QUANTUM EQUIVALENT MAGNETIC FIELDS THAT ARE NOT CLASSICALLY EQUIVALENT
CHAMPS MAGNÉTIQUES QUANTIQUEMENT ÉQUIVALENTS MAIS CLASSIQUEMENT NON-ÉQUIVALENTS

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Dedicated to Pierre Bérard and Sylvestre Gallot on the occasion of their sixtieth birthdays.

ABSTRACT. We construct pairs of compact Kähler-Einstein manifolds (M_i, g_i, ω_i) $(i = 1, 2)$ of complex dimension n with the following properties: The canonical line bundle $L_i = \bigwedge^n T^*M_i$ has Chern class $[\omega_i/2\pi]$, and for each integer k the tensor powers $L_1^{\otimes k}$ and $L_2^{\otimes k}$ are isospectral for the bundle Laplacian associated with the canonical connection, while M_1 and M_2 – and hence T^*M_1 and T^*M_2 – are not homeomorphic. In the context of geometric quantization, we interpret these examples as magnetic fields which are quantum equivalent but not classically equivalent. Moreover, we construct many examples of line bundles L, pairs of potentials Q_1, Q_2 on the base manifold, and pairs of connections ∇_1, ∇_2 on L such that for each integer k the associated Schrödinger operators on $L^{\otimes k}$ are isospectral.

Résumé: On construit des couples de variétés de Kähler-Einstein compactes (M_i, g_i, ω_i) $(i = 1, 2)$ de dimension complexe n avec les propriétés suivantes: La première classe de Chern associée au fibré en droites canonique $L_i = \bigwedge^n T^*M_i$ est $\omega_i/2\pi$, et pour tout entier k, les puissances tensorielles $L_1^{\otimes k}$ et $L_2^{\otimes k}$ sont isospectrales pour le Laplacien associé à la connexion canonique, mais M_1 et M_2 – et, en conséquence, T^*M_1 et T^*M_2 – ne sont pas homéomorphes. Dans le contexte de la quantification géométrique, nous interprétons ces examples comme des champs magnétiques qui sont équivalents au sens quantique mais pas au sens classique. En plus, on construit beaucoup d'exemples de fibrés en droites L, de couples de potentiels Q_1, Q_2 sur la variété de base et de couples de connexions ∇_1, ∇_2 telles que pour tout entier k les opérateurs de Schrödinger associés sur $L^{\otimes k}$ soient isospectraux.

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1. INTRODUCTION

Let L be a Hermitian line bundle over a closed Riemannian manifold (M, g). The Riemannian metric g on M and the connection ∇ on L together give rise to a Laplace operator Δ acting on the space $C^\infty(M, L)$ of smooth sections of L by

$$\Delta = -\mathrm{trace}(\nabla^2), \tag{1.1}$$

where

$$C^\infty(M, L) \xrightarrow{\nabla} C^\infty(T^*M \otimes L) \xrightarrow{\nabla} C^\infty(T^*M \otimes T^*M \otimes L)$$

are the connections on L and on $T^*M \otimes L$ (the latter is obtained from the Levi-Civita connection on T^*M and the given connection ∇ on L; we denote it by ∇ as well) and the trace is with respect to the Riemannian metric g. The connection ∇ gives rise to a connection, and thus also a Laplacian, on the kth tensor power $L^{\otimes k}$ of L over M for each integer k. We will denote its spectrum, which is necessarily discrete, by $Spec(L, \nabla, k)$.

How much information is encoded in these spectra? For example, do they determine the connection? The curvature of the connection? The Chern class of the bundle? The geometry of the base manifold? We will primarily focus on a variant of the second question.

A closed 2-form ω on a Riemannian manifold (M, g) is sometimes viewed as a magnetic field. The classical Hamiltonian system for a charged particle moving in the magnetic field is given by (T^*M, Ω, H). Here Ω is the symplectic structure on the phase space T^*M given by $\Omega := \omega_0 + \pi^*\omega$, where ω_0 is the Liouville form and $\pi : T^*M \to M$ is the projection, and the Hamiltonian H is given by $H(q, \xi) = \frac{1}{2} g_q(\xi, \xi)$. If $\frac{1}{2\pi}\omega$ represents an integer cohomology class, then there exists a complex line bundle L with Chern class $[\frac{1}{2\pi}\omega]$. Endow L with a Hermitian structure and a Hermitian connection with curvature $-i\omega$. Through the procedure of geometric quantization, the space of square integrable sections of $L^{\otimes k}$ is viewed as the "quantum Hilbert space," and the quantum Hamiltonian is the operator $\widehat{H}_k = -\frac{\hbar^2}{2}(-\Delta - \frac{1}{6}R)$ with $\hbar = \frac{1}{k}$, where R is the scalar curvature of M. Thus we ask:

- Does the collection of all $Spec(L, \nabla, k)$, $k \in \mathbb{Z}$, determine the symplectic structure Ω on T^*M? That is, does "quantum equivalence" of two magnetic fields imply their "classical equivalence"?

We answer this question negatively by example. We consider the case in which (M, g, ω) is a Kähler manifold; in fact, we focus on Hermitian locally symmetric spaces of noncompact type, normalized such that the Einstein constant is -1. For such spaces, the line bundle with Chern class $[\omega/2\pi]$ is the canonical line bundle of (M, g, ω). We will show that for every normalized, simply-connected irreducible Hermitian symmetric space X of noncompact type of real dimension at least four, there exist arbitrarily large finite families of Hermitian locally symmetric spaces (M_i, g_i, ω_i) covered by X such that *Spec*$(L_i, \nabla_i, k) =$ *Spec*(L_j, ∇_j, k) for all k and all i, j (where ∇_i is the canonical connection on the canonical line bundle) but such that the cotangent bundles of the various M_i are mutually non-homeomorphic. Hence, the phase spaces (T^*M_j, Ω_j) for the magnetic flows of the various (M_j, ω_j) are not symplectomorphic, and yet the measurable quantum energy spectra are the same. Our method is based on Sunada's isospectrality technique along with D. B. McReynolds's recent construction of arbitrarily large finite families of mutually isospectral locally symmetric spaces.

In the example outlined above, the classical phase spaces of the "quantum equivalent" systems fail not only to be symplectomorphic, but even to be homeomorphic. In a companion article, we will construct by a different method an example of quantum equivalent magnetic fields on a fixed manifold M (a torus) for which the associated symplectic structures on T^*M are not symplectomorphic.

Our technique is similar to that of R. Kuwabara [11], who constructed pairs of connections on a fixed line bundle L over, for example, a Riemann surface M such that $Spec(L, \nabla_1, k) = Spec(L, \nabla_2, k)$ for all k. In the final section of this paper, we review and slightly extend his construction.

The paper is organized as follows: In Section 2, we describe some of the relevant framework of geometric quantization, which will allow for a physical interpretation of the isospectrality results. This material is of course well-known to experts in geometric quantization, but we include it here in the hopes that it may be of interest to a wider audience. In Section 3, we describe Sunada's technique in our context and show how it leads to the examples described above of Hermitian locally symmetric spaces (of real dimension four and higher) that are quantum equivalent but not classically equivalent. We also address the case of Riemann surfaces. Finally, in Section 4, we consider isospectral connections and potentials on a fixed line bundle.

This article, like many others of the authors, was influenced by Pierre Bérard's work. We are pleased to celebrate many years of friendship on the occasion of his birthday.

2. Geometric quantization

2.1. **Hamiltonian system associated with a magnetic field.**

On $\mathbb{R}^3$, a magnetic field may be viewed as an exact 2-form ω, identified with the curl of the magnetic potential field A. The 1-form $\alpha = A^\flat$ defines a connection $\nabla := d - i\alpha$ on the (trivial) Hermitian line bundle $\mathbb{R}^3 \times \mathbb{C}$ with curvature $-i\omega = -i\, d\alpha$.[1]

In analogy with the situation in $\mathbb{R}^3$, a closed 2-form ω on a Riemannian manifold (M, g) can be interpreted as a magnetic field. The Hamiltonian system for a charged particle moving in the magnetic field has phase space (T^*M, Ω) with $\Omega := \omega_0 + \pi^*\omega$, where ω_0 is the Liouville form on T^*M (that is, $\omega_0 = -d\lambda$, where λ is the canonical 1-form on the cotangent bundle), and $\pi : T^*M \to M$ is the projection; see [10], for example. The classical trajectories of the particle are given by the Hamiltonian flow of the (kinetic energy) Hamiltonian $H(q, \xi) :=$

[1] The appearance of $i = \sqrt{-1}$ here is a matter of convention. We choose the convention which is common in mathematics, specifically in geometric quantization.

$\frac{1}{2}g_q(\xi,\xi)$. When $\omega = 0$, so that $\Omega = \omega_0$, this flow is just the usual geodesic flow describing a free particle moving on M.

Notation 2.1. We will say that (M_1, g_1, ω_1) and (M_2, g_2, ω_2) are *classically equivalent* if the associated Hamiltonian systems (T^*M_1, Ω_1, H_1) and (T^*M_2, Ω_2, H_2) are equivalent.

Notation and Remarks 2.2. In case $[\omega/2\pi]$ is an integral cohomology class, let L be the line bundle over M with Chern class $[\omega/2\pi]$. Endow L with a Hermitian structure and let P be the associated principal circle bundle. Let ∇^L be a Hermitian connection on L with curvature $-i\omega$. The connection ∇ and the Riemannian metric on M give rise to a Riemannian metric $\widetilde{g}$ on P, sometimes called a Kaluza–Klein metric. Consider the associated geodesic flow on T^*P. The circle action on the principal bundle P gives rise to a Hamiltonian action of the circle S^1 on T^*P. The symplectic reduction of the geodesic flow on P by the S^1 action yields the Hamiltonian system of the magnetic flow on T^*M described above. In this brief description we have followed Kuwabara; see [12] for more information.

In preparation for Subsection 2.2, we note that the connection ∇^L and the Riemannian metric g on M give rise to a Laplace operator Δ on the space $C^\infty(M, L)$ of smooth sections of L given by (1.1). By the usual construction, ∇^L induces a Laplace operator, also denoted Δ, on the space $C^\infty(M, L^{\otimes k})$, where $L^{\otimes k}$ is the kth tensor power of L. The space $C^\infty(M, L^{\otimes k})$ may be identified with the space $C^\infty_k(P)$ of smooth complex-valued functions f on P satisfying the equivariance condition $f(\alpha.x) = \alpha^{-k}f(x)$ for $\alpha \in S^1$ and $x \in P$. The Laplace operator on $C^\infty(M, L^{\otimes k})$ is unitarily equivalent to the restriction of $\Delta_P - 4k^2\pi^2$ to $C^\infty_k(P)$, where Δ_P is the Laplace–Beltrami operator of $(P, \widetilde{g})$.

The space of smooth sections $C^\infty(M, L^{\otimes k})$ is endowed with the standard L^2 inner product given for smooth sections s and t by

$$\langle s, t\rangle := \left(\frac{k}{2\pi}\right)^n \int s.t\,\frac{\omega^n}{n!},$$

where $s.t$ denotes the pointwise Hermitian inner product on each fibre. This inner product defines a Hilbert space consisting of square-integrable sections of $L^{\otimes k}$, of which the space $C^\infty(M, L^{\otimes k})$ of smooth sections is a dense subspace. The operator Δ is an unbounded operator on this Hilbert space with dense domain $C^\infty(M, L^{\otimes k})$. The theory of unbounded operators on Hilbert spaces is well-developed (see, for example, the classic texts [16, Vol II, Chap. 8] and [4]), and we mention here only that Δ admits a self-adjoint extension, still denoted by Δ, with dense domain D containing the space of smooth sections of $L^{\otimes k}$. In the following, when we say that Δ is an operator on the Hilbert space of L^2-sections, it is to be understood in this usual sense of an unbounded operator with dense domain.

2.2. **Quantization of the Hamiltonian system.**

Using geometric quantization, one associates to a classical mechanical system (satisfying suitable requirements) a quantum mechanical system, consisting of a Hilbert space $\mathcal{H}_k$ and a quantum Hamiltonian operator $\widehat{H}_k : \mathcal{H}_k \to \mathcal{H}_k$ for each $k \in \mathbb{Z}$. (Here Planck's constant is given by $\hbar = 1/k$.) For the Hamiltonian system (T^*M, Ω, H) in Subsection 2.1, the quantization may be carried out provided that $\omega/2\pi$ represents an integral cohomology class of M.

In this case, one obtains Theorem 2.3 below. Following the statement of the theorem and related remarks, we will briefly outline the procedure of geometric quantization. For a complete presentation, see the classic references [21] and [17]; see also [9] and [2].

Theorem 2.3. [21, p. 204] *We use Notation 2.2 and assume that $[\omega/2\pi]$ is an integral cohomology class. The quantum Hilbert space associated to the classical Hamiltonian system (T^*M, Ω, H) of Subsection 2.1 is given for each integer k by $\mathcal{H}_k = L^2(M, L^{\otimes k})$ (the space of square-integrable sections of $L^{\otimes k}$) and the quantization of the Hamiltonian H is the (unbounded) operator*

$$\widehat{H}_k = -\frac{\hbar^2}{2}(-\Delta - \tfrac{1}{6}R) \tag{2.1}$$

on $\mathcal{H}_k$, where R is the scalar curvature of the metric g. (Here $\hbar = \frac{1}{k}$).

The allowed energy values of the charged particle in the magnetic field, which are what one would see if one "measured" the energy of the quantum particle, are the eigenvalues of $\widehat{H}_k$.

Remark 2.4. The definition of the Laplacian Δ on $L^{\otimes k}$, and thus of the operators $\widehat{H}_k$, depends on a choice of connection on L with the specified curvature $-i\omega$. However, in the examples that we will give in Subsection 3.2, there will be a natural choice of connection with that curvature.

Notation 2.5. Let (M_i, g_i), $i = 1, 2$, be a compact Riemannian manifold and let ω_i be a closed 2-form on M_i such that $[\omega/2\pi]$ is an integral cohomology class. For each integer k, let $\widehat{H}^i_k : L^2(M, L_i^{\otimes k}) \to L^2(M, L_i^{\otimes k})$ be the associated quantum Hamiltonian as given in Theorem 2.3. We will say that (M_1, ω_1) and (M_2, ω_2) are *quantum equivalent* (with respect to the connections used to define the line bundle Laplacians) if for every k, the operators $\widehat{H}^1_k$ and $\widehat{H}^2_k$ have the same spectrum.

We now outline the quantization procedure. Consider the classical Hamiltonian system (T^*M, Ω, H) given in Subsection 2.1. Recall that $\Omega = \omega_0 + \pi^*\omega$. Let π^*L be the pullback of L to a bundle over T^*M and $\pi^*\nabla^L$ the pullback of the connection. Since the Liouville form ω_0 on T^*M is exact, the Hermitian line bundle L_Ω with Chern class $[\Omega/2\pi]$ may be identified with π^*L. Writing $\omega_0 = d\Theta$, the Hermitian connection $\nabla := \pi^*\nabla^L - i\Theta$ on L_Ω has curvature $-i\Omega$.

The *prequantization* of the Hamiltonian system (T^*M, Ω) is the space of square-integrable sections of $L_\Omega^{\otimes k}$ with respect to the standard inner product

$$\langle s, t\rangle := \left(\frac{k}{2\pi}\right)^n \int_{T^*M} s.t\,\frac{\Omega^n}{n!},$$

where $s.t$ denotes the (pointwise) Hermitian product on $L_\Omega^{\otimes k}$. For a smooth function f on T^*M (we are interested in particular in the Hamiltonian H above), one associates a *prequantum Hamiltonian operator* $\widehat{f}^{preQ}$, given by the Kostant–Souriau construction:

$$\widehat{f}^{preQ} := \frac{i}{k}\nabla^{L_\Omega^{\otimes k}}_{X_f} + f, \tag{2.2}$$

where X_f is the Hamiltonian vector field associated to f, defined by $\Omega(X_f,\cdot) = df(\cdot)$. The Kostant–Souriau prequantization (2.2) satisfies Dirac's quantization conditions:

(1) the map $f \mapsto \widehat{f}^{\,preQ}$is linear,
(2) the quantization $\widehat{1}^{preQ}$ of the constant map 1 is the identity operator, and
(3) $[\widehat{f}^{preQ}, \widehat{g}^{preQ}] = -i\hbar\widehat{\{f,g\}}^{preQ}$, where $\{\cdot,\cdot\}$ is the Poisson bracket, $[\cdot,\cdot]$ is the operator commutator, and $\hbar = 1/k$.

Indeed, the prequantization (2.2) is derived precisely so that it satisfies (1) – (3) above. (See [21, Chap. 8]). Unfortunately, the pair $(L^2(T^*M, L_\Omega^{\otimes k}),\ f \mapsto \widehat{f}^{preQ})$ does not define a "good" quantization, essentially because $L^2(T^*M, L_\Omega^{\otimes k})$ is too big. For example, in the case $\omega = 0$ and $M = \mathbb{R}^n$, which corresponds to a free particle moving in Euclidean space, the line bundle $L_\Omega^{\otimes k}$ is trivial and the prequantum Hilbert space is then $L^2(T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n)$. The variables in the first $\mathbb{R}^n$ factor give the position of the particle, and the variables in the second $\mathbb{R}^n$ factor describe the momentum. But one knows from quantum mechanics that a wave function cannot be simultaneously a function of both position *and* momentum.

In order to obtain a Hilbert space of the "correct" size, one first chooses a polarization of (T^*M, Ω). A *polarization* of a symplectic manifold is an integrable (real or complex) Lagrangian distribution. If, as is the case here, the phase space is a cotangent bundle, one may take the *vertical polarization*, i.e., the distribution given by the tangent spaces to the fibers of T^*M. (Note that this distribution is indeed Lagrangian with respect to Ω as well as ω_0.) This means that we are considering wave functions which depend only on position, not on momentum. Thus, the vertical polarization corresponds to the "position-space representation" in quantum mechanics.

Once a choice of polarization $\mathcal{P}$ is made, one would ideally like to define the quantum Hilbert space to be the subspace $L^2_{\mathcal{P}}(T^*M, L_\Omega^{\otimes k})$ of the prequantum Hilbert space $L^2(T^*M, L_\Omega^{\otimes k})$ consisting of those sections that are covariantly constant in the $\mathcal{P}$ directions and then restrict the Kostant-Souriau prequantum Hamiltonians $\widehat{f}^{preQ}$ to this subspace. However, there are two problems here. First, $L^2_{\mathcal{P}} = \{0\}$! Secondly, even for a polarization $\mathcal{P}$ that yields a nontrivial quantum Hilbert space,[2] the Kostant-Souriau quantization $\widehat{f}^{preQ}$ does not in general preserve the quantum Hilbert space. Indeed, $\widehat{f}^{preQ}$ will only preserve $\mathcal{H}_k$ if the Hamiltonian flow of f preserves the polarization $\mathcal{P}$. One can show that the Ω-Hamiltonian flow of H does not preserve *any* polarization.

Fortunately, there is only one more piece of the puzzle which will remedy both of the remaining problems at once: the so-called half-form correction. The half-form correction boils down to tensoring $L^{\otimes k}$ with a square root of the canonical bundle associated to the polarization. Sections of such a bundle are called half-forms.

The half-form correction, due essentially to Blattner, Kostant and Sternberg, allows one to quantize a larger set of functions than just those whose flows preserve the polarization, and in particular one can quantize the standard Hamiltonians which appear in wave mechanics,

[2] A typical example of such a polarization is available whenever T^*M admits a Kähler structure, for example when M is a compact Lie group. In this case, one can take $\mathcal{P}$ to be the holomorphic tangent bundle.

of which our $H = \frac{1}{2}\|\xi\|^2$ is an example. Moreover, the quantum Hilbert space associated to the vertical polarization, in the presence of the half-form correction, will turn out to be just $L^2(M, L^{\otimes k})$, which is exactly what one would naively expect for the position-space representation.[3]

The BKS construction in our setting is as follows. (We refer the interested reader to [21], Chap. 9, for more details and proofs; see also [9].) Choose a line bundle δ such that $\delta \otimes \delta = \bigwedge^n TM$ (this is possible because $\bigwedge^n TM$ is trivializable), and let ν be a section of δ with $\nu^2 = \mathrm{vol}_g(M)$, where $\mathrm{vol}_g(M)$ is the Riemannian volume form on M. Sections of δ are called half-forms (associated to the vertical polarization), and the half-form corrected quantum Hilbert space is defined to be

$$\widehat{H}_k := L^2_{\mathcal{P}}(T^*M, L_\Omega^{\otimes k} \otimes \pi^*\delta)$$

where the inner product is defined by the canonical pairing of half-forms. In particular, a section of $L_\Omega \to T^*M$ which is vertically covariantly constant is uniquely determined by its value on the zero-section M, and the inner product of two such sections is therefore given by[4]

$$\langle s\nu, t\nu \rangle = \left(\frac{k}{2\pi}\right)^n \int_M s.t \ \mathrm{vol}_g(M). \tag{2.3}$$

Hence, we see that the quantum Hilbert space associated to the vertical polarization can be identified with $L^2(M, L^{\otimes k})$.

Now that we have the correct quantum Hilbert space, we need to quantize the Hamiltonian flow of the kinetic energy H. Let ρ_t denote the Hamiltonian flow of H on T^*M. In order to define the quantization of the Hamiltonian H, we evolve $\psi\nu$ for a short time (that is, apply $\exp(-ikt\widehat{H}^{preQ})$ to the first factor, and the pull-back ρ_t^* to the second factor), and then project the result back into $\widehat{\mathcal{H}}_k$.

The projection is achieved by a generalization of the half-form pairing (2.3). One can show that the pushforward of the vertical polarization by ρ_t is an integrable Lagrangian distribution which is (at least for small t) transverse to the vertical polarization. Hence, there exists some function $f_t \in C^\infty(T^*M)$ such that $\rho_t^*(\mathrm{vol}_g(M)) \wedge \mathrm{vol}_g(M) = f_t \Omega^n/n!$. The generalized (BKS) half-form pairing is then defined to be

$$(\rho_t^*\nu)\,.\nu := \sqrt{f_t}.$$

This pairing can be shown to be nondegenerate (at least for small t), and therefore defines a bijection between $(\exp(-ikt\,\widehat{H}^{preQ}) \otimes \rho_t^*)\widehat{\mathcal{H}}_k$ and $\widehat{\mathcal{H}}_k$. The quantum Hamiltonian $\widehat{Q}_k(H)$ is

[3] There are several further advantages, from both the mathematical and physical viewpoints, though they are not relevant to our current purposes. One, which is easy to describe, is that when using the BKS construction to quantize the simple harmonic oscillator (a well-known example from physics), a shift is introduced which results in the physically correct energy spectrum. Specifically, without the BKS construction, one obtains an energy spectrum consisting of integer multiples of $\hbar$. The physically correct spectrum, which is obtained using the BKS construction, is $\{(n + \frac{1}{2})\hbar : n \in \mathbb{Z}\}$.

[4] We will abuse notation slightly and not distinguish between ν (or $\mathrm{vol}_g(M)$) and its pullback $\pi^*\nu$ (resp. $\pi^* \mathrm{vol}_g(M)$).

obtained by computing the derivative with respect to t, evaluated at $t = 0$, of the operator on $\widehat{\mathcal{H}}_k$ given by first applying $(\exp(-ikt\widehat{H}^{preQ}) \otimes \rho_t^*)$ and then projecting the result back into $\widehat{\mathcal{H}}_k$ using the BKS pairing. At the end of the day, we are really interested only in sections of $L^{\otimes k}$; thus, we can take a section ψ in $\mathcal{H}_k$, multiply it by ν, apply the BKS construction, and write the result in the form $\psi'\nu$. The quantization of the Hamiltonian H is then defined to be $\widehat{H}_k\psi := \psi'$. In our case, this yields the expression in Theorem 2.3.

3. You can't hear a magnetic field

3.1. **The Sunada technique.** We will use a variant of Sunada's technique [18].

Definition 3.1. Let G be a finite group and let Γ_1 and Γ_2 be subgroups of G. We will say that Γ_1 is *almost conjugate* to Γ_2 in G if there is a bijection $\Gamma_1 \to \Gamma_2$ carrying each element of Γ_1 to a conjugate element in Γ_2; equivalently, each G-conjugacy class $[g]_G$ intersects Γ_1 and Γ_2 in the same number of elements.

Sunada's Theorem states that if a finite group G acts by isometries on a compact Riemannian manifold M and if Γ_1 and Γ_2 are almost conjugate subgroups of G acting freely on M, then $\Gamma_1\backslash M$ and $\Gamma_2\backslash M$ are isospectral.

Remarks 3.2.

(1) The almost conjugacy condition is equivalent to a representation theoretic condition as follows. The right multiplication of G on the cosets in $\Gamma_i\backslash G$ gives rise to a natural action of G on the finite-dimensional vector space $\mathbf{R}[\Gamma_i\backslash G]$. The subgroups Γ_1 and Γ_2 of G are almost conjugate if and only if there exists an isomorphism
$$\tau : \mathbf{R}[\Gamma_1\backslash G] \to \mathbf{R}[\Gamma_2\backslash G]$$
intertwining the actions of G.

(2) Assume that Γ_1 and Γ_2 are almost conjugate in G and let τ be the intertwining map in (i). Let W be any vector space on which G acts on the right. For $i = 1, 2$, let W^{Γ_i} be the subspace of vectors fixed by all elements of Γ_i. Then τ gives rise to a linear isomorphism, called "transplantation"
$$\mathcal{T} : W^{\Gamma_2} \to W^{\Gamma_1}.$$
Transplantation was first introduced in an example in [7] and systematized in [3] to give a new proof of Sunada's Theorem; see also [22]. We are following the presentation in [8].

(3) Transplantation is functorial: if V and W are right G-spaces and $\psi : V \to W$ is a G-equivariant map, then the following diagram commutes:
$$\begin{array}{ccc} W^{\Gamma_2} & \xrightarrow{\mathcal{T}_W} & W^{\Gamma_1} \\ \psi\downarrow & & \downarrow\psi \\ V^{\Gamma_2} & \xrightarrow{\mathcal{T}_V} & V^{\Gamma_1} \end{array}$$

Moreover, if W is an inner product space and if the G action is unitary, then the transplantation map is unitary.

Notation 3.3. Given a Hermitian line bundle L over a closed Riemannian manifold (M, g) and a Hermitian connection ∇ on L we denote by *Spec*(L, ∇, k) the spectrum of the associated Laplace operator Δ on $C^\infty(M, L^{\otimes k})$ (recall Notation and Remarks 2.2). For a potential $Q \in C^\infty(M)$, we denote by *Spec*$(Q; L, \nabla, k)$ the spectrum of $\Delta + Q$ on $C^\infty(M, L^{\otimes k})$.

Proposition 3.4. *Let (M, g) be a compact Riemannian manifold, let L be a Hermitian line bundle over M, and let ∇ be a Hermitian connection on L. Let G be a finite group that acts on L carrying fibers to fibers, preserving ∇, and such that the induced action on M is by isometries. For $i = 1, 2$, suppose that Γ_i is a subgroup of G whose action on M is free. Thus $L_i := \Gamma_i \backslash L$, $i = 1, 2$ is a Hermitian line bundle over $M_i := \Gamma_i \backslash M$, and ∇ induces a connection ∇_i on L_i. If Γ_1 and Γ_2 are almost conjugate in G, then*:

(i)
$$\text{Spec}(L_1, \nabla_1, k) = \text{Spec}(L_2, \nabla_2, k)$$
for all positive integers k.

(ii) *If, moreover, $Q \in C^\infty(M)$ is a G-invariant function, then*
$$\text{Spec}(Q; L_1, \nabla_1, k) = \text{Spec}(Q; L_2, \nabla_2, k)$$
for all positive integers k, where we use the same notation Q for the smooth potentials on M_1 and M_2 induced by the potential Q on M.

This variant of Sunada's Theorem is essentially contained in R. Kuwabara [11], although his interest was in pairs of connections on the same underlying bundle and in the case $Q = 0$.

For a proof by transplantation, observe that G acts on the right on the space $C^\infty(M, L^{\otimes k})$ of smooth sections of $L^{\otimes k}$ by $(f.g)(x) = g^{-1}.f(g.x)$ for $f \in C^\infty(M, L^{\otimes k})$, $g \in G$, and $x \in M$. The space $C^\infty(M_i, L_i^{\otimes k})$ of smooth sections of $L_i^{\otimes k}$ may be identified with the space $C^\infty(M, L^{\otimes k})^{\Gamma_i}$ of Γ_i-invariant elements of $C^\infty(M, L^{\otimes k})$. Thus by Remark 3.2, we obtain a transplantation map $\mathcal{T} : C^\infty(M_2, L_2^{\otimes k}) \to C^\infty(M_1, L_1^{\otimes k})$. Moreover, with this identification, the Schrödinger operator $\Delta_i + Q$ on $C^\infty(M_i, L_i^{\otimes k})$ (associated with the Riemannian metric on M_i, the connection ∇_i, and the potential Q) is the restriction to $C^\infty(M, L^{\otimes k})^{\Gamma_i}$ of the Schrödinger operator $\Delta + Q$ of $L^{\otimes k}$. Since Δ commutes with the action of G and since Q is G-invariant, we may let $\Delta + Q$ play the role of ψ in Remark 3.2. It follows that $\mathcal{T}$ intertwines the Schrödinger operators $\Delta_1 + Q$ and $\Delta_2 + Q$ on $L_1^{\otimes k}$ and $L_2^{\otimes k}$, thus proving the theorem.

Theorem 3.5. *We use the notation and hypotheses of Proposition 3.4, part (i). Let $-i\omega_j$ be the curvature of the connection ∇_j on L_j, $j = 1, 2$. Then in the language of Notation 2.5 and Remark 2.4, (M, ω_1) and (M_2, ω_2) are quantum equivalent with respect to the connections ∇_1 and ∇_2.*

Proof. We apply part (ii) of Proposition 3.4 with the scalar curvature R of M in the role of Q. Note that R is necessarily G-invariant since G acts by isometries on M. □

3.2. **Construction of examples.**

Let (M, g, ω) be a Kähler manifold of complex dimension n. (Here ω is the Kähler form.) The canonical line bundle L_M over M is defined to be the nth exterior power of the holomorphic cotangent bundle. Since M is Kähler, the Levi-Civita connection on TM commutes with the complex structure and thus defines a holomorphic connection on the holomorphic tangent bundle. This connection gives rise to a holomorphic connection on L_M that we will call the *canonical connection*.

If X is a simply-connected Hermitian symmetric space of non-compact type and M is a compact locally symmetric space with universal covering X, we will call M an X-space. Every X-space M is a Hodge manifold, i.e., M is a Kähler manifold and a suitable real multiple of the Kähler form ω of M represents an integer cohomology class. More precisely, if the metric is rescaled such that X (and hence each X-space M) has Einstein constant -1 then the Chern class of the canonical bundle L_M is $[\omega/2\pi]$ (see [1], formulas (4.68) and (4.59); compare also [20], p. 219.) As in Remark 2.4, the notion of "quantum equivalence" of (M_1, ω_1) and (M_2, ω_2), where the M_i are X-spaces and ω_i their Kähler forms, will mean with respect to the canonical connections on the canonical bundles L_{M_i}.

Theorem 3.6. *Let X be a simply-connected Hermitian symmetric space of noncompact type of real dimension at least four. Then there exist arbitrarily large families of non-isometric X-spaces M_i such that the (M_i, ω_i) are all mutually quantum equivalent but not classically equivalent. In fact the phase spaces (T^*M_i, Ω_i) of the classical Hamiltonian systems are not symplectomorphic* (*or even homeomorphic*).

Proof. D.B. McReynolds [13] showed, using Sunada's Theorem, that for every simply-connected symmetric space X of non-compact type, there exist arbitrarily large collections of non-isometric X-spaces M_i whose Laplace-Beltrami operators are mutually isospectral. For each such collection, there exists an X-space M and a finite group G of isometries of M such that $M_i = \Gamma_i \backslash M$, where the Γ_i are almost conjugate subgroups of G. In the setting that X is Hermitian symmetric, the isometries are holomorphic. Since all holomorphic isometries of M preserve both the canonical bundle and the canonical connection, we can now apply Theorem 3.5 to see that (M_i, ω_i) and (M_j, ω_j) are quantum equivalent for all i, j.

Mostow Strong Rigidity tells us that the various M_i have non-isomorphic fundamental groups, i.e., $M_i = \Lambda_i \backslash X$ with Λ_i and Λ_j non-isomorphic discrete uniform subgroups of the group of isometries of X when $i \neq j$. The cotangent bundle T^*M_i is the quotient of the (trivial) bundle T^*X by the action of Λ_i and thus the various T^*M_i are also non-homeomorphic. $\square$

The assumption on the dimension of X in Theorem 3.6, equivalently the exclusion of the case that X is the real hyperbolic plane, was needed only so that the phase spaces for the classical Hamiltonian systems would not be homeomorphic. In fact, we have the following:

Proposition 3.7. *Let (M_1, g_1) and (M_2, g_2) be* any *pair of hyperbolic Riemann surfaces which are isospectral with respect to the Laplace-Beltrami operator acting on functions. Then (M_1, ω_1) and (M_2, ω_2) are quantum equivalent, where ω_i is the Kähler form of (M_i, g_i).*

We emphasize that, in contrast to the Hermitian locally symmetric spaces in Theorem 3.6, the Riemann surfaces are *not* required to satisfy the conditions of Sunada's Theorem.

Proof. H. Pesce [14] proved that every pair of isospectral compact Riemann surfaces (M_1, g_1) and (M_2, g_2) is *strongly* isospectral in the following sense: Let $G = PSL(2, \mathbb{R})$. A Hermitian vector bundle E over the hyberbolic plane X is said to be homogeneous if G acts on E, carrying fibers to fibers, such that the induced action on X is the standard action by isometries. The actions of G on M and E give rise to an action of G on the space of smooth sections of E. A self-adjoint elliptic differential operator D on E (i.e., on smooth sections of E) is said to be *natural* if it commutes with the G-action. In that case, if Γ is a discrete subgroup of G acting freely and properly discontinuously on X, then D induces a self-adjoint elliptic differential operator on the bundle $\Gamma\backslash E$ over the Riemann surface $\Gamma\backslash M$. Compact Riemann surfaces $M_1 = \Gamma_1\backslash X$ and $M_2 = \Gamma_2\backslash X$ are said to be *strongly isospectral* if for each homogeneous Hermitian vector bundle E over X and each natural self-adjoint elliptic operator D on E, the induced operators on the bundles $\Gamma_1\backslash E$ over M_1 and $\Gamma_2\backslash E$ over M_2 are isospectral. (Aside: The key point in proving that isospectral compact Riemann surfaces $M_1 = \Gamma_1\backslash X$ and $M_2 = \Gamma_2\backslash X$ are always strongly isospectral is that isospectrality of the Riemann surfaces implies that the representations of G induced by the trivial representations of Γ_1 and Γ_2 are equivalent. This condition is considerably weaker than the Sunada condition, which requires that Γ_1 and Γ_2 be subgroups of some finite subgroup Γ of G and that the trivial representations of Γ_1 and Γ_2 induce equivalent representations of Γ.)

The proposition follows from that fact that the canonical Hermitian line bundle over X and all its tensor powers are homogeneous, and the Laplacian associated with the canonical connection is natural. ☐

Remark 3.8. Using Sunada's technique, R. Brooks, R. Gornet, and W. Gustafson [6] constructed arbitrarily large finite families of mutually isospectral, non-isometric Riemann surfaces. (Their work motivated that of D.B. McReynolds cited above.) While the vast majority of known isospectral Riemann surfaces were constructed by Sunada's technique, M.-F. Vignéras's examples [19] and recent examples of C.S. Rajan [15] do not satisfy the Sunada condition.

4. ISOSPECTRAL CONNECTIONS AND POTENTIALS ON A LINE BUNDLE AND ITS TENSOR POWERS

Using a trick introduced by R. Brooks [5], we can use Proposition 3.4 to obtain isospectral connections and potentials on a single line bundle and its tensor powers.

Corollary 4.1. *In addition to the hypotheses of Proposition 3.4, assume that there exists a bundle map σ of L, projecting to an isometry (also to be denoted σ) of M, such that σ normalizes G and such that $\sigma\Gamma_1\sigma^{-1} = \Gamma_2$. Continue to denote by σ the induced bundle map from $L_1^{\otimes k}$ to $L_2^{\otimes k}$. Then*

$$\mathit{Spec}(Q; L_1, \nabla_1, k) = \mathit{Spec}(\sigma^* Q; L_1, \sigma^*\nabla_2, k)$$

for all positive integers k.

This corollary is contained in Kuwabara [11] for the case $Q = 0$.

Remark 4.2. One may choose ∇ to be σ-invariant as well as G-invariant, in which case $\nabla_1 = \sigma^*\nabla_2$. We can then conclude that Q_1 and σ^*Q_2 are isospectral potentials for the Schrödinger operator $-\Delta_1$+potential.

We now explain how to use the corollary to obtain examples in which the base manifolds are Riemann surfaces. Brooks [5] gave explicit examples of finite groups G and Riemann surfaces (M, g) (with a hyperbolic Riemannian metric g) such that the following conditions are satisfied:

(i) The group G acts freely by orientation preserving isometries on the oriented Riemann surface (M, g).
(ii) There exists a pair of almost conjugate, nonconjugate subgroups Γ_1, Γ_2 of G.
(iii) There exists an outer automorphism τ of G such that $\Gamma_2 = \tau\Gamma_1\tau^{-1}$ and such that the action of G extends to a free action of the semi-direct product $\widehat{G}$ of G and $\langle\tau\rangle$ on (M, g) by orientation-preserving isometries.

Using these objects we obtain the following class of examples.

Example 4.3. We choose (M, g), G, Γ_1, Γ_2, τ, $\widehat{G}$ as above and consider the Hermitian line bundle L_N over $N := \widehat{G}\backslash M$. Denote its pullback to M by L. The group $\widehat{G}$ acts on L by vector bundle isomorphisms. We choose a $\widehat{G}$-invariant Hermitian connection $\widehat{\nabla}$ on L by pulling back a Hermitian connection from L_N, and we choose a function $f \in C^\infty(M)$ which is G-invariant but not τ-invariant. Denoting the Riemannian volume form on M by ω, we let $\nabla := \widehat{\nabla} + i\, d^*(f\omega)$. Note that ∇ is G-invariant, but not τ-invariant. Moreover, we choose any G-invariant potential $Q \in C^\infty(M)$. Finally, we let σ denote the vector bundle isomorphism of L induced by τ. Applying Proposition 3.4 together with Corollary 4.1 we obtain, for the vector bundle $L_1 := \Gamma_1\backslash L$ over $M_1 := \Gamma_1\backslash M$ and the induced connections ∇_1 on L_1, resp. ∇_2 on $L_2 := \Gamma_2\backslash L$:

$$\textit{Spec}(Q; L_1, \nabla_1, k) = \textit{Spec}(\sigma^*Q; L_1, \sigma^*\nabla_2, k)$$

for all k.

Remark 4.4. (i) The choice of ∇ in the previous example guarantees that the resulting pairs of isospectral connections ∇_1 and $\sigma^*\nabla_2$ have different curvature. In fact, the pullbacks to L of the connections ∇_1 and $\sigma^*\nabla_2$ on L_1 are $\widehat{\nabla} + i\, d^*(f\omega)$ and $\widehat{\nabla} + i\, d^*((\tau^*f)\omega)$, respectively. The pullback to M of the difference of the corresponding curvature forms on M_1 is given by $i\, dd^*((f - \tau^*f)\omega)$. The 2-form $(f - \tau^*f)\omega$ has integral zero over M and is thus exact by Poincaré duality. On the other hand, this form is nonzero by our choice of f, and hence nonharmonic. This immediately implies that $dd^*((f - \tau^*f)\omega) \neq 0$, as claimed.

(ii) Let $\widetilde{\tau}$ denote some lift of τ to the hyperbolic plane H_2, and let $\widetilde{G}$, $\widetilde{\Gamma}_i$ denote the groups of all lifts of elements of G, resp. Γ_i, to H_2. Let $N(\widetilde{\Gamma}_1)$ denote the normalizer of $\widetilde{\Gamma}_1$ within $\mathrm{Isom}(H_2)$. Then $\widetilde{\tau} \notin \widetilde{G}\widetilde{\varphi}$ for any $\widetilde{\varphi} \in N(\widetilde{\Gamma}_1)$ because, otherwise, the relation $\widetilde{\tau}\widetilde{\Gamma}_1\widetilde{\tau}^{-1} = \widetilde{\Gamma}_2$ would imply that Γ_1 and Γ_2 were conjugate in G. Note that $N(\widetilde{\Gamma}_1)$ consists precisely of the lifts of isometries of M_1. Therefore the fact that $\widetilde{\tau} \notin \widetilde{G}\widetilde{\varphi}$ for all $\widetilde{\varphi} \in N(\widetilde{\Gamma}_1)$ implies that it is possible to choose the G-invariant function f subject to the slightly stronger property that the functions f_1 and f_1^τ which are induced by f and $\tau^* f$ on M_1, respectively, do not differ by any isometry of M_1. Then, for any isometry φ of M_1 we can apply the argument of (i) to the the lift of $f_1 - \varphi^* f_1^\tau$ to M and conclude that now the curvature forms associated with ∇_1 and $\sigma^*\nabla_2$ are not related by pullback by any isometry of M_1.

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