

# GAUSS MAPS OF CONSTANT MEAN CURVATURE SURFACES IN THREE-DIMENSIONAL HOMOGENEOUS SPACES

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ABSTRACT. It is well-known that for a surface in a 3-dimensional real space form the constancy of the mean curvature is equivalent to the harmonicity of the Gauss map. However, this is not true in general for surfaces in an arbitrary 3-dimensional ambient space. In this paper we study this problem for surfaces in an important and very natural family of 3-dimensional ambient spaces, namely homogeneous spaces. In particular, we obtain a full classification of constant mean curvature surfaces, whose Gauss map satisfies the more mild condition of vertical harmonicity, in *all* 3-dimensional homogeneous spaces.

## 1. INTRODUCTION

In recent years, there has been done a lot of research concerning surfaces in 3-dimensional homogeneous spaces. Initial work was done in [3]. In particular, the study of constant mean curvature surfaces (CMC surfaces) and minimal surfaces in 3-dimensional homogeneous spaces is paid much attention to by differential geometers. We mention for example the paper [1].

One of the reasons for this success is that homogeneous spaces are among the most natural candidates for the role of ambient space in submanifold theory and that in dimension 3 the classification of these spaces is well-understood. Indeed, we have the following.

**Theorem 1.** *Let  $M^3$  be a 3-dimensional simply connected homogeneous Riemannian manifold with isometry group  $I(M^3)$ , i.e.  $I(M^3)$  acts transitively on  $M^3$ . Then  $\dim I(M^3) \in \{3, 4, 6\}$  and moreover:*

- (i) *if  $\dim I(M^3) = 6$ , then  $M^3$  is a real space form of constant sectional curvature  $c$ , i.e. Euclidean space  $\mathbb{E}^3$ , a hyperbolic space  $\mathbb{H}^3(c)$  or a three-sphere  $\mathbb{S}^3(c)$ ,*
- (ii) *if  $\dim I(M^3) = 4$ , then  $M^3$  is locally isometric to a Bianchi-Cartan-Vranceanu space (different from  $\mathbb{E}^3$  and  $\mathbb{S}^3(c) \setminus \{\infty\}$ ), i.e.  $M^3$  is a Riemannian product  $\mathbb{H}^2(c) \times \mathbb{R}$  or  $\mathbb{S}^2(c) \times \mathbb{R}$ , the Heisenberg group equipped with a left-invariant metric  $\text{Nil}_3$ , or one of following Lie groups equipped with a left-invariant metric yielding a four-dimensional isometry group: the special unitary group  $\text{SU}(2)$  or the universal covering of the special linear group  $\widetilde{\text{SL}}(2, \mathbb{R})$ ,*
- (iii) *if  $\dim I(M^3) = 3$ , then  $M^3$  is a general 3-dimensional Lie group with left-invariant metric.*

Another reason is that the classification above contains the eight model geometries of Thurston (cfr. [15]), namely  $\mathbb{E}^3$ ,  $\mathbb{H}^3$ ,  $\mathbb{S}^3$ ,  $\mathbb{H}^2 \times \mathbb{R}$ ,  $\mathbb{S}^2 \times \mathbb{R}$ ,  $\widetilde{\text{SL}}(2, \mathbb{R})$ ,  $\text{Nil}_3$  and  $\text{Sol}_3$ . The famous geometrization conjecture of Thurston states that these eight spaces are the ‘building blocks’ to construct *any* 3-dimensional manifold.

Gauss maps of surfaces in 3-dimensional real space forms play an important role in surface geometry. In fact, as shown by E. Ruh and J. Vilms in [12], for submanifolds of a Euclidean space the harmonicity of the Gauss map is equivalent to the fact that the mean curvature vector is parallel in the normal bundle. For surfaces in  $\mathbb{E}^3$ , this corresponds to the constancy of the mean curvature. In particular, the minimality of a surface in  $\mathbb{E}^3$  is equivalent to the holomorphicity of Gauss map. The characterization due to Ruh-Vilms was generalized to CMC surfaces in 3-dimensional real space forms (i.e. in spaces of type (i) in Theorem 1) by T. Ishihara in [6].

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On the contrary, in 3-dimensional homogeneous spaces of non-constant curvature, the harmonicity of Gauss map is a very strong restriction for CMC surfaces. In fact, A. Sanini (cfr. [13]) and M. Tamura (cfr. [14]) showed that the only CMC surfaces with harmonic Gauss map in a 3-dimensional homogeneous space with 4-dimensional isometry group are inverse images of geodesics under the Hopf-fibration or totally geodesic leaves. The latter case only occurs if the ambient space is a direct product space. Instead of harmonicity, Sanini and Tamura studied a more mild condition, namely “vertical harmonicity” of the Gauss map for CMC surfaces in 3-dimensional homogeneous spaces with 4-dimensional isometry group (i.e. in spaces of type (ii) of Theorem 1).

In this paper we classify CMC surfaces with vertically harmonic Gauss map in 3-dimensional homogeneous spaces with 3-dimensional isometry group (i.e. in spaces of type (iii) of Theorem 1). For this purpose, we have to treat *all* 3-dimensional Lie groups, for which we use methods based on results of J. Milnor from [10]. Our result finishes the classification of CMC surfaces with vertically harmonic Gauss map in *all* 3-dimensional homogeneous spaces.

## 2. CMC SURFACES AND GAUSS MAPS

**2.1. CMC hypersurfaces.** Consider Riemannian manifolds  $(M^n, g_M)$  and  $(Q^{n+1}, g_Q)$  with Levi Civita connections  $\nabla^M$  and  $\nabla^Q$  respectively. Let  $f : M^n \rightarrow Q^{n+1}$  be an isometric immersion with unit normal  $N$ . The *second fundamental form*  $h$  is a field of symmetric bilinear forms on  $M^n$ , defined by

$$\nabla_{df(X)}^Q df(Y) = df(\nabla_X^M Y) + h(X, Y)N$$

for vector fields  $X, Y$  on  $M^n$ . The *shape operator*  $S$  is a field of symmetric operators on  $M^n$ , defined by  $SX = -df^{-1}(\nabla_{df(X)}^Q N)$  for  $p \in M^n$  and  $X \in T_p M^n$ , and it is related to the second fundamental form by  $h(X, Y) = g_M(SX, Y) = g_M(X, SY)$  for  $p \in M^n$  and  $X, Y \in T_p M^n$ . The mean curvature  $H$  is a function on  $M^n$  defined by

$$H(p) = \text{tr}_{g_M|_p}(h|_p).$$

The immersion is said to have *constant mean curvature* (CMC) if  $H$  is constant, and it is said to be *minimal* if  $H$  vanishes identically.

**2.2. Grassmannian bundle.** Let  $(Q^n, g_Q)$  be a Riemannian  $n$ -manifold. Denote by  $\text{Gr}_\ell(TQ^n)$  the *Grassmannian bundle* of  $\ell$ -planes in the tangent bundle  $TQ^n$ :

$$\text{Gr}_\ell(TQ^n) := \bigcup_{q \in Q^n} \text{Gr}_\ell(T_q Q^n).$$

Here  $\text{Gr}_\ell(T_q Q^n)$  denotes the *Grassmannian manifold* of  $\ell$ -planes in the tangent space  $(T_q Q^n, g_Q|_q)$  at  $q$ . The Grassmannian bundle  $\text{Gr}_\ell(TQ^n)$  is a fiber bundle over  $Q^n$  associated to the orthonormal frame bundle  $\text{O}(Q^n)$ . The standard fiber of this bundle is the Grassmannian manifold  $\text{Gr}_\ell(\mathbb{R}^n)$ . The canonical 1-form of  $\text{O}(Q^n)$  and the Levi Civita connection of  $g_Q$  induce an invariant Riemannian metric on  $\text{Gr}_\ell(TQ^n)$ . With respect to this metric, the natural projection  $\pi : \text{Gr}_\ell(TQ^n) \rightarrow Q^n$  is a Riemannian submersion with totally geodesic fibers. For more details on the Riemannian structure of  $\text{Gr}_\ell(TQ^n)$ , we refer to [7] and [13].

Let  $f : M^m \rightarrow Q^n$  be an immersed submanifold. Then the (*tangential*) *Gauss map*  $\psi$  of  $f$  is a smooth map of  $M^m$  into  $\text{Gr}_m(TQ^n)$  defined by

$$\psi(p) := df_p(T_p M^m) \in \text{Gr}_m(T_{f(p)} Q^n).$$

*Remark 1.* The case  $(Q^3, g_Q) = \mathbb{E}^3$  is exceptional. In fact, since  $\mathbb{R}^3$  has absolute parallelism, the Grassmannian bundle is a trivial fiber bundle:  $\text{Gr}_2(T\mathbb{R}^3) = \mathbb{R}^3 \times \text{Gr}_2(\mathbb{R}^3) = \mathbb{R}^3 \times \mathbb{R}P^2$ . If we consider the Grassmannian bundle  $\text{Gr}_2^+(T\mathbb{R}^3)$  of all *oriented* 2-planes, we have  $\text{Gr}_2^+(T\mathbb{R}^3) = \mathbb{R}^3 \times \mathbb{S}^2$ . If  $f : M^2 \rightarrow \mathbb{E}^3$  is an isometric immersion of a surface with unit normal  $N$ , then the oriented tangential Gauss map  $\psi : M^2 \rightarrow \text{Gr}_2^+(T\mathbb{R}^3)$  is given by  $\psi = (f, N)$ . Hence we may ignore the first component and we obtain the classical Gauss map  $\psi = N : M^2 \rightarrow \mathbb{S}^2$ .

*Remark 2.* For a non-empty subset  $\Sigma$  of  $\text{Gr}_\ell(TQ^n)$ , the totality of  $\ell$ -submanifolds all of whose tangent spaces belong to  $\Sigma$  is called the  $\Sigma$ -geometry. The *Grassmann geometry* on  $Q^n$  is the collection of such  $\Sigma$ -geometries in  $Q^n$ . Let  $G$  be the identity component of the isometry group of  $Q^n$ . Then  $G$  acts isometrically on  $\text{Gr}_\ell(TQ^n)$ . If  $\Sigma$  is a  $G$ -orbit in  $\text{Gr}_\ell(TQ^n)$ , then the  $\Sigma$ -geometry is said to be of *orbit type*. Note that if  $Q^n$  is a homogeneous Riemannian manifold, then an orbit  $\Sigma$  is a subbundle of  $\text{Gr}_\ell(TQ^n)$ . H. Naitoh has developed the general theory of Grassmann geometries of orbit type on Riemannian symmetric spaces. Grassmann geometries on the 3-dimensional Heisenberg group and on the motion groups  $E(1,1)$  and  $E(2)$  are investigated by Naitoh, Kuwabara and the first named author in [4] and [9].

**2.3. Harmonic maps.** Next, we recall some fundamental ingredients of harmonic map theory from the lecture notes [2].

Let  $(M^m, g_M)$  and  $(Q^n, g_Q)$  be Riemannian manifolds, with Levi Civita connections  $\nabla^M$  and  $\nabla^Q$  respectively. Let  $f : M^m \rightarrow Q^n$  be a smooth map. Then the *energy density*  $e(f)$  of  $f$  is a function on  $M^m$  defined by  $e(f) = |df|^2/2$ . One can see that  $f$  is constant if and only if  $e(f) = 0$  and that  $e(f) = m/2$  if  $f$  is an isometric immersion. The *energy*  $E(f; \mathfrak{D})$  of  $f$  over a region  $\mathfrak{D} \subset M^m$  is

$$E(f; \mathfrak{D}) = \int_{\mathfrak{D}} e(f) dv_M.$$

A smooth map  $f$  is said to be *harmonic* if it is a critical point of the energy over every compactly supported region of  $M$ .

The second fundamental form  $\nabla df$  of  $f$  is in fact an extension of the second fundamental form for isometric immersions and is defined by

$$\nabla df(X, Y) = \nabla_{df(X)}^Q df(Y) - df(\nabla_X^M Y)$$

for vector fields  $X, Y$  on  $M^m$ . The second fundamental form  $\nabla df$  is a symmetric  $TQ^n$ -valued tensor field, i.e.,  $(\nabla df)(X, Y) = (\nabla df)(Y, X)$ . The trace  $\tau(f) := \text{tr}_{g_M}(\nabla df)$  is called the *tension field* of  $f$ . It is known that  $f$  is harmonic if and only if  $\tau(f) = 0$ . Remark that an isometric immersion is harmonic if and only if it is minimal.

**2.4. Vertically harmonic maps.** Let  $(P, g_P)$  be a Riemannian manifold and  $\pi : (P, g_P) \rightarrow (Q, g_Q)$  a Riemannian submersion. With respect to the metric  $g_P$ , the tangent bundle  $TP$  of  $P$  has a splitting

$$(1) \quad T_u P = \mathcal{H}_u \oplus \mathcal{V}_u, \quad u \in P.$$

Here  $\mathcal{V}_u = \text{Ker}(d\pi_u)$  and  $\mathcal{H}_u$  is the orthogonal complement of  $\mathcal{V}_u$  in  $T_u P$ . The linear subspaces  $\mathcal{V}_u$  and  $\mathcal{H}_u$  are called the *vertical subspace* and the *horizontal subspace* of  $T_u P$ .

Now let  $f : M \rightarrow P$  be a smooth map. Then its tension field  $\tau(f)$  is decomposed as

$$\tau(f) = \tau^{\mathcal{H}}(f) + \tau^{\mathcal{V}}(f)$$

according to the splitting (1). A smooth map  $f$  is said to be *vertically harmonic* if the vertical component  $\tau^{\mathcal{V}}(f)$  of the tension field vanishes. C. M. Wood has shown that vertical harmonicity of  $f$  is equivalent to the criticality of  $f$  with respect to the vertical energy through vertical variations in [16].

As announced in the introduction, the purpose of the present paper is to study vertical harmonicity of Gauss maps for CMC surfaces in 3-dimensional Lie groups with left-invariant metric, yielding a 3-dimensional isometry group. For our purpose, we recall the following result due to Sanini.

**Lemma 1.** ([13]) *Let  $(Q^3, g_Q)$  be a Riemannian 3-manifold and  $f : M^2 \rightarrow Q^3$  a CMC surface with unit normal  $N$ . Take a principal frame field  $\{\epsilon_1 = df(E_1), \epsilon_2 = df(E_2), \epsilon_3 = N\}$ , where  $\{E_1, E_2\}$  is an orthonormal frame field on  $M^2$  which diagonalizes the shape operator associated to  $N$ . Put  $\mathfrak{R}_{ijkl} := g_Q(R^Q(\epsilon_i, \epsilon_j)\epsilon_k, \epsilon_l)$ . Then we have the following.*

- (i) *The Gauss map  $\psi$  is vertically harmonic if and only if  $\mathfrak{R}_{1213} = \mathfrak{R}_{2123} = 0$ . Moreover, when  $f$  is minimal,  $\psi$  is harmonic if and only if, in addition,  $\mathfrak{R}_{3113} = \mathfrak{R}_{3223}$ .*
- (ii)  *$\psi$  is conformal if and only if  $f$  is minimal or totally umbilical.*

The following statement is an immediate corollary of Lemma 1.

**Lemma 2.** *Let  $(Q^3, g_Q)$  be a Riemannian 3-manifold and let  $f : M^2 \rightarrow Q^3$  be a CMC surface. The Gauss map is vertically harmonic if and only if the normal component of  $R^Q(df(X), df(Y))df(Z)$  vanishes for all  $p \in M^2$  and for all  $X, Y, Z \in T_p M^2$ .*

In [5], we have proven the following lemma.

**Lemma 3.** ([5]) *Consider a Riemannian manifold  $(Q^3, g_Q)$  and an orthonormal frame field  $\{e_1, e_2, e_3\}$  on  $Q^3$  which diagonalizes the Ricci tensor. Denote by  $K_{ij}$  the sectional curvature of the plane spanned by  $\{e_i, e_j\}$ , for  $i, j \in \{1, 2, 3\}$ . Let  $f : M^2 \rightarrow Q^3$  be a surface such that the normal component of  $R^Q(df(X), df(Y))df(Z)$  vanishes identically for all vector fields  $X, Y, Z$  on  $M^2$  and suppose that  $N = \alpha e_1 + \beta e_2 + \gamma e_3$  is a unit normal. Then every point of  $M^2$  has an open neighbourhood in  $M^2$  on which at least one of the following holds:*

- (i)  $\alpha = \beta = 0$ ,
- (ii)  $\alpha = \gamma = 0$ ,
- (iii)  $\beta = \gamma = 0$ ,
- (iv)  $\alpha = 0$  and  $K_{12} = K_{13}$ ,
- (v)  $\beta = 0$  and  $K_{12} = K_{23}$ ,
- (vi)  $\gamma = 0$  and  $K_{13} = K_{23}$ ,
- (vii)  $K_{12} = K_{23} = K_{13}$ .

### 3. THREE-DIMENSIONAL LIE GROUPS

**3.1. Unimodular Lie groups.** A Lie group  $G$  is said to be *unimodular* if its left-invariant Haar measure is right-invariant. We refer to the work of Milnor [10] for an infinitesimal reformulation of the unimodularity property.

Given a 3-dimensional unimodular Lie group  $G$  with a left-invariant metric, there exists an orthonormal basis  $\{e_1, e_2, e_3\}$  for the Lie algebra  $\mathfrak{g}$ , satisfying

$$(2) \quad [e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad c_i \in \mathbb{R}.$$

To describe the Levi Civita connection and the curvature of  $G$ , we introduce the following constants:

$$\mu_i = \frac{1}{2}(c_1 + c_2 + c_3) - c_i.$$

**Proposition 1.** *Let  $G$  be a 3-dimensional unimodular Lie group with a left-invariant metric  $\langle \cdot, \cdot \rangle$  and use the notations introduced above. The Levi Civita connection  $\tilde{\nabla}$  of  $G$  is given by*

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= 0, & \tilde{\nabla}_{e_1} e_2 &= \mu_1 e_3, & \tilde{\nabla}_{e_1} e_3 &= -\mu_1 e_2, \\ \tilde{\nabla}_{e_2} e_1 &= -\mu_2 e_3, & \tilde{\nabla}_{e_2} e_2 &= 0, & \tilde{\nabla}_{e_2} e_3 &= \mu_2 e_1, \\ \tilde{\nabla}_{e_3} e_1 &= \mu_3 e_2, & \tilde{\nabla}_{e_3} e_2 &= -\mu_3 e_1, & \tilde{\nabla}_{e_3} e_3 &= 0. \end{aligned}$$

*The frame  $\{e_1, e_2, e_3\}$  diagonalizes the Ricci tensor and the Riemann-Christoffel curvature tensor  $\tilde{R}$  is determined by the following sectional curvatures:*

$$\begin{aligned} K_{12} &= \langle \tilde{R}(e_1, e_2)e_2, e_1 \rangle = c_3 \mu_3 - \mu_1 \mu_2, \\ K_{23} &= \langle \tilde{R}(e_2, e_3)e_3, e_2 \rangle = c_1 \mu_1 - \mu_2 \mu_3, \\ K_{13} &= \langle \tilde{R}(e_1, e_3)e_3, e_1 \rangle = c_2 \mu_2 - \mu_1 \mu_3. \end{aligned}$$

Milnor classified 3-dimensional unimodular Lie groups based on the signature of the constants  $(c_1, c_2, c_3)$ .

Signature of $(c_1, c_2, c_3)$	Simply connected Lie group	Property
$(+, +, +)$	$\text{SU}(2)$	compact and simple
$(+, +, -)$	$\tilde{\text{SL}}(2, \mathbb{R})$	non-compact and simple
$(+, +, 0)$	$\tilde{\text{E}}(2)$	solvable
$(+, -, 0)$	$\text{E}(1, 1)$	solvable
$(+, 0, 0)$	Heisenberg group	nilpotent
$(0, 0, 0)$	$(\mathbb{R}^3, +)$	Abelian

It is easy to see that any left-invariant metric on  $(\mathbb{R}^3, +)$  gives rise to an isometry group of dimension 6, and in [11] it was proven that any left-invariant metric on the Heisenberg group gives rise to an isometry group of dimension 4. Hence, for our purpose, we may exclude these. The following examples cover the other cases.

*Example 1 (The special unitary group  $SU(2)$ ).* The group  $SU(2)$  is diffeomorphic to the sphere  $\mathbb{S}^3(1)$  since

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

The Lie algebra of this group is explicitly given by

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} iu & -v + iw \\ v + iw & -iu \end{pmatrix} \mid u, v, w \in \mathbb{R} \right\}.$$

To construct an orthonormal basis  $\{e_1, e_2, e_3\}$  as mentioned above, we proceed as follows. We take the following quaternionic basis  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  of  $\mathfrak{su}(2)$ :

$$\mathbf{i} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and we denote the left-translated vector fields of  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  by  $\{E_1, E_2, E_3\}$ . Choose strictly positive real constants  $\lambda_1, \lambda_2, \lambda_3$  and define

$$e_1 = \frac{1}{\lambda_2 \lambda_3} E_1, \quad e_2 = \frac{1}{\lambda_3 \lambda_1} E_2, \quad e_3 = \frac{1}{\lambda_1 \lambda_2} E_3.$$

Then  $[e_1, e_2] = c_3 e_3$ ,  $[e_2, e_3] = c_1 e_1$  and  $[e_3, e_1] = c_2 e_2$ , with  $c_i = 2/\lambda_i^2$ . The left-invariant metric  $g(c_1, c_2, c_3)$ , defined by the condition that  $\{e_1, e_2, e_3\}$  is an orthonormal frame, is

$$g(c_1, c_2, c_3) = 4 \left( \frac{1}{c_2 c_3} \omega_1^2 + \frac{1}{c_1 c_3} \omega_2^2 + \frac{1}{c_1 c_2} \omega_3^2 \right),$$

where  $\{\omega_1, \omega_2, \omega_3\}$  is the dual coframe field of  $\{E_1, E_2, E_3\}$ .

*Proposition 2 ([11]).* Any left-invariant metric on  $SU(2)$  is isometric to one of the metrics  $g(c_1, c_2, c_3)$ , with  $c_1, c_2, c_3 \geq 0$ . Moreover, the dimension of the isometry group is  $\geq 4$  if and only if at least two of the parameters  $c_i$  coincide.

In particular, if  $c_1 = c_2 = c_3 = c > 0$ , then the space is of constant curvature  $c^2/4$ .

*Example 2 (The real special linear group  $SL(2, \mathbb{R})$ ).* The group  $SL(2, \mathbb{R})$  is defined as the following subgroup of  $GL(2, \mathbb{R})$ :

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}.$$

First note that this group is isomorphic to the following subgroup of  $GL(2, \mathbb{C})$ :

$$SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\},$$

via the isomorphism

$$SL(2, \mathbb{R}) \rightarrow SU(1, 1) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}.$$

The Lie algebra of  $SU(1, 1)$  is explicitly given by

$$\mathfrak{su}(1, 1) = \left\{ \begin{pmatrix} iu & v - iw \\ v + iw & -iu \end{pmatrix} \mid u, v, w \in \mathbb{R} \right\}.$$

We take the following split-quaternionic basis of the Lie algebra  $\mathfrak{su}(1, 1)$ :

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j}' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{k}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Denote the left-translated vector fields of  $\{\mathbf{j}', \mathbf{k}', \mathbf{i}\}$  by  $\{E_1, E_2, E_3\}$ . Choose strictly positive real constants  $\lambda_1, \lambda_2, \lambda_3$  and define

$$e_1 = \frac{1}{\lambda_2 \lambda_3} E_1, \quad e_2 = \frac{1}{\lambda_3 \lambda_1} E_2, \quad e_3 = \frac{1}{\lambda_1 \lambda_2} E_3.$$

Then  $[e_1, e_2] = c_3 e_3$ ,  $[e_2, e_3] = c_1 e_1$  and  $[e_3, e_1] = c_2 e_2$ , with  $c_1 = 2/\lambda_1^2$ ,  $c_2 = 2/\lambda_2^2$  and  $c_3 = -2/\lambda_3^2$ . The left-invariant metric  $g(c_1, c_2, c_3)$ , defined by the condition that  $\{e_1, e_2, e_3\}$  is an orthonormal basis, is

$$g(c_1, c_2, c_3) = 4 \left( -\frac{1}{c_2 c_3} \omega_1^2 - \frac{1}{c_3 c_1} \omega_2^2 + \frac{1}{c_1 c_2} \omega_3^2 \right),$$

where  $\{\omega_1, \omega_2, \omega_3\}$  is the dual coframe field of  $\{E_1, E_2, E_3\}$ .

*Proposition 3* ([11]). *Any left-invariant metric on  $\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1)$  is isometric to one of the metrics  $g(c_1, c_2, c_3)$  with  $c_1 \geq c_2 > 0 > c_3$ . Moreover, this metric gives rise to an isometry group of dimension 4 if and only if  $c_1 = c_2$ .*

*Example 3 (The Minkowski motion group  $\mathrm{E}(1, 1)$ ).* Let  $\mathrm{E}(1, 1)$  be the group of orientation preserving isometries of the Minkowski plane:

$$\mathrm{E}(1, 1) = \left\{ \left( \begin{array}{ccc} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{R} \right\}.$$

Consider the following left-invariant frame on  $\mathrm{E}(1, 1)$ :

$$(3) \quad e_1 = \frac{1}{\lambda_1 \sqrt{2}} (-e^z \partial_x + e^{-z} \partial_y), \quad e_2 = \frac{1}{\lambda_2 \sqrt{2}} (e^z \partial_x + e^{-z} \partial_y), \quad e_3 = \frac{1}{\lambda_3} \partial_z,$$

where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are strictly positive constants. Remark that  $\{e_1, e_2, e_3\}$  satisfies the commutation relations  $[e_1, e_2] = 0$ ,  $[e_2, e_3] = c_1 e_1$  and  $[e_3, e_1] = c_2 e_2$ , with  $c_1 = \lambda_1/(\lambda_2 \lambda_3) > 0$  and  $c_2 = -\lambda_2/(\lambda_1 \lambda_3) < 0$ . We equip  $\mathrm{E}(1, 1)$  with a left-invariant Riemannian metric such that  $\{e_1, e_2, e_3\}$  is orthonormal. The resulting Riemannian metric is

$$g_{(\lambda_1, \lambda_2, \lambda_3)} = \frac{\lambda_1^2}{2} (-e^{-z} dx + e^z dy)^2 + \frac{\lambda_2^2}{2} (e^{-z} dx + e^z dy)^2 + \lambda_3^2 dz^2.$$

This metric is 4-symmetric (cfr. [8]) if and only if  $\lambda_1 = \lambda_2$ . Moreover, we have the following:

*Proposition 4* ([11]). *Any left invariant metric on  $\mathrm{E}(1, 1)$  is isometric to one of the metrics  $g_{(\lambda_1, \lambda_2, \lambda_3)}$  with  $\lambda_1 \geq \lambda_2 > 0$  and  $\lambda_3 = 1/(\lambda_1 \lambda_2)$ .*

For simplicity of notation, we put  $g(\lambda_1, \lambda_2) = g_{(\lambda_1, \lambda_2, 1/(\lambda_1 \lambda_2))}$ .

Remark that for  $\lambda_1 = \lambda_2$  we have

$$\begin{aligned} \langle \tilde{R}(X, Y)Z, W \rangle &= \lambda_1^4 (\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \\ &\quad + 2 \langle X, Z \rangle \langle Y, e_3 \rangle \langle W, e_3 \rangle + 2 \langle Y, W \rangle \langle X, e_3 \rangle \langle Z, e_3 \rangle \\ &\quad - 2 \langle X, W \rangle \langle Y, e_3 \rangle \langle Z, e_3 \rangle - 2 \langle Y, Z \rangle \langle X, e_3 \rangle \langle W, e_3 \rangle). \end{aligned}$$

The Riemannian homogeneous manifold  $\mathrm{Sol}_3 = (\mathrm{E}(1, 1), g(1, 1))$  is the model space of solve-geometry in the sense of Thurston. Thus, we have obtained the fact that  $\mathrm{Sol}_3$  has a natural 2-parametric deformation family  $\{(\mathrm{E}(1, 1), g(\lambda_1, \lambda_2)) \mid \lambda_1 \geq \lambda_2 > 0\}$ . Note that this deformation preserves the unimodularity property, because all these spaces have common underlying Lie group  $\mathrm{E}(1, 1)$ .

*Example 4 (The universal covering of the Euclidean motion group  $\tilde{\mathrm{E}}(2)$ ).* The group  $\mathrm{E}(2)$  of orientation-preserving rigid motions of Euclidean plane is given explicitly by the following matrix group:

$$\mathrm{E}(2) = \left\{ \left( \begin{array}{ccc} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y \in \mathbb{R}, \theta \in \mathbb{S}^1 \right\}.$$

Let  $\tilde{\mathbb{E}}(2)$  denote the universal covering group of  $\mathbb{E}(2)$ . Then  $\tilde{\mathbb{E}}(2)$  is isomorphic to  $\mathbb{R}^3$  with group operation

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x} \cos z - \bar{y} \sin z, y + \bar{x} \sin z + \bar{y} \cos z, z + \bar{z}).$$

Take strictly positive constants  $\lambda_1, \lambda_2$  and  $\lambda_3$  and a left-invariant frame

$$e_1 = \frac{1}{\lambda_2} (-\sin z \partial_x + \cos z \partial_y), \quad e_2 = \frac{1}{\lambda_3} \partial_z, \quad e_3 = \frac{1}{\lambda_1} (\cos z \partial_x + \sin z \partial_y).$$

Then this frame satisfies the commutation relations  $[e_1, e_2] = c_3 e_3$ ,  $[e_2, e_3] = c_1 e_1$  and  $[e_3, e_1] = 0$  with  $c_3 = \lambda_1/(\lambda_2 \lambda_3) > 0$  and  $c_1 = \lambda_2/(\lambda_1 \lambda_3) > 0$ . The left-invariant Riemannian metric determined by the condition that  $\{e_1, e_2, e_3\}$  is orthonormal, is given by

$$g_{(\lambda_1, \lambda_2, \lambda_3)} = \lambda_1^2 (\cos z dx + \sin z dy)^2 + \lambda_2^2 (-\sin z dx + \cos z dy)^2 + \lambda_3^2 dz^2.$$

Also in this case, we have:

*Proposition 5 ([11]). Any left-invariant metric on  $\tilde{\mathbb{E}}(2)$  is isometric to one of the metrics  $g_{(\lambda_1, \lambda_2, \lambda_3)}$  with  $\lambda_1 > \lambda_2 > 0$  and  $\lambda_3 = 1/(\lambda_1 \lambda_2)$ , or  $\lambda_1 = \lambda_2 = \lambda_3 = 1$ . Clearly,  $\tilde{\mathbb{E}}(2)$  with metric  $g_{(1,1,1)}$  is isometric to Euclidean three-space  $\mathbb{E}^3$ .*

For simplicity of notation, we put  $g(\lambda_1, \lambda_2) = g_{(\lambda_1, \lambda_2, 1/(\lambda_1 \lambda_2))}$ .

**3.2. Non-unimodular Lie groups.** Let  $G$  be a non-unimodular 3-dimensional Lie group with a left-invariant metric. Then the *unimodular kernel*  $\mathfrak{u}$  of the Lie algebra  $\mathfrak{g}$  of  $G$  is defined by

$$\mathfrak{u} = \{X \in \mathfrak{g} \mid \text{tr}(\text{ad}(X)) = 0\}.$$

Here  $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  is a homomorphism defined by  $\text{ad}(X)Y = [X, Y]$ . One can see that  $\mathfrak{u}$  is an ideal of  $\mathfrak{g}$  which contains the ideal  $[\mathfrak{g}, \mathfrak{g}]$ .

It is proven in [10] that we can take an orthonormal basis  $\{e_1, e_2, e_3\}$  of  $\mathfrak{g}$  such that

$$[e_1, e_2] = ae_2 + be_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = ce_2 + de_3,$$

with  $a + d \neq 0$  and  $ac + bd = 0$ . It is crucial that  $G$  is non-unimodular, since  $e_1$  is perpendicular to  $\mathfrak{u}$ . In the same article, it is remarked that after a suitable homothetic change of the metric, we may assume that  $a + d = 2$ . Then the constants  $a, b, c$  and  $d$  are represented as

$$a = 1 + \xi, \quad b = (1 + \xi)\eta, \quad c = -(1 - \xi)\eta, \quad d = 1 - \xi,$$

with  $\xi, \eta \geq 0$ . From now on, we work under this normalization. We refer to the constants  $(\xi, \eta)$  as the *structure constants* of the non-unimodular Lie group.

**Proposition 6.** *Let  $G$  be a 3-dimensional non-unimodular Lie group with left-invariant metric  $\langle \cdot, \cdot \rangle$  and use the notations introduced above. The Levi Civita connection  $\tilde{\nabla}$  of  $G$  is given by*

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= 0, & \tilde{\nabla}_{e_1} e_2 &= \eta e_3, & \tilde{\nabla}_{e_1} e_3 &= -\eta e_2, \\ \tilde{\nabla}_{e_2} e_1 &= -(1 + \xi)e_2 - \xi\eta e_3, & \tilde{\nabla}_{e_2} e_2 &= (1 + \xi)e_1, & \tilde{\nabla}_{e_2} e_3 &= \xi\eta e_1, \\ \tilde{\nabla}_{e_3} e_1 &= -\xi\eta e_2 - (1 - \xi)e_3, & \tilde{\nabla}_{e_3} e_2 &= \xi\eta e_1, & \tilde{\nabla}_{e_3} e_3 &= (1 - \xi)e_1. \end{aligned}$$

*The frame  $\{e_1, e_2, e_3\}$  diagonalizes the Ricci tensor and the Riemann-Christoffel curvature tensor  $\tilde{R}$  is determined by the following sectional curvatures:*

$$\begin{aligned} K_{12} &= \langle \tilde{R}(e_1, e_2)e_2, e_1 \rangle = -(\xi\eta^2 + (1 + \xi)^2 + \xi\eta^2(1 + \xi)), \\ K_{23} &= \langle \tilde{R}(e_2, e_3)e_3, e_2 \rangle = \xi^2(1 + \eta^2) - 1, \\ K_{13} &= \langle \tilde{R}(e_1, e_3)e_3, e_1 \rangle = \xi\eta^2 - (1 - \xi)^2 + \xi\eta^2(1 - \xi). \end{aligned}$$

Remark that for  $\xi = 0$ , the group  $G$  has constant sectional curvature  $-1$ . If we assume  $G$  to be simply connected, this implies that  $G$  is diffeomorphic to  $\mathbb{H}^3(-1)$ . On the other hand, if  $\xi = 1$ , then  $G$  is locally isometric to a space with 4-dimensional isometry group. Hence, in the non-unimodular case, we may restrict to groups with  $\xi \notin \{0, 1\}$ .

## 4. CLASSIFICATION RESULTS

4.1. **In  $SU(2)$ .** The following theorem characterizes CMC surfaces with vertically harmonic Gauss map in  $SU(2)$  with 3-dimensional isometry group.

**Theorem 2.** *Consider the Riemannian manifold  $(SU(2), g(c_1, c_2, c_3))$ , with a 3-dimensional isometry group. Then we may assume that  $c_1 > c_2 > c_3 > 0$ . If  $f : M^2 \rightarrow SU(2)$  is a CMC surface with vertically harmonic Gauss map, then  $c_1 = c_2 + c_3$ , i.e.  $K_{12} = K_{13}$ , and  $e_1$  is tangent to  $f(M^2)$ . Moreover, the surface is minimal and the Gauss map is harmonic.*

*Proof.* Let  $f : M^2 \rightarrow (SU(2), g(c_1, c_2, c_3))$  be a CMC surface with vertically harmonic Gauss map and unit normal  $N = \alpha e_1 + \beta e_2 + \gamma e_3$ , where  $\{e_1, e_2, e_3\}$  is the orthonormal frame constructed in section 3. It follows from Lemma 2 and Lemma 3 that there are 7 cases to consider. Cases (i), (ii), and (iii) are impossible due to the theorem of Frobenius and formulae (2). From Proposition 1 and the assumption  $c_1 > c_2 > c_3 > 0$ , it follows that only case (iv) of Lemma 3 can occur.

In this case, we have  $K_{12} = K_{13}$  or equivalently

$$(4) \quad c_1 = c_2 + c_3,$$

and the unit normal on  $f(M^2)$  takes the form  $N = \beta e_2 + \gamma e_3$ , with  $\beta^2 + \gamma^2 = 1$ . Then  $E_1 = df^{-1}(e_1)$  and  $E_2 = df^{-1}(-\gamma e_2 + \beta e_3)$  are an orthonormal frame on  $M^2$ . A straightforward computation using Proposition 1 yields

$$df([E_1, E_2]) = \beta\gamma(c_2 - c_3)df(E_2) + (\beta E_1[\gamma] - \gamma E_1[\beta] - \beta^2 c_2 - \gamma^2 c_3)N.$$

Hence, the distribution spanned by  $\{e_1, -\gamma e_2 + \beta e_3\}$  is integrable if and only if

$$(5) \quad \beta E_1[\gamma] - \gamma E_1[\beta] = \beta^2 c_2 + \gamma^2 c_3$$

and in this case, we have  $[E_1, E_2] = \beta\gamma(c_2 - c_3)E_1$ .

We can compute the shape operator  $S$  associated to  $N$  by using the definition of  $S$ , Proposition 1, (4) and (5), to be

$$S = \begin{pmatrix} 0 & \beta^2 c_2 + \gamma^2 c_3 \\ \beta^2 c_2 + \gamma^2 c_3 & \gamma E_2[\beta] - \beta E_2[\gamma] \end{pmatrix}.$$

Hence, the surface is CMC if and only if  $\gamma E_2[\beta] - \beta E_2[\gamma] = C$  is constant. Together with  $\beta^2 + \gamma^2 = 1$ , we obtain

$$\begin{cases} E_2[\beta] = C\gamma, \\ E_2[\gamma] = -C\beta. \end{cases}$$

Similarly, from (5) and  $\beta^2 + \gamma^2 = 1$ , we obtain

$$\begin{cases} E_1[\beta] = \gamma(\beta^2 c_2 + \gamma^2 c_3), \\ E_1[\gamma] = -\beta(\beta^2 c_2 + \gamma^2 c_3). \end{cases}$$

By expressing the compatibility condition for the equations for  $\beta$ , i.e.  $E_1[E_2[\beta]] - E_2[E_1[\beta]] = [E_1, E_2][\beta]$ , we obtain  $3C\beta\gamma^2(c_3 - c_2) = 0$ . Since the distributions spanned by  $\{e_1, e_2\}$  and  $\{e_1, e_3\}$  are not integrable, we have  $\beta \neq 0$  and  $\gamma \neq 0$ . Hence we obtain  $C = 0$ , or equivalently, the immersion is minimal. Remark that the compatibility condition for the equations for  $\gamma$  is then also satisfied.

Finally, using Lemma 1, we see that the Gauss map is harmonic. Indeed, the vector fields  $U_1 = (E_1 + E_2)/\sqrt{2}$  and  $U_2 = (E_1 - E_2)/\sqrt{2}$  diagonalize the shape operator and a straightforward computation yields  $\langle R^{SU(2)}(N, df(U_i))df(U_i), N \rangle = 0$  for  $i = 1, 2$ .  $\square$

4.2. **In  $SL(2, \mathbb{R})$ .** The following theorem can be proven analogously as Theorem 2

**Theorem 3.** *Consider the Riemannian manifold  $(SL(2, \mathbb{R}), g(c_1, c_2, c_3))$ , with a 3-dimensional isometry group. Then we may assume that  $c_1 > c_2 > 0 > c_3$ . If  $f : M^2 \rightarrow SL(2, \mathbb{R})$  is a CMC surface with vertically harmonic Gauss map, then  $c_2 = c_1 + c_3$ , i.e.  $K_{12} = K_{23}$  and  $e_2$  is tangent to  $f(M^2)$ . Moreover, the surface is minimal and the Gauss map is harmonic.*



4.3. **In  $E(1, 1)$ .** Since formulae (3) give the orthonormal frame field  $\{e_1, e_2, e_3\}$  on  $(E(1, 1), g(\lambda_1, \lambda_2))$  explicitly in terms of the natural coordinate vector fields  $\{\partial_x, \partial_y, \partial_z\}$ , we can describe the CMC surfaces with vertically harmonic Gauss map explicitly in these coordinates.

**Theorem 4.** *Consider the Riemannian manifold  $(E(1, 1), g(\lambda_1, \lambda_2))$  and let*

$$f : U \subseteq \mathbb{R}^2 \rightarrow E(1, 1) : (u, v) \mapsto \begin{pmatrix} e^{f_3(u,v)} & 0 & f_1(u, v) \\ 0 & e^{-f_3(u,v)} & f_2(u, v) \\ 0 & 0 & 1 \end{pmatrix}$$

be a CMC surface with vertically harmonic Gauss map. Then the Gauss map is harmonic, the surface is minimal and there exists an open subset  $V \subseteq U$  on which  $f$  is, up to reparametrization and isometries of the ambient space, given by one of the following:

- (i)  $f_1 = u$ ,  $f_2 = v$  and  $f_3 = c$ , with  $c \in \mathbb{R}$ ,
- (ii)  $f_1 = -\frac{ue^v}{\lambda_1\sqrt{2}}(\sin\theta(v) + \cos\theta(v)) + a_1$ ,  $f_2 = \frac{ue^{-v}}{\lambda_1\sqrt{2}}(\sin\theta(v) - \cos\theta(v)) + a_2$  and  $f_3 = v$ , with
 
$$\theta(v) = \arctan\left(\frac{1 + e^{2v+c}}{1 - e^{2v+c}}\right)$$

and  $a_1, a_2, c \in \mathbb{R}$ .

The latter case only occurs if  $\lambda_1 = \lambda_2$ . The surfaces of the first type are flat. Conversely, all surfaces described above are minimal and have harmonic Gauss map.

*Proof.* Let  $f : U \subseteq \mathbb{R}^2 \rightarrow (E(1, 1), g(\lambda_1, \lambda_2))$  be a CMC surface with vertically harmonic Gauss map and suppose that  $N = \alpha e_1 + \beta e_2 + \gamma e_3$  is a unit normal on the surface. From Lemma 2, Lemma 3, the theorem of Frobenius, formulae (2) and Proposition 1, it follows that there are only two cases to consider, namely ‘ $\alpha = \beta = 0$ ’ and ‘ $\gamma = 0$  and  $K_{13} = K_{23}$ ’.

If  $\alpha = \beta = 0$ , we may assume that  $N = e_3$ . A straightforward calculation shows that

$$S = -\frac{\lambda_1^2 + \lambda_2^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis  $\{E_1 = df^{-1}(e_1), E_2 = df^{-1}(e_2)\}$ . Hence the surface is minimal, such that Lemma 2 implies that it has vertically harmonic Gauss map. Since  $\text{span}\{e_1, e_2\} = \text{span}\{\partial_x, \partial_y\}$ , we obtain the first family of surfaces described in the theorem. Remark that, using the equation of Gauss, the Gaussian curvature of the surface is given by  $K = \langle R^{E(1,1)}(e_1, e_2)e_2, e_1 \rangle + \det S = 0$ . From Lemma 1, we obtain that the Gauss map is harmonic. Indeed, the vector fields  $U_1 = (E_1 + E_2)/\sqrt{2}$  and  $U_2 = (E_1 - E_2)/\sqrt{2}$  form an orthonormal frame which diagonalizes the shape operator and  $\langle R^{E(1,1)}(N, df(U_i))df(U_i), N \rangle = (K_{13} + K_{23})/2$  for  $i = 1, 2$ .

In the second case, it follows from  $K_{13} = K_{23}$  that  $\lambda_1 = \lambda_2$ . Moreover, the tangent plane to  $M^2$  is at every point spanned by the orthonormal vector fields  $E_1 = df^{-1}(\beta e_1 - \alpha e_2)$  and  $E_2 = df^{-1}(e_3)$ . The Lie bracket of these vector fields is given by

$$df([E_1, E_2]) = -2\lambda_1^2\alpha\beta df(E_1) + (\beta E_2[\alpha] - \alpha E_2[\beta] - \lambda_1^2(\alpha^2 - \beta^2))N.$$

Hence, the integrability condition for the distribution spanned by  $\{\beta e_1 - \alpha e_2, e_3\}$  is

$$(6) \quad \beta E_2[\alpha] - \alpha E_2[\beta] = \lambda_1^2(\alpha^2 - \beta^2)$$

and we have  $[E_1, E_2] = -2\lambda_1^2\alpha\beta E_1$ .

The shape operator associated to  $N = \alpha e_1 + \beta e_2$  with respect to the basis  $\{E_1, E_2\}$  is given by

$$S = \begin{pmatrix} \alpha E_1[\beta] - \beta E_1[\alpha] & \lambda_1^2(\beta^2 - \alpha^2) \\ \lambda_1^2(\beta^2 - \alpha^2) & 0 \end{pmatrix}.$$

We are interested in the case that  $M^2$  is a CMC surface. Hence assume that

$$(7) \quad \alpha E_1[\beta] - \beta E_1[\alpha] = C$$

for some real constant  $C$ . The fact that  $\alpha^2 + \beta^2 = 1$ , together with (6) and (7) gives the following equations:  $E_1[\alpha] = -C\beta$ ,  $E_1[\beta] = C\alpha$ ,  $E_2[\alpha] = \lambda_1^2\beta(\alpha^2 - \beta^2)$ ,  $E_2[\beta] = -\lambda_1^2\alpha(\alpha^2 - \beta^2)$ . If we

introduce a function  $\theta$  which is locally defined on the surface by  $\alpha = \cos \theta$ ,  $\beta = \sin \theta$ , these equations reduce to

$$(8) \quad \begin{cases} E_1[\theta] = C, \\ E_2[\theta] = -\lambda_1^2 \cos(2\theta). \end{cases}$$

The compatibility condition for this system is  $C\lambda_1^2 \sin(2\theta) = 0$ , from which we conclude  $C = 0$ . Hence, the surface is minimal.

We can take coordinates  $(u, v)$  on the surface, with  $\partial_u = E_1$  and  $\partial_v = pE_1 + E_2$  for a suitable function  $p : M^2 \rightarrow \mathbb{R}$ . Indeed, the condition  $[\partial_u, \partial_v] = 0$  is equivalent to the equation

$$(9) \quad \partial_u p = \lambda_1^2 \sin(2\theta).$$

Moreover, the system of equations (8) is equivalent to

$$\begin{cases} \partial_u \theta = 0, \\ \partial_v \theta = -\lambda_1^2 \cos(2\theta), \end{cases}$$

which can be solved as

$$\theta = \arctan \left( \frac{e^{-2\lambda_1^2 v + c} + 1}{e^{-2\lambda_1^2 v + c} - 1} \right),$$

where  $c$  is a real constant. Remark that (9) yields  $p(u, v) = \lambda_1^2 u \sin(2\theta(v)) + C(v)$  for some function  $C(v)$ . Since we are only interested in one coordinate system, we may assume that  $C(v) = 0$  and hence

$$p(u, v) = \lambda_1^2 u \sin(2\theta(v)).$$

In order to find an explicit expression for  $f$ , we need to integrate the formulae

$$\begin{aligned} (\partial_u f_1, \partial_u f_2, \partial_u f_3) &= df(E_1) = \sin \theta e_1 - \cos \theta e_2, \\ (\partial_v f_1, \partial_v f_2, \partial_v f_3) &= p df(E_1) + df(E_2) = p \sin \theta e_1 - p \cos \theta e_2 + e_3, \end{aligned}$$

with

$$\begin{aligned} e_1 &= \frac{1}{\lambda_1 \sqrt{2}} (-e^{f_3}, e^{-f_3}, 0), \\ e_2 &= \frac{1}{\lambda_1 \sqrt{2}} (e^{f_3}, e^{-f_3}, 0), \\ e_3 &= \lambda_1^2 (0, 0, 1). \end{aligned}$$

A direct computation yields

$$\begin{aligned} f_1 &= -\frac{ue^{\lambda_1^2 v}}{\lambda_1 \sqrt{2}} (\sin \theta(v) + \cos \theta(v)) + a_1, \\ f_2 &= \frac{ue^{-\lambda_1^2 v}}{\lambda_1 \sqrt{2}} (\sin \theta(v) - \cos \theta(v)) + a_2, \\ f_3 &= \lambda_1^2 v, \end{aligned}$$

where  $a_1$  and  $a_2$  are real constants. The reparametrization  $\lambda_1^2 v \mapsto v$  gives the result stated in the theorem.

The vector fields  $U_1 = (E_1 + E_2)/\sqrt{2}$  and  $U_2 = (E_1 - E_2)/\sqrt{2}$  form an orthonormal frame which diagonalizes the shape operator. It follows from Lemma 1 and the observation  $\langle R^{E(1,1)}(N, df(U_i)) df(U_i), N \rangle = 0$  for  $i = 1, 2$ , that the Gauss map is harmonic.

Conversely, one can check that both families of surfaces given in the theorem are minimal and satisfy the conditions of Lemma 1. Hence they have harmonic Gauss map.  $\square$

As a corollary, we obtain that there are two families of CMC surfaces in  $\text{Sol}_3$  with harmonic Gauss map. Both of them are minimal and explicit parametrizations are given in Theorem 4, if we put  $\lambda_1 = \lambda_2 = 1$ .

4.4. In  $\tilde{\mathbb{E}}(2)$ . Also in this case, we can give the solutions to our classification problem by means of an explicit parametrization. The proof is again similar.

**Theorem 5.** *Up to isometries of the ambient space, all CMC surfaces in  $(\tilde{\mathbb{E}}(2), g(\lambda_1, \lambda_2))$  with vertically harmonic tangential Gauss map can be locally parametrized as*

$$f : U \subseteq \mathbb{R}^2 \rightarrow \tilde{\mathbb{E}}(2) : (u, v) \mapsto \begin{pmatrix} \cos c & -\sin c & u \\ \sin c & \cos c & v \\ 0 & 0 & 1 \end{pmatrix},$$

with  $c \in \mathbb{R}$ . All these surfaces are minimal and their Gauss map is harmonic.

4.5. **In non-unimodular Lie groups.** As mentioned at the end of section 3, we may restrict ourselves to non-unimodular groups with  $\xi \notin \{0, 1\}$ . The following theorem can be proven again with similar methods.

**Theorem 6.** *Let  $G$  be a 3-dimensional non-unimodular Lie group with left-invariant metric. We may assume that the first structure constant satisfies  $\xi \notin \{0, 1\}$ . If  $f : M^2 \rightarrow G$  is a CMC surface with vertically harmonic Gauss map, then, up to isometries of  $G$ , one of the following holds:*

- (i)  $f(M^2)$  is an integral surface of the distribution spanned by  $\{e_2, e_3\}$ . The surface is flat and has constant mean curvature  $H = 1$ .
- (ii)  $f(M^2)$  is an integral surface of the distribution spanned by  $\{e_1, e_2\}$  or of the distribution spanned by  $\{e_1, e_3\}$ . These surfaces are totally geodesic and hence minimal.

The second case only occurs if  $\eta = 0$ .

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