EXTREMAL SPECTRAL PROPERTIES OF LAWSON TORI AND THE LAMÉ EQUATION

ALEXEI V. PENSKOI

Abstract. Extremal spectral properties of the Lawson tori are studied. A Lawson torus carries an extremal metric for some eigenvalue λ_j of the Laplace-Beltrami operator. We prove that the metric on a Lawson torus $\tau_{m,k}$ is extremal for $j = 2\left(\left[\sqrt{m^2+k^2}\right] + m + k\right) - 1$.

Introduction

Let M be a closed surface equipped with a Riemannian metric g. Let us consider the associated Laplace-Beltrami operator $\Delta : C^\infty(M) \longrightarrow C^\infty(M)$,

$$\Delta f = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\right),$$

and its eigenvalues

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \lambda_3(M,g) \leqslant \ldots.$$

The functional

$$\Lambda_i(M,g) = \lambda_i(M,g)\,\mathrm{Area}(M,g)$$

is invariant under rescaling transformations $g \mapsto tg$ because for $t > 0$ we have

$$\lambda_i(M,tg) = \frac{1}{t}\lambda_i(M,g), \quad \mathrm{Area}(M,tg) = t\,\mathrm{Area}(M,g).$$

It turns out that the question about the supremum $\sup \Lambda_i(M,g)$ of the functional $\Lambda_i(M,g)$ over the space of Riemannian metrics g on a fixed manifold M is very difficult and only few results are known.

It was proven in the paper [1] by Yang and Yau that for an orientable surface M of genus γ the following inequality holds,

$$\Lambda_1(M,g) \leqslant 8\pi(\gamma+1).$$

Korevaar proved in the paper [2] that there exists a constant C such that for any $i > 0$ and any compact surface M of genus γ the functional $\Lambda_i(M,g)$ is bounded,

$$\Lambda_i(M,g) \leqslant C(\gamma+1)i.$$

However, Colbois and Dodziuk proved in the paper [3] that for a manifold M of dimension $\dim M \geqslant 3$ the functional $\Lambda_i(M,g)$ is not bounded on the space of Riemannian metrics g on M.

The functional $\Lambda_i(M,g)$ depends continuously on the metric g, but this functional is not differentiable. However, Berger proved in the paper [4] that for analytic deformations g_t the left and right derivatives of the functional $\Lambda_i(M,g_t)$ with respect to t exist. This is motivation for the following definition, see the paper [5] by Nadirashvili and the paper [6] by El Soufi and Ilias.

2000 *Mathematics Subject Classification.* 58E11, 58J50.
Key words and phrases. Lawson torus, Lawson minimal surfaces, extremal metric, Lamé equation.

Definition 1. *A Riemannian metric g on a closed surface M is called extremal metric for the functional $\Lambda_i(M,g)$ if for any analytic deformation g_t such that $g_0 = g$ the following inequality holds,*

$$\frac{d}{dt}\Lambda_i(M,g_t)\Big|_{t=0+} \leqslant 0 \leqslant \frac{d}{dt}\Lambda_i(M,g_t)\Big|_{t=0-}.$$

Only few examples of maximal or extremal metrics are known.

Hersch proved in the paper [7] that $\sup\Lambda_1(\mathbb{S}^2,g) = 8\pi$ and the maximum is reached on the canonical metric on $\mathbb{S}^2$. This metric is the unique extremal metric for the functional $\Lambda_1(\mathbb{S}^2,g)$. Nadirashvili proved in the paper [8] that $\sup\Lambda_2(\mathbb{S}^2,g) = 16\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of two spheres of equal radius with canonical metric glued together.

Li and Yau proved in the paper [9] that $\sup\Lambda_1(\mathbb{R}P^2,g) = 12\pi$ and the maximum is reached on the canonical metric on $\mathbb{R}P^2$. This metric is the unique extremal metric for the functional $\Lambda_1(\mathbb{R}P^2,g)$.

Nadirashvili proved in the paper [5] that $\sup\Lambda_1(\mathbb{T}^2,g) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is reached on the flat equilateral torus. El Soufi and Ilias proved in the paper [6] that the only extremal metric for $\Lambda_1(\mathbb{T}^2,g)$ different from the maximal one is the metric on the Clifford torus.

Jakobson, Nadirashvili and Polterovich proved in the paper [10] that the metric on a Klein bottle realized as the Lawson bipolar surface $\tilde\tau_{3,1}$ is extremal. El Soufi, Giacomini and Jazar proved in the paper [11] that this metric is the unique extremal metric and the maximal one.

Let $r,k \in \mathbb{N}$, $0 < k < r$, $(r,k) = 1$. Lapointe studied in the paper [12] Lawson bipolar surfaces $\tilde\tau_{r,k}$ and proved the following result.

(1) If $mk \equiv 0 \mod 2$ then $\tilde\tau_{m,k}$ is a torus and it carries an extremal metric for $\Lambda_{4m-2}(\mathbb{T}^2,g)$.
(2) If $mk \equiv 1 \mod 4$ then $\tilde\tau_{m,k}$ is a torus and it carries an extremal metric for $\Lambda_{2m-2}(\mathbb{T}^2,g)$.
(3) If $mk \equiv 3 \mod 4$ then $\tilde\tau_{m,k}$ is a Klein bottle and it carries an extremal metric for $\Lambda_{m-2}(\mathbb{K},g)$.

It is shown in the paper [13] by Jakobson, Levitin, Nadirashvili, Nigam and Polterovich that the maximal metric on the surface of genus two is the metric on the Bolza surface $\mathcal{P}$ induced from the canonical metric on the sphere using the standard covering $\mathcal{P} \longrightarrow \mathbb{S}^2$. In fact, the authors state this result as a conjecture, because a part of the argument is based on a numerical calculation.

The goal of the present paper is to continue investigation of extremal spectral properties of metrics on Lawson tori $\tau_{m,k}$ started in the paper [14].

Definition 2. *Lawson surface $\tau_{m,k} \subset \mathbb{S}^3$ is defined by the doubly-periodic immersion $\Psi_{m,k} : \mathbb{R}^2 \longrightarrow \mathbb{S}^3 \subset \mathbb{R}^4$ given by the following explicit formula,*

$$\Psi_{m,k}(x,y) = (\cos mx\cos y, \sin mx\cos y, \cos kx\sin y, \sin kx\sin y). \tag{1}$$

This family of surfaces is introduced by Lawson in the paper [15]. He proved that for each unordered pair of positive integers (m,k) with $(m,k) = 1$ the surface $\tau_{m,k}$ is a distinct compact minimal surface in $\mathbb{S}^3$. If both integers m and k are odd then $\tau_{m,k}$ is a torus. If one of integers m or k is even then $\tau_{m,k}$ is a Klein bottle (both m and k cannot be even due to the condition $(m,k) = 1$). The torus $\tau_{1,1}$ is the Clifford torus.

Since $\tau_{m,k} \cong \tau_{k,m}$, let us suppose that always $m > k > 0$ except the special case of the Clifford torus $\tau_{1,1}$. We deal only with Lawson tori, hence m and k are odd positive integers such that $(m,k) = 1$. When it is necessary to have uniquely defined coordinates of a point on a Lawson torus, we consider coordinates $(x, y) \in [0, 2\pi) \times [0, 2\pi)$.

The main result of the paper [14] is the following Proposition.

Proposition 1. *Let $\tau_{m,k}$ be a Lawson torus such that*

$$\frac{k}{m} > \frac{1}{\sqrt{20} + \sqrt{21}} = 0.11044\ldots \tag{2}$$

Let $p(y) = \sqrt{k^2 + (m^2 - k^2)\cos^2 y}$. Let

$$\lambda_0(l) < \lambda_1(l) \leqslant \lambda_2(l) < \lambda_3(l) \leqslant \lambda_4(l) < \lambda_5(l) \leqslant \lambda_6(l) < \ldots$$

be eigenvalues of a family of periodic Sturm-Liouville problems

$$(p(y)\varphi'(y))' + \left(\lambda p(y) - \frac{l^2}{p(y)}\right)\varphi(y) = 0, \tag{3}$$

$$\varphi(y + 2\pi) \equiv \varphi(y), \tag{4}$$

where $l = 0, 1, 2, 3, \ldots$. Let $J = \#\{\lambda_0(l) | \lambda_0(l) < 2\} = \min\{l \in \mathbb{Z} | \lambda_0(l) \geqslant 2\}$.

Then the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\Lambda_j(\mathbb{T}^2, g)$, where $j = 2(J + m + k) - 3$.

The important component of the proof of Proposition 1 is a beautiful result relating extremal metrics to minimal immersions in spheres proved by El Soufi and Ilias in the paper [16]. This result reduces calculating j to counting the eigenvalues of the Laplace-Beltrami operator Δ on the Lawson torus $\tau_{m,k}$. In Proposition 1 we reduce this counting to counting the fundamental tones $\lambda_0(l)$ of the family of auxiliary periodic Sturm-Liouville problems (3), (4).

Several methods for estimating J were proposed in the paper [14]. Using these methods an explicit answer was found for several Lawson tori including $\tau_{1,1}$ (the Clifford torus), $\tau_{3,1}$, $\tau_{9,7}$ etc. The metric on $\tau_{1,1}$ is an extremal metric for the functional $\Lambda_5(\mathbb{T}^2, g)$, the metric on $\tau_{3,1}$ is an extremal metric for the functional $\Lambda_{13}(\mathbb{T}^2, g)$ etc. However, these methods based on estimates were insufficient to find J as an explicit function of m and k. We also remarked in the paper [14] that numerical experiments shows that condition (2) could possibly be dropped.

An explicit formula for J is obtained in the present paper. We also show that we do not need condition (2). The main result of the present paper is the following Theorem.

Theorem. *Let $\tau_{m,k}$ be a Lawson torus. Then the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\Lambda_j(\mathbb{T}^2, g)$, where*

$$j = 2\left(\left[\sqrt{m^2 + k^2}\right] + m + k\right) - 1, \tag{5}$$

and $[z]$ denotes the integer part of z.

The corresponding value of the functional is

$$\Lambda_j(\tau_{m,k}) = 16\pi m E\left(\frac{\sqrt{m^2 - k^2}}{m}\right), \tag{6}$$

where E is the complete elliptic integral of the second kind,

$$E(k) = \int_0^1 \frac{\sqrt{1 - k^2\alpha^2}}{\sqrt{1 - \alpha^2}} d\alpha.$$

The proof is based on the properties of the Lamé equation (7), see Section 1.

The developed approach could also be used for the Lawson Klein bottles. This is the subject of a subsequent paper.

1. The Lamé equation

In this Section we recall some properties of the Lamé equation usually written as

$$\frac{d^2\varphi}{dz^2} + (h - n(n+1)[\hat{k}\,\mathrm{sn}(z)]^2)\varphi = 0. \tag{7}$$

see e.g. the book [17] or the book [18]. We denote the modulus of the elliptic function $\mathrm{sn}\,z$ by $\hat{k}$ since we already use k in $\tau_{m,k}$.

The Lamé equation could be written in different forms, we will use the trigonometric form of the Lamé equation

$$[1-(\hat{k}\cos y)^2]\frac{d^2\varphi}{dy^2} + \hat{k}^2\cos y\sin y\frac{d\varphi}{dy} + [h - n(n+1)(\hat{k}\cos y)^2]\varphi = 0. \tag{8}$$

The equation (8) could be obtained from (7) using the change of variable

$$\mathrm{sn}\,z = \cos y. \tag{9}$$

We are interested in the 2π-periodic solutions of the Lamé equation (8). Usually $0 < \hat{k} < 1$ and n are fixed parameters and h plays the role of an eigenvalue. The following Proposition holds.

Proposition 2. *Given $0 < \hat{k} < 1$ and n, there exist an infinite sequence of values*

$$h_0 < h_1 \leqslant h_2 < h_3 \leqslant h_4 < \dots$$

of the parameter h such that if $h = h_i$ then the the Lamé equation (8) has a 2π-periodic solution $\varphi_i(y) \neq 0$.

For $h = h_0$ a solution $\varphi_0(y)$ is unique up to a multiplication by a non-zero constant.

If $h_{2i+1}(l) < h_{2i+2}(l)$, then solutions $\varphi_{2i+1}(y)$ and $\varphi_{2i+2}(y)$ are unique up to a multiplication by a non-zero constant.

If $h_{2i+1}(l) = h_{2i+2}(l)$, then there exist two independent solutions $\varphi_{2i+1}(y)$ and $\varphi_{2i+2}(y)$ corresponding to $h = h_{2i+1} = h_{2i+1}$.

The solution $\varphi_0(y)$ has no zero on $[0, 2\pi)$. For $i \geqslant 0$ both solutions $\varphi_{2i+1}(y)$ and $\varphi_{2i+2}(y)$ have exactly $2i+2$ zeroes on $[0, 2\pi)$.

Our main interest is the case $n = 1$. In this case three wonderful solutions of the Lamé equation (7) are known,

$$\mathrm{Ec}_1^0(z) = \mathrm{dn}\,z, \quad \mathrm{Ec}_1^1(z) = \mathrm{cn}\,z, \quad \mathrm{Es}_1^1(z) = \mathrm{sn}\,z$$

where we use the notation used by Ince in the paper [19]. Using standard properties of the Jacobi elliptic functions and change of variable (9) we obtain three solutions of the Lamé equation in the trigonometric form (8),

$$\mathrm{Ec}_1^0(y) = \sqrt{1-\hat{k}^2\cos^2 y}, \quad \mathrm{Ec}_1^1(y) = \sin y, \quad \mathrm{Es}_1^1(y) = \cos y.$$

Proposition 3. *If $n = 1$ then we have*

$$\begin{aligned} \varphi_0(y) = \mathrm{Ec}_1^0(y) = \sqrt{1-\hat{k}^2\cos^2 y}, &\quad h_0 = \hat{k}^2, \\ \varphi_1(y) = \mathrm{Ec}_1^1(y) = \sin y, &\quad h_1 = 1, \\ \varphi_2(y) = \mathrm{Es}_1^1(y) = \cos y, &\quad h_2 = 1 + \hat{k}^2. \end{aligned} \tag{10}$$

Proof. The function $\mathrm{Ec}_1^0(y) = \sqrt{1-\hat{k}^2\cos^2 y}$ has no zeroes, hence by Proposition 2 it is $\varphi_0(y)$. Direct check by substitution shows that $h_0 = \hat{k}^2$. The same argument works for $\varphi_1(y)$ and $\varphi_2(y)$. $\square$

We should remark that in general h_i are roots of a very complicated transcendental equation with parameters n and $\hat{k}$ and cannot be found explicitly.

Using the same approach as in Proposition 25 from the paper [14] we can prove the following Proposition 4.

Proposition 4. *Let us fix $n = 1$ and consider h_3 as a function of $\hat{k}^2$, where $0 < \hat{k}^2 \leqslant 1$. Then $h_3(\hat{k}^2)$ is a decreasing function.*

When $\hat{k} = 1$ the Lamé equation (7) is called degenerate because in this case we have $\operatorname{sn} z = \tanh z$.

Proposition 5. *Let $n = 1$ and $\hat{k} = 1$. Then we have*

$$h_0 = h_1 = 1, \quad h_2 = h_3 = 2.$$

Proof follows from the formulae in [19].

Propositions 4 and 5 implies the following Proposition 6.

Proposition 6. *Let $n = 1$. Then for $0 < \hat{k}^2 < 1$ we have*

$$h_3 > 2.$$

2. Proof of the Theorem

Let us now consider not only integer values of l but also real values of l. We proved in [14] that $\lambda_0(l)$ is a strictly increasing function of l. Let us denote by l_c the solution of the equation

$$\lambda_0(l) = 2.$$

Then

$$J = \min\{l \in \mathbb{Z} | \lambda_0(l) \geqslant 2\} = \lceil l_c \rceil,$$

where $\lceil \cdot \rceil$ denotes the ceiling function, i.e. $\lceil x \rceil = \min\{a \in \mathbb{Z} | a \geqslant x\}$.

Let us remark that after the shift $y \mapsto y + \frac{\pi}{2}$ equation (3) could be written as the Lamé equation in the trigonometric form (8) with

$$\hat{k} = \frac{\sqrt{m^2 - k^2}}{m}, \quad h = \lambda - \frac{l^2}{m^2}, \quad n(n+1) = \lambda. \tag{11}$$

Let us recall that we need condition (2) in Proposition 1 because it was used in the proof of Proposition 1 in the paper [14] in order to prove that $\lambda_3(0) > 2$.

Let us show that $\lambda_3(0) > 2$ for any $m > k > 0$. If this is true, we can drop condition (2).

Let us suppose that $\lambda_3(0) \leqslant 2$. We proved in the paper [14] that

$$\lambda_3(k) > \lambda_2(k) = 2.$$

It follows that there exists some value $l_2 \geqslant 0$ such that $\lambda_3(l_2) = 2$. We know that $\lambda = 2$ corresponds to $n = 1$ and λ_3 corresponds to h_3 and we see from formulae (11) and Proposition 6 that

$$2 - \frac{l_2^2}{m^2} = \lambda_3 - \frac{l_2^2}{m^2} = h_3 > 2.$$

This implies

$$\frac{l_2^2}{m^2} < 0,$$

but this is impossible. Hence, $\lambda_3(0) > 2$ and we do not need condition 2.

We know that $\lambda = 2$ corresponds to $n = 1$ and λ_0 corresponds to h_0. Hence we obtain from (11) and Proposition 3 the identities

$$\frac{m^2 - k^2}{m^2} = \hat{k}^2 = h_0 = \lambda_0(l) - \frac{l^2}{m^2}, \quad 2 = n(n+1) = \lambda_0(l). \tag{12}$$

But we denoted the solution of the equation $\lambda_0(l) = 2$ by l_c and we obtain from identities (12) the equation

$$\frac{m^2-k^2}{m^2} = 2 - \frac{l_c^2}{m^2}.$$

It follows that

$$l_c = m\sqrt{2-\frac{m^2-k^2}{m^2}} = \sqrt{m^2+k^2}.$$

This implies $J = \lceil l_c \rceil = \left\lceil \sqrt{m^2+k^2} \right\rceil$. It is easy to see that $\sqrt{m^2+k^2}$ is not integer since m and k are both odd. It follows that

$$J = \left\lceil \sqrt{m^2+k^2} \right\rceil = \left[\sqrt{m^2+k^2} \right] + 1.$$

Substituting into Proposition 1, we obtain formula (5) from the statement of Theorem.

We proved in the paper [14] that the induced metric on a Lawson torus is equal to $p^2(y)dx^2 + dy^2$ and $\lambda_j(\tau_{m,k}) = 2$. This implies formula (6) from the statement of Theorem because

$$\Lambda_j(\tau_{m,k}) = \lambda_j(\tau_{m,k}) \operatorname{Area}(\tau_{m,k}) = 2\int_0^{2\pi}\int_0^{2\pi} p(y)dxdy =$$

$$= 2\cdot 2\pi \cdot 4kE\left(i\frac{\sqrt{m^2-k^2}}{k}\right) = 16\pi k\frac{m}{k}E\left(-\frac{\sqrt{m^2-k^2}}{m}\right) = 16\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

□

Acknowledgments

The author is very indebted to I. V. Polterovich who inspired the interest to spectral geometry.

This work was partially supported by the Russian Foundation for Basic Research (grant no. 08-01-00541) and by the Russian State Programme for the Support of Leading Scientific Schools (grant no. 5413.2010.1).

References

[1] P. C. Yang, S.-T. Yau, Eigenvalues of the laplacian of compact Riemann surfaces and minimal submanifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **7**:1 (1980), 55–63.

[2] N. Korevaar, Upper bounds for eigenvalues of conformal metrics. *J. Differential Geom.* **37**:1 (1993), 73–93.

[3] B. Colbois, J. Dodziuk, Riemannian metrics with large λ_1, *Proc. Amer. Math. Soc.* **122**:3 (1994), 905-906.

[4] M. Berger, Sur les premières valeurs propres des variétés Riemanniennes, *Compositio Math.* **26** (1973), 129-149.

[5] N. Nadirashvili, Berger's isometric problem and minimal immersions of surfaces. *Geom. Funct. Anal.* **6**:5 (1996), 877-897.

[6] A. El Soufi, S. Ilias, Riemannian manifolds admitting isometric immersions by their first eigenfunctions. *Pacific J. Math.* **195**:1 (2000), 91-99.

[7] J. Hersch, Quatre propriétés isopérimétriques de membranes sphériques homogènes. *C. R. Acad. Sci. Paris Sér A-B* **270** (1970), A1645-A1648.

[8] N. Nadirashvili, Isoperimetric inequality for the second eigenvalue of a sphere. *J. Differential Geom.* **61**:2 (2002), 335-340.

[9] P. Li, S.-T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Invent. Math.* **69**:2 (1982), 269-291.

[10] D. Jakobson, N. Nadirashvili, I. Polterovich, Extremal Metric for the First Eigenvalue on a Klein Bottle. *Canad. J. Math.* **58**:2 (2006), 381-400. Preprint `arXiv:math/0311484`.

[11] A. El Soufi, H. Giacomini, M. Jazar, A unique extremal metric for the least eigenvalue of the Laplacian on the Klein bottle. *Duke Math. J.* **135**:1 (2006), 181-202. Preprint `arXiv:math/0701773`.

[12] H. Lapointe, Spectral properties of bipolar minimal surfaces in $\mathbb{S}^4$. *Differential Geom. Appl.* **26**:1 (2008), 9-22. Preprint `arXiv:math/0511443.`

[13] D. Jakobson, M. Levitin, N. Nadirashvili, N. Nigam, I. Polterovich, How large can the first eigenvalue be on a surface of genus two? *Int. Math. Res. Not.* **63** (2005), 3967-3985. Preprint `arXiv:math/0509398.`

[14] A. V. Penskoi, Extremal spectral properties of Lawson tori. Preprint `arXiv:1008.2954.`

[15] H. B. Lawson, Complete minimal surfaces in S^3. *Ann. of Math.* **92** (1970), 335-374.

[16] A. El Soufi, S. Ilias, Laplacian eigenvalues functionals and metric deformations on compact manifolds. *J. Geom. Phys.* **58**:1 (2008), 89-104. Preprint `arXiv:math/0701777.`

[17] A. Erdelyi, W. Magnus, F. Oberhettinger, F. Tricomi. *Higher transcendental functions,* Vol. III. McGraw-Hill Book Company Inc., New York-Toronto-London, 1955.

[18] F. M. Arscott, *Periodic differential equations. An introduction to Mathieu, Lamé and allied functions.* International Series of Monographs in Pure in Applied Mathematics, Vol. 66. A Pergamon Press book. The McMillan Co., New York, 1964.

[19] E. L. Ince, The periodic Lamé functions. *Proc. Royal Soc. Edinburgh* **60** (1940), 47-63.

Department of Geometry and Topology, Faculty of Mathematics and Mechanics, Moscow State University, Leninskie Gory, GSP-1, 119991, Moscow, Russia
and
Independent University of Moscow, Bolshoy Vlasyevskiy pereulok 11, 119002, Moscow, Russia
and
Department of Mathematical Modelling (FN-12), Faculty of Fundamental Sciences, Bauman Moscow State Technical University, 2-ya Baumanskaya 5, 105005, Moscow, Russia

E-mail address: `penskoi@mccme.ru`