ON THE CONNECTIVITY OF THE REALIZATION SPACES OF LINE ARRANGEMENTS

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ABSTRACT. We prove that under certain combinatorial conditions, the realization spaces of line arrangements on the complex projective plane are connected. We also give several examples of arrangements with eight, nine and ten lines which have disconnected realization spaces.

1. INTRODUCTION

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a line arrangement in the complex projective plane $\mathbb{P}^2$ and denote by $M = M(\mathcal{A})$, the corresponding arrangement complement. An arrangement $\mathcal{A}$ determines the incidence data $I(\mathcal{A})$ (equivalently the intersection lattice $L(\mathcal{A})$). This combinatorial data possesses the topological information, e.g. the cohomology algebra of M are determined by the intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$. However, not all geometric information is determined by the incidence $I(\mathcal{A})$. In 1993, Rybnikov [11] posed an example of arrangements $\mathcal{A}_1$, $\mathcal{A}_2$ which have the same incidence but their fundamental groups are not isomorphic (see also [2]). Nevertheless, in many cases the topological structures are determined by the combinatorial ones. They includes:

(1) Combining results of Fan [5],[6], Garber, Teicher and Vishne [7] and an unpublished work by Falk and Sturmfels (see [3]), if $n \leq 8$, then the fundamental group $\pi_1(M(\mathcal{A}))$ is determined by the combinatorics.
(2) In 2009, Nazir-Raza [9] introduced a complexity hierarchy of lattice: class $\mathcal{C}_k$, and proved that if $\mathcal{A}$ is in $\mathcal{C}_{\leq 2}$, then the cohomology $H^*(M, \mathcal{L})$ with coefficients in a rank one local system $\mathcal{L}$, is combinatorially determined.

In this paper, we generalize these results by using the connectivity of the realization space $\mathcal{R}(I)$ of an incidence relation I. Indeed, the connectivity of realization spaces is related to the topology of the complements by Randell's lattice isotopy theorem.

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Theorem 1.1. *(Randell* [10]*) If two arrangements are connected by a one-parameter family of arrangements which have the same lattice, then the complements are diffeomorphic, hence of the same homotopy type.*

Once the connectivity of the realization space $\mathcal{R}(I)$ is proved, then for any arrangements $\mathcal{A}_1$, $\mathcal{A}_2$ having the same incidences $I(\mathcal{A}_1) = I(\mathcal{A}_2) = I$, we can conclude that $M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$ by Theorem 1.1. Since the realization space $\mathcal{R}(I)$ is a (quasi-projective) algebraic variety over $\mathbb{C}$, the irreducibility of $\mathcal{R}(I)$ implies the connectivity. (Note an irreducible algebraic variety is connected in the classical topology. For the proof, see [12] chapter VII.) For our purposes, the following is useful.

Corollary 1.2. *If $\mathcal{R}(I)$ is irreducible (in Zariski topology) and $I(\mathcal{A}_1) = I(\mathcal{A}_2) = I$, then $M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$.*

As far as the authors know, a systematic study of the connectivity of the realization space $\mathcal{R}(I)$ of line arrangements was initiated by Jiang and Yau [8] and subsequently by Wang and Yau [13]. They introduce the notion of graph associated to a line arrangement and under certain combinatorial conditions ("nice" and "simple" arrangements), it is proved that $\mathcal{R}(I)$ is connected. In particular, the structure of fundamental groups are combinatorially determined. Explicit presentations for a class of combinatorially determined fundamental groups are also studied in [4].

The purpose of this paper is to develop these ideas more. We will prove the connectivity of $\mathcal{R}(I)$ for "inductively connected arrangement" (Definition 3.4) and "$\mathcal{C}_{\leq 3}$ of simple type" (Definition 3.13). The relations between "nice"([8]), "simple"([13]) and our classes are not clear at the moment. However up to 8 lines, we will prove that all arrangements except for MacLane arrangement are contained in our class (§4, Proposition 4.6). We also give a complete classification of disconnected realization space up to 9 lines in §5.

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2. Generality on the realization spaces of arrangements

From now, we assume that $\mathcal{A}$ contains $H_i, H_j, H_k \in \mathcal{A}$ such that $H_i \cap H_j \cap H_k = \emptyset$ (thus excluding $n < 3$ and pencil cases). Let $H_i \in \mathcal{A}$. H_i is defined by

$$H_i = \{(x : y : z) \in \mathbb{P}^2 \mid a_i x + b_i y + c_i z = 0\}.$$

We may consider $(a_i : b_i : c_i) \in (\mathbb{P}^2)^*$ as an element of dual projective plane. We call a triple (H_i, H_j, H_k) an *intersecting triple* if $H_i \cap H_j \cap H_k \neq \emptyset$, or equivalently,

$$\det(H_i, H_j, H_k) := \det \begin{pmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{pmatrix} = 0.$$

Definition 2.1. Define the *Incidence* of $\mathcal{A}$ by

$$I(\mathcal{A}) := \left\{ \{i,j,k\} \in \binom{[n]}{3} \,\middle|\, H_i \cap H_j \cap H_k \neq \emptyset \right\},$$

where $\binom{[n]}{3} = \{\{i,j,k\} \mid i,j,k \in \{1,2,\ldots,n\}$ mutually distinct$\}$.

The set of all arrangements which have prescribed incidence I is called the realization space of the incidence I. Let us define

$$\mathcal{R}(I) := \left\{ (H_1,\ldots,H_n) \in ((\mathbb{P}^2)^*)^n \,\middle|\, \begin{array}{l} H_i \neq H_j \text{ for } i \neq j, \text{ and} \\ \det(H_i,H_j,H_k) = 0 \text{ for } \{i,j,k\} \in I, \\ \det(H_i,H_j,H_k) \neq 0 \text{ for } \{i,j,k\} \notin I \end{array} \right\}.$$

It can be seen that $(H_1,\ldots,H_n)$ and $(gH_1,\ldots,gH_n)$ for $g \in PGL_3(\mathbb{C})$ have the same incidence. Hence $PGL_3(\mathbb{C})$ acts on $\mathcal{R}(I)$. Now, we will discuss the irreducibility of $\mathcal{R}(I)$.

Definition 2.2. Define

$$\overline{\mathcal{R}}(I) := \left\{ (H_1,\ldots,H_n) \in ((\mathbb{P}^2)^*)^n \,\middle|\, \begin{array}{l} H_i \neq H_j \text{ for } i \neq j, \text{ and} \\ \det(H_i,H_j,H_k) = 0 \text{ for } \{i,j,k\} \in I \end{array} \right\}.$$

Example 2.3. Consider the incidence $I = \{\{1,2,3\}\}$ of 4 lines $\{H_1,H_2,H_3,H_4\}$. Then

$$\mathcal{R}(I) := \left\{ (H_1,\ldots,H_4) \in ((\mathbb{P}^2)^*)^4 \,\middle|\, \begin{array}{l} H_i \neq H_j \text{ for } i \neq j, \text{ and} \\ \det(H_1,H_2,H_3) = 0 \\ \det(H_1,H_2,H_4) \neq 0 \\ \det(H_1,H_3,H_4) \neq 0 \\ \det(H_2,H_3,H_4) \neq 0 \end{array} \right\},$$

and,

$$\overline{\mathcal{R}}(I) := \left\{ (H_1,\ldots,H_4) \in ((\mathbb{P}^2)^*)^4 \,\middle|\, \begin{array}{l} H_i \neq H_j \text{ for } i \neq j, \text{ and} \\ \det(H_1,H_2,H_3) = 0 \end{array} \right\}.$$

By definition, $\mathcal{R}(I)$ is a Zariski open subset of $\overline{\mathcal{R}}(I)$. Hence, $\overline{\mathcal{R}}(I)$ is irreducible implies that $\mathcal{R}(I)$ is irreducible and hence $\mathcal{R}(I)$ is connected (unless $\mathcal{R}(I)$ is empty).

Proposition 2.4. *Assume that $\overline{\mathcal{R}}(I)$ is irreducible. Then $I = I(\mathcal{A}_1) = I(\mathcal{A}_2)$ implies that $M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$.*

3. Connectivity and field of realization

In this section we establish several conditions on the incidence I for the realization space $\mathcal{R}(I)$ to be connected. We also discuss field of definition, since in the case of ≤ 9 lines, it is related to the connectivity of $\mathcal{R}(I)$.

Definition 3.1. Let $\mathcal{A}$ be a line arrangement on $\mathbb{P}^2_{\mathbb{C}}$. Denote by

$$\mathrm{mult}(\mathcal{A}) = \{p \in \mathbb{P}^2 \mid p \text{ is contained in} \geq 3 \text{ lines of } \mathcal{A}\}.$$

We call $p \in \mathrm{mult}(\mathcal{A})$ a multiple point.

The next lemma will be used frequently.

Lemma 3.2. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$. Assume that $|H_n \cap \mathrm{mult}(\mathcal{A})| \leq 2$. Set $\mathcal{A}' = \{H_1, \dots, H_{n-1}\}$, $I = I(\mathcal{A})$ and $I' = I(\mathcal{A}')$. If $\mathcal{R}(I')$ is irreducible, then $\mathcal{R}(I)$ is also irreducible.*

Proof. Let $\mu = |H_n \cap \mathrm{mult}(\mathcal{A})|$. By assumption $\mu \in \{0, 1, 2\}$. We claim that $\mathcal{R}(I)$ is a Zariski open subset of $\mathbb{P}^{2-\mu}_{\mathbb{C}}$-fibration over $\mathcal{R}(I')$. Consider the projection $\pi : \mathcal{R}(I) \to \mathcal{R}(I')$ defined as $(H_1, \dots, H_n) \mapsto (H_1, \dots, H_{n-1})$. Let $p \in H_n \cap \mathrm{mult}(\mathcal{A})$. Then p is a (possibly normal crossing) intersection point of $\mathcal{A}' = \mathcal{A} \setminus H_n$.

Case 1: $\mu = 2$. Let $p_1, p_2 \in H_n$ be multiple points of $\mathcal{A}$. In this case, H_n can be uniquely determined by $\mathcal{A}'$ as H_n is the line connecting p_1 and p_2. Hence π is an inclusion $\mathcal{R}(I) \hookrightarrow \mathcal{R}(I')$. The defining conditions of $\mathcal{R}(I)$ concerning H_n other than "$p_1, p_2 \in H_n$" are of the form $\det(H_i, H_j, H_n) \neq 0$, that is Zariski open conditions. Thus, in this case, $\pi : \mathcal{R}(I) \to \mathcal{R}(I')$ is a Zariski open embedding.

Case 2: $\mu = 1$. In this case, $H_n \cap \mathrm{mult}(\mathcal{A}) = \{p\}$. Suppose $p \in H_1, \dots, H_t$ and $p \notin H_{t+1}, \dots, H_{n-1}$. Then the realization space can be described as

$$\mathcal{R}(I) = \left\{ (H', H_n) \in \mathcal{R}(I') \times (\mathbb{P}^2)^* \;\middle|\; \begin{array}{l} H_i \neq H_n, \text{ for } 1 \leq i \leq n-1, \\ \det(H_i, H_j, H_n) = 0 \text{ for } 1 \leq i < j \leq t, \\ \det(H_i, H_j, H_n) \neq 0 \text{ for others} \end{array} \right\}.$$

Note that the Zariski closed condition in the second line ($\det(H_i, H_j, H_n) = 0$) indicates that H_n goes through $p = H_1 \cap \cdots \cap H_t$, which is equivalent to say that H_n is contained in the dual projective line $p^{\perp}(\simeq \mathbb{P}^1) \subseteq (\mathbb{P}^2)^*$. Hence, $\mathcal{R}(I)$ is a Zariski open subset of $\mathbb{P}^1$-fibration over $\mathcal{R}(I')$.

Case 3: $\mu = 0$. In this case H_n is generic to $\mathcal{A}'$. Hence $\mathcal{R}(I)$ is a Zariski open subset of $\mathcal{R}(I') \times (\mathbb{P}^2)^*$. □

Lemma 3.2 allows us to prove the irreducibility of $\mathcal{R}(I)$ by the inductive arguments.

Proposition 3.3. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be lines on $\mathbb{P}^2_{\mathbb{C}}$. Define the subarrangements $\mathcal{A}_t = \{H_1, \dots, H_t\}$ ($t = 1, \dots, n$). If $|H_t \cap \mathrm{mult}(\mathcal{A}_t)| \leq 2$ for all t, then $\mathcal{R}(I(\mathcal{A}))$ is irreducible.*

Proof. Induction on n using Lemma 3.2. □

Definition 3.4. A line arrangement $\mathcal{A}$ is said to be *inductively connected* ("i.c." for brevity) if there exists an appropriate numbering $\mathcal{A} = \{H_1, \dots, H_n\}$ of $\mathcal{A}$ which satisfies the assumption of Proposition 3.3.

Inductive connectedness is a combinatorial property. We also say the incidence $I = I(\mathcal{A})$ is i.c. By Proposition 3.3, $R(I)$ is irreducible for i.c. incidence I.

Corollary 3.5. *If $|\mathrm{mult}(\mathcal{A}) \cap H| \leq 2$ for all $H \in \mathcal{A}$, then $\mathcal{A}$ is i.c., hence $R(I(\mathcal{A}))$ is irreducible.*

Corollary 3.6. *If $\mathcal{R}(I(\mathcal{A}))$ is disconnected, then there exists subarrangement $\mathcal{A}' \subset \mathcal{A}$ such that*

$$|\mathrm{mult}(\mathcal{A}') \cap H| \geq 3,$$

for all $H \in \mathcal{A}'$.

Proof. If not, $\mathcal{A}$ is i.c. for any ordering. □

Remark 3.7. It is easily seen that if the characteristic of the field is $\neq 2$ and $|\mathcal{A}| \leq 7$, every line arrangement is an i.c. arrangement. Obviously the set of all $\mathbb{F}_2$-lines on $\mathbb{P}^2_{\mathbb{F}_2}$ is not i.c. In the case of characteristic zero, MacLane arrangement (Example 4.3) is the smallest one which is not i.c.

Example 3.8. Let $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) be a line arrangement defined as left of Figure 1 (resp. right). Then $\mathcal{A}_1$ is i.c., but $\mathcal{A}_2$ is not i.c. (Each line $H \in \mathcal{A}_2$ has at least 3 multiple points.)

Let $K \subset \mathbb{C}$ be a subfield, I an incidence. The incidence I is realizable over the field K if the the set of K-valued points $\mathcal{R}(I)(K)$ is nonempty. (Equivalently, there exists an arrangement $\mathcal{A}$ with the coefficients of defining linear forms in K satisfying $I = I(\mathcal{A})$.) The next Lemma can also be proved similarly as Lemma 3.2.

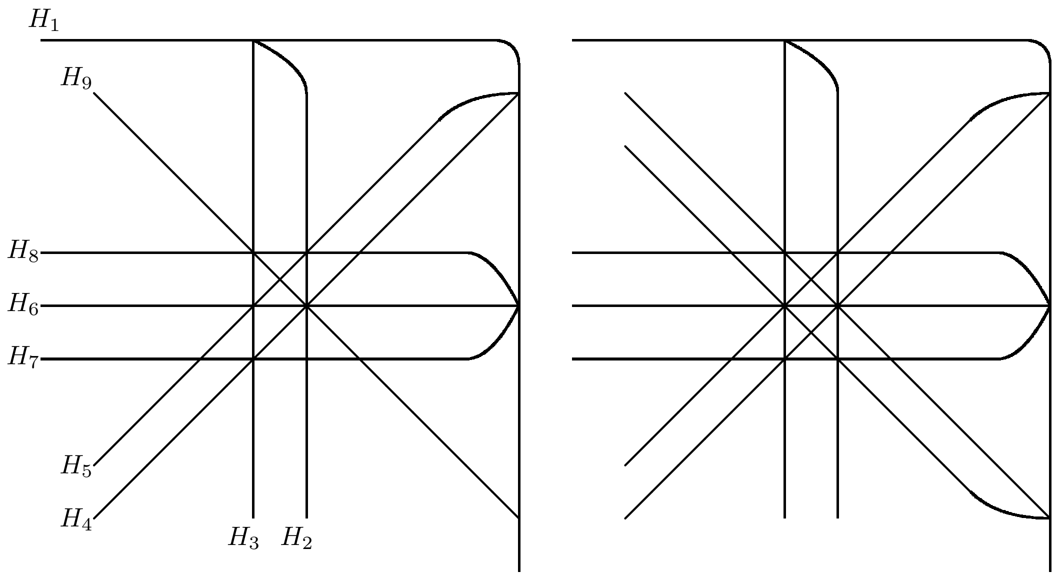

FIGURE 1. An i.c. arrangement $\mathcal{A}_1$ and non i.c. arrangement $\mathcal{A}_2$. Both are $\mathcal{C}_3$ of simple type.

Proposition 3.9. *Under the assumption of Lemma 3.2. If the K-valued points $\mathcal{R}(I')(K)$ is Zariski dense in $\mathcal{R}(I')(\mathbb{C})$, then $\mathcal{R}(I)(K)$ is Zariski dense in $\mathcal{R}(I)(\mathbb{C})$. In particular, $\mathcal{R}(I)(K) \neq \emptyset$, I is realizable over K. Every i.c. arrangement is realizable over $\mathbb{Q}$.*

Next we discuss connectivity of $\mathcal{R}(I)$ for another type of incidence.

Definition 3.10. Let k be a non-negative integer. We say that a line arrangement $\mathcal{A}$ (or its incidence $I(\mathcal{A})$) is of type $\mathcal{C}_k$ if k is the minimal number of lines in $\mathcal{A}$ containing all the multiple points.

For instance $k = 0$ corresponds to nodal arrangements, while $k = 1$ corresponds to the case of a nodal *affine* arrangement. Note that $k = k(\mathcal{A})$ is combinatorially defined, i.e. depends only on the intersection lattice $L(\mathcal{A})$.

Theorem 3.11. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$ of class $\mathcal{C}_{\leq 2}$ (i.e., either $\mathcal{C}_0, \mathcal{C}_1$ or $\mathcal{C}_2$). Then $\mathcal{A}$ is i.c. In particular, the realization space $\mathcal{R}(I(\mathcal{A}))$ is irreducible.*

Proof. By assumption, we may say that all multiple points are on $H_1 \cup H_2$. For $i \geq 3$, as $|H_i \cap (H_1 \cup H_2)| \leq 2$, there are at most two multiple points on H_i. Hence the subarrangements $\mathcal{A}_t := \{H_1, \dots, H_t\}$ satisfy the assumption of Proposition 3.3. Thus $\mathcal{R}(I(\mathcal{A}))$ is irreducible. $\square$

Remark 3.12. Under the same assumption with Theorem 3.11, using Proposition 3.9, we can prove that $I(\mathcal{A})$ is realizable over $\mathbb{Q}$.

The irreducibility of the realization spaces are not guaranteed for class $\mathcal{C}_3$ in general (see Example 5.1). Now we introduce a subclass of $\mathcal{C}_3$.

Definition 3.13. Let $\mathcal{A}$ be an arrangement of type $\mathcal{C}_3$. Then $\mathcal{A}$ is called $\mathcal{C}_3$ of simple type if there are $H_1, H_2, H_3 \in \mathcal{A}$ such that all multiple points are in $H_1 \cup H_2 \cup H_3$ and one of the following holds:

(i): $H_1 \cap H_2 \cap H_3 = \emptyset$ and there is only one multiple point on $H_1 \setminus (H_2 \cup H_3)$;
(ii): $H_1 \cap H_2 \cap H_3 \neq \emptyset$.

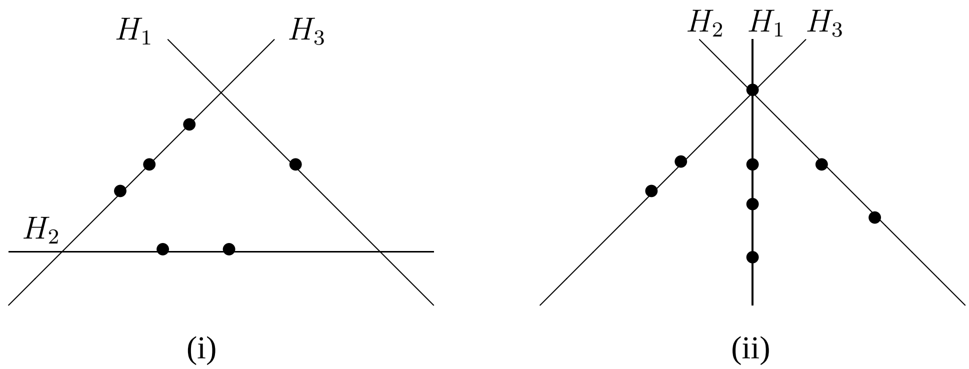

FIGURE 2. $\mathcal{C}_3$ of simple type

Example 3.14. The both line arrangements defined in Figure 1 are $\mathcal{C}_3$ of simple type. (E.g. $\mathrm{mult}(\mathcal{A}) \subset H_1 \cup H_2 \cup H_3$.)

Theorem 3.15. *Let $\mathcal{A}$ be an arrangement of $\mathcal{C}_3$ of simple type. Then $\mathcal{R}(I(\mathcal{A}))$ is irreducible.*

Proof. The proof is divided into two parts according to (i) and (ii) of the definition of $\mathcal{C}_3$ of simple type.

Case: (i). By the assumption, there exist $H_1, H_2, H_3 \in \mathcal{A}$ which satisfy the condition (i). Let $p \in H_1 \setminus (H_2 \cup H_3)$ be the unique multiple point. Let us assume that $H_4, \dots, H_t$ contain p and $H_{t+1}, \dots, H_n$ do not contain p. For $i \geq t+1$, H_i has at most two multiple points. By Lemma 3.2, it suffices to prove the irreducibility for $\mathcal{A}' = \{H_1, \dots, H_t\}$. However in this case, there are at most two multiple points: one is p and the other possibility is $H_2 \cap H_3$. Hence by Theorem 3.11, $\mathcal{R}(I(A'))$ is irreducible and so is $\mathcal{R}(I(\mathcal{A}))$.

Case: (ii). By the assumption, there exist $H_1, H_2, H_3 \in \mathcal{A}$ which satisfy the condition (ii) of the definition. Let $O = H_1 \cap H_2 \cap H_3$. If H_i $(i \geq 4)$ passes through O, then there is only one multiple point on H_i. Thus, by Lemma 3.2, the irreducibility of $\mathcal{R}(I(\mathcal{A}))$ is reduced to $\mathcal{R}(I(\mathcal{A}'))$, where $\mathcal{A}' = H_1 \cup H_2 \cup H_3 \cup \bigcup_{O \notin H_j} H_j$. We shall prove the irreducibility of $\mathcal{R}(I(\mathcal{A}'))$ by describing $\overline{\mathcal{R}}(I(\mathcal{A}))/PGL_3(\mathbb{C})$ explicitly. By the $PGL_3(\mathbb{C})$-action, we may fix as follows: $H_1 = \{(x:y:z) \mid x = 0\}, H_2 = \{(x:y:z) \mid x = z\}$ and $H_3 = \{(x:y:z) \mid z = 0\}$, so $O = H_1 \cap H_2 \cap H_3 = (0:1:0)$. We list all intersections on $H_i \setminus \{O\}$, (i=1, 2, 3):

$$P_\alpha(0:a_\alpha:1) \in H_1, (\alpha = 1, \dots, r, a_\alpha \in \mathbb{C}),$$

$$Q_\beta(1:b_\beta:1) \in H_2, (\beta = 1, \dots, s, b_\beta \in \mathbb{C}),$$

$$R_\gamma(1:c_\gamma:0) \in H_3, (\gamma = 1, \dots, t, c_\gamma \in \mathbb{C}).$$

Every line H_i $(i \geq 4)$ in $\mathcal{A}'$, can be described as a line connecting P_{α_i} and Q_{β_j}. Hence, the quotient space $\mathcal{R}(I(\mathcal{A}))/PGL_3(\mathbb{C})$ can be embedded in the space $\mathbb{C}^{r+s+t} = \{(a_\alpha, b_\beta, c_\gamma)\}$. (More precisely, here we consider $X := \mathbb{C} \times \mathbb{C}^* \times \mathcal{R}(I(\mathcal{A}))/PGL_3(\mathbb{C})$. Because we fix only H_1, H_2, H_3 and the isotropy subgroup is $\{g \in PGL_3(\mathbb{C}) \mid gH_i = H_i, i = 1, 2, 3\} \simeq \mathbb{C} \times \mathbb{C}^*$.) Thus, we can describe the realization space by using the parameters $a_\alpha, b_\beta, c_\gamma$.

Suppose H_i $(i \geq 4)$ passes through $P_{\alpha_i}, Q_{\beta_i}, R_{\gamma_i}$. These three points are collinear if and only if

$$\det \begin{pmatrix} 0 & a_{\alpha_i} & 1 \\ 1 & b_{\beta_i} & 1 \\ 1 & c_{\gamma_i} & 0 \end{pmatrix} = a_{\alpha_i} - b_{\beta_i} + c_{\gamma_i} = 0.$$

Collecting these linear equations together, we have

$$(a_1, \dots, a_r, b_1, \dots, b_s, c_1, \dots, c_t) \cdot A = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

where A is a $(r + s + t) \times (n - 3)$ matrix with entries ± 1 or 0. Thus the space X can be described as

$$X = \left\{ (a_\alpha, b_\beta, c_\gamma) \in \mathbb{C}^{r+s+t} \;\middle|\; \begin{array}{l} (a_\alpha, b_\beta, c_\gamma) \cdot A = 0, \\ a_\alpha \neq a_{\alpha'}, b_\beta \neq b_{\beta'}, c_\gamma \neq c_{\gamma'}, \\ \text{and other Zariski open conditions.} \end{array} \right\}.$$

Since $\ker A$ is isomorphic to $\mathbb{C}^K$ for some $K \geq 0$, the Zariski open subset $X \subset \mathbb{C}^K$ is irreducible. $\square$

Thus we have proved that if $\mathcal{A}$ is either in the class $\mathcal{C}_{\leq 2}$ or $\mathcal{C}_3$ of simple type ("$\mathcal{C}_{\leq 3}$ *of simple type*" for short), $\mathcal{R}(I(\mathcal{A}))$ is connected. As is mentioned, there are arrangements in $\mathcal{C}_3$ of non-simple type which have disconnected realization spaces (Example 5.1).

By lattice isotopy theorem, we have

Corollary 3.16. *Let $\mathcal{A}_1, \mathcal{A}_2$ be arrangements in $\mathbb{P}^2$ of $\mathcal{C}_{\leq 3}$ of simple type. If $I(\mathcal{A}_1) = I(\mathcal{A}_2)$, then the pairs $(\mathbb{P}^2, \cup_{H\in\mathcal{A}_1} H)$ and $(\mathbb{P}^2, \cup_{H\in\mathcal{A}_2} H)$ are homeomorphic.*

Remark 3.17. Under the same assumption with Theorem 3.15, $\mathcal{R}(I(\mathcal{A}))(\mathbb{Q})$ is Zariski dense in $\mathcal{R}(I(\mathcal{A}))(\mathbb{C})$, hence realizable over $\mathbb{Q}$. The proof is similar. Case (i) uses Proposition 3.9 and in case (ii), we note that the matrix A is with $\mathbb{Q}$-coefficients. Hence $\ker A$ has $\mathbb{C}$-valued points if and only if it has $\mathbb{Q}$-valued points.

4. Application to the fundamental groups

In this section, as an application of the connectivity theorem, we prove the following:

Theorem 4.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two line arrangements in $\mathbb{P}^2_{\mathbb{C}}$. Suppose that $|\mathcal{A}_1| = |\mathcal{A}_2| \leq 8$ and $I(\mathcal{A}_1) = I(\mathcal{A}_2)$. Then*

$$(\mathbb{P}^2_{\mathbb{C}}, \mathcal{A}_1) \cong (\mathbb{P}^2_{\mathbb{C}}, \mathcal{A}_2).$$

Corollary 4.2. *Under the same assumption, we have*

$$\pi_1(M(\mathcal{A}_1)) \simeq \pi_1(M(\mathcal{A}_2)).$$

Thus the isomorphism classes of the fundamental groups are combinatorial for $n \leq 8$.

The proof is done by using Theorem 3.15 in §3. Indeed, for almost all cases, $\mathcal{A}$ is of class $\mathcal{C}_{\leq 3}$ of simple type. Hence the realization space is connected. However there is exceptions (unique up to the PGL-action and the complex conjugation).

Example 4.3. (MacLane arrangement $\mathcal{M}^{\pm}$) Let $\omega_{\pm} := \frac{1\pm\sqrt{-3}}{2}$ be the roots of the quadratic equation $x^2 - x + 1 = 0$. Consider 8 lines $\mathcal{M}^{\pm} = \{H_1, \dots, H_8\}$ defined by:

$$\begin{array}{lll} H_1 : x = 0, & H_2 : x = z, & H_3 : x = \omega_{\pm} z, \\ H_4 : y = 0, & H_5 : y = z, & H_6 : y = \omega_{\pm} z, \\ H_7 : x = y, & H_8 : \omega_{\pm} x + y = \omega_{\pm}. & \end{array}$$

The MacLane arrangement is not of type $\mathcal{C}_{\leq 3}$, but of type $\mathcal{C}_4$ (e.g. all multiple points are contained in $H_1 \cup H_2 \cup H_3 \cup H_4$), and the realization

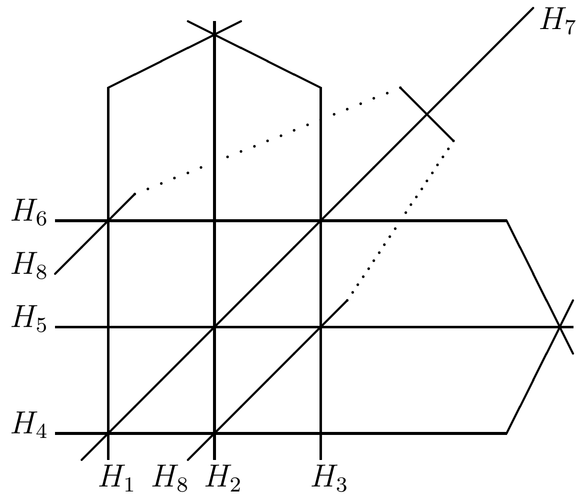

FIGURE 3. MacLane Arrangement $\mathcal{M}^{\pm}$

space has two connected components.

$$\mathcal{R}(I)/PGL_3(\mathbb{C}) = \{\mathcal{M}^+, \mathcal{M}^-\}.$$

However the corresponding complements $M(\mathcal{M}^+)$ and $M(\mathcal{M}^-)$ are diffeomorphic under complex conjugation. Hence the complements have isomorphic fundamental groups.

To prove Theorem 4.1, it is suffices to prove the following.

(1) If $n \leq 5$, then $\mathcal{A}$ is in class $\mathcal{C}_{\leq 1}$;
(2) $n \leq 6$, then $\mathcal{A}$ is in class $\mathcal{C}_{\leq 2}$;
(3) $n \leq 7$, then $\mathcal{A}$ is in class $\mathcal{C}_{\leq 3}$ of simple type;
(4) $n = 8$, then $\mathcal{A}$ is either in class $\mathcal{C}_{\leq 3}$ of simple type or isomorphic to the MacLane arrangement $\mathcal{M}^{\pm}$.

Proof of (1) and (2):

(1) If a line arrangement is in class $\mathcal{C}_2$, then it is clear that there should be at least six lines. Thus, for $n \leq 5$, $\mathcal{A}$ is in class $\mathcal{C}_1$.
(2) Let $H \in \mathcal{A}$ and $\mathcal{A}' = \mathcal{A} \setminus \{H\}$. Then by (1), there is a line $H' \in \mathcal{A}'$ such that all multiple points of $\mathcal{A}'$ are contained in H', therefore, all multiple points of $\mathcal{A}$ are in $H \cup H'$. Thus, (2) holds.

The following is the key lemma for our classification.

Lemma 4.4. *Let $\mathcal{A}$ be a line arrangement which is not in class $\mathcal{C}_{\leq 3}$ of simple type. Then there exist $H_1, H_2, \dots, H_6 \in \mathcal{A}$ satisfying $H_1 \cap H_2 \cap H_3 \neq \emptyset$, $H_4 \cap H_5 \cap H_6 \neq \emptyset$, and $(H_1 \cup H_2 \cup H_3) \cap (H_4 \cup H_5 \cup H_6)$ consists of 9 points. (Figure 4.)*

Proof. Suppose $H_1 \cap H_2 \cap H_3 \neq \emptyset$. Then there exists a multiple point which is not contained in $H_1 \cup H_2 \cup H_3$, otherwise, $\mathcal{A}$ will be in class $\mathcal{C}_{\leq 3}$

of simple type. Suppose $H_4 \cap H_5 \cap H_6 \neq \emptyset$ be such a multiple point. Then $H_1, \dots, H_6$ satisfy the conditions. □

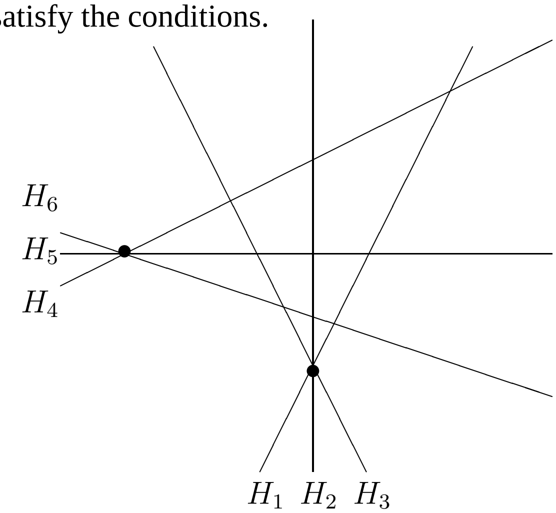

FIGURE 4. 6 lines contained in a non $\mathcal{C}_{\leq 3}$-simple type $\mathcal{A}$

Proposition 4.5. *Let $\mathcal{A}$ be a line arrangement with $|\mathcal{A}| = 7$. Then $\mathcal{A}$ is in class $\mathcal{C}_{\leq 3}$ of simple type.*

Proof. Suppose that $\mathcal{A}$ is not in class $\mathcal{C}_{\leq 3}$ of simple type. Then there exist 6 lines $H_1, \dots, H_6 \in \mathcal{A}$ satisfying the conditions of Lemma 4.4. So, all multiple points of $\mathcal{A}$ are either $H_1 \cap H_2 \cap H_3$, $H_4 \cap H_5 \cap H_6$ or contained in the line H_7.

Hence, all multiple points are contained in $H_1 \cup H_4 \cup H_7$. Moreover, as multiple points on $H_1 \setminus (H_4 \cup H_7)$ are at most one, $\mathcal{A}$ is in $\mathcal{C}_{\leq 3}$ of simple type, which is a contradiction. □

Proposition 4.6. *Let $\mathcal{A}$ be a line arrangement with $|\mathcal{A}| = 8$. Then $\mathcal{A}$ is either in class $\mathcal{C}_{\leq 3}$ of simple type or $\mathcal{A} = \mathcal{M}^{\pm}$, the MacLane arrangement.*

Proof. Suppose that $\mathcal{A}$ is not in class $\mathcal{C}_{\leq 3}$ of simple type. Then by Lemma 4.4, we have six lines $L_1, L_2, L_3, K_1, K_2, K_3 \in \mathcal{A}$ such that

- $L_1 \cap L_2 \cap L_3 \neq \emptyset$, $K_1 \cap K_2 \cap K_3 \neq \emptyset$, and
- Let $Q_{ij} := L_i \cap K_j$. Then $Q_{ij} = Q_{i'j'}$ only if $i = i', j = j'$.

Let us denote by $\mathcal{Q} := \{Q_{ij} \mid i, j = 1, 2, 3\}$ the set of 9 intersections of $(L_1 \cup L_2 \cup L_3) \cap (K_1 \cup K_2 \cup K_3)$. Suppose $\mathcal{A} = \{L_1, L_2, L_3, K_1, K_2, K_3, H_7, H_8\}$. We divide the cases according to the cardinality of $H_7 \cap \mathcal{Q}$ and $H_8 \cap \mathcal{Q}$. We may assume that $0 \leq |H_7 \cap \mathcal{Q}| \leq |H_8 \cap \mathcal{Q}| \leq 3$.

Case 1: $|H_7 \cap \mathcal{Q}| = 0$ (Fig. 5). In this case, every multiple point of $\mathcal{A}$ is contained in $K_1 \cup L_1 \cup H_8$ and there are at most one multiple point in $K_1 \setminus (L_1 \cup H_8)$. Hence, $\mathcal{A}$ is in $\mathcal{C}_{\leq 3}$ of simple type.

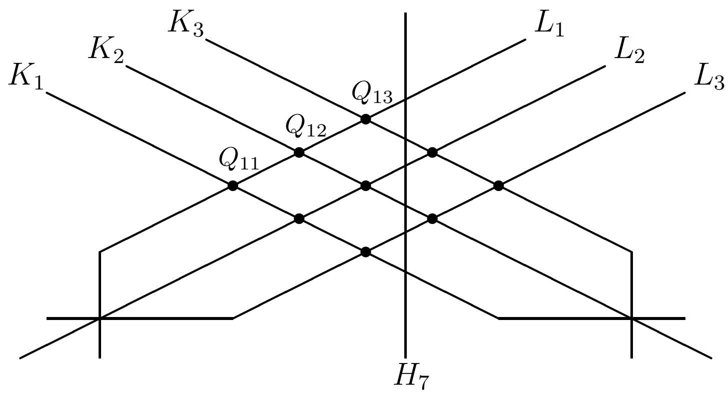

FIGURE 5. Case 1: $\mathcal{Q} \cap H_7 = \emptyset$.

Case 2: $|H_7 \cap Q| = 1$ (Fig. 6). Let $H_7 \cap \mathcal{Q} = L_i \cap K_j = \{Q_{ij}\}$. Then every multiple point of $\mathcal{A}$ is contained in $K_j \cup L_i \cup H_8$ and there are at most one multiple point in $K_j \setminus (L_i \cup H_8)$. Hence, $\mathcal{A}$ is in $\mathcal{C}_3$ of simple type.

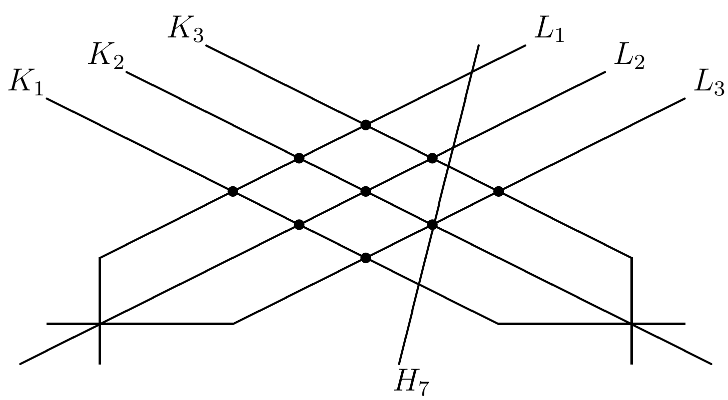

FIGURE 6. Case 2: $\mathcal{Q} \cap H_7 = \{Q_{32}\}$.

The rest cases are $2 \leq |H_7 \cap \mathcal{Q}| \leq |H_8 \cap \mathcal{Q}| \leq 3$.

Case 3: $|H_7 \cap \mathcal{Q}| = 2$ and $|H_8 \cap \mathcal{Q}| = 3$ (Fig. 7). By changing the numbering of K_i, L_j, we may assume $H_8 \cap \mathcal{Q} = \{Q_{11}, Q_{22}, Q_{33}\}$. Set $H_7 \cap \mathcal{Q} = \{Q_{i_1 j_1}, Q_{i_2 j_2}\}$. It can be noted that $i_1 \neq i_2$ and $j_1 \neq j_2$. As $\{i_1, i_2\}$ and $\{j_1, j_2\}$ are subsets of $\{1, 2, 3\}$, so the intersection is non-empty. Let $k \in \{1, 2, 3\}$ such that $k \in \{i_1, i_2\} \cap \{j_1, j_2\}$. Then $H_8 \cup K_k \cup L_k$ contains all multiple points of $\mathcal{A}$ and $H_8 \cap L_k \cap K_k \neq \emptyset$.

Case 4: $|H_7 \cap Q| = |H_8 \cap Q| = 2$.

We may assume that $H_8 \cap \mathcal{Q} = \{Q_{11}, Q_{22}\}$. We can check one-by-one, for any H_8, it is $\mathcal{C}_{\leq 3}$ of simple type.

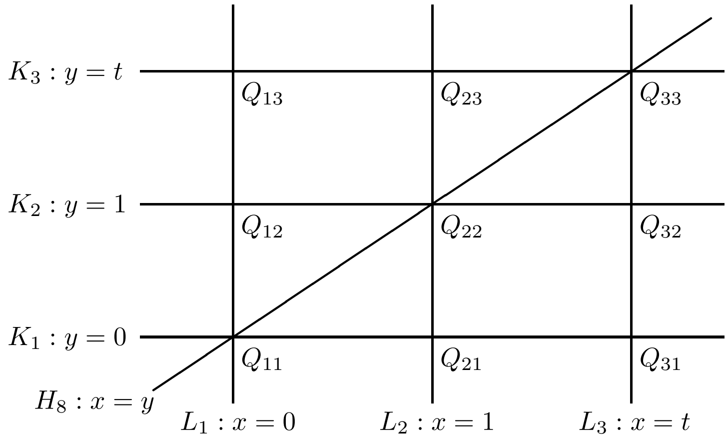

FIGURE 7. Case 3 and 5: $\mathcal{Q} \cap H_8 = \{Q_{11}, Q_{22}, Q_{33}\}$.

Case 5: $|H_7 \cap \mathcal{Q}| = |H_8 \cap \mathcal{Q}| = 3$ (Fig. 7). We may assume that $H_8 \cap \mathcal{Q} = \{Q_{11}, Q_{22}, Q_{33}\}$. We set $H_7 \cap \mathcal{Q} = \{Q_{1j_1}, Q_{2j_2}, Q_{3j_3}\}$. Hence there are six possibilities corresponding to the permutation (j_1, j_2, j_3) of $(1, 2, 3)$. We fix an affine coordinate as Figure 7.

(1) If $(j_1, j_2, j_3) = (1, 2, 3)$, then $H_7 = H_8$.
(2) If $(j_1, j_2, j_3) = (1, 3, 2)$. (This implies that $t = -1$.) $L_2 \cup K_2 \cup H_8$ covers all multiple points.
(3) If $(j_1, j_2, j_3) = (2, 1, 3)$. (This implies $t = \frac{1}{2}$.) $L_1 \cup K_1 \cup H_8$ covers all multiple points.
(4) If $(j_1, j_2, j_3) = (3, 2, 1)$. (This implies $t = 2$.) $L_1 \cup K_1 \cup H_8$ covers all multiple points.
(5) If $(j_1, j_2, j_3) = (3, 1, 2)$. Then $Q_{13}(0, t), Q_{21}(1, 0), Q_{32}(1, t)$ are collinear if and only if $t = \frac{1 \pm \sqrt{-3}}{2}$. Hence $\mathcal{A} = \mathcal{M}^{\pm}$.
(6) If $(j_1, j_2, j_3) = (2, 3, 1)$. Similarly, $\mathcal{A} = \mathcal{M}^{\pm}$.

□

5. EXAMPLES OF 9 AND 10 LINES

In this section, we will see several examples of 9 and 10 lines on $\mathbb{P}^2$ which are not covered by previous results.

Example 5.1. Let $\mathcal{M}^{\pm}$ be the MacLane arrangement with defining equations as in Example 4.3. Consider

$$\widetilde{\mathcal{M}}^{\pm} := \mathcal{M}^{\pm} \cup \{H_9\},$$

where $H_9 = \{z = 0\}$ is the line at infinity (Fig. 8).

The arrangement $\widetilde{\mathcal{M}}^{\pm}$ is of class $\mathcal{C}_3$. Indeed, all multiple points are contained in $H_7 \cup H_8 \cup H_9$. However since the realization space is not connected, it is not $\mathcal{C}_3$ of simple type.

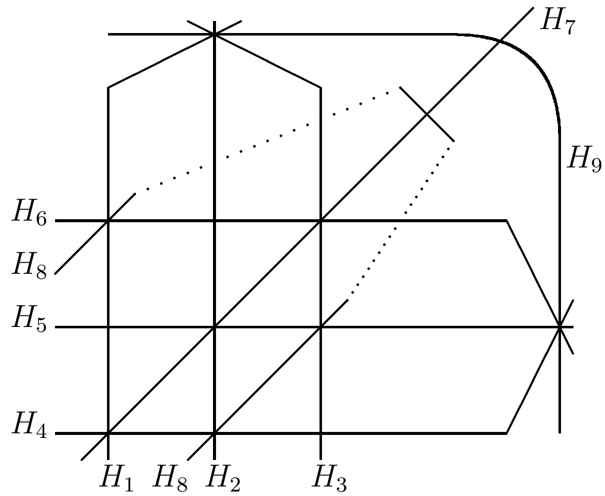

FIGURE 8. $\widetilde{\mathcal{M}}^{\pm} := \mathcal{M}^{\pm} \cup \{H_9\}$.

Example 5.2. (Falk-Sturmfels arrangements $\mathcal{FS}^{\pm}$.) Let $\gamma_{\pm} = \frac{1\pm\sqrt{5}}{2}$, and define

$$\mathcal{FS}^{\pm} = \{L_i^{\pm}, K_i^{\pm}, H_9^{\pm}, i = 1, 2, 3, 4\}$$

of 9 lines as follows (Fig. 9):

$$\begin{array}{llll} L_1^{\pm} : x = 0, & L_2^{\pm} : x = \gamma_{\pm}(y-1), & L_3^{\pm} : y = z, & L_4^{\pm} : x + y = z, \\ K_1^{\pm} : x = z, & K_2^{\pm} : x = \gamma_{\pm} y, & K_3^{\pm} : y = 0, & K_4^{\pm} : x + y = (\gamma_{\pm} + 1)z, \\ H_9^{\pm} : z = 0. & & & \end{array}$$

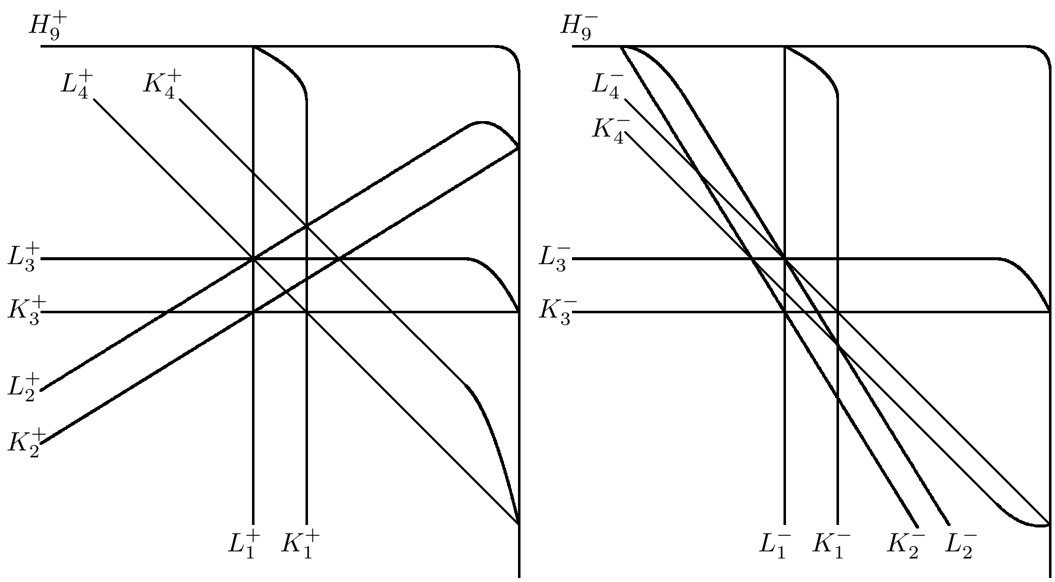

FIGURE 9. Falk-Sturmfels arrangements $\mathcal{FS}^+$ and $\mathcal{FS}^-$

$\mathcal{FS}^+$ and $\mathcal{FS}^-$ have isomorphic incidence relations, which are in $\mathcal{C}_4$ (e.g., multiple points are covered by $L_1^\pm \cup L_2^\pm \cup L_3^\pm \cup L_4^\pm$). The realization space consists of 2 connected components $\mathcal{R}(I(\mathcal{FS}^\pm))/PGL_3(\mathbb{C}) = \{\mathcal{FS}^+, \mathcal{FS}^-\}$. Thus it is the minimal example of $\mathbb{R}$-realizable arrangement with disconnected realization space (Falk-Sturmfels). The Galois group action $\sqrt{5} \mapsto -\sqrt{5}$ does not induce a continuous map of $M(\mathcal{FS}^\pm)$. However there is a $PGL_3(\mathbb{C})$ action $(\mathbb{P}^2, \bigcup_{H\in\mathcal{FS}^+} H) \to (\mathbb{P}^2, \bigcup_{H\in\mathcal{FS}^-} H)$ which maps

$$\begin{array}{llll} L_1^+ \longmapsto L_3^-, & L_2^+ \longmapsto L_4^-, & L_3^+ \longmapsto L_2^-, & L_4^+ \longmapsto L_1^-, \\ K_1^+ \longmapsto K_3^-, & K_2^+ \longmapsto K_4^-, & K_3^+ \longmapsto K_2^-, & K_4^+ \longmapsto K_1^-, \\ H_9^+ \longmapsto H_9^-. & & & \end{array}$$

(In the affine plane the unit square $(L_1^+, K_1^+, L_3^+, K_3^+)$ is mapped to the parallelogram $(L_3^-, K_3^-, L_2^-, K_2^-)$.) In particular, $M(\mathcal{FS}^+)$ and $M(\mathcal{FS}^-)$ are homeomorphic and having the isomorphic fundamental groups.

Example 5.3. (Arrangements $\mathcal{A}^{\pm i}$) Define the arrangement

$$\mathcal{A}^{\pm i} = \{A_j^\pm, B_j^\pm, C_j^\pm \mid j = 1, 2, 3\},$$

of 9 lines as follows (Fig. 10):

$$\begin{array}{lll} A_1^\pm : x = 0, & A_2^\pm : x = z, & A_3^\pm : x + y = z, \\ B_1^\pm : y = 0, & B_2^\pm : y = z, & B_3^\pm : z = 0, \\ C_1^\pm : y = \pm\sqrt{-1}x, & C_2^\pm : y = \mp\sqrt{-1}x + (1 \pm \sqrt{-1})z, & C_3^\pm : x + y = (1 \pm \sqrt{-1})z. \end{array}$$

It is also in $\mathcal{C}_4$ (e.g., $A_1^\pm \cup A_2^\pm \cup A_3^\pm \cup B_1^\pm$). The realization space consists

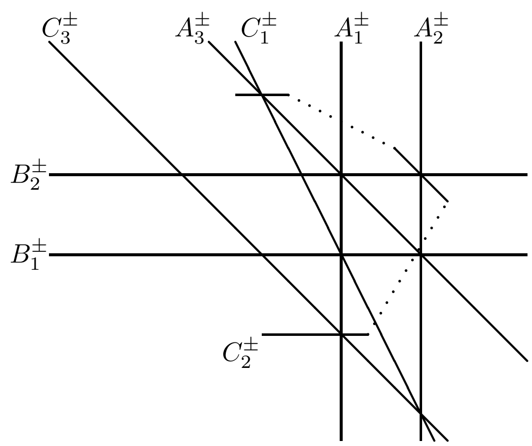

FIGURE 10. $\mathcal{A}^{\pm i}$, where $B_3^\pm$ is the line at infinity.

of 2 connected components. As in the case of MacLane arrangement (Example 4.3), the complements $M(\mathcal{A}^{\pm i})$ are homeomorphic by the complex conjugation.

Remark 5.4. Recently the authors verified that, up to 9 lines, these are the complete list of disconnected realization spaces. Namely, when $|\mathcal{A}| \leq 9$, after appropriate re-numbering of $H_1, \dots, H_n$, one of the following holds:

(i) The realization space $\mathcal{R}(I(\mathcal{A}))$ is irreducible (but not necessarily $\mathcal{C}_{\leq 3}$ of simple type, e.g., Pappus arrangements),
(ii) $\mathcal{A}$ contains the MacLane arrangement $\mathcal{M}^{\pm}$ (Example 4.3, 5.1),
(iii) $\mathcal{A}$ is isomorphic to the Falk-Sturmfels arrangement $\mathcal{FS}^{\pm}$ (Example 5.2),
(iv) $\mathcal{A}$ is isomorphic to $\mathcal{A}^{\pm i}$ (Example 5.3).

(Cases (ii), (iii), and (iv) are characterized by the minimal field of the realization, $\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{5})$, and $\mathbb{Q}(\sqrt{-1})$, respectively. It is also concluded from (i) that if I is realizable over $\mathbb{Q}$ (with $|\mathcal{A}| \leq 9$), then $\mathcal{R}(I)$ is irreducible.) The idea of the proof is very similar to that of Proposition 4.6 which is based on Lemma 4.4.

Consequently, if $I(\mathcal{A}_1) = I(\mathcal{A}_2)$ (with $|\mathcal{A}_1| = |\mathcal{A}_2| \leq 9$), $\mathcal{A}_1$ and $\mathcal{A}_2$ are transformed to each other by the composition of the following operations:

(α) change of numbering,
(β) lattice isotopy,
(γ) complex conjugation.

In particular, $M(\mathcal{A}_1)$ and $M(\mathcal{A}_2)$ are homeomorphic. Rybnikov type pairs of arrangements require at least 10 lines.

Example 5.5. (Extended Falk-Sturmfels arrangements $\widetilde{\mathcal{FS}}^{\pm}$.) Define an arrangement $\widetilde{\mathcal{FS}}^{\pm}$ of 10 lines by adding a line $H_{10}^{\pm} = \{x = 5z\}$ to Falk-Sturmfels arrangements $\mathcal{FS}^{\pm}$:

$$\widetilde{\mathcal{FS}}^{\pm} := \mathcal{FS}^{\pm} \cup \{H_{10}^{\pm}\}.$$

$\widetilde{\mathcal{FS}}^{\pm}$ have the same incidence, however there are no ways to transform from $\widetilde{\mathcal{FS}}^{+}$ to $\widetilde{\mathcal{FS}}^{-}$ by operations (α), (β) and (γ). (This fact can be proved as follows. First we prove that the identity is the only permutation of $\{1, \dots, 10\}$ which preserves the incidence. Hence if $\widetilde{\mathcal{FS}}^{+}$ is transformed to $\widetilde{\mathcal{FS}}^{-}$, it sends $L_i^+ \mapsto L_i^-, K_i^+ \mapsto K_i^-, H_i^+ \mapsto H_i^-$. Deleting $H_{10}^{\pm}$, $\mathcal{FS}^+$ can be transformed to $\mathcal{FS}^-$ with preserving the numbering. Note that $\mathcal{FS}^{\pm}$ are defined over $\mathbb{R}$ and there is no isotopy except for PGL action. There should exist a PGL action sending $\mathcal{FS}^+$ to $\mathcal{FS}^-$ which preserves the numbering. However it is impossible.) The pair $\{\widetilde{\mathcal{FS}}^{\pm}\}$ is a minimal one with such property. Thus at this moment the authors do not know whether if the fundamental groups $\pi_1(M(\widetilde{\mathcal{FS}}^{\pm}))$ are isomorphic.

Remark 5.6. We should point out that $\widetilde{\mathcal{FS}}^+$ is closer in spirit to examples in [1, §5].

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