MODULES WITH 1-DIMENSIONAL SOCLE AND COMPONENTS OF LUSZTIG QUIVER VARIETIES IN TYPE A

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1. Introduction

For any simply-laced Kac-Moody Lie algebra $\mathfrak{g}$, Lusztig [L] has constructed canonical bases for its representations using the geometry of quiver varieties. In particular, Lustzig considered the variety $Rep(w)_v$ of representations of the preprojective algebra Λ on a fixed vector space of dimension v and having dimension of socle bounded by w. The irreducible components of this variety index Lusztig's canonical basis for a particular weight space of a highest weight representation of $\mathfrak{g}$. The components of $Rep(w)_v$ are also in natural bijection with the components of Nakajima's Lagrangian quiver varieties. This is shown in the work of Saito [Sai, section 4.6], who also studied a crystal structure on these components jointly with Kashiwara [KS].

Because the components of $Rep(w)_v$ index the canonical basis, it would be interesting to descibe them in an explicit fashion using known combinatorics. In certain special cases (including $\mathfrak{g} = \mathfrak{sl}_n$), this has been done by Savage [Sav], using ad-hoc methods. In a forthcoming paper [BK], Pierre Baumann and the first author will use module-theoretic means to give a uniform description of the components using the theory of MV polytopes [K]. In our description, a key role is played by certain Λ-modules with one dimensional socle.

In the current paper, we focus on the case $\mathfrak{g} = \mathfrak{sl}_n$. Using elementary means, we classify Λ-modules with one dimensional socle and explain how these modules can be used to describe components of $Rep(w)_v$. Similar results (and more) will be formulated and proved for general $\mathfrak{g}$ in [BK].

More specifically in section 3, we classify Λ-modules with one dimensional socle by showing that they are all isomorphic to certain Maya modules introduced by Savage [Sav]. These Maya modules are in bijection with subsets of $\{1, \dots, n\}$ (other than $\{1, \dots, i\}$). Next, we compute the space of homomorphisms between two such modules, obtaining an explicit combinatorial formula. We show that this formula is related to a truncated permutahedron, which is the MV polytope for this situation.

In section 4, we show how Maya modules can be used to describe the components of $Rep(w)_v$. We begin by computing the space of homomorphisms between certain Maya modules and modules associated to tableaux by Savage [Sav]. We use this to rephrase Savage's description of the components in a module-theoretic fashion.

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2. Background

2.1. Notation. Let Q denote the root lattice of SL_n. So

$$Q = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : \sum x_i = 0\}.$$

For $i = 1, \dots, n-1$, let $\alpha_i = (\dots, 0, 1, -1, 0, \dots)$ denote the simple roots (the 1 is in the ith position). Let Q_+ be the subset of Q given by non-negative sums of the α_i. Let $\omega_i = (1, \dots, 1, 0, \dots, 0)$ denote the fundamental weights (the first i entries are 1s).

If A and B are i element subsets of $\{1, \dots, n\}$, then we define

$$A - B := 1_A - 1_B \in Q$$

where 1_A is the n-tuple which is 1 in positions indexed by numbers in A and 0 in the other positions. We write $A \geq B$ if $A - B \in Q_+$.

2.2. The preprojective algebra. Let Ω be a simply-laced Dynkin quiver (that is a Dynkin diagram with orientation) with edge set Ω and vertex set I. Let Λ denote the preprojective algebra of the quiver Ω. By definition Λ is the quotient

$$P(\Omega \oplus \overline{\Omega}) / (\sum_{\tau \in \Omega} \tau\overline{\tau} - \overline{\tau}\tau)$$

of the path algebra of the doubled quiver $\Omega \oplus \overline{\Omega}$ by the preprojective relation.

For this paper, we will work exclusively with the type A_{n-1} quiver with the leftward orientation.

$$\begin{array}{ccccccc} 1 & & 2 & & \cdots & & n-1 \\ \bullet & \longleftarrow & \bullet & \longleftarrow & \cdots & \longleftarrow & \bullet \end{array}$$

For this quiver we have vertex set $I = \{1, \dots, n-1\}$ and edge sets

$$\Omega = \{2 \to 1, 3 \to 2, \dots, n-1 \to n-2\} \quad \overline{\Omega} = \{1 \to 2, 2 \to 3, \dots, n-2 \to n-1\}$$

So a Λ-module M consists of an I-graded vector space $M = \oplus_{i \in I} M_i$ with linear maps

$$(i \to i+1) : M_i \to M_{i+1} \quad (i \to i-1) : M_i \to M_{i-1}$$

such that the preprojective relations

$$(i+1 \to i)(i \to i+1) = (i-1 \to i)(i \to i-1) \text{ for } i = 1, \dots, n-1$$

are satisfied. Here and later, we adopt the convention that $(1 \to 0) : M_1 \to 0$ and $(n-1 \to n) : M_{n-1} \to 0$ are 0.

If M is a Λ-module, then it has a dimension vector $v = (v_i)_{i \in I} \in \mathbb{N}^I$, where $v_i = \dim(M_i)$. It will be convenient to encode this as an element of Q_+ as $\alpha_v = \sum_i v_i \alpha_i$.

2.3. Socle of modules. The only simple Λ-modules are the one-dimensional modules S_i, which have dimension 1 in the ith slot and 0 elsewhere.

If M is any Λ-module, then the socle of M is defined to be the maximal semisimple submodule of M. The S_ith isotypic component of the socle of M is called the i-socle of M and is denoted $\mathrm{soc}_i(M)$.

More explicitly, $\mathrm{soc}(M)$ is the submodule of M whose ith graded piece is

$$\mathrm{soc}_i(M) = \{w \in M_i : (i \to i+1)(w) = 0 \text{ and } (i \to i-1)(w) = 0\}$$

All arrows act by 0 in $\mathrm{soc}(M)$.

2.4. **Lusztig quiver varieties.** If $v \in \mathbb{N}^I$, then we may consider the variety Rep_v of representations of Λ on a fixed I-graded vector space of dimension v. By the work of Lusztig [L], the irreducible components of Rep_v index the canonical basis for $(U\mathfrak{n})_{\alpha_v}$, where $U\mathfrak{n}$ denotes the universal envelopping algebra of the upper triangular subalgebra of $\mathfrak{g}$. In particular, the number of irreducible components of Rep_v is equal to the Kostant partition function of α_v.

For each point $x \in Rep_v$, we can consider the corresponding abstract Λ-module M_x. For $w \in \mathbb{N}^I$, we consider the variety $Rep(w)_v$ of consisting of those points $x \in Rep_v$ with $\dim \mathrm{soc}_i(M_x) \leq w_i$ for all i. Under Lusztig's construction the components of these varieties are related to the irreducible representations as follows. Let $\lambda = \lambda_w := \sum_i w_i \omega_i$ and $\mu = \lambda_w - \alpha_v$ (here ω_i are the fundamental weights). The irreducible components of $Rep(w)_v$ index the canonical basis for the μ weight space of the irreducible representation $V(\lambda)$ of SL_n.

3. Modules with one-dimensional socle

3.1. **The Maya modules.** Let A be a proper subset of $\{1, \dots, n\}$ of size i, other than $\{1, \dots, i\}$.

The Maya module $N(A)$ has the following description. If $A = \{a_1 < \cdots < a_i\}$, then $N(A)$ has basis

$$w_{1,1}, \dots, w_{a_1-1,1}, \dots, w_{k,k}, \dots, w_{a_k-1,k}, \dots, w_{i,i}, \dots, w_{a_i-1,i}$$

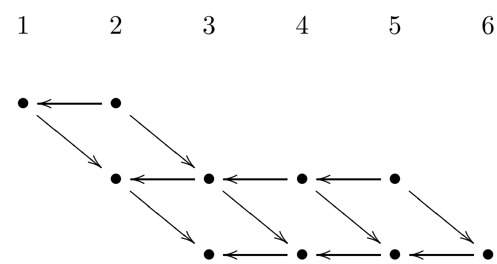
where $w_{j,k} \in N(A)_j$.

We define

$$(1) \qquad \begin{aligned} (j \to j-1)(w_{j,k}) &= w_{j-1,k} \\ (j \to j+1)(w_{j,k}) &= w_{j+1,k+1} \end{aligned}$$

Note that $N(A)$ has a 1-dimensional socle S_i, spanned by $w_{i,i}$.

Let us call the span of $w_{k,k}, \dots, w_{a_k-1,k}$ the kth "row" of $N(A)$ and let us call $N(A)_j$ the jth "column". So the kth row starts at column k and extends to column $a_k - 1$. This can be seen in the following picture of the module $N(\{3, 6, 7\})$.



Lemma 3.1. *Let $v = \dim(N(A))$. Then $\alpha_v = \{1, \dots, i\} - A$.*

Proof. Since we have an explicit basis for $N(A)$ it is easy to see that

$$\dim N(A)_j = |\{r \in \{1, \dots, i\} : r \leq j < a_r\}|$$

From this, the desired result follows immediately. $\square$

3.2. **The uniqueness theorem.** We will now show that every Λ-module with 1-dimensional socle is isomorphic to a Maya module.

We start by characterizing the dimension vectors of modules with 1-dimensional socle. If v is a dimension vector, we will extend v by defining $v_0 = 0 = v_n$ (this will eliminate some special cases below).

Lemma 3.2. *Let M be a module with socle S_i. Let $v = \dim(M)$. Then*

$$v_j = v_{j+1} \text{ or } v_j + 1 = v_{j+1}, \text{ for all } j < i \tag{2}$$

$$v_{j-1} = v_j \text{ or } v_{j-1} = v_j + 1, \text{ for all } j > i \tag{3}$$

Proof. Suppose that (2) does not hold for some $j < i$. Then either $v_j > v_{j+1}$ or $v_j + a = v_{j+1}$ for some $a > 1$.

Suppose that $v_j > v_{j+1}$. Then $\dim\ker(j \to j+1) > 0$.

Consider a non-zero element $w \in \ker(j \to j+1)$. Then

$$(j+1 \to j) \circ (j \to j+1)(w) = 0 \Rightarrow (j-1 \to j) \circ (j \to j-1)(w) = 0$$

But, $(j \to j-1)(w) \neq 0$, since M has no j-socle. Hence

$$\dim\ker(j-1 \to j) > 0.$$

Continuing in this manner we see that $\dim\ker(1 \to 2) > 0$. This means that M has 1-socle, a contradiction.

Now, suppose that $v_j + a = v_{j+1}$ for some $a > 1$. Assume $j+1 < i$. In this case,

$$\dim\ker(j+1 \to j) \geq a > 1$$

Let $w \in \ker(j+1 \to j)$, then

$$(j \to j+1) \circ (j+1 \to j)(w) = 0 \Rightarrow (j+2 \to j+1) \circ (j+1 \to j+2)(w) = 0$$

But $(j+1 \to j+2)(w) \neq 0$, since M has no $j+1$-socle. Therefore $(j+1 \to j+2)$ gives us an injective map from $\ker(j+1 \to j)$ to $\ker(j+2 \to j+1)$ and so

$$\dim\ker(j+2 \to j+1) \geq a$$

Continuing in this manner, when we reach i, we see that since the socle is one-dimensional, it must be the case that $\dim\ker\left((i \to i+1)|_{\ker(i \to i-1)}\right) = 1$ and hence we find

$$\dim\ker(i+1 \to i) \geq a - 1 > 0$$

Again, we consider an element $w \in \ker(i+1 \to i)$. Then

$$(i \to i+1) \circ (i+1 \to i)(w) = 0 \Rightarrow (i+2 \to i+1) \circ (i+1 \to i+2)(w) = 0$$

Again, since M does not have $i+1$-socle,

$$\dim\ker(i+2 \to i+1) > 0$$

Continuing in this manner, we see that $\dim\ker(n-1 \to n-2) > 0$, which implies that M has $(n-1)$-socle. This is a contradiction.

The proof of (3) follows similarly. □

Lemma 3.3. *Suppose that $v \in \mathbb{N}^I$ satisfies the condition (2) and (3). Then $\alpha_v = \{1, \dots, i\} - A$ for some i element subset of $\{1, \dots, n\}$, $A \neq \{1, \dots, i\}$.*

Proof. Let $\alpha_v = \sum_{j=1}^{n-1} v_j \alpha_j$, and let x_j be the j^{th} coordinate of α_v. Then for each $j = 1, \dots, n$, we have

$$x_j = 1 \iff v_j = v_{j-1} + 1,$$
$$x_j = -1 \iff v_j + 1 = v_{j-1},$$
$$x_j = 0 \iff v_j = v_{j-1}.$$

Also note that $x_j = 1 \Rightarrow j \leq i$ and $x_j = -1 \Rightarrow j > i$. So define

$$A := \{j \leq i : x_j = 0\} \cup \{j > i : x_j = 1\}$$

and then it is easily seen that A has the desired properties. □

Now we formulate and prove the uniqueness statement.

Theorem 3.4. *Let M be a module with socle S_i and dimension v. Let A be such that $\alpha_v = \{1, \ldots, i\} - A$. Then $M \cong N(A)$.*

This result is well-known to experts. For example, it follows from the fact that certain Nakajima quiver varieties are 0-dimensional. It can also be proved using the crystal structure on components of quiver varieties (due to Kashiwara-Saito [KS]). Here we prefer to give an elementary argument.

Proof. Our goal is to find a basis for M whose module structure matches the Maya module structure (1). Let $A = \{a_1 < \cdots < a_i\}$.

Let $w_{i,i} \in M_i$ be a basis for the socle of M. Assume that $a_i > i + 1$. We claim that there exists $w_{i+1,i} \in M_{i+1}$ such that $(i+1 \to i)(w_{i+1,i}) = w_{i,i}$.

Suppose that no such $w_{i+1,i}$ exists. From the proof of Lemma 3.2, we see that $w_{i,i}$ spans the kernel of $(i \to i+1)$. Hence if $w_{i,i}$ is not in the image of $(i+1 \to i)$, then $(i \to i+1) \circ (i+1 \to i)$ is an isomorphism. By the preprojective relations, this means that $(i+2 \to i+1) \circ (i+1 \to i+2)$ is an isomorphism. From the proof of Lemma 3.2, we know that $(i+2 \to i+1)$ is injective, so both $(i+2 \to i+1)$ and $(i+1 \to i+2)$ are isomorphisms. Hence $(i+1 \to i+2) \circ (i+2 \to i+1)$ is an isomorphism. Continuing in this fashion, we find that all $(j \to j+1)$ are isomorphisms for $j \geq i$ and so we see that $v_{i+1} = \cdots = v_n = 0$. This contradicts $a_i > i + 1$.

By a similar argument, there exist $w_{i+2,i}, \ldots, w_{a_i-1,i}$ such that

$$w_{i,i} \xleftarrow{i+1 \to i} w_{i+1,i} \xleftarrow{i+2 \to i+1} \cdots \xleftarrow{a_i-1 \to a_i-2} w_{a_i-1,i}.$$

Since $(i \to i-1)(w_{i,i}) = 0$, from the preprojective relations we find that $(k \to k+1)(w_{k,i}) = 0$ for all k. Thus $w_{i,i}, \ldots, w_{a_i-1,i}$ spans a submodule which we denote by N. Note that $N \cong N(\{1, \ldots, i-1, a_i\})$.

If $M = N$, then we are done. Suppose that $N \neq 0$ and consider the quotient module M/N. Since $\dim M/N = \dim M - \dim N$, we see that if $v' = \dim M/N$, then

$$\alpha_{v'} = \{1, \ldots, i-1\} - \{a_1, \ldots, a_{i-1}\}.$$

We claim that $\operatorname{soc}(M/N) = S_{i-1}$. As above, there exists $w \in M_{i-1}$ such that $(i-1 \to i)(w) = w_{i,i}$ and as above $(i-1 \to i-2)(w) = 0$. Hence $[w] \in \operatorname{soc}(M/N)$.

To see that there is no other socle, note that if $[u] \in \operatorname{soc}(M/N)_j$, then $(j \to j-1)(u) \in N$ and $(j \to j+1)(u) \in N$. Suppose that $j < i-1$, then $(j \to j+1)(u) \in N$ implies that $(j \to j+1)(u) = 0$ which implies that $u = 0$ since $(j \to j+1)$ is injective (as in the proof of Lemma 3.2). Suppose that $j = i-1$, then the injectivity of $(j \to j+1)$ forces $u = w$. If $j = i$, then the $[u] = 0$, since the kernel of $(i \to i-1)$ is spanned by $w_{i,i}$. Similarly if $j > i$, then $[u] = 0$, since $(j \to j-1)$ is injective (as in the proof of Lemma 3.2), so u must be a multiple of $w_{j,i}$.

Thus, we have shown that $\operatorname{soc}(M/N) = S_{i-1}$. Thus by the induction hypothesis, we see that $M/N \cong N(\{a_1, \ldots, a_{i-1}\})$ and we obtain a short exact sequence of Λ-modules

$$0 \to N \to M \to N(\{a_1, \ldots, a_{i-1}\}) \to 0.$$

Let us pick a vector space splitting. Thus combining the standard basis of $N(\{a_1, \ldots, a_{i-1}\})$ with the above basis of N, we obtain a basis $w_{k,l}$ for M with $l = 1, \ldots, i$ and $k = l, \ldots, a_l - 1$. This module structure with respect to this basis does not match (1), since extra terms involving the basis for N may enter into the result of applying quiver arrows to the basis elements of $N(\{a_1, \ldots, a_{i-1}\})$. Hence we will now adjust our basis.

In particular, for each $l = 1, \dots, i-1$ and $k = i+1, \dots a_l - 1$, we see that there is a scalar $c_{k,l}$ such that

$$(k \to k-1)(w_{k,l}) = w_{k-1,l} + c_{k,l} w_{k-1,i}$$

We may eliminate this scalar by setting $w'_{k,l} = w_{k,l} - c_{k,l} w_{k,i}$ for these (k,l) and $w'_{k,l} = w_{k,l}$ otherwise.

Next, note that $(i-1 \to i)(w_{i-1,i-1}) = 0$ in $N(\{a_1, \dots, a_{i-1}\})$ and thus $(i-1 \to i)(w'_{i-1,i-1}) = cw'_{i,i}$ in M for some scalar c. Since M has no $i-1$ socle, c is non-zero. Scaling all $w'_{k,l}$ by $1/c$ (for $l < i$), we may assume that $c = 1$. It then follows from the preprojective relations that

$$(k \to k+1)(w'_{k,i-1}) = w'_{k+1,i} \text{ for all } k = i, \dots, a_{i-1} - 1.$$

Now consider some $w'_{k,l}$ for $l < i-1$ and $k \geq i-1$. Then

$$(k \to k+1)(w'_{k,l}) = w'_{k+1,l+1} + c_l w'_{k+1,i}$$

for some scalar c_l. By the preprojective relations c_l depends only on l. Then we make the adjustment $w''_{k,l} = w'_{k,l} - c_l w'_{k,i-1}$ for all $k = i-1, \dots, a_{l-1} - 1$ and $w''_{k,l} = w'_{k,l}$ for all other (k,l).

After all these adjustments, we see that $w''_{k,l}$ satisfy the Maya module structure (1). Thus $M \cong N(A)$ as desired. □

3.3. **Computation of hom spaces.** Now we compute the space of homomorphisms between Maya modules.

Theorem 3.5. *Let A, B be i, j element subsets respectively. Then we have*

$$\begin{aligned} \dim \operatorname{Hom}(N(A), N(B)) = \#\ &\textit{of } r \in \{1, \dots, i\}, \textit{ such that } r \leq j < a_r, \\ &\textit{and } a_{r-l} \leq b_{j-l} \textit{ for } l = 0, \dots, r-1 \end{aligned}$$

where $A = \{a_1 < \cdots < a_i\}$ and $B = \{b_1 < \cdots < b_j\}$.

Proof. Let

$$R := \{ r \in \{1, \dots, i\} : r \leq j < a_r, \text{ and } a_{r-l} \leq b_{j-l} \text{ for } l = 0, \dots, r-1 \}$$

We construct a map $\varphi : R \to \operatorname{Hom}(N(A), N(B))$, and then show that it gives a bijection between R and a basis for $\operatorname{Hom}(N(A), N(B))$. This will yield the desired result.

For simplicity of notation, we will use $w_{k,l}$ for the basis for $N(A)$ and $w'_{k,l}$ for the basis for $N(B)$.

For each $r \in R$, let us define $\varphi(r) = \phi_r$ to be the homomorphism which takes the rth row of $N(A)$ to the bottom row of $N(B)$ and then extended to higher rows in the obvious way. More explicitly, we define ϕ_r by

$$\phi_r(w_{k,r-l}) = \begin{cases} w'_{k,j-l}, & \text{if } l \geq 0, \text{ and } k \geq j-l \\ 0, & \text{otherwise} \end{cases}$$

Such a $w'_{k,j-l}$ will always exist since $j - l \leq k < a_{r-l} \leq b_{j-l}$. A simple check using the structure of Maya modules (1) shows that ϕ_r is a homomorphism.

Now, suppose that ψ in any element of $\operatorname{Hom}(N(A), N(B))$. Since we have explicit bases for $N(A)$ and $N(B)$ we may consider the matrix coeffients involving $w'_{j,j}$, the generator of the socle of $N(B)$. ψ takes $N(A)_j$ to $N(B)_j$ so for each $r \in \{1, \dots, i\}$ such that $r \leq j < a_r$, we get a matrix coefficient s_r, such that

$$\psi(w_{j,r}) = s_r w'_{j,j} + \cdots .$$

Note that if all the s_r are zero, then $\psi = 0$. This is because every submodule of $N(B)$ must contain $w'_{j,j}$ (since $w'_{j,j}$ spans the socle of $N(B)$) and so any non-zero homomorphism from $N(A)$ to $N(B)$ must hit $w'_{j,j}$. Thus, the collection s_r completely determines ψ.

Also note that if $r \notin R$, then $a_{r-l} > b_{j-l}$ for some l. This means that we can find some non-zero $w \in N(A)$ and $p \in \Lambda$ such that $pw = w_{j,r}$ but $\psi(w) = 0$ (in fact we can choose $w = w_{a_{r-l}-1, r-l}$). Hence for $r \notin R$, we see that $s_r = 0$.

Combining these observations, we see that $\psi = \sum_{r \in R} s_r \phi_r$. Thus the ϕ_r span $\mathrm{Hom}(N(A), N(B))$. These ϕ_r are linearly independent since ϕ_r vanishes on $a_{j,r'}$ for $r > r'$. Thus the ϕ_r form a basis for $\mathrm{Hom}(N(A), N(B))$ as desired. □

3.4. **Connection with MV polytopes.** We now make the connection between Theorem 3.5 and MV polytopes.

For each subset B of $\{1, \ldots, n\}$ of size i, we may consider the truncated permutahedron $P(B)$ which is defined as

$$P(B) := conv(\{1_C - 1_{\{1,\ldots,j\}} : \ C \text{ is a subset of } \{1, \ldots, n\} \text{ of size } j \text{ and } C \le B \ \})$$

These polytopes $P(B)$ are relevant since Naito-Sagaki [NS] have shown that these are the MV polytopes associated to the vertices of the crystal corresponding to the minuscule representation $\Lambda^i \mathbb{C}^n$. These vertices are precisely labelled by subsets B of size i.

Corollary 3.6. *For each subset B of $\{1, \ldots, n\}$, the max value of $\langle 1_A, \rangle$ on the polytope $P(B)$ is given by* $\dim \mathrm{Hom}(N(A), N(B))$.

Proof. Assume for simiplicity that $i \le j$. A similar proof holds in the $i > j$ case.

By Theorem 3.5, $\dim \mathrm{Hom}(N(A), N(B)) = r - s$ where r is the maximal element of $\{1, \ldots, i\}$ such that $a_{r-l} \le b_{j-l}$ for $j = 0, \ldots, r-1$ and $s = |\{1, \ldots, j\} \cap A|$.

Now, we claim that $r = \max_{C \le B} |C \cap A|$. First note that if we choose C to be the smallest possible j element subset of $\{1, \ldots, n\}$ such that $\{a_1, \ldots, a_r\} \subset C$, then $C \le B$ and $|C \cap A| \ge r$. On the other hand, for any $C \le B$, we claim that $|C \cap A| \le r$. To see why this is the case, note that by the definition of r, not all the inequalities

$$a_{r+1} \le b_j, a_r \le b_{j-1}, \ldots, a_1 \le b_{j-r}. \tag{4}$$

can hold. So now suppose that $C \le B$ and $C \cap A$ contains at least $r+1$ elements. Let us choose $r+1$ of these elements and order them $a_{i_1} < \cdots < a_{i_{r+1}}$. Then since $C \le B$, we find that

$$a_{i_{r+1}} \le b_j, \ldots, a_{i_1} \le b_{j-r}.$$

But since $a_{i_l} \ge a_l$ for all l, this implies that all the inequalities (4) hold — a contradiction. Hence we conclude that $r = \max_{C \le B} |C \cap A|$.

Thus

$$\begin{aligned} \dim \mathrm{Hom}(N(A), N(B)) = r - s &= \max_{C \le B} |C \cap A| - |\{1, \ldots, j\} \cap A| \\ &= \max_{C \le B} \langle 1_A, 1_C - 1_{\{1,\ldots,j\}} \rangle \end{aligned}$$

as desired. □

4. Description of irreducible components

4.1. **Savage's description of the components.** Alistair Savage has given a description of the components of $Rep(w)_v$ in terms of tableaux. We would like to reformulate his description in terms of Hom spaces.

Let $Tab_\mu(\lambda)$ denote the set of semistandard Young tableaux (SSYT) of shape λ and content μ. If X is a box in a SSYT T, then we will write $r(X)$ for the row of X and $c(X)$ for the content of X.

For each $T \in Tab(\lambda)_\mu$, Savage has identified a component C_T of $Rep(w)_v$.

Let $T \in Tab(\lambda)_\mu$. A Λ-module is said to be of type T if there exists a basis for M with the following properties. For each box X in T, there are vectors

$$w^X_{r(X)}, \dots, w^X_{c(X)-1} \in M$$

and the collection of all these vectors (over all boxes X) forms a basis for M. Moreover, we have

$$(j \to j-1)(w^X_j) = w^X_{j-1}, \quad (j \to j+1)(w^X_j) = \sum_Y d^X_Y w^Y_{j+1} \tag{5}$$

for some scalars d^X_Y, where the sum varies over all those boxes Y such that $r(Y) < r(X) \leq c(Y) < c(X)$.

Let $C_T = \overline{\{x \in Rep(w)_v : M_x \text{ is of type } T\}}$ denote the closure of the locus of those modules of type T.

Theorem 4.1 ([Sav, Section 5]). *C_T is a component of $Rep(w)_v$ and this provides a bijection between the components of $Rep(w)_v$ and $Tab(\lambda)_\mu$.*

4.2. **Description of components by Hom spaces.** We would like to reformulate Savage's description. The key will be the following generalization of Theorem 3.5. A connnected subset of $\{1, \dots, n\}$ is one of the form $\{t-i+1, t-i+2, \dots, t\}$.

Theorem 4.2. *Let M be a module of type T and let $A = \{t-i+1, \dots, t\}$ be a connected subset of $\{1, \dots, n\}$. Then*

$$\dim \operatorname{Hom}(M, N(A)) = \#\ \textit{of boxes } X \textit{ in } T\textit{, such that } r(X) \leq i < c(X) \leq t.$$

Proof. The idea is similar to the proof of Theorem 3.5.

To each box X of T in the above set, we can define a homomorphism $\phi_X : M \to N(A)$ by taking the row indexed by X to the bottom row of $N(A)$. We then extend to all of M.

More explicitly, we define

$$\phi_X(w^X_k) = w'_{k,i}$$

for $k \geq i$ (note that such $w'_{k,i}$ exists since $i \leq k < c(X) \leq t = a_i$). We define $\phi_X(w^X_k) = 0$ for $k < i$. We also define $\phi_X(w^Y_k) = 0$ for all $Y \neq X$ with $r(Y) \geq r(X)$ or $c(Y) \geq c(X)$.

Now we proceed to define $\phi_X(w^Y_k)$ for those boxes Y with $r(Y) < r(X)$ and $c(Y) < c(X)$. We do so by an inductive procedure on $r(Y)$. Suppose that Y is a box with $r(Y) = r(X) - 1$ and $c(Y) < c(X)$. Then we define

$$\phi_X(w^Y_k) = d^X_Y w'_{k,i-1}.$$

Note that such $w'_{k,i-1}$ exists since $k < c(Y) < c(X) \leq t$ and so $k < t-1 = a_{i-1}$.

Next, suppose that Y is a box with $r(Y) = r(X) - 2$ and $c(Y) < c(X)$. Then we define

$$\phi_X(w^Y_k) = d^X_Y w'_{k,i-1} + \sum_Z d^Z_Y d^X_Z w'_{k,i-2},$$

where the sum ranges over all those boxes Z such that $r(Z) = r(X) - 1$ and $c(Y) < c(Z) < c(X)$.

Continuing in this fashion, we define ϕ_X on all of M. The structure of the module as given in (5) ensures that ϕ_X is a Λ-module homomorphism.

The fact that these ϕ_X form a basis for $\operatorname{Hom}(M, N(A))$ follows along the same lines as in the proof of Theorem 3.5. $\square$

We now combine this result and Savage's theorem. For each $A = \{t-i+1, \dots, t\}$, we define a constructible function

$$f_A : Rep(w)_v \to \mathbb{N}$$
$$x \mapsto \dim(\mathrm{Hom}(M_x, N(A))$$

Since this is a constructible function it takes a constant value on a constructible dense subset of each component of $Rep(w)_v$. For each component $Z \subset Rep(w)_v$, let $f_A(Z)$ denote this constant value.

Also, for each $T \in Tab(\lambda)_\mu$, let $g_A(T)$ denote the number of boxes X in T such that $r(X) \leq i < c(X) \leq t$. Note that the collection $\{g_A(T)\}$ (where A ranges over all connected subsets) determines T.

Theorem 4.3. *For each component $Z \subset Rep(w)_v$, there exists a tableau $T \in Tab(\lambda)_\mu$ such that $f_A(Z) = g_A(T)$ for all connected subsets $A \subset \{1, \dots, n\}$. This provides a bijection between the components of $Rep(w)_v$ and the SSYT of shape λ and filling μ.*

Proof. Theorem 4.2 shows that if M is of type T, then $f_A(M) = g_A(T)$. Theorem 4.1 shows that for each component Z, there exists a unique tableau T such that there is a dense subset of Z consisting of modules of type T. Combining these two results, we obtain the desired result. □

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