$\text{diam}(\Pi(B(h(z_{n+q}), \tau))) \geq k_\gamma^q \text{diam}(\Pi(B(h(z_n), \tau))),$$

where  $k_\gamma \asymp \exp(-d(0, \gamma(0)))$ . □

Let us remark that the constant  $\kappa$  in [RRP] and the constant  $\tau$  in [TRP] are independent of each other. Also, the statements in [RRP] continue to hold if we replace  $\kappa$  by a smaller positive number, and the same holds for  $\tau$  in [TRP]. Therefore, for the remainder of this paper, when applying [RRP] and [TRP], we use

$$\sigma := \min\{\kappa, \tau\}$$

instead of  $\kappa$  and  $\tau$ .

## 4 Proof of the theorem

Let us first observe that it is sufficient to prove the assertion in the theorem for the case in which

$$\delta(N) = \dim_H(L(G)).$$

Indeed, this can immediately be seen by way of contradiction as follows. Suppose that  $\delta(N) < \dim_H(L(G))$ . Since diminishing a set by a subset of smaller Hausdorff dimension does not alter the Hausdorff dimension of that set and using the well known fact that  $\delta(N) = \dim_H(L_r(N))$  (see [2] and [7]), we obtain

$$\dim_H(L(G)) = \dim_H(L(N)) = \dim_H(L_t(N) \cup L_r(N)) = \dim_H(L_t(N)).$$

Therefore, we can now assume, without loss of generality, that

$$\delta(N) = \dim_H(L(G)) = \delta(G).$$

The rough strategy for proving the main theorem in this case is as follows. For some arbitrary given  $0 < s < \delta(N)$ , we construct a certain Cantor set  $\mathcal{C}_s \subset L_t(N)$ , and then show that  $\dim_H(\mathcal{C}_s) \geq s$ . By the arbitrary choice of  $s$ , the theorem then follows. The idea of the Cantor set construction is to start at the origin and then to perform an alternating inductive process using both renormalisation procedures. The building block of this process is that we first apply the recurrent renormalisation procedure [RRP] sufficiently many times until the resulting power of  $K_s$  is large enough (in fact, this number of times depends on the outcome of the step to come). After that, we perform the transient renormalisation procedure [TRP] sufficiently many times, without losing the control on the distortion (in particular, this step will guarantee that our Cantor set contains only transient limit points). More precisely, let  $0 < s < \delta(N)$  be given. Then  $s$  determines the width  $\ell_s$  of the recurrent renormalisation procedure [RRP]. Having fixed  $\ell_s$ , we choose  $q \in \mathbb{N}$ , the number of times we are going to apply the transient renormalisation procedure [TRP], so that

$$q d(0, \gamma(0)) \geq 4\ell_s. \quad (2)$$

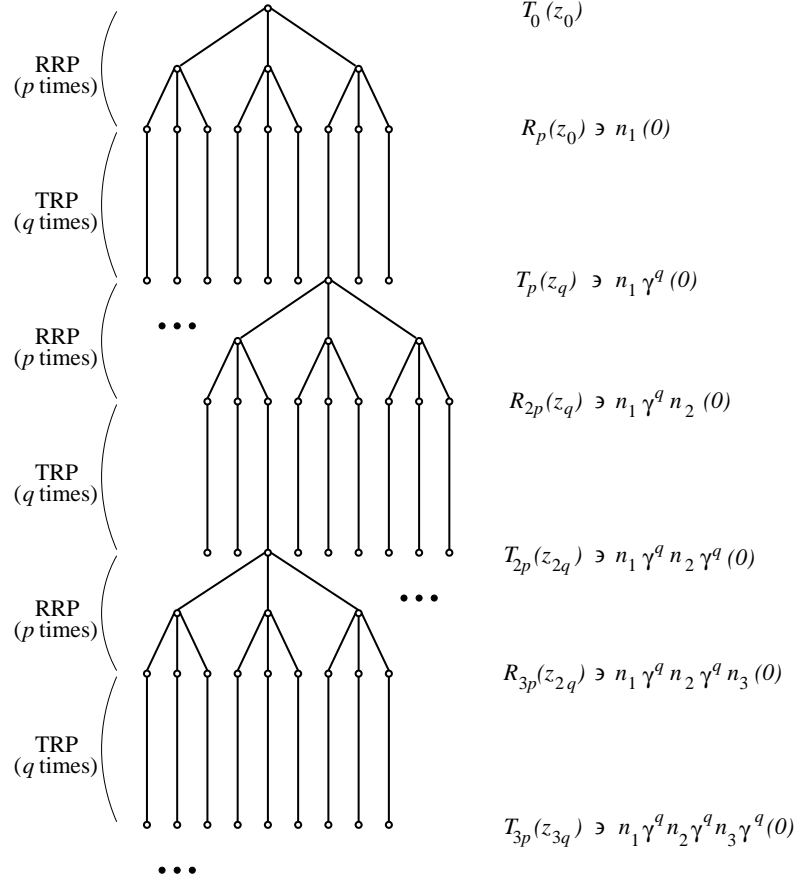
This choice of  $q$  will guarantee that the Cantor set  $\mathcal{C}_s$  we are going to construct will be contained in  $L_t(N)$ . Finally, we choose  $p \in \mathbb{N}$  to be minimal with respect to the property

$$K_s^p k_\gamma^q > 1. \quad (3)$$

Let us now come to the explicit construction of  $\mathcal{C}_s$ . As already mentioned, the construction starts at the origin, and we set  $T_0(z_0) = T_0(0) := \{0\}$ . Then, the first step is to apply the recurrent renormalisation procedure [RRP]  $p$  times, starting at  $z_0$ . According to [RRP], this gives rise to a set of hyperbolic balls  $B(v, \sigma)$  whose radial projections to the boundary  $\mathbb{S}^m$  are pairwise disjoint and of comparable diameter. The set of centres of these balls in  $\mathbb{D}^{m+1}$  will be denoted by  $R_p(z_0)$ . Then, the second step is to apply the transient renormalisation procedure [TRP]  $q$  times to each element in  $R_p(z_0)$ . The set of centres of the so obtained hyperbolic balls will be denoted by  $T_p(z_q)$ . This represents the start of the induction, and we then continue as follows. Assume that the sets  $R_{np}(z_{(n-1)q})$  and  $T_{np}(z_{nq})$  have been constructed, for some  $n \in \mathbb{N}$ . To each of the points in  $T_{np}(z_{nq})$  we then apply the recurrent renormalisation procedure [RRP]  $p$  times. The set of centres of these so obtained hyperbolic balls gives the set  $R_{(n+1)p}(z_{nq})$ . Next, we apply the transient renormalisation procedure [TRP]  $q$  times to each of the elements in  $R_{(n+1)p}(z_{nq})$ . The set of centres of these so obtained hyperbolic balls gives rise to the set  $T_{(n+1)p}(z_{(n+1)q})$ . This finishes our alternating inductive argument (see also Figure 3), and we can now



Figure 3: The inductive construction of  $\mathcal{C}_s$ .



use it to define our desired Cantor set  $\mathcal{C}_s$  by

$$\mathcal{C}_s := \bigcap_{n \in \mathbb{N}} \bigcup_{v \in T_{np}(z_{nq})} \Pi(B(v, \sigma)).$$

Here,  $\sigma > 0$  refers to the constant which we specified at the end of Section 3. Next, observe that in this Cantor set construction we have good control over the distortion, when going from one generation in the construction to the next. That is, by using [RRP] (iv), [TRP] and the condition in (3), we have the following crucial

estimate, for each  $n \in \mathbb{N}$ ,

$$\begin{aligned}
\sum_{v \in T_{np}(z_{nq})} (\text{diam}(\Pi(B(v, \sigma))))^s &\geq k_\gamma^q \sum_{v \in R_{np}(z_{(n-1)q})} (\text{diam}(\Pi(B(v, \sigma))))^s \\
&\geq K_s^p k_\gamma^q \sum_{v \in T_{(n-1)p}(z_{(n-1)q})} (\text{diam}(\Pi(B(v, \sigma))))^s \\
&> \sum_{v \in T_{(n-1)p}(z_{(n-1)q})} (\text{diam}(\Pi(B(v, \sigma))))^s.
\end{aligned}$$

Using a straightforward generalisation of the folklore arguments from fractal geometry of [7, Lemma 2.5] and [7, Corollary 2.6], the latter estimate immediately gives that

$$\dim_H(\mathcal{C}_s) \geq s.$$

It remains to show that the set  $\mathcal{C}_s$  is contained in  $L_t(N)$ . For this, note that, by viewing the construction of  $\mathcal{C}_s$  from within  $\mathbb{D}^{m+1}$ , the set  $\mathcal{C}_s$  gives rise to a geodesic  $G$ -tree which is rooted at the origin and whose vertex set is equal to  $\bigcup_{n \in \mathbb{N}} (T_{(n-1)p}(z_{(n-1)q}) \cup R_{np}(z_{(n-1)q}))$ . By construction, this tree has the property that the lengths of the constituting geodesic edges and the angles formed by adjacent edges are uniformly bounded away from zero. Therefore, each path in this tree starting at the origin is a quasi-geodesic heading towards a uniquely determined point at infinity. Clearly, the projection of each of these quasi-geodesics onto the manifold  $\mathcal{M}_N$  gives some piecewise geodesic movement in  $\mathcal{M}_N$  which has the following properties. If an edge in the tree starts at a vertex in  $T_{(n-1)p}(z_{(n-1)q})$  and ends at a point in  $R_{np}(z_{(n-1)q})$ , then in  $\mathcal{M}_N$  this edge is represented by a geodesic loop of hyperbolic length at most  $\ell_s$ . Obviously, this loop must then be contained in a bounded region of  $\mathcal{M}_N$  of diameter at most  $\ell_s$ . Whereas, if an edge in the tree starts at a vertex in  $R_{np}(z_{(n-1)q})$  and ends at a point in  $T_{np}(z_{nq})$ , using [TRP], we then have that in  $\mathcal{M}_N$  this edge represents a geodesic segment in  $\mathcal{M}_N$  which starts in the previous bounded region and then heads straight towards the end of  $\mathcal{M}_N$  associated with the attractive fixed point  $\eta_+$  of  $\gamma$ . Moreover, the condition in (2) guarantees that the hyperbolic length of that segment is at least equal to  $4\ell_s$ , and this shows that it's end point is separated by at least  $3\ell_s$  from the previous bounded region.  $\square$

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