Nielsen equivalence of generating sets for closed surface groups

Larsen Louder

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Abstract

Generating sets for closed surfaces are either reducible or Nielsen equivalent to a standard generating set.

1 Introduction

Let G be a finitely generated group and let $S = \{s_1, \ldots, s_n\} \subset G \sqcup G \sqcup \cdots$ be a set of generators for G. Let $F_S = \langle x_i, \ldots, x_n \rangle$ be a free group with one generator for each element of S and denote the natural map $F_S \to G$ by φ_S. Two generating sets S and S' of the same cardinality are *Nielsen equivalent* if there is an isomorphism $\varepsilon \colon F_S \to F_{S'}$ such that $\varphi_{S'} \circ \varepsilon = \varphi_S$. If S contains the identity element then it is *reducible*. If, for all S' equivalent to S, S' does not contain the identity element then S is *irreducible*.

Theorem 1 ([Zie70])**.** *Sei $\mathfrak{F}$ die Fundamentalgruppe einer orientierbaren geschlossenen Fläche vom Geschlecht $g \neq 3$[1], und es seien $\{t_1, u_1, \ldots, t_g, u_g\}$ kanonische Erzeugende mit der definierenden Relation $\Pi_{i=1}^{g}[t_i, u_i] = 1$. Ist $\{x_1, \ldots, x_{2g}\}$ ein anderes Erzeugendensystem für $\mathfrak{F}$, so gibt es einen freien Übergang von $\{t_1, \ldots, u_g\}$ zu $\{x_1, \ldots, x_{2g}\}$.*

Translation: any minimal generating set for a closed, orientable, surface of genus not 3 is equivalent to a standard generating set. According to Zieschang, his method does not apply for genus 3 surfaces. We generalize Zieschang's theorem to all closed surfaces and generating sets:

[1] Die Einschränkung $g \neq 3$ ist nur durch unseren Beweis bedingt. Die Aussage ist wohl auch für $g = 3$ richtig.

Theorem 2 (Generating sets are either standard or reducible)**.** *Let S be a generating set for the fundamental group of a closed surface Σ. Then S is either reducible or equivalent to a standard generating set. Equivalently,* $\mathrm{Aut}(\mathbb{F}_n)$ *acts transitively on* $\mathrm{Epi}(\mathbb{F}_n, \pi_1(\Sigma))$.

The theorem is obvious for $\mathbb{S}^2$ and $\mathbb{RP}^2$, and so we ignore these cases.

Rather than work with generating sets for $\pi_1(Y)$ algebraically, we consider pairs (X, φ), where X is a compact, aspherical, two-dimensional CW–complex, $\pi_1(X)$ is free, and $\varphi\colon X \to Y$ is a continuous, π_1–onto map. A morphism $\varepsilon\colon (X, \varphi) \to (X', \varphi')$ is a continuous map ε such that ε_* is surjective and $\varphi' \circ \varepsilon$ is homotopic to φ. The *rank* of X is the rank of $\pi_1(X)$.

A morphism is an *equivalence* if it is a homotopy equivalence, and is a *reduction* if it is surjective but not injective on fundamental groups. If (X, φ) factors through a lower rank (X', φ') then (X, φ) is *reducible*: Let $\{x_i\}$ be a minimal generating set for a reducible (X, φ), and let $S = \{\varphi_*(x_i)\}$. Then S is Nielsen equivalent to some S' containing the identity element.

Let Σ be a closed surface. The standard generating set for a surface may be represented by $\Sigma \setminus D^\circ \hookrightarrow \Sigma$, where D is an embedded closed disk: If Σ is orientable, there is the standard presentation

$$\pi_1(\Sigma) = \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle$$

and if Σ is nonorientable, there is the standard presentation

$$\pi_1(\Sigma) = \langle x_1, \ldots, x_n \mid x_1^2 \cdots x_n^2 \rangle$$

Consider the presentation 2–complex homeomorphic to Σ associated to the above presentation. There is an embedded subgraph $\Gamma \hookrightarrow \Sigma$ representing the standard generating set $\{x_i, y_i\}$ (or $\{x_i\}$) with the property that if $D \subset \Sigma \setminus \Gamma$ is a closed disk then $\Sigma \setminus D^\circ$ deformation retracts to Γ. We may therefore consider the "standard" generating systems for surfaces to be represented by the maps $\Sigma \setminus D^\circ \to \Sigma$. All standard generating systems for a fixed surface are clearly equivalent by a homeomorphism, with the homotopy provided by the point, or rather disk, pushing map.

Our approach to Theorem 2 is a variation on the Dunwoody-Stallings folding sequences for morphisms of graphs of groups. For the most part we ignore the group theoretic interpretations, except in Lemma 6 and Theorem 9, and construct a class of spaces with maps to a closed surface and a family of "moves" on these spaces. The moves are either homotopy equivalences or reductions. A theorem of Stallings/Dunwoody and an elementary proposition about surfaces with boundary (concealed in Lemma 14) will guarantee that the moves can be applied. First, we review one dimensional spaces.

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2 Graphs

An oriented graph in a category $\mathcal{C}$ is a category G with two sets of objects $\mathcal{E}$ and $\mathcal{V}$ from $\mathcal{C}$, the edges and vertices of G, respectively, for each $e \in \mathcal{E}$ morphisms $\tau_e \colon e \to \tau(e) \in \mathcal{V}$ and $\iota_e \colon e \to \iota(e) \in \mathcal{V}$, and a fixed point free involution $\bar{\ } \colon \mathcal{E} \to \mathcal{E}$ such that $\bar{\ } \colon e \to \overline{e}$ is an isomorphism. We usually drop, without introducing any ambiguity, the subscripts from τ and ι. If e is an edge then $\tau = \iota \circ \bar{\ }$ and $\iota = \tau \circ \bar{\ }$. The orbits of $\bar{\ }$ (on elements of $\mathcal{E}$) are the *unoriented* edges of G. If e is an edge of V then $\tau(e)$ and $\iota(e)$ are the *terminal* and *initial* endpoints of e, respectively.

If the elements of $\mathcal{V}$ and $\mathcal{E}$ are points, then G is an ordinary graph. If, for instance, the elements are topological spaces, then G is a description of a graph of spaces. Graphs of groups are also graphs in this sense: The vertex and edge spaces are groups, and the morphisms are injective homomorphisms. To avoid ambiguity we always include the modifier when the graph under consideration is not an ordinary graph. A graph X in some category always has an *underlying graph*, Γ_X, the graph obtained by regarding all edge and vertex spaces as points.

A *link* in a category $\mathcal{C}$ is a category $(V, \mathcal{E}_V, \tau_V)$ where

- V and the members of $\mathcal{E}$ are objects in $\mathcal{C}$. We call V the *core*.
- for each $E \in \mathcal{E}_V$ there is $\tau_E \in \tau_V$ such that $\tau_E \colon E \to V$ is a morphism in $\mathcal{C}$.

The spaces $E \in \mathcal{E}_V$ are *incident edges* or *incident edge spaces*. Let $\mathcal{V} = (V, \mathcal{E}_V, \tau_V)$ and $\mathcal{W} = (W, \mathcal{E}_W, \tau_W)$ be links. A morphism $\varphi \colon \mathcal{V} \to \mathcal{W}$ consists of a morphism $\varphi \colon V \to W$, and maps of incident edge spaces respecting the peripheral structure: for each E, there is some $\varphi(E) \in \mathcal{E}_W$ and a map $\varphi_E \colon E \to \varphi(E)$ such that

$$\varphi \circ \tau_E = \tau_{\varphi(E)} \circ \varphi_E$$

The *link* of a vertex V in a graph is the tuple $\mathrm{lk}(V) := (V, \mathcal{E}_V, \tau_V)$ where $\mathcal{E}_V$ is the set of edges E such that $\tau(E) = V$, and τ_V is the associated collection of maps.

A morphism of graphs in $\mathcal{C}$ is a functor $V \to V'$ carrying vertices to vertices and edges to edges, such that maps of objects are morphisms in $\mathcal{C}$. A map $V \to V'$ induces both a map of underlying graphs $\Gamma_V \to \Gamma_{V'}$ and maps $\varphi_v \colon \mathrm{lk}(v) \to \mathrm{lk}(\varphi(v))$ for each vertex $v \in \mathcal{V}$.

3 Graphs of graphs

A *graph of graphs* is a finite graph in the category of finite graphs.

A link $\mathcal{V}$ in the category of finite graphs is *decomposable* if $\pi_1(V)$ admits a free product decomposition

$$\pi_1(V) = (*_{E \in \mathcal{E}} G_E) * H$$

with G_E a conjugate of $\mathrm{Im}(\tau_E)$ and H a complementary free factor. Given $\varphi \colon \mathcal{V} \to \mathcal{W}$, the *type* of an incident edge E is the image $\varphi(E) \in \mathcal{E}_W$, the *type* of a vertex v or edge e of E is the image $\varphi_E(v)$. Two incident edges E and E' *intersect* if they have the same type and there are vertices $v \in E$ and $v \in E'$ of the same type and image in V. An edge E *self-intersects* if two vertices in E of the same type have the same image in V. Likewise, two edges E and E' *intersect in an edge* if they have the same type and there are edges $f \subset E$ and $f' \subset E'$ with the same type and image in V. Similarly for self intersections.

An edge f of V is *traversed* by $E \in \mathcal{E}$ if there is an edge f' of E such that $\tau_E(f') = e$.

If two incident edges in a link intersect then we may fold them together. Suppose E and E' intersect. Let v and v' be vertices such that $\tau_E(v) = \tau_{E'}(v')$; φ factors through the link

$$\mathcal{V}' = (V, \mathcal{E} \setminus \{E, E'\} \cup E \vee_{v=v'} E', \tau \setminus \{\tau_E, \tau_{E'}\} \cup \tau_E \vee_{v=v'} \tau_{E'})$$

obtained by *folding E and E' along v and v'*.

Let $\mathcal{V}$ be a link. Suppose e is an edge of $\mathcal{V}$ traversed *exactly* once by $\mathcal{E}$. Suppose that E traverses e, and that $\tau_E(e') = e$. Denote the link obtained by removing e, $\overline{e}$, e', and $\overline{e}'$ by $\mathcal{V}_e$.

A morphism of graphs of graphs induces, at each vertex, a morphism of links. Let $\varphi\colon X \to Y$, and let $\varphi_v\colon \mathrm{lk}(X_v) \to \mathrm{lk}(Y_{\varphi(v)})$ be a morphism of vertex spaces: if $\tau\colon X_e \to X_v$ we have, by definition, $\varphi_v \circ \tau = \tau \circ \varphi_e$.

We think of X as a description of a square complex, also called X, which is the realization of X as a topological space:

$$X = ((\sqcup_v X_v) \sqcup (\sqcup_e X_e \times I))/\{(x,1) \sim \tau(x), (x,0) \sim (\overline{x},0)\}$$

Similarly, let $\mathcal{V}$ be a link. Define the mapping cylinder

$$M_{\mathcal{V}} := V \sqcup_{F\in\mathcal{F}} (F \times I)/(x,1) \sim \tau_F(x)$$

and its boundary

$$\partial M_{\mathcal{V}} := \sqcup_{F\in\mathcal{F}} F \times \{0\} \subset M_{\mathcal{V}}$$

The realization of a graph of graphs is obtained by gluing the realizations of links along their boundaries via $\bar{\ }$.

If $\mathcal{V} \to \mathcal{W}$ is a morphism of links then there is a natural map $M_{\mathcal{V}} \to M_{\mathcal{W}}$, and if $X \to Y$ is a morphism of graphs of graphs there is a natural continuous map of realizations $X \to Y$ obtained by gluing maps on links.

For ordinary graphs, a *fold* is a surjective morphism $\varphi\colon V \to V'$ such that φ *only* identifies two edges which share a common endpoint. A morphism injective on all links is an *immersion*. Immersions of graphs are π_1–injective.

Theorem 3 ([Sta83])**.** *A morphism of finite graphs $\varphi\colon V \to V'$ factors as*

$$V = V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \to \cdots \to V_k \looparrowright V'$$

where ϕ_i is a fold and $V_k \looparrowright V'$ is an immersion.

Let Y be a graph of graphs. Of special interest in this paper is the collection of graphs of graph over Y, that is, the collection $\{(X,\varphi) \mid \varphi\colon X \to Y\}$, such that if $\varphi\colon X \to Y$ and $\psi\colon (X,\varphi) \to (X',\varphi')$, then $\varphi' \circ \psi = \varphi$.

3.1 Moves on graphs of graphs

We define some elementary transformations of graphs of graphs. These are all standard (un)foldings adapted to the setting of graphs of graphs.

Let (X,φ) be a graph of graphs over Y. The vertex map $\varphi_v\colon X_v \to Y_{\varphi(v)}$, by Theorem 3, factors through a folding sequence

$$X_v = W_0 \xrightarrow{\phi_0} W_1 \xrightarrow{\phi_1} W_2 \to \cdots \to W_k \looparrowright Y_{\varphi(v)}$$

By replacing X_v by W_i and composing maps, we obtain a sequence of graphs of graphs (X^i, φ_i) over Y. At each stage, there is a morphism $\varepsilon_i \colon X^i \to X^{i+1}$ induced by ϕ_i. If ε_i is a homotopy equivalence then ϕ_i is, and vice-versa. The last map $\varphi_k \colon X^k \to Y$ is *an immersion at* v.

Let (X, φ) be a graph of graphs over Y. Let $\mathcal{V} = \mathrm{lk}(X_v)$ and let $\mathcal{W} = \mathrm{lk}(Y_{\varphi(v)})$. Suppose that two edges X_e and X_f incident to X_v intersect at a and b. Fold X_e and X_f together along a and b, identify $\iota(a)$ and $\iota(b)$ in $X_{\iota(e)}$ and $X_{\iota(f)}$, and identify $\overline{a}$ and $\overline{b}$ in $X_{\overline{e}}$ and $X_{\overline{f}}$. If $\iota(a) = \iota(b)$ then the induced map $X \to X'$ is not a homotopy equivalence, though if either X_e or X_f is contractible the map is simple on the level of fundamental group: $\pi_1(X)$ has a free splitting as $G * \langle x \rangle$ such that the homomorphism $\pi_1(X) \to \pi_1(X')$ simply kills the generator x. In this case the map $X \to X'$ is a reduction.

The following three moves are collapsing free faces/removing valence one vertices (or the reverse), hence are homotopy equivalences. The letter X always represents a generic graph of graphs.

Suppose that f is an edge of X_v traversed exactly once by incident edges, without loss $\tau(f') = f$, f' an edge of X_e. Let X' be the graph of graphs obtained by removing f, f', $\overline{f}$, and $\overline{f}'$. There is a natural homotopy equivalence $X \hookrightarrow X'$. The graph of spaces X is said to be obtained from X' by *unpulling* X_e. There may some ambiguity in that X_e may be unpulled in multiple ways, but it should be clear from the context which one we mean.

Suppose that X_e is a point. Let $\rho \colon E \to X_{\tau(e)}$ be a morphism of graphs, and suppose that $\rho(v) = \tau(X_e)$. Let X' be the graph of spaces obtained by replacing X_e and $X_{\overline{e}}$ by E and $X_{\iota(e)}$ by $X_{\iota(e)} \vee_{\iota(X_e)=v} E$. If $\varphi \colon X \to Y$ and there is a map $\psi \colon E \to Y_{\varphi(e)}$ such that $\psi(v) = \varphi(X_e)$ then there is an extension of φ to $\varphi' \colon X' \to Y$. The space X' is obtained by *pulling E across e*.

For each edge space X_e, if X_e is not contractible, let X'_e be the graph obtained by trimming all trees, and if X_e is contractibile, let X'_e be a vertex of X_e. There is an inclusion map from the graph of spaces X' obtained by replacing X_e by X'_e into X. Let X'_v be a vertex space of X', and let X''_v be the union of all non-backtracking paths with endpoints in the images of incident edge spaces. Let X'' be the space obtained by replacing each vertex space X'_v by X''_v. Then X deformation retracts to the image of X''. We say that X'' is obtained from X by *trimming trees*. Note that X'' has the same number of edge spaces as X, and that if X is a graph of graphs over Y, then the inclusion $X'' \hookrightarrow X$ is a homotopy equivalence of graphs of graphs over Y.

4 Links over surfaces with boundary

A surface with boundary Σ will be thought of as a link in the following way: let $\Gamma \subset \Sigma$ be an embedded subgraph in Σ° such that complementary components of Γ in Σ are half-open annuli. There is a deformation retraction $\Sigma \to \Gamma$ immersing the boundary components of Σ as closed paths in Γ. Each edge of Γ is traversed exactly twice. Subdivide.

Definition 4. Let Σ be a surface with boundary and let $(\mathcal{V}, \varphi)$ be a link over Σ. If φ_E is an immersion for all E, no two circular incident edges intersect in an edge, and no incident edge self-intersects in an edge, then $\mathcal{V}$ is *surfacelike*.

Definition 5. A graph with peripheral structure $(\mathcal{V}, \varphi)$ over Σ is a *pseudosurface with boundary* if it is surfacelike and every edge traversed by a circular incident edge is traversed by another circular incident edge.

A pseudosurface with boundary $\mathcal{V}$ over a surface with boundary Σ is never decomposable. If the map of core graphs is an immersion then, after forgetting incident edge spaces that are not circles, $\mathcal{V}$ is a finite sheeted cover of Σ.

Let $\varphi_i \colon V_i \to W$, $i = 1, 2$, be immersions. The pullback of V_1 and V_2 is the graph $\{(x, y) \in V_1 \times V_2 \mid \varphi_1(x) = \varphi_2(y)\}$. Suppose that $\mathcal{V} \to \Sigma$ is surfacelike. Let $p \in E$ be a vertex, and let $q = \varphi_E(p)$ be the type of E. Consider the maps $\varphi \colon V \looparrowright \Gamma$ and $\tau_{\varphi(E)} \colon \varphi(E) \looparrowright \Gamma$. Let ∂E be the connected component of the pullback containing (p, q). Since $\mathcal{V}$ is surfacelike, if E is a circle then $\partial E = E$, and otherwise ∂E is a point, an interval, or a circle. If ∂E and $\partial E'$ intersect, then $\partial E = \partial E'$, furthermore, if E is a circle and $\partial E' = \partial E$, then surfacelikeness of $\mathcal{V}$ guarantees that E' is *not* a circle. The *degree* of E is the degree of the map $\partial E \looparrowright \varphi(E)$.

Lemma 6. *$\varphi \colon \mathcal{V} \to \Sigma$ be a surfacelike immersion. Suppose that if ∂E is a circle then $E = \partial E$. If*

$$\varphi_* \colon \pi_1(M_\tau, \partial M_\tau) \to \pi_1(\Sigma, \partial\Sigma)$$

is not injective then there are two incident edges E and E' of the same type, $\partial E = \partial E'$, and at least one of E or E' is not circular.

Proof. By pulling we may assume that for all E, $E = \partial E$. Suppose that φ_* is not injective. Let α be a shortest edge path in the one skeleton of $M_{\mathcal{V}}$ such that $\varphi \circ \alpha$ is homotopic rel endpoints into $\partial\Sigma$. Since α is shortest, it consists of a path from E to V, a path β in V, which we may assume is an immersion, and a path from V to E'. The path β intersects E and E', and, since it is an immersion, factors through the pullback of $M_{\mathcal{V}}$ and $\varphi(E)$ (or, equivalently, $\varphi(E')$). Since β and E intersect, they correspond to the same component of the pullback, likewise for β and E'. Since E and E' factor through the same component of the pullback, they intersect, hence $\partial E = \partial E'$. Since $\mathcal{V}$ is surfacelike, one of E or E' is not circular. ☐

Lemma 7. *Let Σ be a surface with boundary and let $\varphi\colon \mathcal{V} \to \Sigma$ be decomposable and surfacelike. If there is a circular incident edge then there is an edge of V traversed only once by circular incident edges. If e is the edge traversed only once then $\mathcal{V}_e$ is decomposable.*

Proof. If every edge traversed once by a circular incident edge is traversed twice by circular incident edges then the union of edges traversed by circular incident edge spaces along with the traversing edge spaces is a pseudosurface with boundary and there is a relation in homology, contradicting decomposability of $\mathcal{V}$. ☐

5 Surfaces

We need to show that closed surfaces are graphs of graphs.

Theorem 8. *Let S be a closed surface with $\chi(S) \leq 0$. Then S is a graph of graphs. The links of vertices are surfaces with boundary. We may assume that no link is a Möbius strip.*

Proof. See Figure 1. ☐

For the remainder of the paper, "surface" will mean a graph of graphs such that every link has at least two boundary components, and whose topological realization is homeomorphic to a surface.

Let $\varphi\colon \mathbb{F} \to \pi_1(S)$ be a homomorphism. We represent φ as a map of graphs of graphs as follows. Let R be a rose with fundamental group $\mathbb{F}$, and identify the petals of R with the generators of $\mathbb{F}$. After identifying fundamental groups appropriately, there is a map from R to the one-skeleton of S inducing φ. We may assume, by subdividing, that this map is a morphism

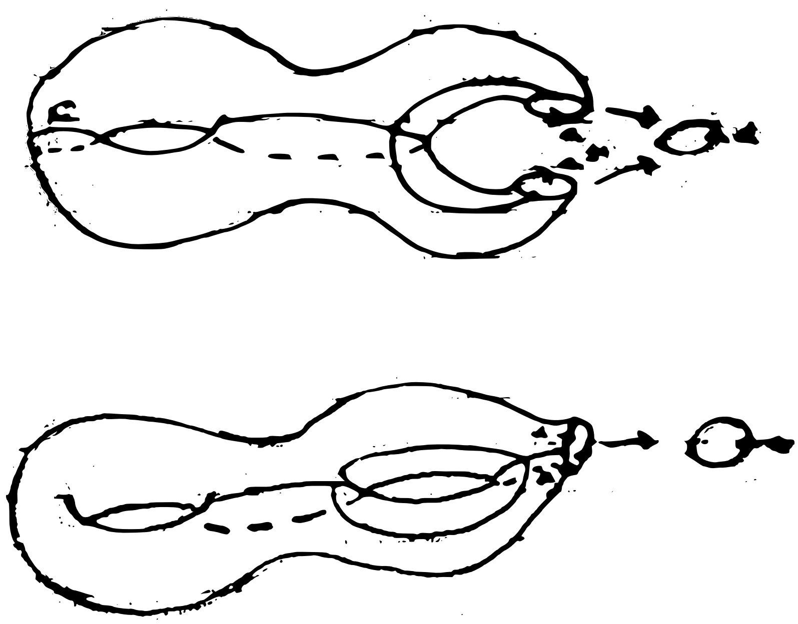

Figure 1: Writing a surface, $\chi \leq 0$, as a graph of graphs. The top figure is for even Euler characteristic, and the bottom is for odd Euler characteristic. The odd Euler characteristic surfaces are all nonorientable.

of graphs. We regard R as a graph of graphs over S by declaring the connected components of preimages of vertex spaces of S to be vertex spaces. Midpoints of edges mapping to edge spaces of S are the edge spaces.

Let Σ be a closed surface written as a graph of graphs, and consider an edge space Σ_e. Let v be a vertex of Σ_e, and let Σ' be the graph of graphs obtained by restricting Σ_e and $\Sigma_{\overline{e}}$ to v and $\overline{v}$, respectively. Then $\Sigma' \subset \Sigma$ is a deformation retract of $\Sigma \setminus D^\circ$, for some embedded closed disk $D \subset \Sigma$; Σ' is a minimal generating set.

5.1 Surfacelike graphs of graphs

A graph of graphs X over Σ is *surfacelike* if all links of vertex spaces of X are surfacelike and $\pi_1(X)$ is free. A morphism $\varphi\colon X \to \Sigma$ of a surfacelike X is *locally injective* if, for all v, φ_{v*} is injective. If, for all v, $\varphi_v\colon X_v \to \Sigma_{\varphi(v)}$

is an immersion, then φ is a *local immersion*. Suppose $\varphi\colon X \to \Sigma$ is locally injective. Then folding in vertices yields a homotopy equivalent surfacelike local immersion of graphs of graphs over Σ.

Theorem 9 (*cf.* [Dun98])**.** *Let $\varphi\colon X \to \Sigma$ be a local immersion of a surfacelike X. If φ_* is not injective then either*

- *there is a vertex space X_v of X and two incident edge spaces, one of which is* not *a circle, $X_e, X_{e'} \to X_v$ such that $\partial X_e = \partial X_{e'}$, or*
- *there is X_e such that X_e is not a circle but ∂X_e is a circle.*

Proof. Consider the map $\varphi\colon X \to \Sigma$ of topological realizations. Suppose that the second bullet doesn't hold. Let α be a reduced edge path in the one skeleton of X, with nullhomotopic image in Σ, and least intersection with edge spaces of X. Let $\eta\colon D \to \Sigma$ be a continuous map of a disk such that $\eta|_{\partial D}$ lifts to α. Homotope η on the interior of D so that η is transverse to the collection edge curves $\mathcal{C}$ of Σ. Since Σ is aspherical, we may assume, after homotopy, that $\Lambda = \eta^{-1}(\mathcal{C})$ consists of arcs connecting ∂D to itself. Let D' be an innermost disk bounded by an arc β of ∂D and an arc of Λ. Then D' represents a nullhomotopy of $\varphi \circ \beta$ in $(M_{X_v}, \partial M_{X_v}) \to (M_{\Sigma_{\varphi(v)}}, \partial M_{X_v}) = (\Sigma_{\varphi(v)}, \partial\Sigma_{\varphi(v)})$. Suppose β connects the incident edge spaces X_e and $X_{e'}$. If $e = e'$ then either α may be homotoped to have smaller intersection with X_e or X_e is not a circle and ∂X_e is a circle. If $e \neq e'$ then, by Lemma 6, $\partial X_e = \partial X_e'$. ∎

Rather than work with arbitrary graphs of graphs over a surface, we restrict our attention to the following special classes. Let Σ be a closed surface, and define

- $\mathcal{S}_\Sigma$, the collection of surfacelike graphs of graphs over Σ,
- $\mathcal{LI}_\Sigma \subset \mathcal{S}_\Sigma$, the subcollection of local immersions,
- $\mathcal{FD}_\Sigma \subset \mathcal{S}_\Sigma$, the subcollection such that each vertex map is either a finite sheeted cover or is decomposable, and
- $\mathcal{PP}_\Sigma \subset \mathcal{LI}_\Sigma \cap \mathcal{FD}_\Sigma$, the subcollection such that there is an edge space X_e such that X_e is a point and ∂X_e and $\partial X_{\overline{e}}$ are both circles.

Let $|X|$ be the number of edges in Γ_X and $|X|_c$ be the number of circular edge spaces of X. The *complexity* of a surfacelike graph of graphs (X, φ) over Σ is the pair $c(x) := (|X|, |X|_c)$, ordered lexicographically.

Lemma 10 (*cf.* [She55])**.** *Let $(X, \varphi) \in \mathcal{S}_\Sigma$. Then $(X, \varphi) \cong (X', \varphi') \in \mathcal{LI}_\Sigma$, such that $|X'| \leq |X|$ and $|X'|_c = 0$.*

Proof. By trimming trees we may assume that all edge spaces of X are points or circular. If X has a circular edge space then, since $\pi_1(X)$ is free, there must be a vertex space X_v of X with an edge f traversed by exactly one incident edge, otherwise $\pi_1(X)$ contains a surface subgroup. Suppose X_e traverses f. Unpull X_e and trim trees to produce $X' \hookrightarrow X$. Since X_e is a circle, X'_e is connected, and X' has the same number of edges spaces as X. Since unpulling and trimming trees are homotopy equivalences, the fundamental group of X' is free. The vertex spaces of X' are still surfacelike. Repeat until there are no circular edge spaces remaining. Now fold in vertex spaces. □

Lemma 11. *Let $X \in \mathcal{S}_\Sigma$. Suppose X is irreducible and has a vertex where two incident edges, one of which is not circular, intersect. Then X is equivalent to $Y \in \mathcal{LI}_\Sigma$ such that $|Y| < |X|$.*

Proof. Suppose that $X_e, X_{e'} \to X_v$ intersect. Let X' be the result of folding X_e and $X_{e'}$ and trimming trees. Since one of X_e or $X_{e'}$ is not circular, all edge spaces of X' are either circular or points. There is a morphism $X \to X'$ over Σ, $|X'| < |X|$, and since X is irreducible, $X \to X'$ is an equivalence.

Let Y be the result of applying Lemma 10 to X'. Since Y has no circular edges and X is irreducible, the map $X \to Y$ is an equivalence. Clearly $|Y| < |X|$. □

Lemma 12. *Let $(X, \varphi) \in \mathcal{LI}_\Sigma$. Suppose that X is irreducible, that φ_* is not injective, and that for all edges e, if X_e is not a circle then ∂X_e is not a circle. Then X is equivalent to Y such that $|Y| < |X|$.*

Proof. By Theorem 9 there is an edge e such that X_e is not a circle and ∂X_e intersects another incident edge space $X_{e'}$. Suppose that $X_{e'}$ is a circle. Since φ_v is an immersion, $X_{e'}$ and X_e intersect. If neither X_e nor $X_{e'}$ is a circle, adjust X by pulling ∂X_e across e. In either case X_e and $X_{e'}$ intersect and one is not circular. Apply Lemma 11. □

Local immersions over surfaces can be transmuted to maps which are not injective at some vertex. Once this is done, Lemma 14 guarantees that every generating set is equivalent to one with some obvious relations.

Lemma 13. *Let $(X, \varphi) \in \mathcal{LI}_\Sigma$. Suppose that X is irreducible. Then either*

- *X is equivalent to $X' \in \mathcal{S}_\Sigma$ such that X' has strictly fewer edges than X, or*
- *X is equivalent to $X' \in \mathcal{FD}_\Sigma$ such that X' has the same number of edges as X and there is a decomposable vertex X'_v such that the map $\varphi_v \colon X'_v \to \Sigma_{\varphi'(v)}$ is not π_1–injective.*

Proof. Suppose that case one doesn't hold. By Lemma 12 there is an edge e such that X_e is a point or an interval, and ∂X_e is a circle. By trimming trees, we may assume that X_e is a point.

Suppose $X_{\iota(e)}$ is not decomposable. Then $X_{\overline{e}}$ and X_f, for some $f \neq \overline{e}$, $\tau(f) = \iota(e)$, intersect. By Lemma 11 the first bullet holds. We may therefore assume that $X_{\iota(e)}$ is decomposable.

Let X' be the space obtained by pulling ∂X_e across e. Then X' is free, surfacelike, and $X'_{\iota(e)}$ is also decomposable. If φ' is injective at $\iota(e)$ fold $X'_{\iota(e)} \to \Sigma_{\varphi(\iota(e))}$ to an immersion. Then X' is irreducible, $X' \in \mathcal{LI}_\Sigma$, and $|X|_c < |X'|_c \leq |X|$. Repeat using X' as the initial data. $\square$

Lemma 14. *Suppose $(X, \varphi) \in \mathcal{FD}_\Sigma$ is irreducible and that φ is not locally injective. Then either*

- *X is equivalent to $X' \in \mathcal{S}_\Sigma$ such that X' has strictly fewer edges than X, or*
- *X is equivalent to $X' \in \mathcal{PP}_\Sigma$, $|X'| \leq |X|$, and $|X'|_c \leq |X|_c$.*

Associated to an element of $\mathcal{PP}_\Sigma$ is the obvious relation that $\partial X_e \vee_{X_e} \partial X_{\overline{e}}$ has abelian image in Σ. Failure of this lemma for arbitrary groups that split over $\mathbb{Z}$ is partially responsible for the existence of nonequivalent generating sets.

Proof of Lemma 14. Since φ is not locally injective, there is a decomposable vertex space X_v such that $\varphi \colon X_v \to \Sigma_{\varphi(v)}$ is not π_1–injective. Let

$$X_v = V_0 \to V_1 \to \cdots \to V_k$$

be a folding sequence for $\varphi_v \colon X_v \to \Sigma_{\varphi(v)}$ maximal with respect to the property that V_i is surfacelike for all i, that for $i < k$, $V_{i-1} \to V_i$ is a homotopy equivalence. The last map $V_{k-1} \to V_k$ may or may not be a homotopy equivalence. Let X^i be the space obtained by replacing X_v by V_i.

Suppose that for all i, and at all vertices, there are no pairs of incident edges that intersect or incident edges that self intersect. Let e and f be the two edges identified under $V_{k-1} \to V_k$. Suppose that one of e or f is not traversed by a circular incident edge. Then $X^{k-1} \to X^k$ simply kills a primitive element and the map $X \to X^k$ is a reduction. Suppose that both e and f are crossed by a circular incident edge. Then neither is crossed by two, otherwise there is an (self) intersection of incident edges. Suppose that X_g^{k-1} traverses e and that X_h^{k-1} traverses f. Since e is traversed exactly once, we may write $X_g^{k-1} \to X_v^{k-1}$ as composition of paths $S \cdot e$. The path $S \cdot f$ is indivisible, intersects X_g^{k-1}, and has the same image as X_g^{k-1} in Σ. Unpull X_g^{k-1} at e (collapse a free face) to obtain X'. If φ'_v is injective, fold to an immersion. Then both $\partial X'_g$ and $\partial X'_{\overline{g}}$ have positive degree and X' has lower complexity than X. Start over if φ'_{v*} is not injective.

Suppose, for some $i < k$, a circular pair of incident edges X_e^i and $X_{e'}^i$ intersect, say at $a \in X_e^i$ and $b \in X_{e'}^i$. Let f be the edge in V_i traversed by exactly one circular incident edge, and let X_g^i be the incident edge space traversing f. Unpull X_g^k and call the resulting space X'. If $g \neq e, e'$, then X' is surfacelike, free, and not injective at v: $X'_e \vee_{a=b} X'_{e'}$ has nonabelian image in X'_v, but abelian image in Σ. Furthermore, $\mathrm{lk}(X'_v)$ is decomposable and surfacelike. Thus X' satisfies the hypothesis of the lemma, but $|X'|_c < |X|_c$. Repeat. If $g = e$ or e', suppose, by relabeling, that $g = e$. Then X'_e and $X'_{e'}$ intersect and X'_e is not a circle. By Lemma 11 X' is equivalent to $X'' \in \mathcal{LI}_\Sigma$ with strictly fewer edge spaces than X.

Suppose, for some $i < k$, there is an incident edge X_e^i that self intersects. If X_e^i traverses f, then unpull X_e^i to obtain X'. If $X'_v \to \Sigma_{\varphi(v)}$ is π_1–injective then fold to an immersion.

Let S be the segment of X_e^i from a to b that *doesn't* traverse f. The closed curve obtained by identifying the endpoints of S intersects X_e^i at b, immerses in $\Sigma_{\varphi(e)}$, is sent to the same boundary component of $\Sigma_{\varphi(v)}$ that X_e^i is, and covers $\partial X'_e$, which is therefore a circle. In X', both X'_e and $X'_{\overline{e}}$ are circles. Start over if φ'_v is not injective.

If X_e^i doesn't traverse f, unpull X_g^i. Suppose that X_e^i self intersects at a and b. The edge map $X_e^i \to X_v^i$ factors through $X_e^i/a \sim b$. The image of $X_e^i/a \sim b$ in $X_v^i (:= V_i)$ is nonabelian but its image in Σ is, hence φ'_v is not injective. Repeat, using X' as the initial data. Since $|X'|_c < |X|_c$ the process stops in finitely many steps.

If φ_* is not injective at some other vertex repeat the entire process. □

6 Proof of main theorem

Let $(Y, \psi) \in \mathcal{S}_\Sigma$, and suppose that Y is irreducible. Choose, out of all (Z, ρ) equivalent to (Y, ψ), $(X, \varphi) \in \mathcal{PP}_\Sigma$, the existence of which is guaranteed by Lemmas 13 and 14, with minimal complexity. We may assume, by trimming trees, that if X_e is not a circle then it is a point.

Since X has as few circular edge spaces as possible, if f is an edge of a vertex space traversed by only one incident edge space, then f is traversed either by ∂X_e or $\partial X_{\overline{e}}$, otherwise the number of circular edge spaces can be reduced by unpulling. Hence the links other than $\mathrm{lk}(X_{\iota(e)})$ and $\mathrm{lk}(X_{\tau(e)})$ are surfaces with boundary and are finite sheeted covers of the associated vertex spaces of Σ. Suppose that ∂X_e traverses an edge f of $X_{\tau(e)}$ exactly once, and that no other incident edge traverses f. Form X' by pulling ∂X_e, and then unpulling $X'_{\tau(e)}$ at f. Then, since both ∂X_e and $\partial X_{\overline{e}}$ were circles, $X'_{\iota(e)}$ is decomposable and $X'_{\iota(e)} \to \Sigma_{\varphi(\iota(e))}$ is not π_1–injective. By Lemma 14, either X' is reducible, or X was not of minimal complexity.

Every edge traversed by ∂X_e is then either traversed twice by circular incident edges, ∂X_e, and $\partial X_{\overline{e}}$. Since φ is a local immersion, $X_{\tau(e)} \to \Sigma_{\varphi(\tau(e))}$ is a finite sheeted cover. Likewise for $\iota(e)$. Furthermore, since φ is a local immersion, there are no foldable pairs of edges at any vertex. In conclusion, every edge of X except for $X_e(= X_{\overline{e}})$ is circular.

There are two cases to consider, depending on the degrees of ∂X_e and $\partial X_{\overline{e}}$. Suppose that ∂X_e and $\partial X_{\overline{e}}$ have different degrees. One of them, say ∂X_e, is smaller. Let $\iota\colon X_e \to X_{\iota(e)}$ and $\tau\colon X_e \to X_{\tau(e)}$ be the two edge maps associated to X_e.

Lemma 15. *$\partial X_{\overline{e}}$ traverses some edge of $X_{\iota(e)}$ that is traversed either by some other circular incident edge or by ∂X_e.*

Proof. Suppose that every edge traversed by $\partial X_{\overline{e}}$ is traversed by $\partial X_{\overline{e}}$ twice. Then $\mathrm{lk}(X_{\iota(e)})$ is a finite sheeted cover of $\mathrm{lk}(\Sigma_{\varphi(\iota(e))})$ having only one boundary component, hence $\mathrm{lk}(\Sigma_{\varphi(\iota(e))})$ has only one boundary component, contrary to hypothesis. □

If $\mathrm{lk}(\Sigma_{\varphi(\iota(e))})$ is orientable then we may, by Riemann-Hurwitz, drop the assumption that it have only one boundary component, since the number of critical points of a nontrivial branched cover of an orientable surface is at least two.

Let f be an edge of $X_{\iota(e)}$ traversed by $\partial X_{\overline{e}}$ and either by some other incident edge space or ∂X_e. Pull ∂X_e across e to form X'. In X', f is traversed by a circular incident edge space X'_g, and it is possible that $g = e$. Unpull X'_g to form Z. By construction $Z \in \mathcal{FD}_\Sigma$, $c(Z) = c(X)$, and $Z_{\iota(e)}$ is decomposable.

Lemma 16. *The vertex map* $\varphi_{\iota(e)} \colon Z_{\iota(e)} \to \Sigma_{\varphi(\iota(e))}$ *is not injective.*

Proof. Folding in the vertex $Z_{\iota(e)}$ yields after some stage a morphism of graphs $Z^k_{\iota(e)} \to \Sigma_{\varphi(\iota(e))}$ which is surjective on all links of vertices, but not injective on at least one link. □

Since $\varphi \colon Z \to \Sigma$ is not injective at $\iota(e)$, and Z_g is not circular, folding in the vertex $\iota(e)$, by Lemma 14, yields an equivalent surfacelike $Y' \to \Sigma$ which either has fewer edges, is a reduction of X, or is in $\mathcal{PP}_\Sigma$. Since X was irreducible and Y'_g is not circular, $|Y'|_c < |X|_c$ and X was not the minimal complexity element of $\mathcal{PP}_\Sigma$ equivalent to (Y, ψ), contrary to hypothesis.

Thus ∂X_e and $\partial X_{\overline{e}}$ have the same degree. Let X' be the space obtained by replacing X_e by the cover of $\Sigma_{\varphi(e)}$ associated to ∂X_e or, equivalently, since the degrees are the same, $\partial X_{\overline{e}}$. In X', every link is a surface with boundary and the map $X' \to \Sigma$ is a finite sheeted cover. Since X generates $\pi_1(\Sigma)$, $X' \to \Sigma$ must be an isomorphism. Since X is obtained by replacing X'_e by a vertex, X represents a minimal generating set.

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Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043
USA
*email:*llouder@umich.edu, lars@d503.net