INTERSECTION THEORY FOR GENERIC DIFFERENTIAL POLYNOMIALS AND DIFFERENTIAL CHOW FORM

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Abstract. In this paper, an intersection theory for generic differential polynomials is presented. The intersection of an irreducible differential variety of dimension d and order h with a generic differential hypersurface of order s is shown to be an irreducible variety of dimension $d-1$ and order $h+s$. As a consequence, the dimension conjecture for generic differential polynomials is proved. Based on the intersection theory, the Chow form for an irreducible differential variety is defined and most of the properties of the Chow form in the algebraic case are extended to its differential counterpart. Furthermore, the generalized differential Chow form is defined and its properties are proved. As an application of the generalized differential Chow form, the differential resultant of $n+1$ generic differential polynomials in n variables is defined and properties similar to that of the Sylvester resultant of two univariate polynomials are proved.

Contents

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1. Introduction

Differential algebra founded by Ritt and Kolchin aims to study algebraic differential equations in a similar way that polynomial equations are studied in algebraic geometry. Therefore, most of the results in differential algebra can be considered as generalizations of their algebraic counterparts to the differential case [29, 18, 23]. However, for many algebraic properties, the differential counterparts are much more difficult to prove and some of them are still open. An excellent survey on this subject can be found in [4].

In this paper, two naturally connected problems in differential algebra are studied: the differential dimension conjecture for generic differential polynomials and the differential Chow form.

The first part of the paper is concerned with the differential dimension conjecture which is one of the problems proposed by Ritt in his classic book *Differential Algebra*: Let $F_1, \ldots, F_r$ be differential polynomials in $\mathcal{F}\{y_1, \ldots, y_n\}$ with $r < n$, where $\mathcal{F}$ is a differential field. If the differential variety of the system $\{F_1, \ldots, F_r\}$ is nonempty, then each of its component is of dimension at least $n - r$ [29, p.178].

Ritt proved that the conjecture is correct when $r = 1$, that is, any component of a differential polynomial equation in $\mathcal{F}\{y_1, \ldots, y_n\}$ is of dimension $n-1$ [29, p.57]. The general dimension conjecture is still open. In [9], it is shown that the dimension conjecture is closely related with Jacobi's bound for the order of differential polynomial systems, which is another well-known conjecture in differential algebra.

In this paper, we consider the dimension and order for the intersection of a differential variety with generic differential hypersurfaces. A differential polynomial f is said to be generic of order s and degree m, if f contains all the monomials with degree less than or equal to m in $y_1, \ldots, y_n$ and their derivatives of order up to s, and the coefficients of f are differential indeterminates. A generic differential hypersurface is the set of solutions of a generic differential polynomial. We show that for generic differential hypersurfaces, we can determine the dimension and order of their intersection with an irreducible differential variety explicitly. More precisely, we will prove

Theorem 1.1. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$ with dimension d and order h and f a generic differential polynomial with order s and degree greater than zero. If $d > 0$, then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime differential ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle\{y_1, \ldots, y_n\}$ with dimension $d-1$ and order $h+s$, where $\mathbf{u}_f$ is the set of coefficients of f. And if $d = 0$, $\mathcal{I}_1$ is the unit ideal in $\mathcal{F}\langle \mathbf{u}_f \rangle\{y_1, \ldots, y_n\}$.*

As a direct consequence of this result, we show that the dimension conjecture is valid for a system of generic differential polynomials. Furthermore, the order of the system is also given explicitly.

Another purpose of studying the intersection of an irreducible variety with generic differential hypersurfaces is to establish the theory of the differential Chow form, which is the concern of the second part of the paper consisting of Sections 4 to 6.

The Chow form, also known as the Cayley form, is a basic concept in algebraic geometry [37, 16]. More recently, the Chow form also becomes a powerful tool in elimination theory. This is not surprising, since the Chow form is a resultant in certain sense. The Chow form was used as a tool to obtain deep results in transcendental number theory by Nesterenko [24] and Philippon [26]. Brownawell made a major breakthrough in elimination theory by developing new properties of the Chow form and proving an effective version of the Nullstellensatz with optimal bounds [3]. Gel'fand et al and Sturmfels started the sparse elimination theory which is to study the Chow form and the resultant associated with toric varieties [13, 36]. Eisenbud et al proposed a new expression for the Chow form via exterior algebra and used it to give explicit formulas in many new cases [11]. Jeronimo et al gave a bounded probabilistic algorithm which can be used to compute the Chow form, whose complexity is polynomial in the size and the geometric degree of the input equation system [17]. Other properties of the Chow form can be found in [5, 25, 27, 34]. Giving the fact that the Chow form plays an important role in both theoretic and algorithmic aspects of algebraic geometry and has applications in many fields, it is worthwhile to develop the theory of the differential Chow form and hope that it will play a similar role as its algebraic counterpart.

Let V be an irreducible differential variety of dimension d in an n-dimensional affine space and

$$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 0, \ldots, d)$$

$d+1$ generic primes in variables $y_1, \ldots, y_n$, where $u_{ij} \quad (i = 0, \ldots, d; j = 1, \ldots, n)$ are differential indeterminates. The differential Chow form of V is roughly defined to be the elimination differential polynomial in u_{ij} by intersecting V with $\mathbb{P}_i = 0 (i = 0, \ldots, d)$. More intuitively, the differential Chow form of V can be roughly considered as the condition on the coefficients of $\mathbb{P}_i$ such that these $d+1$ primes will meet V. We will show that most of the properties of the Chow form in the algebraic case presented in [16, 37] can be generalized to the differential case. Precisely, we will prove

Theorem 1.2. *Let V be an irreducible differential variety with dimension d and order h over a differential field $\mathcal{F}$ and $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \in \mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d\}$ the Chow form of V where $\mathbf{u}_i = (u_{i0}, u_{i1}, \ldots, u_{in})$ $(i = 0, 1, \ldots, d)$. Then $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ has the following properties:*

1. $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is differentially homogenous of the same degree in each $\mathbf{u}_i$ and $\mathrm{ord}(F, u_{ij}) = h$ for all u_{ij} occurring in F.

2. $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ can be factored uniquely into the following form

$$\begin{aligned} F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) &= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} (u_{00}^{(h)} + \sum_{\rho=1}^{n} u_{0\rho}^{(h)} \xi_{\tau\rho} + t_\tau) \\ &= A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} (u_{00} + \sum_{\rho=1}^{n} u_{0\rho} \xi_{\tau\rho})^{(h)} \end{aligned}$$

where $g = \deg(F, u_{00}^{(h)})$ and $\xi_{\tau\rho}$ are in an extension field of $\mathcal{F}$. The first " = " is obtained by factoring $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ as an algebraic polynomial in the variables $u_{00}^{(h)}, u_{01}^{(h)}, \ldots, u_{0n}^{(h)}$, while the second one is a differential expression by defining *derivatives of $\xi_{\tau\rho}$ to be*

$$\xi_{\tau\rho}^{(m)} = (\delta_u \xi_{\tau\rho}^{(m-1)})|_{u_{00}^{(h)} = -\sum_{\rho=1}^{n} u_{0\rho}^{(h)} \xi_{\tau\rho} - t_\tau} \, (m \geq 1)$$

recursively, where δ_u is the natural derivation over $\mathcal{F}\langle \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d \rangle$.

3. $\Xi_\tau = (\xi_{\tau 1}, \ldots, \xi_{\tau n})\,(\tau = 1, \ldots, g)$ are generic points of V. And they are the only elements of V lying on the primes $\mathbb{P}_\sigma = 0\,(\sigma = 1, \ldots, d)$ as well as on ${}^a\mathbb{P}_0^{(l)} = 0\,(l = 0, \ldots, h-1)$, where ${}^a\mathbb{P}_0^{(l)} = 0$ are algebraic equations.

4. Suppose that $\mathbf{u}_i (i = 0, \ldots, d)$ specialize to sets $\mathbf{v}_i$ of specific elements in an extension field of $\mathcal{F}$ and $\overline{\mathbb{P}}_i\,(i = 0, \ldots, d)$ are obtained by substituting $\mathbf{u}_i$ by $\mathbf{v}_i$ in $\mathbb{P}_i$. If $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, d)$ meet V, then $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$. Furthermore, if $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$ and $S_F(\mathbf{v}_0, \ldots, \mathbf{v}_d) \neq 0$, then the $d+1$ primes $\overline{\mathbb{P}}_i = 0$ $(i = 0, \ldots, d)$ meet V, where $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$.

The number g in the above theorem is called the leading differential degree of V. From the third statement of the theorem, we see that V intersects with $\mathbb{P}_\sigma = 0\,(\sigma = 1, \ldots, d)$ and ${}^a\mathbb{P}_0^{(l)} = 0\,(l = 0, \ldots, h-1)$ in exactly g points.

Furthermore, we prove that the four conditions given in Theorem 1.2 are also the sufficient conditions for a differential polynomial $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ to be the Chow form for an irreduicble differential variety. As a consequence of this result, we define the Chow quasi-variety for a special class of differential varieties in the sense that each point in the Chow quasi-variety represents a differential variety V in that class via the Chow form of V. This is clearly a generalization of the algebraic Chow variety.

In [26], Philippon considered the intersection of a variety of dimension d with $d+1$ homogeneous polynomials with generic coefficients and developed the theory for an elimination form which can be regarded as a type of generalized Chow form. In [2], Bost, Gillet, and Soulé further generalized the concept to generalized Chow divisors of cycles and estimated their heights. In this paper, we will introduce the generalized differential Chow form which is roughly defined to be the elimination differential polynomial obtained by intersecting an irreducible differential variety V of dimension d with $d+1$ generic differential hypersurfaces. We show that the generalized differential Chow form satisfies similar properties to that given in Theorem 1.2.

As an application of the generalized differential Chow form, we can define the differential resultant. The differential resultant for two nonlinear differential polynomials in one variable was studied by Ritt in [28, p.47]. General differential resultants were defined by Carra' Ferro [6, 7] using Macaulay's definition of algebraic resultant of polynomials. But, the treatment in [6] is not complete because a key fact, "the differential resultant is not identically zero for generic differential polynomials", is not established. Differential resultants for linear ordinary differential polynomials were studied by Rueda and Sendra in [32]. In this paper, a rigorous definition for the differential resultant of $n+1$ generic differential polynomials in n variables is given as the generalized differential Chow form of the prime ideal $I = [0]$. In this way, we obtain the following properties for differential resultants, which are similar to that of the Sylvester resultant for two algebraic univariate polynomials.

Theorem 1.3. *Let $\mathbb{P}_i (i = 0, \ldots, n)$ be generic differential polynomials in n variables $y_1, \ldots, y_n$ with orders s_i, degrees m_i, and constant terms u_{i0} respectively. Let*

$R(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ be the differential resultant of $\mathbb{P}_0, \ldots, \mathbb{P}_n$, where $\mathbf{u}_i$ is the set of coefficients of $\mathbb{P}_i$. Then

a) $R(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ is differentially homogeneous in each $\mathbf{u}_i$ and is of order $h_i = s - s_i$ in $\mathbf{u}_i$ $(i = 0, \ldots, n)$ with $s = \sum_{l=0}^{n} s_l$.

b) There exist $\xi_{\tau\rho}(\tau = 1, \ldots, t_0\,; \rho = 1, \ldots, n)$ in an extension field of $\mathcal{F}$ such that

$$R(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{t_0} \mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h_0)}$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is a differential polynomial in $\mathbf{u}_i$, $t_0 = \deg(R, u_{00}^{(h_0)})$, $\mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h_0)}$ is the (h_0)-th derivative of $\mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})$, and $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ $(\tau = 1, \ldots, t_0)$ are certain generic points of the zero dimensional prime ideal $[\mathbb{P}_1, \ldots, \mathbb{P}_n]$.

c) The differential resultant can be written as a linear combination of $\mathbb{P}_i$ and their derivatives up to the order $s - s_i$ $(i = 0, \ldots, n)$. Precisely, we have

$$R(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij} \mathbb{P}_i^{(j)}.$$

In the above expression, $h_{ij} \in \mathcal{F}\langle \mathbf{u} \rangle [y_1, \ldots, y_n, \ldots, y_1^{(s)}, \ldots, y_n^{(s)}]$ have degrees at most $(sn + n)^2 D^{sn+n} + D(sn + n)$, where $\mathbf{u} = \cup_{i=0}^{n} \mathbf{u}_i \setminus \{u_{00}, \ldots, u_{n0}\}$, $y_i^{(j)}$ is the j-th derivative of y_i, and $D = \max\{m_0, m_1, \ldots, m_n\}$.

d) Suppose that $\mathbf{u}_i (i = 0, \ldots, n)$ specialize to sets $\mathbf{v}_i$ of specific elements in an extension field of $\mathcal{F}$ and $\overline{\mathbb{P}}_i$ $(i = 0, \ldots, n)$ are obtained by substituting $\mathbf{u}_i$ by $\mathbf{v}_i$ in $\mathbb{P}_i$. If $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, n)$ have a common solution, then $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$. On the other hand, if $R(\mathbf{v}_0, \ldots, \mathbf{v}_n) = 0$ and $S_R(\mathbf{v}_0, \ldots, \mathbf{v}_n) \neq 0$, then $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, n)$ have a common solution in an extension field of $\mathcal{F}$, where $S_R = \frac{\partial R}{\partial u_{00}^{(h_0)}}$.

Properties a) and b) of differential resultants are first proved in this paper. Properties similar to c) and d) were proved in [28, p.47] and in [6, 7].

As a prerequisite result, we prove a useful property of differential specializations, which roughly says that if a set of differential polynomial functions in a set of indeterminates are differentially dependent, then they are still differentially dependent when the indeterminates are specialized to any concrete values. This property plays a key role throughout this paper. The algebraic version of this result is also a key result in algebraic elimination theory ([15, p.168], [38, p.161]).

It is not straightforward to extend the intersection theory for generic polynomials and the theory of Chow forms from the algebraic case to the differential case. Due to the complicated structure of differential polynomials, most proofs in the algebraic case cannot be directly used in the differential case. In particular, we need to consider the orders of differential polynomials, which is not an issue in the algebraic case. For instance, the second property of the differential Chow form in Theorem 1.2 has a different form as its algebraic counterpart.

One of the main tools used in the paper is the theory of characteristic set developed by Ritt [29, p.47]. The algorithmic character of Ritt's work on differential algebra is mainly due to the usage of characteristic sets. Properties of characteristic sets proved more recently in [1, 8, 12, 10, 38] will be also used in this paper.

The rest of this paper is organized as follows. In Section 2, we will present the notations and preliminary results used in this paper. In Section 3, the intersection theory for generic differential polynomials is given and Theorem 1.1 is proved. In Section 4, the Chow form for an irreducible differential variety is defined and its properties will be proved. Basically, we will prove Theorem 1.2. In Section 5, necessary and sufficient conditions for a differential polynomial to be the Chow form of an differential variety is given and the Chow quasi-variety for a class of differential varieties is defined. In Section 6, we present the theory of the generalized differential Chow form and the differential resultant. Theorem 1.3 will be proved. In Section 7, we present the conclusion.

2. Preliminaries

In this section, some basic notations and preliminary results in differential algebra will be given. For more details about differential algebra, please refer to [29, 18, 23, 35].

2.1. **Characteristic set of a differential polynomial set.**

Let $\mathcal{F}$ be an ordinary differential field of characteristic zero, with derivation δ. Let Θ denote the free commutative semigroup with unit (written multiplicatively) generated by δ. Let S be a subset of a differential ring $\mathcal{R}$ which contains $\mathcal{F}$. We will denote respectively by $\mathcal{F}[S]$ and $\mathcal{F}\{S\}$ the smallest subring and the smallest differential subring of $\mathcal{R}$ containing $\mathcal{F}$ and S. If we denote $\Theta(S)$ to be the smallest subset of $\mathcal{R}$ containing S and stable under δ, we have $\mathcal{F}\{S\} = \mathcal{F}[\Theta(S)]$. A differential ideal $\mathcal{I}$ of a differential ring is an ordinary algebraic ideal closed under derivation, i.e. $\delta(\mathcal{I}) \subset \mathcal{I}$. A prime differential ideal is a differential ideal distinct from the unit ideal, which is prime as an ordinary algebraic ideal. And a differential ideal is perfect(radical) if whenever some power of a differential polynomial f belongs to $\mathcal{I}$, f itself belongs to $\mathcal{I}$.

Now suppose $\mathbb{Y} = \{y_1, y_2, \ldots, y_n\}$ is a set of differential indeterminates. Let $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}[\Theta(\mathbb{Y})]$ be the ring of differential polynomials over $\mathcal{F}$, where $\Theta(\mathbb{Y}) = (\theta y_j)_{\theta\in\Theta, y_j\in\mathbb{Y}}$ is the set of derivatives of y_i. Throughout the paper, we also use $\eta^{(k)}$ to represent $\delta^k\eta$. Let f be a differential polynomial. We define the order of f w.r.t. y_i to be the greatest number k such that $y_i^{(k)}$ appears effectively in f, which is denoted by $\mathrm{ord}(f, y_i)$. And if y_i does not appear in f, then we set $\mathrm{ord}(f, y_i) = -\infty$. The *order* of f is defined to be $\max_i \mathrm{ord}(f, y_i)$, that is, $\mathrm{ord}(f) = \max_i \mathrm{ord}(f, y_i)$.

A *ranking* $\mathscr{R}$ is a total order over $\Theta(\mathbb{Y})$, which is compatible with the derivations over the alphabet:

1) $\delta\theta y_j > \theta y_j$ for all derivatives $\theta y_j \in \Theta(\mathbb{Y})$.
2) $\theta_1 y_i > \theta_2 y_j \Longrightarrow \delta\theta_1 y_i > \delta\theta_2 y_j$ for $\theta_1 y_j, \theta_2 y_j \in \Theta(\mathbb{Y})$.

By convention, $1 < \theta y_j$ for all $\theta y_j \in \Theta(\mathbb{Y})$.

Two important kinds of rankings are the followings:

1) *Elimination ranking*: $y_i > y_j \Longrightarrow \delta^k y_i > \delta^l y_j$ for any $k, l \geq 0$.
2) *Orderly ranking*: $k > l \Longrightarrow \delta^k y_i > \delta^l y_j$, for any $i, j \in \{1, 2, \ldots, n\}$.

Let p be a differential polynomial in $\mathcal{F}\{\mathbb{Y}\}$ and $\mathscr{R}$ a ranking endowed on it. The greatest derivative w.r.t. $\mathscr{R}$ which appears effectively in p is called the *leader* of p, which will be denoted by u_p or $\mathrm{ld}(p)$. The two conditions mentioned above imply that the leader of θp is θu_p for $\theta \in \Theta$. Let the degree of p in u_p be d. We rewrite p as an algebraic polynomial in u_p. Then

$$p = I_d u_p^d + I_{d-1} u_p^{d-1} + \cdots + I_0.$$

We call I_d the *initial* of p and denote it by I_p. The partial derivative of p w.r.t. u_p is called the *separant* of p, which will be denoted by S_p. Clearly, S_p is the initial of any proper derivative of p. The *rank* of p is u_p^d, and we denote it by $\mathrm{rk}(p)$. For any two differential polynomials p, q in $\mathcal{F}\{\mathbb{Y}\}\backslash\mathcal{F}$, p is said to be of *lower rank* than q if either $u_p < u_q$ or $u_p = u_q = u$ and $\deg(p, u) < \deg(q, u)$. By convention, any element of $\mathcal{F}$ is of lower rank than elements of $\mathcal{F}\{\mathbb{Y}\}\backslash\mathcal{F}$. We denote $p \preceq q$ if and only if either p is of lower rank than q or they have the same rank. Clearly, $\preceq$ is a totally ordering of $\mathcal{F}\{\mathbb{Y}\}$.

Let p and q be two differential polynomials and u_p^d the rank of p. q is said to be *partially reduced* w.r.t. p if no proper derivatives of u_p appears in q. q is said to be *reduced* w.r.t. p if q is partially reduced w.r.t. p and $\deg(q, u_p) < d$. Let $\mathcal{A}$ be a set of differential polynomials. $\mathcal{A}$ is said to be an *auto-reduced set* if each polynomial of $\mathcal{A}$ is reduced w.r.t. any other element. Every auto-reduced set is finite.

Let $\mathcal{A}$ be an auto-reduced set. We denote $\mathrm{H}_{\mathcal{A}}$ to be the set of all the initials and separants of $\mathcal{A}$ and $\mathrm{H}_{\mathcal{A}}^{\infty}$ to be the minimal multiplicative set containing $\mathrm{H}_{\mathcal{A}}$. The *saturation ideal* of $\mathcal{A}$ is defined to be $\mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} = \{p : \exists h \in H_{\mathcal{A}}^{\infty} s.t.\, hp \in [A]\}$.

Let $\mathcal{A}{=}A_1, A_2, \ldots, A_s$ and $\mathcal{B} = B_1, B_2, \ldots, B_l$ be two auto-reduced sets with the A_i, B_j arranged in nondecreasing ordering. $\mathcal{A}$ is said to be of *lower rank* than $\mathcal{B}$, if either 1) there is some k ($\leq$min{s,l}) such that for each $i < k$, A_i has the same rank as B_i, and $A_k \prec B_k$ or 2) $s > l$ and for each $i \in \{1, 2, \ldots, l\}$, A_i has the same rank as B_i. It is easy to see that the above definition introduces really a partial ordering among all auto-reduced sets. Any sequence of auto-reduced sets steadily decreasing in ordering $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \cdots \mathcal{A}_k \succ \cdots$ is necessarily finite.

Let $\mathcal{A} = A_1, A_2, \ldots, A_p$ be an auto-reduced set with S_i and I_i as the separant and initial of A_i, and f any differential polynomial. Then there exists an algorithm, called Ritt's algorithm of reduction, which reduces f w.r.t. $\mathcal{A}$ to a polynomial r that is reduced w.r.t. $\mathcal{A}$, satisfying the relation

$$\prod_{i=1}^{p} \mathrm{S}_i^{d_i} \mathrm{I}_i^{e_i} \cdot f \equiv r, \mathrm{mod}[\mathcal{A}],$$

for nonnegative integers d_i, e_i $(i = 1, 2, \ldots, p)$. We call r the *pseudo remainder* of f w.r.t. $\mathcal{A}$.

Definition 2.1. For an auto-reduced set $\mathcal{A} = A_1, A_2, \ldots, A_p$, with $\mathrm{ld}(A_i) = y_{c_i}^{(o_i)}$, the order of $\mathcal{A}$ is defined to be $\mathrm{ord}(\mathcal{A}) = \sum_{i=1}^{p} o_i$, and the set $\mathbb{Y}\backslash\{y_{c_1}, \ldots, y_{c_p}\}$ is called a parametric set of $\mathcal{A}$.

An auto-reduced set $\mathcal{C}$ contained in a differential polynomial set $\mathcal{S}$ is said to be a *characteristic set* of $\mathcal{S}$, if $\mathcal{S}$ does not contain any nonzero element reduced w.r.t. $\mathcal{C}$. All the characteristic sets of $\mathcal{S}$ have the same and minimal rank among all auto-reduced sets contained in $\mathcal{S}$. A characteristic set $\mathcal{C}$ of an ideal $\mathcal{J}$ reduces to zero all elements of $\mathcal{J}$. If the ideal is prime, $\mathcal{C}$ reduces to zero only the elements of $\mathcal{J}$ and we have $\mathcal{J} = \mathrm{sat}(\mathcal{C})$ ([18, Lemma 2, p.167]).

In ordinary differential algebra, we can define an auto-reduced set to be *irreducible* if when considered as an algebraic auto-reduced set in the underlying polynomial ring, it is irreducible. We have ([29, p.107])

Theorem 2.2. *Let $\mathcal{A}$ be an auto-reduced set. Then a necessary and sufficient condition for $\mathcal{A}$ to be a characteristic set of a prime differential ideal is that $\mathcal{A}$ is* *irreducible. Moreover, in the case $\mathcal{A}$ is irreducible,* $\mathrm{sat}(\mathcal{A})=[\mathcal{A}]:H_{\mathcal{A}}^{\infty}$ *is prime with $\mathcal{A}$ being a characteristic set of it.*

Remark 2.3. A set of differential polynomials $\mathcal{A}=\{A_1\ldots,A_p\}$ is called a *differential chain* if the following conditions are satisfied,

1) the leaders of A_i are differentially auto-reduced,
2) each A_i is partially reduced w.r.t. all the others,
3) no initial of an element of $\mathcal{A}$ is reduced to zero by $\mathcal{A}$.

Similar properties to auto-reduced sets can be developed for differential chains [10]. In particular, we can define a differential characteristic set of a differential ideal $\mathcal{I}$ to be a differential chain contained in $\mathcal{I}$ of minimal rank among all the differential chains contained in $\mathcal{I}$. So, in this paper we will not distinguish auto-reduced sets and differential chains. Note that we can also use the weak differential chains introduced in [8].

2.2. **Dimension and order of a prime differential ideal.** A subset Σ of a differential extension field $\mathcal{G}$ of $\mathcal{F}$ is said to be *differentially dependent* over $\mathcal{F}$ if the set $(\theta\alpha)_{\theta\in\Theta,\alpha\in\Sigma}$ is algebraically dependent over $\mathcal{F}$, and is said to be *differentially independent* over $\mathcal{F}$, or to be a family of *differential indeterminates* over $\mathcal{F}$, in the contrary case. In the case Σ consists of one element α, we say that α is differentially algebraic or differentially transcendental over $\mathcal{F}$ respectively. The maximal subset Ω of $\mathcal{G}$ which are differentially independent over $\mathcal{F}$ is said to be a differential transcendence basis of $\mathcal{G}$ over $\mathcal{F}$. We use $\mathrm{d.tr.deg}\,\mathcal{G}/\mathcal{F}$ (see [18, p.105-109]) to denote the *differential transcendence degree* of $\mathcal{G}$ over $\mathcal{F}$, which is the cardinal number of Ω. Considering $\mathcal{F}$ and $\mathcal{G}$ as ordinary algebraic fields, we denote the transcendence degree of $\mathcal{G}$ over $\mathcal{F}$ by $\mathrm{tr.deg}\,\mathcal{G}/\mathcal{F}$.

Let $\mathcal{E}$ be a universal extension field of $\mathcal{F}$ [18, p.107]. In this paper, by differential indeterminates, we mean they are differentially independent over $\mathcal{E}$ unless mentioned otherwise. Let Σ be a subset of differential polynomials and D a differential polynomial in $\mathcal{F}\{\mathbb{Y}\}$ respectively. A point $\eta=(\eta_1,\ldots,\eta_n)\in\mathcal{E}^n$ is called a differential zero of Σ if $f(\eta)=0$ for any $f\in\Sigma$. The set of differential zeros of Σ is denoted by $\mathbb{V}(\Sigma)$, which is called a *differential variety.* And $\mathbb{V}(\Sigma/D)=\mathbb{V}(\Sigma)\setminus\mathbb{V}(D)$ is called a *differential quasi-variety.* By convenience, we also call $\cup_{i=1}^m\mathbb{V}(\Sigma_i/D_i)$ a differential quasi-variety, where Σ_i and D_i are differential polynomial sets and differential polynomials respectively.

For a differential variety V, we denote $\mathbb{I}(V)$ to be the perfect differential ideal corresponding to the differential variety V. Let $\mathcal{I}$ be a prime differential ideal and $\xi=(\xi_1,\ldots,\xi_n)$ a generic point of $\mathcal{I}$ [18, p.19]. Then $\mathcal{F}\langle\xi_1,\ldots,\xi_n\rangle$ is called a differential function field of $\mathcal{I}$. The *dimension* of $\mathcal{I}$ or of $\mathbb{V}(\mathcal{I})$ is defined to be the differential transcendence degree of its differential function field of $\mathcal{I}$ over $\mathcal{F}$, that is, $\dim(\mathcal{I})=\mathrm{d.tr.deg}\,\mathcal{F}\langle\xi_1,\ldots,\xi_n\rangle/\mathcal{F}$.

In [29], Ritt gave another definition of the dimension of $\mathcal{I}$. An independent set modulo $\mathcal{I}$ is defined to be a variable set $\mathbb{U}\subset\{\mathbb{Y}\}$ such that $\mathcal{I}\cap\mathcal{F}\{\mathbb{U}\}=\{0\}$, and in this case $\mathbb{U}$ is also said to be differentially independent modulo $\mathcal{I}$. And a *parametric set* of $\mathcal{I}$ is a maximal independent set modulo $\mathcal{I}$. Then Ritt defined the dimension of $\mathcal{I}$ to be the cardinal number of its parametric set. Clearly, the two definitions are equivalent.

Definition 2.4. [21] Let $\mathcal{I}$ be a prime differential ideal of $\mathcal{F}\{\mathbb{Y}\}$ with a generic point $\eta=(\eta_1,\ldots,\eta_n)$. Then there exists a unique numerical polynomial $\omega_{\mathcal{I}}(t)$ such that $\omega_{\mathcal{I}}(t) = \text{tr.deg}\,\mathcal{F}(\eta_i^{(j)} : i = 1, \ldots, n; j \leq t)/\mathcal{F}$ for all sufficiently large $t \in \mathbb{N}$. $\omega_{\mathcal{I}}(t)$ is called the *differential dimension polynomial* of $\mathcal{I}$.

Theorem 2.5. [30, Theorem 13] *Let $\mathcal{I}$ be a prime differential ideal of dimension d. Then the differential dimension polynomial has the form $\omega_{\mathcal{I}}(t) = d(t+1) + h$, where h is defined to be the order of $\mathcal{I}$ or of $\mathbb{V}(\mathcal{I})$, that is,* $\text{ord}(\mathcal{I}) = h$. *Let $\mathcal{A}$ be a characteristic set of $\mathcal{I}$ under any orderly ranking. Then,* $\text{ord}(\mathcal{I}) = \text{ord}(\mathcal{A})$.

In [29], Ritt introduced the concept of relative order for a prime ideal w.r.t. a particular parametric set.

Definition 2.6. Let $\mathcal{I}$ be a prime differential ideal of $\mathcal{F}\{\mathbb{Y}\}, \mathcal{A}$ a characteristic set of $\mathcal{I}$ w.r.t. any elimination ranking, and $\{u_1, \ldots, u_d\} \subset \mathbb{Y}$ the parametric set of $\mathcal{A}$. The *relative order* of $\mathcal{I}$ w.r.t. $\{u_1, \ldots, u_d\}$, denoted by $\text{ord}_{u_1,\ldots,u_d}\mathcal{I}$, is defined to be $\text{ord}(\mathcal{A})$.

The relative order of a prime ideal $\mathcal{I}$ can be computed from its generic points as shown by the following result ([20]).

Corollary 2.7. Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with a generic point $(\xi_1, \ldots, \xi_n)$. If $\{y_1, \ldots, y_d\}$ is a parametric set of $\mathcal{I}$, then $\text{ord}_{y_1,\ldots,y_d}(\mathcal{I}) = \text{tr.deg}\,\mathcal{F}\langle\xi_1, \ldots, \xi_d\rangle\langle\xi_{d+1}, \ldots, \xi_n\rangle/\mathcal{F}\langle\xi_1, \ldots, \xi_d\rangle$.

Ritt's definition of relative order is based on the elimination ranking. Hubert proved that all characteristic sets of $\mathcal{I}$ admitting the same parametric set have the same order [10].

Theorem 2.8. [10] *Let $\mathcal{A}$ be a characteristic set of a prime differential ideal $\mathcal{I}$ in $\mathcal{F}\{\mathbb{Y}\}$ endowed with any ranking. The parametric set $\mathbb{U}$ of $\mathcal{A}$ is a maximal independent set modulo $\mathcal{I}$. Its cardinal gives the differential dimension of $\mathcal{I}$. Furthermore, the order of $\mathcal{I}$ relative to $\mathbb{U}$ is the order of $\mathcal{A}$.*

Corollary 2.9. Let $\mathcal{I}$ be a prime differential ideal with dimension zero, and $\mathcal{A}$ a characteristic set of $\mathcal{I}$ w.r.t. any ranking $\mathscr{R}$. Then $\text{ord}(\mathcal{I}) = \text{ord}(\mathcal{A})$.

The following result gives the relation between the order and relative order for a prime ideal.

Theorem 2.10. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$. Then* $\text{ord}(\mathcal{I})$ *is the maximum of all the relative orders of $\mathcal{I}$, that is,* $\text{ord}(\mathcal{I}) = \max_{\mathbb{U}}\text{ord}_{\mathbb{U}}(\mathcal{I})$, *where $\mathbb{U}$ is a parametric set of $\mathcal{I}$.*

Proof: Let $\mathcal{C}$ be a characteristic set of $\mathcal{I}$ w.r.t. some orderly ranking. Firstly, we claim that any relative order of $\mathcal{I}$ is less than or equal to $\text{ord}(\mathcal{C})$. Let $\mathbb{U} = \{u_1, \ldots, u_q\}$ be any parametric set of $\mathcal{I}$, $\{y_1, \ldots, y_p\}(p + q = n)$ the set of the remaining variables, and $\mathcal{B}$ any characteristic set of $\mathcal{I}$ w.r.t. the elimination ranking $u_1 \prec \ldots \prec u_q \prec y_1 \prec \ldots \prec y_p$. By Theorem 2.8, it suffices to prove $\text{ord}_{\mathbb{U}}(\mathcal{I}) \leq \text{ord}(\mathcal{C})$.

Let $\eta = (\overline{u_1}, \ldots, \overline{u_q}, \overline{y_1}, \ldots, \overline{y_p})$ be a generic point of $\mathcal{I}$. Then for sufficiently large t, the differential dimension polynomial of $\mathcal{I}$ is

$$
\begin{aligned}
& \omega_{\mathcal{I}}(t) \\
= \ & \omega_{\eta/\mathcal{F}}(t) \\
= \ & \mathrm{tr.deg}\, \mathcal{F}(\delta^s \overline{u_i}, \delta^k \overline{y_j} : s, k \leq t; i = 1, \ldots, q; j = 1, \ldots, p)/\mathcal{F} \\
= \ & \mathrm{tr.deg}\, \mathcal{F}(\delta^s \overline{u_i} : s \leq t)(\delta^k \overline{y_j} : k \leq t)/\mathcal{F}(\delta^s \overline{u_i} : s \leq t) + \mathrm{tr.deg}\, \mathcal{F}(\delta^s \overline{u_i} : s \leq t)/\mathcal{F} \\
= \ & q(t+1) + \mathrm{tr.deg}\, \mathcal{F}(\delta^s \overline{u_i} : s \leq t)(\delta^k \overline{y_j} : k \leq t)/\mathcal{F}(\delta^s \overline{u_i} : s \leq t)
\end{aligned}
$$

Since $\omega_{\mathcal{I}}(t) = q(t+1) + \mathrm{ord}(\mathcal{C})$, $\mathrm{tr.deg}\, \mathcal{F}(\delta^s \overline{u_i} : s \leq t)(\delta^k \overline{y_j} : k \leq t)/\mathcal{F}(\delta^s \overline{u_i} : s \leq t) = \mathrm{ord}(\mathcal{C})$. By Corollary 2.7, we have

$$
\begin{aligned}
& \mathrm{ord}_{\mathbb{U}}(\mathcal{I}) \\
= \ & \mathrm{tr.deg}\, \mathcal{F}\langle \overline{u_1}, \ldots, \overline{u_q} \rangle (\delta^k \overline{y_l} : k \geq 0)/\mathcal{F}\langle \overline{u_1}, \ldots, \overline{u_q} \rangle \\
= \ & \mathrm{tr.deg}\, \mathcal{F}\langle \overline{u_1}, \ldots, \overline{u_q} \rangle (\delta^k \overline{y_l} : k \leq t)/\mathcal{F}\langle \overline{u_1}, \ldots, \overline{u_q} \rangle \quad (\text{for } t \geq \mathrm{ord}(\mathcal{B})) \\
\leq \ & \mathrm{tr.deg}\, \mathcal{F}(\delta^s \overline{u_i} : s \leq t)(\delta^k \overline{y_j} : k \leq t)/\mathcal{F}(\delta^s \overline{u_i} : s \leq t) \\
= \ & \mathrm{ord}(\mathcal{C}).
\end{aligned}
$$

Thus, the claim is proved.

Now, let $\mathbb{U}^*$ be the parametric set of $\mathcal{C}$. Then, by Theorem 2.8, $\mathrm{ord}(\mathcal{I}) = \mathrm{ord}(C) = \mathrm{ord}_{\mathbb{U}^*}(\mathcal{I})$. That is, for any parametric set $\mathbb{U}$ of $\mathcal{I}$, we have $\mathrm{ord}_{\mathbb{U}}(\mathcal{I}) \leq \mathrm{ord}(\mathcal{I})$ and there exists one parametric set $\mathbb{U}^*$ of $\mathcal{I}$ such that $\mathrm{ord}_{\mathbb{U}^*}(\mathcal{I}) = \mathrm{ord}(\mathcal{I})$. As a consequence, $\mathrm{ord}(\mathcal{I}) = \max_{\mathbb{U}} \mathrm{ord}_{\mathbb{U}}(\mathcal{I})$. □

The following well known result about the adjoining indeterminates to the base field will be used in this paper [29, p.18].

Lemma 2.11. *Let $\mathbb{U} = \{u_1, \ldots, u_r\}$ be a set of differential indeterminates over $\mathcal{F}$, $\mathcal{I}_0$ a prime differential ideal of dimension d and order h in $\mathcal{F}\{\mathbb{Y}\}$, and $\mathcal{I}$ the differential ideal generated by $\mathcal{I}_0$ in $\mathcal{F}\langle\mathbb{U}\rangle\{\mathbb{Y}\}$. Then $\mathcal{I}$ is a prime ideal of dimension d and order h.*

2.3. **A property on differential specialization.** The following lemma is a key result in algebraic elimination theory, which is also used to develop the theory of Chow form ([15, p.168-169], [38, p.161]).

Lemma 2.12. *Let $P_i \in \mathcal{F}[\mathbb{U}, \mathbb{Y}]\ (i = 1, \ldots, m)$ be polynomials in the indeterminates $\mathbb{U} = (u_1, \ldots, u_r)$ and $\mathbb{Y} = (y_1, \ldots, y_d)$. Let $\mathbb{Y}^0 = (y_1^0, \ldots, y_d^0)$, with y_i^0 elements of some extension field of $\mathcal{F}$. If $P_i(\mathbb{U}, \mathbb{Y}^0)\ (i = 1, \ldots, m)$ are algebraically dependent over $\mathcal{F}(\mathbb{U})$, then for any specialization $\mathbb{U}^0$ of $\mathbb{U}$ over $\mathcal{F}$, $P_i(\mathbb{U}^0, \mathbb{Y}^0)\ (i = 1, \ldots, m)$ are algebraically dependent over $\mathcal{F}$.*

To generalize the above result to the differential case, we need the following lemma [29, p35].

Lemma 2.13. *Suppose $\mathcal{F}$ contains at least one nonconstant element. If $G \in \mathcal{F}\{u\}$ is a nonzero differential polynomial with order r, then for any nonconstant $\eta \in \mathcal{F}$, there exists an element $c_0 + c_1\eta + c_2\eta^2 + \cdots + c_r\eta^r$ which does not annul G, where $c_0, \ldots, c_r$ are constants in $\mathcal{F}$.*

Now we prove the following result, which is crucial throughout the paper.

Theorem 2.14. *Let $\mathbb{U} = \{u_1, \ldots, u_r\}$ be a set of differential indeterminates, and $P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}$ $(i = 1, \ldots, m)$ differential polynomials in the differential indeterminates $\mathbb{U} = (u_1, \ldots, u_r)$ and $\mathbb{Y} = (y_1, \ldots, y_n)$. Let $\mathbb{Y}^0 = (y_1^0, y_2^0, \ldots, y_n^0)$, where y_i^0 are in some differential extension field of $\mathcal{F}$. If $P_i(\mathbb{U}, \mathbb{Y}^0)$ $(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any specialization $\mathbb{U}$ to $\mathbb{U}^0$ in $\mathcal{F}$, $P_i(\mathbb{U}^0, \mathbb{Y}^0)\,(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}$.*

Proof: It suffices to prove the case $r = 1$. Denote u_1 by u. Firstly, we suppose $\mathcal{F}$ contains at least one nonconstant element.

Since $P_i(u, \mathbb{Y}^0)\,(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}\langle u\rangle$, there exists a nonzero $G(z_1, \ldots, z_m) \in \mathcal{F}\langle u\rangle\{z_1, \ldots, z_m\}$ such that $G(P_1(u, \mathbb{Y}^0), \ldots, P_m(u, \mathbb{Y}^0))$ $= 0$. We can take $G \in \mathcal{F}\{u, z_1, \ldots, z_m\}$ by clearing the denominators when necessary.

Since $G(u, z_1, \ldots, z_m) \neq 0$, by Lemma 2.13, for any nonconstant $\eta \in \mathcal{F}$, there exist constants $c_0, \ldots, c_s$ $(s = \mathrm{ord}(G, u))$ of $\mathcal{F}$, such that $G(\overline{u}, z_1, \ldots, z_m) \neq 0$ where $\overline{u} = \sum_{i=0}^{s} c_i\eta^i$. Now regarding the c_i as arbitrary constants over $\mathcal{F}$, then $\overline{G} = G(\sum_{i=0}^{s} c_i\eta^i, z_1, \ldots, z_m) \neq 0$ and $\overline{G}(P_1(\overline{u}, \mathbb{Y}^0), \ldots, P_m(\overline{u}, \mathbb{Y}^0)) = G(\overline{u}, P_1(\overline{u}, \mathbb{Y}^0), \ldots,$ $P_m(\overline{u}, \mathbb{Y}^0)) = 0$. Regarding $\overline{G}$ as an algebraic polynomial in $c_i\,(i = 0, \ldots, s)$ and $z_i^{(j)}$ $(i = 1, \ldots, m; j \geq 0)$ which appear effectively, we have

$$\overline{G}(c_0, \ldots, c_s, \ldots, z_i^{(j)}, \ldots) \neq 0$$

and

$$\overline{G}(c_0, \ldots, c_s, \ldots, (P_i(\overline{u}, \mathbb{Y}^0))^{(j)}, \ldots) = 0.$$

So $P_i(\overline{u}, \mathbb{Y}^0))^{(j)}$ $(i = 1, \ldots, m; j \geq 0)$ are algebraically dependent over $\mathcal{F}(c_0, \ldots, c_s)$, by Lemma 2.12, when the c_i specialize to constants c_i^0 in $\mathcal{F}$, the corresponding $P_i(\overline{u}^0, \mathbb{Y}^0)^{(j)}\,(i = 1, \ldots, m)$ are algebraically dependent over $\mathcal{F}$, where $\overline{u}^0 = \sum_{i=0}^{s} c_i^0\eta^i$. That is, $P_i(\overline{u}^0, \mathbb{Y}^0)\,(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}$. To complete the proof, if u^0 is a nonconstant, as above we take $\eta = u^0$, and specialize $c_1 \to 1$ and other c_i to zero; else we take η as an arbitrary nonconstant and specialize $c_0 \to u^0$ and other c_i to zero. Then in either case, u specializes to u^0, and we have completed the proof in the case that $\mathcal{F}$ contains at least one nonconstant element.

If $\mathcal{F}$ consists of constant elements, take a differential indeterminate v independent over $\mathcal{F}\langle\mathbb{U}^0, \mathbb{Y}^0\rangle$. Now we consider in the differential field $\mathcal{F}\langle v\rangle$. Following the first case, for any specialization $\mathbb{U}$ to $\mathbb{U}^0 \subset \mathcal{F}$, we can show that $P_i(\mathbb{U}^0, \mathbb{Y}^0)\,(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}\langle v\rangle$. Since v is differentially independent with $P_i(\mathbb{U}^0, \mathbb{Y}^0)\,(i = 1, \ldots, m)$, $P_i(\mathbb{U}^0, \mathbb{Y}^0)\,(i = 1, \ldots, m)$ are differentially dependent over $\mathcal{F}$. $\square$

From the proof above, we can obtain the following result easily:

Corollary 2.15. Let $\mathbb{U} = \{u_1, \ldots, u_r\}$ be a set of differential indeterminates, and $P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\{\mathbb{U}, \mathbb{Y}\}\,(i = 1, \ldots, m)$ differential polynomials in the indeterminates $\mathbb{U}$ and $\mathbb{Y} = (y_1, \ldots, y_n)$. Let $\mathbb{Y}^0 = (y_1^0, y_2^0, \ldots, y_n^0)$, where y_i^0 are in some differential extension field of $\mathcal{F}$. If the set $(P_i(\mathbb{U}, \mathbb{Y}^0))^{(\sigma_{ij})}$ $(i = 1, \ldots, m; j = 1, \ldots, n_i)$ are algebraically dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any specialization $\mathbb{U}$ to $\mathbb{U}^0$ in $\mathcal{F}$, $(P_i(\mathbb{U}^0, \mathbb{Y}^0))^{(\sigma_{ij})}(i = 1, \ldots, m; j = 1, \ldots, n_i)$ are algebraically dependent over $\mathcal{F}$.

For convenience, we will assume that $\mathcal{F}$ contains at least one nonconstant element in the rest of this paper.

3. Intersection theory for generic differential polynomials

In this section, we will develop an intersection theory for generic differential polynomials by proving Theorem 1.1. As a consequence, the dimension conjecture is shown to be true for generic differential polynomials. These results will also be used in Sections 4 and 6 to compute the order of the differential Chow form.

3.1. **Generic dimension theorem.**

In this section, we will show that the dimension conjecture is valid for certain generic differential polynomials. To prove the dimension conjecture in the general case, one simple idea is to generalize the following theorem ([18, p.43]) in algebra to the differential case.

Theorem 3.1. *Let $\mathcal{I}$ be a prime ideal of dimension $d > 0$ and $f \in \mathcal{F}[\mathbb{Y}]$. If $(\mathcal{I}, f) \neq (1)$, then every prime component of $(\mathcal{I}, f)$ has dimension not less than $d-1$. Moreover, if f is not in $\mathcal{I}$, then every prime component of $(\mathcal{I}, f)$ has dimension $d-1$.*

Unfortunately, in the differential case, the above theorem does not hold. Ritt gave a counter example.

Example 3.2. [29, p.133] $F = y_1^5 - y_2^5 + y_3(y_1y_2' - y_2y_1')^2 \in \mathcal{F}\{y_1, y_2, y_3\}$ and $f = y_3$, where $\mathcal{F}$ is the field of complex numbers. Then $\mathrm{sat}(F)$ is a prime ideal of dimension two. But, $\{\mathrm{sat}(F), f\} = [y_1, y_2, y_3]$ which is a prime ideal of dimension zero.

It could also happen that when adding a differential polynomial to a prime ideal, the dimension is still the same.

Example 3.3. Let $F = y_1'y_2'' - y_1''y_2'$. Then $\mathrm{sat}(F) = [F] : y_1'^\infty$ is a prime ideal of dimension one. It is clear that $y_2' \notin \mathrm{sat}(F)$ and $[\mathrm{sat}(F), y_2'] = [y_2']$ is still a prime ideal of dimension one.

In this section, we will prove that Theorem 3.1 is valid for certain generic differential polynomials, which will lead to the solution to the dimension conjecture in these generic cases.

A *generic primal* of degree m in an algebraic polynomial ring $\mathcal{F}[x_1, \ldots, x_n]$ is of the form

$$\sum u_{i_1\ldots i_n} x_1^{i_1} \cdots x_n^{i_n} = 0 \, (i_1 + \cdots + i_n \leq m)$$

where $u_{i_1\ldots i_n}$ are indeterminates.

Definition 3.4. A differential polynomial f is said to be a generic differential polynomial over $\mathcal{F}\{\mathbb{Y}\}$ of order s and degree m, if

1) f is a differential polynomial in $y_1, \ldots, y_n$ of order s with differential indeterminates as coefficients.

2) Regarded as an algebraic polynomial in $y_i, y_i', \ldots, y_i^{(s)}$ $(i = 1, \ldots, n)$, it is a generic primal of degree m.

A generic differential hypersurface is the set of solutions of a generic differential polynomial. Throughout this paper, a generic differential polynomial is assumed to be of degree greater than zero. We use $\mathbf{u}_f$ to denote the set of coefficients of a generic differential polynomial f.

Lemma 3.5. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d and f a generic differential polynomial of degree greater than zero. Then $\mathcal{I}_0 = [\mathcal{I}, f]$ is a* *prime differential ideal in* $\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{y_1,\ldots,y_n,u_0\}$ *with dimension* d*, where* u_0 *is the constant term of* f *and* $\widetilde{\mathbf{u}}=\mathbf{u}_f\setminus\{u_0\}$*. Furthermore,* $\mathcal{I}_0\cap\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{u_0\}=\{0\}$ *if and only if* $d>0$*.*

Proof: Let $(\xi_1,\ldots,\xi_n)$ be a generic point of $\mathcal{I}$ and $f=u_0+\sum_{i=1}^n\sum_{j=0}^s u_{ij}y_i^{(j)}+f_0$, where f_0 denotes the nonlinear part of f in $y_i^{(j)}$. Clearly, $(\xi_1,\ldots,\xi_n,-\sum_{i=1}^n\sum_{j=0}^s u_{ij}\xi_i^{(j)}-f_0(\xi_1,\ldots,\xi_n))$ is a generic point of $\mathcal{I}_0$, so $\mathcal{I}_0$ is a prime differential ideal. By Lemma 2.11,

$$\begin{aligned}\dim\mathcal{I}_0 &= \text{d.tr.deg}\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\langle\xi_1,\ldots,\xi_n,-\sum_{i=1}^n\sum_{j=0}^s u_{ij}\xi_i^{(j)}-f_0(\xi_1,\ldots,\xi_n)\rangle/\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\\ &= \text{d.tr.deg}\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\langle\xi_1,\ldots,\xi_n\rangle/\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\\ &= \text{d.tr.deg}\,\mathcal{F}\langle\xi_1,\ldots,\xi_n\rangle/\mathcal{F}=d.\end{aligned}$$

Now consider the second part of the lemma. If $d=0$, then $\dim\mathcal{I}_0=0$, so $\mathcal{I}_0\cap\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{u_0\}\neq\{0\}$. Thus, if $\mathcal{I}_0\cap\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{u_0\}=\{0\}$, then $d>0$. It remains to show that if $d>0$, then $\mathcal{I}_0\cap\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{u_0\}=\{0\}$. Suppose the contrary, then there exists a nonzero differential polynomial $p(\widetilde{\mathbf{u}},u_0)\in\mathcal{I}_0\cap\mathcal{F}\{\widetilde{\mathbf{u}},u_0\}$. So $p(\widetilde{\mathbf{u}},-\sum_{i=1}^n\sum_{j=0}^s u_{ij}\xi_i^{(j)}-f_0(\xi_1,\ldots,\xi_n))=0$. Then, $\phi=-\sum_{i=1}^n\sum_{j=0}^s u_{ij}\xi_i^{(j)}-f_0(\xi_1,\ldots,\xi_n)$ is differentially algebraic over $\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle$. So for any fixed i when u_{i0} specializes to -1 and all the other $u\in\widetilde{\mathbf{u}}$ specialize to zero, by Theorem 2.14, we conclude that $\overline{\phi}=\xi_i(i=1,\ldots,n)$ is differentially algebraic over $\mathcal{F}$, which contradicts to the fact that $\mathcal{I}$ has a positive dimension. So $\mathcal{I}_0\cap\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{u_0\}=\{0\}$. □

We will prove the first key result of this paper. The following result shows that by adding a generic differential polynomial to a prime ideal, the new ideal is still prime and its dimension decreases by one. This is generally not valid if the polynomial is not generic as shown in Examples 3.2 and 3.3.

Theorem 3.6. *Let* $\mathcal{I}$ *be a prime differential ideal in* $\mathcal{F}\{\mathbb{Y}\}$ *with dimension* d *and* f *a generic differential polynomial with degree greater than zero. If* $d>0$*, then* $\mathcal{I}_1=[\mathcal{I},f]$ *is a prime differential ideal in* $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ *with dimension* $d-1$*, where* $\mathbf{u}_f$ *is the set of coefficients of* f*. And if* $d=0$*, then* $\mathcal{I}_1$ *is the unit ideal in* $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$*.*

Proof: Firstly, we consider the case $d>0$. Let $(\xi_1,\ldots,\xi_n)$ be a generic point of $\mathcal{I}$, $f=u_0+\sum_{i=1}^n\sum_{j=0}^s u_{ij}y_i^{(j)}+f_0$ where f_0 denotes the nonlinear part of f in $y_i^{(j)}$. Let $\mathcal{I}_0=[\mathcal{I},f]$ in $\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{y_1,\ldots,y_n,u_0\}$ where $\widetilde{\mathbf{u}}=\mathbf{u}_f\setminus\{u_0\}$. By Lemma 3.5, $\mathcal{I}_0\cap\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{u_0\}=\{0\}$. So $\mathcal{I}_1=[\mathcal{I}_0]$ in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ is not the unit ideal and consequently $\mathcal{I}_1$ is prime with $\mathcal{I}_1\cap\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{y_1,\ldots,y_n,u_0\}=\mathcal{I}_0$.

Suppose $\xi_1,\ldots,\xi_d$ are differentially independent over $\mathcal{F}$. Then, $\{y_1,\ldots,y_d\}$ is a parametric set of $\mathcal{I}$. Thus each $y_{d+i}(i=1,\ldots,n-d)$ is differentially dependent with $y_1,\ldots,y_d$ modulo $\mathcal{I}_1$, since $\mathcal{I}\subset\mathcal{I}_1$. By Lemma 3.5, $\dim\mathcal{I}_0=d$. Then $u_0,y_1,\ldots,y_d$ are differentially dependent modulo $\mathcal{I}_0$, so $\{y_1,\ldots,y_d\}$ is differentially dependent modulo $\mathcal{I}_1$. Thus $\dim\mathcal{I}_1\leq d-1$. Now we claim $y_1,\ldots,y_{d-1}$ are differentially independent modulo $\mathcal{I}_1$, which proves $\dim\mathcal{I}_1=d-1$. Suppose the contrary: $y_1,\ldots,y_{d-1}$ are differentially dependent modulo $\mathcal{I}_1$. Thus there exists a nonzero differential polynomial $p(y_1,\ldots,y_{d-1})\in\mathcal{I}_1$. Take $p\in\mathcal{F}\{\widetilde{\mathbf{u}},y_1,\ldots,y_{d-1},u_0\}$, then

$$p(\widetilde{\mathbf{u}},\xi_1,\ldots,\xi_{d-1},-\sum_{i=1}^n\sum_{j=0}^s u_{ij}\xi_i^{(j)}-f_0(\xi_1,\ldots,\xi_n))=0.$$

That is, $\xi_1, \ldots, \xi_{d-1}, -\sum_{i=1}^n \sum_{j=0}^s u_{ij}\xi_i^{(j)} - f_0(\xi_1, \ldots, \xi_n)$ are differentially dependent over $\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle$. Now we specialize u_{d0} to -1, and the other $u \in \widetilde{\mathbf{u}}$ to zero. Then $-\sum_{i=1}^n \sum_{j=0}^s u_{ij}\xi_i^{(j)} - f_0(\xi_1, \ldots, \xi_n)$ specializes to ξ_d. By Theorem 2.14, $\xi_1, \ldots, \xi_d$ are differentially dependent over $\mathcal{F}$, which is a contradiction. So in this case $\dim\mathcal{I}_1 = d - 1$.

Now, it remains to show the case $d = 0$. Since $d = 0$, by Lemma 3.5, $\mathcal{I}_0 \cap \mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{u_0\} \neq \{0\}$. So $\mathcal{I}_0 \cap \mathcal{F}\langle\mathbf{u}_f\rangle \neq \{0\}$, and consequently $\mathcal{I}_1 = [\mathcal{I}_0]$ in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ is the unit ideal. □

A special case of Theorem 3.6 is particularly interesting and its algebraic counterpart is often listed as a theorem in algebraic geometry textbooks [16, p.54, p.110].

Theorem 3.7. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with differential dimension $d > 0$. Let $u_0, u_1, \ldots, u_n$ be differential indeterminates. Then $\mathcal{I}_1 = [\mathcal{I}, u_0 + u_1y_1 + \cdots + u_ny_n]$ is a prime differential ideal in $\mathcal{F}\langle u_0, u_1, \ldots, u_n\rangle\{\mathbb{Y}\}$ with dimension $d - 1$.*

Theorem 3.6 is also valid for a wider class of polynomials. A differential polynomial f is said to be *quasi-generic* in $\mathcal{F}\{\mathbb{Y}\}$, if 1) the coefficients of f as a differential polynomial in $y_1, \ldots, y_n$ are differential indeterminates and 2) in addition to the constant term, for each $1 \leq i \leq n$, f also contains at least one differential monomial in $\mathcal{F}\{y_i\} \setminus \mathcal{F}$. For instance, $f = u_0 + u_1y_1 + u_2y_1y_2$ is not quasi-generic, because f contains no monomials in $\mathcal{F}\{y_2\} \setminus \mathcal{F}$.

The proof for Theorem 3.6 can be easily adapted to prove the following result.

Corollary 3.8. Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with differential dimension d and f a quasi-generic differential polynomial with $\mathbf{u}_f$ as the set of coefficients. If $d > 0$, then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime differential ideal in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ with dimension $d - 1$. And if $d = 0$, then $\mathcal{I}_1$ is the unit ideal in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$.

As a direct consequence, we can show that the dimension conjecture is valid for quasi-generic polynomials.

Theorem 3.9 (Generic Dimension Theorem)**.** *Let $f_1, \ldots, f_r$ be quasi-generic differential polynomials in $\mathcal{F}\langle\mathbf{u}\rangle\{\mathbb{Y}\}$ with $r \leq n$ and $\mathbf{u}$ the set of coefficients of all f_i. Then $[f_1, \ldots, f_r]$ is a prime ideal with dimension $n - r$. And if $r > n$, $[f_1, \ldots, f_r]$ is the unit ideal.*

Proof: We prove the theorem by induction. When $r = 1$, let $\mathcal{I} = [0]$. Then by Corollary 3.8, $[f_1]$ is prime with dimension $n - 1$. Supposing this holds for $r - 1$, now consider the case $r \leq n$. By the hypothesis, $[f_1, \ldots, f_{r-1}]$ is a prime ideal with dimension $n - r + 1$. Note that the coefficients of f_r are new indeterminates. Using Corollary 3.8 again, $[f_1, \ldots, f_r]$ is a prime ideal with dimension $n - r$. When $r > n$, since $[f_1, \ldots, f_n]$ is of dimension zero, by Corollary 3.8, $[f_1, \ldots, f_r]$ is the unit ideal. □

3.2. **Order of a system of generic differential polynomials.** In this section, we consider the order of the intersection of a differential variety by a generic differential hypersurface. Before proving the main result, we give a series of lemmas and theorems.

Lemma 3.10. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension $n - 1$. Suppose $\{f\}$ is a characteristic set of $\mathcal{I}$ w.r.t. some ranking $\mathscr{R}$ and f is irreducible. Then for any other ranking $\overline{\mathscr{R}}$, $\{f\}$ is also a characteristic set of $\mathcal{I}$.*

Proof: Denote f to be $\overline{f}$ under the ranking $\overline{\mathscr{R}}$. By Theorem 2.2, $\mathcal{I} = \mathrm{sat}(f)$ and $\overline{\mathcal{I}} = \mathrm{sat}(\overline{f})$ are prime ideals with f and $\overline{f}$ as characteristic sets respectively. We need to show that $\mathcal{I} = \overline{\mathcal{I}}$. Let S be the separant of f. Then for $g \in \mathrm{sat}(f)$, we have $S^m g = hf + h_1 f' + \ldots + h_s f^{(s)}$ for $m, s \in \mathbb{N}$. Then, $S^m g \in \mathrm{sat}(\overline{f})$. Since $\mathrm{sat}(\overline{f})$ is prime, we need only to show that S is not in $\mathrm{sat}(\overline{f})$. Suppose the contrary, $S \in \mathrm{sat}(\overline{f})$. Since S is partially reduced w.r.t. $\overline{f}$, we have $S = h\overline{f}$ for a differential polynomial h, which is impossible since $S = \frac{\partial f}{\partial u_f}$. So $\mathcal{I} \subseteq \overline{\mathcal{I}}$. Similarly, we can prove that $\mathcal{I} \supseteq \overline{\mathcal{I}}$, thus $\mathcal{I} = \overline{\mathcal{I}}$. □

The following lemma generalizes a result in [9, p.5] to the case of positive dimensions.

Lemma 3.11. *Let the variety of a system $\mathcal{S}$ of differential polynomials in $\mathcal{F}\{\mathbb{Y}\}$ have a component V of differential dimension d and order h. Let $\overline{\mathcal{S}}$ be obtained from $\mathcal{S}$ by replacing $y_1^{(k)}$ by $y_1^{(k+1)} (k = 0, 1, \ldots)$ in all of the polynomials of $\mathcal{S}$. Then the variety of $\overline{\mathcal{S}}$ has a component $\overline{V}$ of dimension d and order $h \leq \mathrm{ord}(\overline{V}) \leq h + 1$. Moreover, if there exists a parametric set $\mathbb{U}$ not containing y_1 such that the relative order of $\mathbb{I}(V)$ w.r.t. $\mathbb{U}$ is h, then the order of $\overline{V}$ is $h+1$; otherwise, the order of $\overline{V}$ is h. In particular, if $d = 0$, then $\mathrm{ord}(\overline{V}) = h + 1$.*

Proof: Let $(\xi_1, \ldots, \xi_n)$ be a generic point of V and $\mathcal{I} = \mathbb{I}(V) \in \mathcal{F}\{\mathbb{Y}\}$. Let $\{z' - \xi_1\}$ be a prime differential ideal in $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle\{z\}$ and η a generic point of it. Then $(\eta, \xi_2, \ldots, \xi_n)$ is a point of $\overline{\mathcal{S}}$. Suppose this point lies in a component $\overline{V}$ of $\overline{\mathcal{S}}$, which has a generic point $(\eta_1, \ldots, \eta_n)$. Then $(\eta_1, \eta_2, \ldots, \eta_n)$ specializes to $(\eta, \xi_2, \ldots, \xi_n)$ and correspondingly $(\eta_1', \eta_2, \ldots, \eta_n)$ specializes to $(\xi_1, \xi_2, \ldots, \xi_n)$. Since $(\xi_1, \ldots, \xi_n)$ is a generic point of V and $(\eta_1', \eta_2, \ldots, \eta_n)$ is a zero of $\mathcal{S}$, the latter specialization is generic, that is, $(\eta_1', \eta_2, \ldots, \eta_n)$ is a generic point of V. We claim that any parametric set $\mathbb{U}$ of $\mathcal{I}$ is a parametric set of $\mathbb{I}(\overline{V})$, and $\mathrm{ord}_{\mathbb{U}}\mathcal{I} \leq \mathrm{ord}_{\mathbb{U}}\mathbb{I}(\overline{V}) \leq \mathrm{ord}_{\mathbb{U}}\mathcal{I} + 1$, which follows that $\dim(\overline{V}) = d$ and by Theorem 2.10, $h \leq \mathrm{ord}(\overline{V}) \leq h + 1$. Let $\mathbb{U}$ be any parametric set of $\mathcal{I}$. We consider the following two cases.

Case 1: $y_1 \notin \mathbb{U}$. Suppose $\mathbb{U}$ is the set of $y_2, \ldots, y_{d+1}$. By Corollary 2.7 we have

$$\mathrm{ord}_{y_2,\ldots,y_{d+1}}\mathcal{I} = \mathrm{tr.deg}\mathcal{F}\langle\xi_1, \ldots, \xi_d, \xi_{d+1}, \ldots, \xi_n\rangle/\mathcal{F}\langle\xi_2, \ldots, \xi_{d+1}\rangle.$$

Since $\xi_2, \ldots, \xi_{d+1}$ are differentially independent over $\mathcal{F}$, $\eta_2, \ldots, \eta_{d+1}$ must be differentially independent over $\mathcal{F}$, i.e. $\mathbb{I}(\overline{V}) \cap \mathcal{F}\{\mathbb{U}\} = \{0\}$.

$$\begin{aligned}
& \mathrm{tr.deg}\mathcal{F}\langle\eta_1, \eta_2, \ldots, \eta_n\rangle/\mathcal{F}\langle\eta_2, \ldots, \eta_{d+1}\rangle \\
\geq\ & \mathrm{tr.deg}\mathcal{F}\langle\eta, \xi_2, \ldots, \xi_n\rangle/\mathcal{F}\langle\xi_2, \ldots, \xi_{d+1}\rangle \\
& \quad (\text{for } (\eta_1, \eta_2, \ldots, \eta_n) \text{ specializes to } (\eta, \xi_2, \ldots, \xi_n)) \\
=\ & \mathrm{tr.deg}\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle/\mathcal{F}\langle\xi_2, \ldots, \xi_{d+1}\rangle + \mathrm{tr.deg}\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle\langle\eta\rangle/\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle \\
=\ & \mathrm{ord}_{y_2,\ldots,y_{d+1}}\mathcal{I} + 1
\end{aligned}$$

and

$$\begin{aligned}
& \mathrm{tr.deg}\mathcal{F}\langle\eta_1, \ldots, \eta_n\rangle/\mathcal{F}\langle\eta_2, \ldots, \eta_{d+1}\rangle \\
\leq\ & 1 + \mathrm{tr.deg}\mathcal{F}\langle\eta_1', \eta_2 \ldots, \eta_n\rangle/\mathcal{F}\langle\eta_2, \ldots, \eta_{d+1}\rangle \\
=\ & 1 + \mathrm{ord}_{y_2,\ldots,y_{d+1}}\mathcal{I}
\end{aligned}$$

So $\mathrm{tr.deg}\mathcal{F}\langle\eta_1, \ldots, \eta_n\rangle/\mathcal{F}\langle\eta_2, \ldots, \eta_{d+1}\rangle = 1 + \mathrm{ord}_{y_2,\ldots,y_{d+1}}\mathcal{I} < \infty$. Thus $\overline{V}$ is of dimension d and $\{y_2, \ldots, y_{d+1}\}$ is a parametric set of $\mathbb{I}(\overline{V})$. Moreover, the relative order of $\mathbb{I}(\overline{V})$ w.r.t. $y_2, \ldots, y_{d+1}$ is $\mathrm{ord}_{y_2,\ldots,y_{d+1}}\mathcal{I} + 1$.

Case 2: $y_1 \in \mathbb{U}$. Suppose $\mathbb{U} = \{y_1, \ldots, y_d\}$. Then by Corollary 2.7, $\mathrm{ord}_{\mathbb{U}}\mathcal{I} = \mathrm{ord}_{y_1,\ldots,y_d}\mathcal{I} = \mathrm{tr.deg}\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle/\mathcal{F}\langle\xi_1, \ldots, \xi_d\rangle$. We have seen that $(\eta_1', \eta_2, \ldots, \eta_n)$ is a generic point of V. Since $\mathrm{tr.deg}\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle\langle\eta\rangle/\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle = 1$ and $(\eta_1, \eta_2, \ldots, \eta_n)$ can specialize to $(\eta, \xi_2, \ldots, \xi_n)$, η_1 is algebraically independent over $\mathcal{F}\langle\eta_1', \eta_2, \ldots, \eta_n\rangle$. So

$$\begin{aligned}
& \mathrm{tr.deg}\mathcal{F}\langle\eta_1, \ldots, \eta_n\rangle/\mathcal{F}\langle\eta_1, \ldots, \eta_d\rangle \\
= \; & \mathrm{tr.deg}\mathcal{F}(\eta_1)\langle\eta_1', \eta_2 \ldots, \eta_n\rangle/\mathcal{F}(\eta_1)\langle\eta_1', \eta_2, \ldots, \eta_d\rangle \\
= \; & \mathrm{tr.deg}\mathcal{F}\langle\eta_1', \eta_2, \ldots, \eta_n\rangle/\mathcal{F}\langle\eta_1', \eta_2, \ldots, \eta_d\rangle \\
& \quad (\text{for } \mathrm{tr.deg}\,\mathcal{F}\langle\eta_1', \ldots, \eta_n\rangle(\eta_1)/\mathcal{F}\langle\eta_1', \ldots, \eta_n\rangle = 1) \\
= \; & \mathrm{ord}_{y_1,\ldots,y_d}\mathcal{I}
\end{aligned}$$

Since $(\eta_1, \ldots, \eta_d)$ can specialize to $(\eta, \xi_2, \ldots, \xi_d)$ over $\mathcal{F}$, we have $d \geq \mathrm{d.tr.deg}\mathcal{F}\langle\eta_1, \ldots, \eta_d\rangle/\mathcal{F} \geq \mathrm{d.tr.deg}\mathcal{F}\langle\eta, \xi_2, \ldots, \xi_d\rangle/\mathcal{F} \geq \mathrm{d.tr.deg}\mathcal{F}\langle\xi_1, \xi_2, \ldots, \xi_d\rangle/\mathcal{F} = d$. Since $\mathrm{tr.deg}\mathcal{F}\langle\eta_1, \ldots, \eta_n\rangle/\mathcal{F}\langle\eta_1, \ldots, \eta_d\rangle < \infty$, we have $\mathrm{d.tr.deg}\mathcal{F}\langle\eta_1, \ldots, \eta_d, \ldots, \eta_n\rangle/\mathcal{F} = d$. Thus in this case, $\dim(\overline{V}) = d$ and $\mathbb{U} = \{y_1, \ldots, y_d\}$ is a parametric set of $\mathbb{I}(\overline{V})$ with $\mathrm{ord}_{y_1,\ldots,y_d}\mathbb{I}(\overline{V}) = \mathrm{ord}_{y_1,\ldots,y_d}\mathcal{I}$.

Consider the two cases together, we can see $\dim(\overline{V}) = d$. And by Theorem 2.10, $h \leq \mathrm{ord}(\overline{V}) \leq h + 1$. Moreover, if there exists a parametric set $\mathbb{U}$ not containing y_1 such that the relative order of $\mathbb{I}(V)$ w.r.t. $\mathbb{U}$ is h, then the order of $\overline{V}$ is $h + 1$; otherwise, the order of $\overline{V}$ is h. In particular, if $d = 0$, then $y_1 \notin \mathbb{U} = \emptyset$. From case 1, $\mathrm{ord}(\overline{V}) = \mathrm{ord}(V) + 1 = h + 1$. □

Let $\mathcal{G}$ be a differential extension field of $\mathcal{F}$. By a *differential isomorphism* of $\mathcal{G}$ w.r.t. $\mathcal{F}$, we mean a differential isomorphic mapping of $\mathcal{G}$ onto a differential field $\mathcal{G}'$ such that (a) $\mathcal{G}'$ is an extension of $\mathcal{F}$, (b) the differential isomorphic mapping leaves each element of $\mathcal{F}$ invariant, and (c) $\mathcal{G}'$ and $\mathcal{G}$ have a common extension. By means of well-ordering methods, it is easy to show that an isomorphism of $\mathcal{G}$ w.r.t. $\mathcal{F}$ can be extended to an automorphism of a common extension of $\mathcal{G}$ and $\mathcal{G}'$. We will use the following result about differential isomorphism.

Theorem 3.12. [19] *Let $\mathcal{G}$ be a differential extension field of $\mathcal{F}$ and $\gamma \in \mathcal{G}$. A necessary and sufficient condition that γ be a primitive element of $\mathcal{G}$, i.e. $\mathcal{G} = \mathcal{F}\langle\gamma\rangle$, is that no isomorphism of $\mathcal{G}$ w.r.t. $\mathcal{F}$ other than the identity leaves γ invariant.*

The following theorem as well as Theorem 3.6 prove Theorem 1.1.

Theorem 3.13. *Let $\mathcal{I}$ be a prime differential ideal with dimension $d > 0$ and order h, and f a generic differential polynomial of order s. Then $\mathcal{I}_1 = [\mathcal{I}, f]$ is a prime differential ideal in $\mathcal{F}\langle\mathbf{u}_f\rangle\{\mathbb{Y}\}$ with dimension $d - 1$ and order $h + s$, where $\mathbf{u}_f$ is the set of coefficients of f.*

Proof: By Theorem 3.6, $\mathcal{I}_1$ is prime with dimension $d - 1$. Now we prove the order of $\mathcal{I}_1$ is $h + s$.

Let $\mathscr{A}$ be a characteristic set of $\mathcal{I}$ w.r.t. an orderly ranking $\mathscr{R}$ with $y_1, \ldots, y_d$ as a parametric set. By Theorem 2.5, $\mathrm{ord}(\mathscr{A}) = h$. Suppose $(\xi_1, \ldots, \xi_n)$ is a generic point of $\mathcal{I}$. Let $f = u_0 + \sum_{i=1}^n \sum_{j=0}^s u_{ij} y_i^{(j)} + f_0$ where f_0 is the nonlinear part of f in $y_i^{(j)}$ and $\mathcal{I}_0 = [\mathcal{I}, f]$ in $\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{y_1, \ldots, y_n, u_0\}$, where $\widetilde{\mathbf{u}} = \mathbf{u}_f \setminus \{u_0\}$ and u_0 is the constant term of f. By Lemma 3.5, $\mathcal{I}_0$ is a prime ideal of dimension d with a generic zero $(\xi_1, \ldots, \xi_n, -\sum_{i=1}^n \sum_{j=0}^s u_{ij}\xi_i^{(j)} - f_0(\xi_1, \ldots, \xi_n))$, and u_0 is differentially independent modulo $\mathcal{I}_0$. $\mathcal{I}_0$ and $\mathcal{I}_1$ have such relations: Any characteristic set of $\mathcal{I}_0$ with u_0 in the parametric set is a characteristic set of $\mathcal{I}_1$, and conversely, any characteristic set of $\mathcal{I}_1$ is a characteristic set of $\mathcal{I}_0$ with u_0 in the parametric set. By Theorem 2.10, we have $\mathrm{ord}(\mathcal{I}_1) \leq \mathrm{ord}(\mathcal{I}_0)$.

We claim that $\mathrm{ord}(\mathcal{I}_0) \leq h+s$. As a consequence, $\mathrm{ord}(\mathcal{I}_1) \leq h+s$. To prove this claim, let $\mathcal{I}_0^{(i)} = [\mathcal{I}, u_0^{(i)} + \sum_{i=1}^n \sum_{j=0}^s u_{ij} y_i^{(j)} + f_0]\,(i=0,\ldots,s)$ in $\mathcal{F}\langle\widetilde{\mathbf{u}}\rangle\{y_1,\ldots,y_n,u_0\}$. Let $\bar{f}$ be the pseudo remainder of $u_0^{(s)} + \sum_{i=1}^n \sum_{j=0}^s u_{ij} y_i^{(j)} + f_0$ w.r.t. $\mathcal{A}$ under the ranking $\mathscr{R}$. Clearly, $\mathrm{ord}(\bar{f}, u_0) = s$. It is obvious that for some orderly ranking, $\{\mathcal{A}, \bar{f}\}$ is a characteristic set of $\mathcal{I}_0^{(s)}$ with $y_1,\ldots,y_d$ as a parametric set. So $\mathrm{ord}(\mathcal{I}_0^{(s)}) = h+s$. Using Lemma 3.11 s times, we have $\mathrm{ord}(\mathcal{I}_0) \leq \mathrm{ord}(\mathcal{I}_0^{(1)}) \leq \cdots \leq \mathrm{ord}(\mathcal{I}_0^{(s)}) = h+s$.

Now, it suffices to prove $\mathrm{ord}(\mathcal{I}_1) \geq h+s$. Let $w = u_0 + \sum_{i=1}^d \sum_{j=0}^s u_{ij} y_i^{(j)}$ be a new differential indeterminate. We denote $\mathcal{F}_1 = \mathcal{F}\langle\mathbf{u}_g\rangle$, where $\mathbf{u}_g$ is the set of coefficients of $g = w + \sum_{i=d+1}^n \sum_{j=0}^s u_{ij} y_i^{(j)} + f_0$ as a differential polynomial in w and $y_1,\ldots,y_n$. Then $\mathcal{I}_2 = [\mathcal{I}, g]$ in $\mathcal{F}_1\{y_1,\ldots,y_n,w\}$ is a prime differential ideal with a generic point $(\xi_1,\ldots,\xi_n,\gamma)$ where $\gamma = -\sum_{i=d+1}^n \sum_{j=0}^s u_{ij}\xi_i^{(j)} - f_0(\xi_1,\ldots,\xi_n)$. We claim that γ is a primitive element of $\mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle\langle\xi_{d+1},\ldots,\xi_n\rangle$ over $\mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle$. By Theorem 3.12, it needs only to show that no isomorphism of $\mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle\langle\xi_{d+1},\ldots,\xi_n\rangle$ w.r.t. $\mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle$ other than the identity leaves γ invariant. Let φ be any differential isomorphism of $\mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle\langle\xi_{d+1},\ldots,\xi_n\rangle$ w.r.t. $\mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle$ which leaves γ invariant, and $\varphi(\xi_{d+i}) = \eta_{d+i}\,(i=1,\ldots,n-d)$. Since each $\xi_{d+i}\,(i=1,\ldots,n-d)$ is differentially algebraic over $\mathcal{F}\langle\xi_1,\ldots,\xi_d\rangle$ and φ is an isomorphism leaving each element of $\mathcal{F}\langle\xi_1,\ldots,\xi_d\rangle$ invariant, we can see that each $\eta_{d+i}\,(i=1,\ldots,n-d)$ is also differentially algebraic over $\mathcal{F}\langle\xi_1,\ldots,\xi_d\rangle$. So, $\eta_{d+i}\,(i=1,\ldots,n-d)$ are also in the universal field $\mathcal{E}$. From $\varphi(\gamma)=\gamma$, we have $-\sum_{i=d+1}^n \sum_{j=0}^s u_{ij}\eta_i^{(j)} - f_0(\xi_1,\ldots,\xi_d,\eta_{d+1},\ldots,\eta_n) = -\sum_{i=d+1}^n \sum_{j=0}^s u_{ij}\xi_i^{(j)} - f_0(\xi_1,\ldots,\xi_d,\xi_{d+1},\ldots,\xi_n) = 0$ which can be rewritten as:

$$\sum_{i=d+1}^n \sum_{j=0}^s u_{ij}(\xi_i^{(j)} - \eta_i^{(j)}) + f_0(\xi_1,\ldots,\xi_n) - f_0(\xi_1,\ldots,\xi_d,\eta_{d+1},\ldots,\eta_n) = 0.$$

Since the coefficients u_{ij}, u_α of f are differential indeterminates and the coefficients of u_{ij}, u_α in the above equation are in $\mathcal{E}$, we have $\xi_i - \eta_i = 0 (i=d+1,\ldots,n)$. So φ must be the identity map, which proves the claim.

Since $\mathcal{F}_1\langle\xi_1,\ldots,\xi_d,\xi_{d+1},\ldots,\xi_n\rangle = \mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle\langle\gamma\rangle$, γ is differentially algebraic over $\mathcal{F}_1\langle\xi_1,\ldots,\,\xi_d\rangle$ and each $\xi_{d+i} \in \mathcal{F}_1\langle\xi_1,\ldots,\xi_d\rangle\langle\gamma\rangle (i=1,\ldots,n-d)$. Let $R(y_1,\ldots,y_d,w) \in \mathcal{I}_2$ be a characteristic polynomial of γ. Then there exist $A_i \in \mathcal{I}_2$ with the form $A_i = P_i(y_1,\ldots,y_d,w)y_{d+i} + Q_i(y_1,\ldots,y_d,w)\ (i=1,\ldots,n-d)$, which are reduced w.r.t. R. Since $\mathcal{I}_2 \cap \mathcal{F}_1\{y_1,\ldots,y_d,w\}$ is a d-dimensional prime differential ideal, by Lemma 3.10, $\{R\}$ is its characteristic set w.r.t. any ranking. So for the elimination ranking $y_1 \prec \ldots \prec y_d \prec w \prec y_{d+1} \prec \ldots \prec y_n$, a characteristic set of $\mathcal{I}_2$ is $\{R(y_1,\ldots,y_d,w), A_1,\ldots,A_{n-d}\}$. Since $\mathcal{F}_1\langle\xi_1,\ldots,\xi_n,\gamma\rangle = \mathcal{F}_1\langle\xi_1,\ldots,\xi_n\rangle$, by Corollary 2.7, $\mathrm{ord}_{y_1,\ldots,y_d}\mathcal{I}_2 = \mathrm{ord}_{y_1,\ldots,y_d}(\mathcal{I}) = \mathrm{ord}(\mathcal{A}) = h$. Thus, $\mathrm{ord}(R,w) = h$.

Let $\mathbf{u}_d = \{u_{ij} : i=1,\ldots,d; j=0,\ldots,s\}$. In $\mathcal{F}_1\langle\mathbf{u}_d\rangle\{w,y_1,\ldots,y_n\}$, $\mathcal{I}_2$ is also prime with $R(y_1,\ldots,y_d,w), A_1,\ldots,A_{n-d}$ as a characteristic set w.r.t. the elimination ranking $y_1 \prec \ldots \prec y_d \prec w \prec y_{d+1} \prec \ldots \prec y_n$. Let

$$\begin{array}{rcl}\phi:\mathcal{F}_1\langle\mathbf{u}_d\rangle\{y_1,\ldots,y_n,w\} & \longrightarrow & \mathcal{F}_1\langle\mathbf{u}_d\rangle\{y_1,\ldots,y_n,u_0\}\\ w & & u_0+\sum_{i=1}^d\sum_{j=0}^s u_{ij}y_i^{(j)}\\ y_i & & y_i\end{array}$$

be a differential homomorphism over $\mathcal{F}_1\langle\mathbf{u}_d\rangle$. Clearly, this is a differential isomorphism which maps $\mathcal{I}_2$ to $\mathcal{I}_0$. It is obvious that $\mathcal{I}_0$ has $\phi(R),\phi(A_1),\ldots,\phi(A_{n-d})$ as a characteristic set w.r.t. the elimination ranking $y_1\prec\ldots\prec y_d\prec u_0\prec y_{d+1}\prec\ldots\prec y_n$ with $\mathrm{rk}(\phi(A_i))=y_{d+i}(i=1,\ldots,n-d)$. We claim that $\mathrm{ord}(\phi(R),y_1)\geq h+s$. If $\mathrm{ord}(R,y_1)\geq h+s$, rewrite R in the form $R=\sum_{\psi_\nu(w)\neq1}p_\nu(y_1,\ldots,y_d)\psi_\nu(w)+p(y_1,\ldots,y_d)$ where $\psi_\nu(w)$ are monomials in w and its derivatives. Then

$$\begin{aligned}\phi(R) &= \sum_{\psi_\nu\neq1}p_\nu(y_1,\ldots,y_d)\psi_\nu(u_0+\sum_{i=1}^d\sum_{j=0}^s u_{ij}y_i^{(j)})+p(y_1,\ldots,y_d)\\ &= \sum_{\psi_\nu\neq1}p_\nu(y_1,\ldots,y_d)\psi_\nu(u_0)+p(y_1,\ldots,y_d)\\ &\quad +\text{terms involving}\,u_{ij}(i=1,\ldots,d;j=0,\ldots,s)\ \text{and their derivatives.}\end{aligned}$$

Clearly, in this case we have $\mathrm{ord}(\phi(R),y_1)\geq\max\{\mathrm{ord}(p_\nu,y_1),\mathrm{ord}(p,y_1)\}=\mathrm{ord}(R,y_1)\geq h+s$. If $\mathrm{ord}(R,y_1)<h+s$, rewrite R as a polynomial in $w^{(h)}$, that is, $R=I_l(w^{(h)})^l+I_{l-1}(w^{(h)})^{l-1}+\cdots+I_0$. Then $\phi(R)=\phi(I_l)[(u_0+\sum_{i=1}^d\sum_{j=0}^s u_{ij}y_i^{(j)})^{(h)}]^l+\phi(I_{l-1})[(u_0+\sum_{i=1}^d\sum_{j=0}^s u_{ij}y_i^{(j)})^{(h)}]^{l-1}+\cdots+\phi(I_0)$. Since $\mathrm{ord}(\phi(I_k),y_1)<h+s$ $(k=0,\ldots,l)$, we have exactly $\mathrm{ord}(\phi(R),y_1)=h+s$. Thus, consider the two cases together, $\mathrm{ord}(\phi(R),y_1)\geq h+s$.

Since $\mathcal{I}_0\cap\mathcal{F}_1<\mathbf{u}_d>\{y_1,\ldots,y_d,u_0\}$ is a d-dimensional prime differential ideal, by Lemma 3.10, $\{\phi(R)\}$ is its characteristic set w.r.t. any ranking, in particular, for the elimination ranking $u_0\prec y_2\prec\ldots\prec y_d\prec y_1$. So w.r.t. the elimination ranking $u_0\prec y_2\prec\ldots\prec y_d\prec y_1\prec y_{d+1}\prec\ldots\prec y_n$, $\{\phi(R),\phi(A_1),\ldots,\phi(A_{n-d})\}$ is a characteristic set of $\mathcal{I}_0$, thus a characteristic set of $\mathcal{I}_1$. By Theorem 2.10, $\mathrm{ord}(\mathcal{I}_1)\geq\mathrm{ord}_{y_2,\ldots,y_d}\mathcal{I}_1\geq h+s$.

Thus, the order of $\mathcal{I}_1$ is $h+s$. □

As a consequence, Theorem 3.7 can be strengthened as follows.

Theorem 3.14. *Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with differential dimension $d>0$ and order h. Let $u_0,u_1,\ldots,u_n$ be differential indeterminates. Then $\mathcal{I}_1=[\mathcal{I},u_0+u_1y_1+\ldots+u_ny_n]$ is a prime differential ideal in $\mathcal{F}\langle u_0,u_1,\ldots,u_n\rangle\{\mathbb{Y}\}$ with dimension $d-1$ and order h.*

As another consequence, the dimension theorem for generic differential polynomials can be strengthened as follows.

Theorem 3.15. *Let $F_1,\ldots,F_r(r\leq n)$ be generic differential polynomials with each F_i of order s_i. Then $\mathbb{V}(F_1,\ldots,F_r)$ is an irreducible variety with dimension $n-r$ and order $\sum_{i=1}^r s_i$.*

Remark 3.16. When f is a quasi-generic differential polynomial, Theorem 3.13 may not be true. A counter example is as follows. Let $\mathcal{I}=[y_2,\ldots,y_n]\in\mathcal{F}\{\mathbb{R}\}$ and $f=u_0+u_1y_1+u_2y_2''+\cdots+u_ny_n''$. Clearly, f is a quasi-generic differential polynomial and $[\mathcal{I},f]$ is a prime differential ideal of dimension 0. But $\mathrm{ord}([\mathcal{I},f])=\mathrm{ord}(\mathcal{I})=0\neq\mathrm{ord}(\mathcal{I})+\mathrm{ord}(f)=2$.

4. Chow form for an irreducible differential variety

In this section, we define the differential Chow form and establish its properties by proving Theorem 1.2.

4.1. **Definition of differential Chow form.** Let V be an irreducible differential variety of dimension d over $\mathcal{F}$ with $(\xi_1, \ldots, \xi_n)$ as a generic point. We adjoin a set of differential indeterminates

$$\mathbf{u} = \{u_{ij}\,(i = 0, \ldots, d; j = 1, \ldots, n)\}$$

to $\mathcal{F}$ and define $d+1$ elements $\zeta_0, \zeta_1, \ldots, \zeta_d$ of $\mathcal{F}\langle \mathbf{u}, \xi_1, \ldots, \xi_n \rangle$:

$$\zeta_\sigma = -\sum_{\rho=1}^{n} u_{\sigma\rho}\xi_\rho\,(\sigma = 0, \ldots, d). \tag{4.1}$$

The following result shows that the differential transcendence degree of $\zeta_0, \ldots, \zeta_d$ over $\mathcal{F}\langle \mathbf{u} \rangle$ is d.

Lemma 4.1. d.tr.deg$\mathcal{F}\langle \mathbf{u} \rangle\langle \zeta_0, \ldots, \zeta_d \rangle/\mathcal{F}\langle \mathbf{u} \rangle = d$. *Furthermore, if* $d > 0$, $\zeta_1, \ldots, \zeta_d$ *are differentially independent over* $\mathcal{F}\langle \mathbf{u} \rangle$.

Proof: By Lemma 2.11, d.tr.deg$\mathcal{F}\langle \mathbf{u} \rangle\langle \xi_1, \ldots, \xi_n \rangle/\mathcal{F}\langle \mathbf{u} \rangle = d$. Since the $d+1$ elements $\zeta_0, \ldots, \zeta_d$ belong to $\mathcal{F}\langle \mathbf{u}, \xi_1, \ldots, \xi_n \rangle$, they are differentially dependent over $\mathcal{F}\langle \mathbf{u} \rangle$. Then, we have d.tr.deg $\mathcal{F}\langle \mathbf{u} \rangle\langle \zeta_0, \ldots, \zeta_d \rangle/\mathcal{F}\langle \mathbf{u} \rangle \leq d$. Thus, if $d = 0$, we have d.tr.deg $\mathcal{F}\langle \mathbf{u} \rangle\langle \zeta_0, \ldots, \zeta_d \rangle/\ \mathcal{F}\langle \mathbf{u} \rangle = 0$.

Now, suppose $d > 0$. We claim that $\zeta_1, \ldots, \zeta_d$ are differentially independent over $\mathcal{F}\langle \mathbf{u} \rangle$, thus it follows that d.tr.deg$\mathcal{F}\langle \mathbf{u} \rangle\langle \zeta_0, \ldots, \zeta_d \rangle/\mathcal{F}\langle \mathbf{u} \rangle = d$. Suppose the contrary. Since $\zeta_i \in \mathcal{F}\{\mathbf{u}, \xi_1, \ldots, \xi_n\}$, when we specialize u_{ij} to $-\delta_{k_i j}$ $(j = 1, \ldots, n,\ k_i \in \{1, \ldots, n\})$, ζ_i will be specialized to ξ_{k_i}. Then from Theorem 2.14, we conclude that $\xi_{k_1}, \ldots, \xi_{k_d}$ are differentially dependent over $\mathcal{F}$. Since we can choose $k_1, \ldots, k_d$ so that $\xi_{k_1}, \ldots, \xi_{k_d}$ are differentially independent over $\mathcal{F}$, it amounts to a contradiction. Thus the claim is proved. $\square$

Let $\mathbb{I}_\zeta$ be the prime differential ideal in $\mathcal{R} = \mathcal{F}\langle \mathbf{u} \rangle\{z_0, \ldots, z_d\}$ having $\zeta = (\zeta_0, \ldots, \zeta_d)$ as a generic point. By Lemma 4.1, the dimension of $\mathbb{I}_\zeta$ is d. By Theorem 2.8, the characteristic set of $\mathbb{I}_\zeta$ w.r.t. any ranking consists of an irreducible differential polynomial $f(z_0, \ldots, z_d)$ in $\mathcal{R}$ and

$$\mathbb{I}_\zeta = \mathrm{sat}(f). \tag{4.2}$$

Since the coefficients of $f(z_0, \ldots, z_d)$ are elements in $\mathcal{F}\langle \mathbf{u} \rangle$, without loss of generality, we assume that $f(\mathbf{u}; z_0, \ldots, z_d)$ is irreducible in $\mathcal{F}\{\mathbf{u}; z_0, \ldots, z_d\}$. We shall subsequently replace $z_0, \ldots, z_d$ by differential indeterminates $u_{00}, \ldots, u_{d0}$, and obtain

$$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \ldots, u_{d0}), \tag{4.3}$$

where $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$ for $i = 0, \ldots, d$.

Definition 4.2 (Differential Chow form)**.** The differential polynomial defined in (4.3) is called the *Chow form* of V or the prime ideal $\mathbb{I}(V)$.

Intuitively, by Lemma 4.1, there exists an irreducible differential polynomial $G(z_0, \ldots, z_d)$ in $\mathcal{F}\langle \mathbf{u} \rangle\{z_0,\ \ldots, z_d\}$, which satisfies $G(\zeta_0, \ldots, \zeta_d) = 0$ and has the smallest order w.r.t. a given ranking. The differential Chow form can be obtained from G by clearing denominators of its coefficients.

Example 4.3. If V is an irreducible variety of dimension $n-1$ and its corresponding prime ideal is $\mathcal{I} = \mathrm{sat}(p) \subset \mathcal{F}\{\mathbb{R}\}$. Then its differential Chow form is $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}) = D^m p\big(\frac{D_1}{D}, \ldots, \frac{D_n}{D}\big)$, where

$$D = \begin{vmatrix} u_{01} & u_{02} & \ldots & u_{0n} \\ u_{11} & u_{12} & \ldots & u_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n-1,1} & u_{n-1,2} & \ldots & u_{n-1,n} \end{vmatrix}$$

and $D_i (i = 1, \ldots, n)$ is the determinant of the matrix formed by replacing the i-th column of D by the column vector $(-u_{00}, -u_{10}, \ldots, -u_{n-1,0})^T$, and m is the minimal integer such that $D^m p\big(\frac{D_1}{D}, \ldots, \frac{D_n}{D}\big) \in \mathcal{F}\{\mathbf{u}; u_{00}, \ldots, u_{d0}\}$.

Example 4.4. Let V be the general component of $(y')^2 - 4y$ in $\mathbb{Q}(t)\{y\}$. It has dimension 0 and the differential Chow form of V is $F(\mathbf{u}_0) = u_1^2(u_0')^2 - 2u_1u_1'u_0u_0' + (u_1')^2u_0^2 + 4u_1^3u_0$, where $\mathbf{u}_0 = (u_0, u_1)$.

Example 4.5. Let V be the irreducible variety corresponding to $[y_1' + 1, y_2'] \in \mathbb{Q}(t)\{y_1, y_2\}$. It is of dimension 0 and the differential Chow form of V is $F(\mathbf{u}_0) = u_2u_1'u_0'' + u_1''u_1u_2 - 2u_2(u_1')^2 - u_2u_1''u_0' - u_2'u_1u_0'' - u_2''u_1^2 - u_1'u_2''u_0 + 2u_2'u_1u_1' + u_2''u_1u_0' + u_1''u_2'u_0$, where $\mathbf{u}_0 = (u_0, u_1, u_2)$.

A *generic differential prime* is of the form $u_0 + u_1y_1 + \cdots + u_ny_n = 0$ where u_i are differential indeterminates. The following result shows that the Chow form can be obtained by intersecting I with $d+1$ generic primes.

Lemma 4.6. *Using the notations introduced above, let* $\mathcal{I} = \mathbb{I}(V)$ *and*

$$\mathbb{P}_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 0, \ldots, d).$$

Then $\mathbb{I}_{\zeta,\xi} = [\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_d]$ *is a prime ideal in* $\mathcal{F}\langle\mathbf{u}\rangle\{u_{00}, u_{10}, \ldots, u_{d0}, y_1, \ldots, y_n\}$ *and* $\mathbb{I}_{\zeta,\xi} \cap \mathcal{F}\langle\mathbf{u}\rangle\{u_{00}, u_{10}, \ldots, u_{d0}\} = \mathrm{sat}(F)$.

Proof: It is easy to show that $\mathbb{I}_{\zeta,\xi}$ is a prime ideal with a generic zero (ζ, ξ). Then, $\mathbb{I}_{\zeta,\xi} \cap \mathcal{F}\langle\mathbf{u}\rangle\{u_{00}, u_{10}, \ldots, u_{d0}\}$ is a prime ideal with a generic zero ζ, which is $\mathbb{I}_\zeta = \mathrm{sat}(F)$. □

Remark 4.7. From Lemma 4.6, we have three observations. Firstly, the Chow form for a differential variety is independent of the generic point used in (4.1). Secondly, we can see that the Chow form is roughly the condition for the $d+1$ primes to meet V. This property will be further explored in Section 4.5. Thirdly, we can compute the differential Chow form of V with the characteristic set method if we know a set of finitely many generating differential polynomials for V. Furthermore, given a characteristic set $\mathcal{A}$ of $\mathbb{I}(V)$, we can also compute its differential Chow form. Indeed, from Lemma 4.6, it suffices to compute a characteristic set of $\mathbb{I}_{\zeta,\xi}$ w.r.t. a ranking $\mathbb{U} << \mathbb{Y}$ (elimination ranking between elements of $\mathbb{U} = \{u_{00}, \ldots, u_{d0}\}$ and $\mathbb{Y}$). It is clear that $\mathbb{I}_{\zeta,\xi}$ has a characteristic set $\{\mathcal{A}, \mathbb{P}_0, \ldots, \mathbb{P}_d\}$ w.r.t. a ranking $\mathbb{Y} << \mathbb{U}$. Then using the algorithms given by Boulier et al [1] and Golubitsky et al [14] for transforming a regular or characteristic decomposition of a radical differential ideal from one ranking to another, we can obtain the Chow form.

4.2. **Order of differential Chow form.** In this section, we will show that the order of the differential Chow form is the same as that of the corresponding differential variety. Before this, we give the following lemmas.

Lemma 4.8. *Let $\zeta_0, \zeta_1, \ldots, \zeta_d$ be defined in* (1)*, and $f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ the Chow form of V. Then for any $p(\mathbf{u}; u_{00}, \ldots, u_{d0}) \in \mathcal{F}\langle\mathbf{u}\rangle\{u_{00}, \ldots, u_{d0}\}$ with $\mathrm{ord}(f) = \mathrm{ord}(p)$, such that $p(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$, we have $p(\mathbf{u}; u_{00}, \ldots, u_{d0}) = f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ $h(\mathbf{u}; u_{00}, \ldots, u_{d0})$, where $h(\mathbf{u}; u_{00}, \ldots, u_{d0})$ is in $\mathcal{F}\langle\mathbf{u}\rangle\{u_{00}, \ldots, u_{d0}\}$.*

Proof: Since $\{f\}$ is the characteristic set of $\mathbb{I}_\zeta$ w.r.t. an orderly ranking, and $p \in \mathbb{I}_\zeta$ with $\mathrm{ord}(f) = \mathrm{ord}(p)$, so $\mathrm{I}_f^m p = fh$ for some $m \in \mathbb{N}$. Since f is irreducible, we can see that f divides p. $\square$

The differential Chow form $f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ has certain symmetric properties as shown by the following results.

Lemma 4.9. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ be the Chow form of an irreducible differential variety V and $F^*(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ obtained from F by interchanging $\mathbf{u}_\rho$ and $\mathbf{u}_\tau$. Then F^* and F differ at most by a sign. Furthermore, $\mathrm{ord}(F, u_{ij})$ $(i = 0, \ldots, d; j = 0, 1, \ldots, n)$ are the same for all u_{ij} occurring in F. In particular, u_{i0} $(i = 0, \ldots, d)$ appear effectively in F. And a necessary and sufficient condition for some u_{ij} $(j > 0)$ not occurring effectively in F is that $y_j \in \mathbb{I}(V)$.*

Proof: Consider the differential automorphism ϕ of $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle\langle\mathbf{u}\rangle$ over $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle$: $\phi(u_{ij}) = u^*_{ij} = \begin{cases} u_{ij}, & i \neq \rho, \tau \\ u_{\tau j}, & i = \rho \\ u_{\rho j}, & i = \tau \end{cases}$. Of course, $\phi(\zeta_i) = \zeta^*_i = \begin{cases} \zeta_i, & i \neq \rho, \tau \\ \zeta_\tau, & i = \rho \\ \zeta_\rho, & i = \tau \end{cases}$. Then $\phi(f(\mathbf{u}; \zeta_0, \ldots, \zeta_\rho, \ldots, \zeta_\tau, \ldots, \zeta_d)) = f(\mathbf{u}^*; \zeta_0, \ldots, \zeta_\tau, \ldots, \zeta_\rho, \ldots, \zeta_d) = 0$. Instead of $f(\mathbf{u}; z_0, \ldots, z_d)$, we obtain another differential polynomial $p(\mathbf{u}; z_0, \ldots, z_d) = f(\mathbf{u}^*; z_0, \ldots, z_\tau, \ldots, z_\rho, \ldots, z_d) \in \mathbb{I}_\zeta$. Since the two differential polynomials f and p have the same order and degree, and as algebraic polynomials they have the same content, by Lemma 4.8, $f(\mathbf{u}^*; z_0, \ldots, z_\tau, \ldots, z_\rho, \ldots, z_d)$ can only differ by a sign with $f(\mathbf{u}; z_0, \ldots, z_\rho, \ldots, z_\tau, \ldots, z_d)$. So we conclude that $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ produces at most a change of sign if we interchange $\mathbf{u}_\rho$ with $\mathbf{u}_\tau$. In particular, each u_{i0} appears effectively in F and $\mathrm{ord}(F, u_{i0})$ are the same for all $i = 0, 1, \ldots, d$. Suppose $\mathrm{ord}(F, u_{i0}) = s$. For $j \neq 0$, we consider $\mathrm{ord}(F, u_{ij})$. If $\mathrm{ord}(F, u_{ij}) = l > s$, then we differentiate $f(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$ on both sides w.r.t. $u_{ij}^{(l)}$. Thus $\frac{\partial f}{\partial u_{ij}^{(l)}}(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$, which amounts to a contradiction by Lemma 4.8. If $\mathrm{ord}(F, u_{ij}) = l < s$, then we differentiate $f(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$ on both sides w.r.t. $u_{ij}^{(s)}$. Thus $\frac{\partial f}{\partial \zeta_i^{(s)}}(-\xi_j) = 0$. Since $\frac{\partial f}{\partial \zeta_i^{(s)}} = \frac{\partial f}{\partial u_{i0}^{(s)}}(\mathbf{u}; \zeta_0, \ldots, \zeta_d) \neq 0$, we have $\xi_j = 0$. And $y_j \in \mathbb{I}(V) \Longleftrightarrow \xi_j = 0 \Longleftrightarrow \zeta_i$ is free of $u_{ij} \Longleftrightarrow \frac{\partial f}{\partial u_{ij}^{(k)}} = 0$ for all $k \in \mathbb{Z}^+ \Longleftrightarrow u_{ij}$ does not appear in F. From the above, if u_{ij} occurs effectively in F, $\mathrm{ord}(F, u_{ij}) = s$, which completes the theorem. $\square$

The *order of the Chow form* is defined to be $\mathrm{ord}(f) = \mathrm{ord}(F) = \mathrm{ord}(f, u_{i0})$ for any $i \in \{0, \ldots, d\}$. By Lemma 4.9, $\mathrm{ord}(f)$ is equal to $\mathrm{ord}(F, u_{ij})$ for those u_{ij} occurring in F.

The following lemma gives another property for the ideal $\mathbb{I}_{\zeta,\xi}$ defined in Lemma 4.6.

Lemma 4.10. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, u_{10}, \ldots, u_{d0})$ be the Chow form of a prime differential ideal $\mathcal{I}$ and $s = \mathrm{ord}(f)$. Then*

$$\mathcal{A} = \{f, S_f y_1 - \frac{\partial f}{\partial u_{01}^{(s)}}, \ldots, S_f y_n - \frac{\partial f}{\partial u_{0n}^{(s)}}\}$$

is a characteristic set of the prime ideal $\mathbb{I}_{\zeta,\xi} = [\mathcal{I}, \mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_d]$ *in* $\mathcal{F}\langle\mathbf{u}\rangle\{u_{00}, u_{10}, \ldots, u_{d0}, \mathbb{Y}\}$ *w.r.t. the elimination ranking* $u_{d0} \prec \ldots \prec u_{00} \prec y_1 \prec \ldots \prec y_n$*, where* $S_f = \frac{\partial f}{\partial u_{00}^{(s)}}$.

Proof: From Lemma 4.6, $\mathbb{I}_{\zeta,\xi}$ is a prime differential ideal of dimension d with a generic point $(\zeta_0, \ldots, \zeta_d, \xi_1, \ldots, \xi_n)$. From Lemma 4.1, $u_{10}, \ldots, u_{d0}$ is a parametric set of $\mathbb{I}_{\zeta,\xi}$. If we differentiate $f(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$ w.r.t. $u_{0\rho}^{(s)}$ $(\rho = 1, \cdots, n)$, then we have $\overline{\frac{\partial f}{\partial u_{0\rho}^{(s)}}} - \xi_\rho \overline{S}_f = 0$, where $\overline{\frac{\partial f}{\partial u_{0\rho}^{(s)}}}$ and $\overline{S}_f$ are obtained by replacing $(u_{00}, \ldots, u_{d0})$ with $(\zeta_0, \ldots, \zeta_d)$ in $\frac{\partial f}{\partial u_{0\rho}^{(s)}}$ and S_f respectively. So $T_\rho = S_f y_\rho - \frac{\partial f}{\partial u_{0\rho}^{(s)}} \in \mathbb{I}_{\zeta,\xi}$. Since f is irreducible, we have $S_f \notin \mathbb{I}_{\zeta,\xi}$. Also note that T_i is linear in y_i. $\mathcal{A}$ must be a characteristic set[1] of $\mathbb{I}_{\zeta,\xi}$ w.r.t. the elimination ranking $u_{d0} \prec \ldots \prec u_{00} \prec y_1 \prec \ldots \prec y_n$. □

Now we give the first main property of the differential Chow form.

Theorem 4.11. *Let* $\mathcal{I}$ *be a prime differential ideal with dimension* d*, and* $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, u_{10}, \ldots, u_{d0})$ *its differential Chow form. Then* $\mathrm{ord}(f) = \mathrm{ord}(\mathcal{I})$.

Proof: Use the notations such as $\mathbb{P}_i$, ξ_i, ζ_i as above in this section. Let $\mathcal{I}_d = [\mathcal{I}, \mathbb{P}_1, \ldots, \mathbb{P}_d] \subset \mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d\rangle\{\mathbb{Y}\}$. By Theorem 3.14, $\mathcal{I}_d$ is a prime ideal with dimension 0 and the same order as $\mathcal{I}$.

Let $\mathcal{I}_1 = [\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] = [\mathcal{I}_d, \mathbb{P}_0] \subset \mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d; u_{01}, \ldots, u_{0n}\rangle\{u_{00}, y_1, \ldots, y_n\}$. From Lemma 4.10, $\mathcal{A} = \{f, S_f y_1 - \frac{\partial f}{\partial u_{01}^{(s)}}, \ldots, S_f y_n - \frac{\partial f}{\partial u_{0n}^{(s)}}\}$ is a characteristic set of $\mathbb{I}_{\zeta,\xi}$. From Lemma 4.1, $u_{10}, \ldots, u_{d0}$ is a parametric set of $\mathbb{I}_{\zeta,\xi}$. Since $u_{10}, \ldots, u_{d0}$ is a parametric set of $\mathbb{I}_{\zeta,\xi}$, $\mathcal{A}$ is also a characteristic set of $\mathcal{I}_1$ w.r.t. the elimination ranking $u_{00} \prec y_1 \prec \ldots \prec y_n$. Since $\dim(\mathcal{I}_1) = 0$, from Corollary 2.9, we have $\mathrm{ord}(\mathcal{I}_1) = \mathrm{ord}(\mathcal{A}) = \mathrm{ord}(f)$.

On the other hand, if $(\eta_1, \ldots, \eta_n)$ is a generic point of $\mathcal{I}_d$, then $(\eta_1, \ldots, \eta_n, \zeta)$ is a generic point of $\mathcal{I}_1$ with $\zeta = -\sum_{j=1}^n u_{0j}\eta_j$ and $\dim(\mathcal{I}_1) = 0$. Clearly, $u_{0k} (k = 1, \ldots, n)$ are differentially independent over $\mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d, \eta_1, \ldots, \eta_n\rangle$. So for a sufficiently large t,

$$\begin{aligned}
& \omega_{\mathcal{I}_1}(t) = \mathrm{ord}(\mathcal{I}_1) \\
= \;& \mathrm{tr.deg}\,\mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d, u_{01}, \ldots, u_{0n}\rangle(\eta_i^{(j)}, \zeta^{(j)} : i = 1, \ldots, n; j \le t)/ \\
& \quad \mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d, u_{01}, \ldots, u_{0n}\rangle \\
= \;& \mathrm{tr.deg}\,\mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d, u_{01}, \ldots, u_{0n}\rangle(\eta_i^{(j)} : i = 1, \ldots, n; j \le t)/ \\
& \quad \mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d, u_{01}, \ldots, u_{0n}\rangle \\
= \;& \mathrm{tr.deg}\,\mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d\rangle(\eta_i^{(j)} : i = 1, \ldots, n; j \le t)/\mathcal{F}\langle\mathbf{u}_1, \ldots, \mathbf{u}_d\rangle \\
= \;& \omega_{\mathcal{I}_d}(t) = \mathrm{ord}(\mathcal{I}_d)
\end{aligned}$$

Thus, $\mathrm{ord}(\mathcal{I}_1) = \mathrm{ord}(\mathcal{I}_d) = \mathrm{ord}(\mathcal{I})$, and consequently, $\mathrm{ord}(\mathcal{I}) = \mathrm{ord}(f)$. □

As a consequence, we can give an equivalent definition for the order of a prime differential ideal using Chow forms.

Definition 4.12. Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ with dimension d and $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ its Chow Form. The order of $\mathcal{I}$ is defined to be the order of its differential Chow form.

[1]Here $\mathcal{A}$ is a differential chain. See Remark 2.3.

The following result shows that we can recover the generic point $(\xi_1, \ldots, \xi_n)$ of V from its Chow form.

Theorem 4.13. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ be defined as above and h the order of V. Then we have*

$$\xi_\rho = \overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}} \Big/ \overline{S}_f, \rho = 1, \ldots, n$$

where $\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}$ and $\overline{S}_f$ are obtained by replacing $(u_{00}, \ldots, u_{d0})$ by $(\zeta_0, \ldots, \zeta_d)$ in $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$ and $\frac{\partial f}{\partial u_{00}^{(h)}}$ respectively.

Proof: It follows directly from Lemma 4.10 and Theorem 4.11. □

4.3. **Differential Chow form is differentially homogenous.** Following Kolchin [22], we introduce the concept of differentially homogenous polynomials.

Definition 4.14. A differential polynomial $p \in \mathcal{F}\{y_0, y_1, \ldots, y_n\}$ is called differentially homogenous of degree m if for a new differential indeterminate λ, we have $p(\lambda y_0, \lambda y_1 \ldots, \lambda y_n) = \lambda^m p(y_0, y_1, \ldots, y_n)$.

The differential analog of Euler's theorem related to homogenous polynomials is valid.

Theorem 4.15. [18, p.71] *A differential polynomial $p \in \mathcal{F}\{y_0, y_1, \ldots, y_n\}$ is differentially homogenous of degree m if and only if*

$$\sum_{j=0}^{n} \sum_{k \in \mathbb{N}} \binom{k+r}{r} y_j^{(k)} \frac{\partial f(y_0, \ldots, y_n)}{\partial y_j^{(k+r)}} = \begin{cases} mf & r = 0 \\ 0 & r \neq 0 \end{cases}$$

For the differential Chow form, we have the following result.

Theorem 4.16. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ be the differential Chow form of a differential irreducible variety V of dimension d and order h. Then*

1)

$$\sum_{j=0}^{n} u_{\tau j} \frac{\partial f}{\partial u_{\sigma j}} + \sum_{j=0}^{n} u'_{\tau j} \frac{\partial f}{\partial u'_{\sigma j}} + \cdots + \sum_{j=0}^{n} u_{\tau j}^{(h)} \frac{\partial f}{\partial u_{\sigma j}^{(h)}} = \begin{cases} 0 & \sigma \neq \tau \\ rf & \sigma = \tau \end{cases}$$

where r is a nonnegative integer.

2) $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is a differentially homogenous polynomial of degree r in each set $\mathbf{u}_i$ of indeterminates and F is of total degree $(d+1)r$.

Proof: Let $(\xi_1, \ldots, \xi_n)$ be a generic point of V and $\zeta_i = -\sum_{j=1}^{n} u_{ij}\xi_j$ $(i = 0, \ldots, d)$ defined in (4.1). From (4.2), $f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ is the characteristic set of the prime differential ideal $\mathbb{I}_\zeta$. Since $f(\mathbf{u}; \zeta_0, \zeta_1, \ldots, \zeta_d) = 0$, we have

$$\begin{array}{llllllllll}
\overline{\frac{\partial f}{\partial u_{\sigma j}}} & + \frac{\partial f}{\partial \zeta_\sigma}(-\xi_j) & + \frac{\partial f}{\partial \zeta'_\sigma}(-\xi'_j) & + \frac{\partial f}{\partial \zeta''_\sigma}(-\binom{2}{0}\xi''_j) & + \ldots & + \frac{\partial f}{\partial \zeta_\sigma^{(h)}}[-\binom{h}{0}\xi_j^{(h)}] & = 0 & (0*) \\
\overline{\frac{\partial f}{\partial u'_{\sigma j}}} & + \quad 0 & + \frac{\partial f}{\partial \zeta'_\sigma}(-\xi_j) & + \frac{\partial f}{\partial \zeta''_\sigma}(-\binom{2}{1}\xi'_j) & + \ldots & + \frac{\partial f}{\partial \zeta_\sigma^{(h)}}[-\binom{h}{1}\xi_j^{(h-1)}] & = 0 & (1*) \\
\overline{\frac{\partial f}{\partial u''_{\sigma j}}} & + \quad 0 & + \quad 0 & + \frac{\partial f}{\partial \zeta''_\sigma}(-\binom{2}{2}\xi_j) & + \ldots & + \frac{\partial f}{\partial \zeta_\sigma^{(h)}}[-\binom{h}{2}\xi_j^{(h-2)}] & = 0 & (2*) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
\overline{\frac{\partial f}{\partial u_{\sigma j}^{(h)}}} & + \quad 0 & + \quad 0 & + \quad 0 & + \; 0 & + \frac{\partial f}{\partial \zeta_\sigma^{(h)}}[-\binom{h}{h}\xi_j^{(0)}] & = 0 & (h*)
\end{array}$$

In the above equations, $\overline{\frac{\partial f}{\partial u_{\sigma j}^{(l)}}}$ and $\frac{\partial f}{\partial \zeta_\sigma^{(l)}}$ $(l=0,\ldots,h;j=1,\ldots,n)$ are obtained by substituting ζ_i to u_{i0} $(i=0,1,\ldots,d)$ in each $\frac{\partial f}{\partial u_{\sigma j}^{(l)}}$ and $\frac{\partial f}{\partial u_{\sigma 0}^{(l)}}$ respectively.

Now, let us consider the differential polynomial $\sum_{j=0}^n\sum_{k\geq 0}\binom{k+i}{k}u_{\sigma j}^{(k)}\frac{\partial f}{\partial u_{\sigma j}^{(k+i)}}$.

In the case $i=0$, firstly, let $(0*)\times u_{\tau j}+(1*)\times u'_{\tau j}+\cdots+(h*)\times u_{\tau j}^{(h)}$ and add them together for j from 1 to n. We obtain

$$\sum_{j=1}^n u_{\tau j}\overline{\frac{\partial f}{\partial u_{\sigma j}}}+\sum_{j=1}^n u'_{\tau j}\overline{\frac{\partial f}{\partial u'_{\sigma j}}}+\cdots+\sum_{j=1}^n u_{\tau j}^{(h)}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(h)}}}+\zeta_\tau\frac{\partial f}{\partial \zeta_\sigma}+\zeta'_\tau\frac{\partial f}{\partial \zeta'_\sigma}+\cdots+\zeta_\tau^{(h)}\frac{\partial f}{\partial \zeta_\sigma^{(h)}}=0.$$

So the polynomial $\sum\limits_{j=0}^n u_{\tau j}\frac{\partial f}{\partial u_{\sigma j}}+\sum\limits_{j=0}^n u'_{\tau j}\frac{\partial f}{\partial u'_{\sigma j}}+\sum\limits_{j=0}^n u''_{\tau j}\frac{\partial f}{\partial u''_{\sigma j}}+\cdots+\sum\limits_{j=0}^n u_{\tau j}^{(h)}\frac{\partial f}{\partial u_{\sigma j}^{(h)}}$ vanishes at $(u_{00},\ldots,u_{d0})=(\zeta_0,\ldots,\zeta_d)$. Thus in the case $\tau=\sigma$, it can be divisible by f, i.e. $\sum\limits_{j=0}^n\sum\limits_{k=0}^h u_{\sigma j}^{(k)}\frac{\partial f}{\partial u_{\sigma j}^{(k)}}=rf$. By Euler's theorem, f is an algebraic homogenous polynomial of degree r in each set of indeterminates $\mathbf{u}_i=(u_{i0},\ldots,u_{in})$ and their derivatives. But in the case $\tau\neq\sigma$, since this polynomial is of order not greater than f and can not be divisible by f, by Lemma 4.8, it must be identically zero. Thus, we have proved 1) of the Theorem.

In the case $i\neq 0$,

$$\begin{aligned}
0 &= (i*)\times\binom{i}{i}u_{\sigma j}+(i+1*)\times\binom{i+1}{i}u'_{\sigma j}+\cdots+(h*)\times\binom{h}{i}u_{\sigma j}^{(h-i)}\\
&= \binom{i}{i}u_{\sigma j}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(i)}}}+\binom{i+1}{i}u'_{\sigma j}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(i+1)}}}+\cdots+\binom{h}{i}u_{\sigma j}^{(h-i)}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(h)}}}\\
&\quad+\frac{\partial f}{\partial \zeta_\sigma^{(i)}}\Big(-\binom{i}{i}u_{\sigma j}\xi_j\Big)+\frac{\partial f}{\partial \zeta_\sigma^{(i+1)}}\Big(-\binom{i+1}{i}u_{\sigma j}\xi'_j-\binom{i+1}{i}\binom{i+1}{i+1}u'_{\sigma j}\xi_j\Big)\\
&\quad+\cdots\cdots\\
&\quad+\frac{\partial f}{\partial \zeta_\sigma^{(h)}}\Big(-\binom{i}{i}\binom{h}{i}u_{\sigma j}\xi_j^{(h-i)}-\binom{i+1}{i}\binom{h}{i+1}u'_{\sigma j}\xi_j^{(h-i-1)}-\cdots-\binom{h}{i}\binom{h}{h}u_{\sigma j}^{(h-i)}\xi_j\Big)\\
&= \binom{i}{i}u_{\sigma j}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(i)}}}+\binom{i+1}{i}u'_{\sigma j}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(i+1)}}}+\cdots+\binom{h}{i}u_{\sigma j}^{(h-i)}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(h)}}}\\
&\quad+\binom{i}{i}\frac{\partial f}{\partial \zeta_\sigma^{(i)}}(-u_{\sigma j}\xi_j)+\binom{i+1}{i}\frac{\partial f}{\partial \zeta_\sigma^{(i+1)}}(-u_{\sigma j}\xi_j)'+\cdots+\binom{h}{i}\frac{\partial f}{\partial \zeta_\sigma^{(h)}}(-u_{\sigma j}\xi_j)^{(h-i)}
\end{aligned}$$

Therefore, $\sum_{j=1}^n\binom{i}{i}u_{\sigma j}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(i)}}}+\sum_{j=1}^n\binom{i+1}{i}u'_{\sigma j}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(i+1)}}}+\cdots+\sum_{j=1}^n\binom{h}{i}u_{\sigma j}^{(h-i)}\overline{\frac{\partial f}{\partial u_{\sigma j}^{(h)}}}+\binom{i}{i}\zeta_\sigma\frac{\partial f}{\partial \zeta_\sigma^{(i)}}+\binom{i+1}{i}\zeta'_\sigma\frac{\partial f}{\partial \zeta_\sigma^{(i+1)}}+\cdots+\binom{h}{i}\zeta_\sigma^{(h-i)}\frac{\partial f}{\partial \zeta_\sigma^{(h)}}=0.$

Thus, the polynomial $\sum_{j=0}^n\binom{i}{i}u_{\sigma j}\frac{\partial f}{\partial u_{\sigma j}^{(i)}}+\sum_{j=0}^n\binom{i+1}{i}u'_{\sigma j}\frac{\partial f}{\partial u_{\sigma j}^{(i+1)}}+\cdots+\sum_{j=0}^n\binom{h}{i}u_{\sigma j}^{(h-i)}\frac{\partial f}{\partial u_{\sigma j}^{(h)}}$ is identically zero, for it vanishes at $(u_{00},\ldots,u_{d0})=(\zeta_0,\ldots,\zeta_d)$ and can not be divisible by f.

From the above, we conclude that

$$\sum_{j=0}^n\sum_{k\geq 0}\binom{k+i}{i}u_{\sigma j}^{(k)}\frac{\partial f}{\partial u_{\sigma j}^{(k+i)}}=\begin{cases}0 & i\neq 0\\ rf & i=0\end{cases}$$

From Theorem 4.15 and the symmetry property of $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$, the theorem is obtained. □

Lemma 4.9, Theorem 4.11, and Theorem 4.16 together prove the first property of Theorem 1.2.

Definition 4.17. Let $\mathcal{I}$ be a prime differential ideal in $\mathcal{F}\{\mathbb{Y}\}$ of dimension d and $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ its differential Chow form. The *differential degree* of $\mathcal{I}$ is defined to be the homogenous degree r of its differential Chow form in each $\mathbf{u}_i$ $(i = 0, \ldots, d)$.

The following result shows that the differential degree of a variety V is an invariant of V under invertible linear transformations.

Lemma 4.18. *Let $A = (a_{ij})$ be an $n \times n$ invertible matrix with $a_{ij} \in \mathcal{F}$ and $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ the Chow form of an irreducible differential variety V with dimension d. Then the Chow form of the image variety of V under the linear transformation $\mathbb{Y} = A\mathbb{X}$ is $F^A(\mathbf{v}_0, \ldots, \mathbf{v}_d) = F(\mathbf{v}_0 B, \ldots, \mathbf{v}_d B)$, where $B = \begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}$ and $\mathbf{u}_i$ and $\mathbf{v}_i$ are regarded as row vectors.*

Proof: Let $\xi = (\xi_1, \ldots, \xi_n)$ be a generic point of V. Under the linear transformation $\mathbb{Y} = A\mathbb{X}$, ξ is mapped to $\eta = (\eta_1, \ldots, \eta_n)$ with $\eta_i = \sum_{j=1}^n a_{ij}\xi_j$. Under this transformation V is mapped to a differential variety V^A whose generic point is η. Note that $F^A(\mathbf{v}_0, \ldots, \mathbf{v}_d) = f^A(v_{ij}; v_{00}, \ldots, v_{d0}) = f(\sum_{k=1}^n v_{ik}a_{kj}; v_{00}, \ldots, v_{d0})$ and $f^A(v_{ij}; -\sum_{k=1}^n v_{0k}\eta_k, \ldots, -\sum_{k=1}^n v_{dk}\eta_k) = f(\sum_{k=1}^n v_{ik}a_{kj}; -\sum_{k=1}^n v_{0k}\eta_k, \ldots, -\sum_{k=1}^n v_{dk}\eta_k) = f(\sum_{k=1}^n v_{ik}a_{kj}; -\sum_{j=1}^n(\sum_{k=1}^n v_{0k}a_{kj})\xi_j, \ldots, -\sum_{j=1}^n(\sum_{k=1}^n v_{dk} a_{kj})\xi_j) = 0$. Since V^A is of the same dimension and order as V and F^A is irreducible, from the definition of the differential Chow form, the claim is proved. $\square$

Definition 4.19. Let p be a differential polynomial in $\mathcal{F}\{\mathbb{Y}\}$. Then the smallest number r such that $y_0^r p(y_1/y_0, \ldots, y_n/y_0) \in \mathcal{F}\{y_0, y_1, \ldots, y_n\}$ is called the denomination of p, which is denoted by $\mathrm{den}(p)$.

Example 4.20. In the case $d = n-1$ and $n > 1$, if $\{f(y_1, \ldots, y_n)\}$ is a characteristic set of $\mathcal{I}$ w.r.t. any ranking, then the differential degree of $\mathcal{I}$ cannot exceed the denomination of f. So the denomination of f gives an upper bound of differential degree of the variety. But, we do not know whether they are the same.

Example 4.21. In the case $n = 1$ and $d = 0$, if $\{f(y)\}$ is a characteristic set of $\mathcal{I}$ w.r.t. any ranking, then the differential degree of $\mathcal{I}$ is exactly equal to the denomination of f. Now we can give a bound for the differential degree of $\mathcal{I}$ from the original equation of f without computing its denomination.

For a differential monomial $\theta(y) = y^{l_0}(y')^{l_1}(y'')^{l_2}\ldots(y^{(s)})^{l_s}$, define its *weighted degree* to be $l_0+2l_1+\cdots+(s+1)l_s$, denoted by $\mathrm{wdeg}(\theta(y)) = l_0+2l_1+\cdots+(s+1)l_s$. For a differential polynomial $f \in \mathcal{F}\{y\}$, we can define its weighted degree to be the maximum of the weighted degrees of all the differential monomials effectively appearing in f. Clearly, the denomination of f cannot exceed its weighted degree. And we have examples for which $\mathrm{den}(f) < \mathrm{wdeg}(f)$. Let $f = 2y'^2 - yy''$. Then, $\mathrm{den}(f) = 3$ and $\mathrm{wdeg}(f) = 4$. The differential Chow form of $\mathrm{sat}(f)$ is $F(\mathbf{u}) = u_0u_1u_0'' - u_0^2u_1'' - 2u_0'^2u_1 + 2u_0u_0'u_1'$, where $\mathbf{u} = (u_0, u_1)$. So the differential degree of $\mathcal{I} = \mathrm{sat}(f)$ is 3 which is less than $\mathrm{wdeg}(f)$.

Continue from Example 4.4. In this example $F(\mathbf{u}) = u_1^2(u_0')^2 - 2u_1u_1'u_0u_0' + (u_1')^2u_0^2 + 4u_1^3u_0$ is a differentially homogenous polynomial of degree 4 in $\mathbf{u} = (u_0, u_1)$, and its order is 1. And the differential degree of V is 4, which is equal to the weighed degree of $f = (y')^2 - 4y$.

4.4. **Factorization of differential Chow form.** In the algebraic case, the Chow form can be factored into the product of linear polynomials with the generic points of the variety as coefficients. In this section, we will show that there is a differential analog to this result.

Let $\mathbb{U}_0 = \{u_{ij}^{(k)} : i = 0, \ldots, d; j = 0, \ldots, n; k = 0, \ldots, h\} \setminus \{u_{00}^{(h)}\}$, and $\mathcal{F}_0 = \mathcal{F}(\mathbb{U}_0)$. We have

Theorem 4.22. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = f(\mathbf{u}; u_{00}, \ldots, u_{d0})$ be the Chow form of a differential variety of dimension d and order h. Then, there exist $\xi_{\tau 1}, \ldots, \xi_{\tau n}$ ($\tau = 1, \ldots, g$) in an extension field of $\mathcal{F}$ such that*

$$F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{g} (u_{00} + \sum_{\rho=1}^{n} u_{0\rho}\xi_{\tau\rho})^{(h)} \tag{4.4}$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is in $\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ and $g = \deg(f, u_{00}^{(h)})$.

Proof: Consider the irreducible algebraic equation $p(\mathbb{U}_0, u_{00}^{(h)}) = f(\mathbf{u}; u_{00}, u_{10}, \ldots, u_{d0}) = 0$ in $u_{00}^{(h)}$, where p is considered as a polynomial in $\mathcal{F}_0[u_{00}^{(h)}]$. In a suitable algebraic extension field of $\mathcal{F}_0$, this equation has g roots $z_1, \ldots, z_g$. Thus

$$f(\mathbf{u}; u_{00}, u_{10}, \ldots, u_{d0}) = A \prod_{\tau=1}^{g} (u_{00}^{(h)} - z_\tau). \tag{4.5}$$

Notice that z_τ are in an algebraic extension field $\mathcal{F}_0(z_1, \ldots, z_g)$ of $\mathcal{F}_0$. Since we are studying differential polynomials over differential fields, we will define their derivatives by making $\mathcal{F}_0(z_1, \ldots, z_g)$ a differential field. This can be done in a very natural way. Since f, regarded as an algebraic polynomial in $u_{00}^{(h)}$, is a minimal polynomial of z_τ, $\mathrm{S}_f = \frac{\partial f}{\partial u_{00}^{(h)}}$ does not vanish at $u_{00}^{(h)} = z_\tau$. Now, we define the derivatives of z_τ by induction. Firstly, $f' = \mathrm{S}_f u_{00}^{(h+1)} + T$ is linear in $u_{00}^{(h+1)}$. We define $\delta(z_\tau) = z_\tau'$ to be $-\frac{T}{\mathrm{S}_f}\Big|_{u_{00}^{(h)} = z_\tau}$. Supposing the derivatives of z_τ with order less than i have been defined, we now define $\delta^i(z_\tau)$. Since $f^{(i)} = \mathrm{S}_f u_{00}^{(h+i)} + T_i$ is linear in $u_{00}^{(h+i)}$, we define $\delta^i(z_\tau)$ to be $-\frac{T_i}{\mathrm{S}_f}\Big|_{u_{00}^{(h+j)} = z_\tau^{(j)}, j<i}$. Similarly, we can define the derivatives of each element in $\mathcal{F}_0(z_\tau)$ and obtain the differential field $\mathcal{F}_0\langle z_\tau\rangle$. For convenience, by saying a differential polynomial vanishes at $u_{00}^{(h)} = z_\tau$, we mean that as a polynomial in $\mathcal{F}\langle \mathbf{u}_1, \ldots, \mathbf{u}_d, u_{01}, \cdots, u_{0n}\rangle\{u_{00}\}$, it vanishes at $u_{00}^{(h+i)} = z_\tau^{(i)}$ ($i \geq 0$).

Since f is irreducible, we have $f_{\tau 0} = \frac{\partial f}{\partial u_{00}^{(h)}}\Big|_{u_{00}^{(h)} = z_\tau} \neq 0$. Let $\xi_{\tau\rho} = f_{\tau\rho}/f_{\tau 0}$($\rho = 1, \ldots, n$), where $f_{\tau\rho} = \frac{\partial f}{\partial u_{0\rho}^{(h)}}\Big|_{u_{00}^{(h)} = z_\tau}$. We will prove

$$z_\tau = -(u_{01}\xi_{\tau 1} + u_{02}\xi_{\tau 2} + \cdots + u_{0n}\xi_{\tau n})^{(h)}.$$

From $f(\mathbf{u}; \zeta_0, \zeta_1, \ldots, \zeta_d) = 0$, if we differentiate this equality w.r.t. $u_{0\rho}^{(h)}$, we have

$$\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}} + \frac{\partial f}{\partial \zeta_0^{(h)}}(-\xi_\rho) = 0,$$

where $\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}$ and $\frac{\partial f}{\partial \zeta_0^{(h)}}$ are obtained by substituting ζ_i to u_{i0} $(i = 0, 1, \ldots, d)$ in $\frac{\partial f}{\partial u_{0\rho}^{(h)}}$ and $\frac{\partial f}{\partial u_{00}^{(h)}}$ respectively. Multiplying $u_{0\rho}$ to the above equation and for ρ from 1 to n, adding them together, we have

$$\sum_{\rho=1}^{n} u_{0\rho}\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}} + \frac{\partial f}{\partial \zeta_0^{(h)}}(-\sum_{\rho=1}^{n} u_{0\rho}\xi_\rho) = \sum_{\rho=1}^{n} u_{0\rho}\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}} + \zeta_0\frac{\partial f}{\partial \zeta_0^{(h)}} = 0.$$

Thus, $q = \sum_{\rho=1}^{n} u_{0\rho}\frac{\partial f}{\partial u_{0\rho}^{(h)}} + u_{00}\frac{\partial f}{\partial u_{00}^{(h)}} \in \text{sat}(f)$. Since q is of order not greater than f, it must be divisible by f. Since q and f have the same degree, there exists an $a \in \mathcal{F}$ such that $q = af$. Setting $u_{00}^{(h)} = z_\tau$ in both sides of $q = af$, we have $\sum_{\rho=1}^{n} u_{0\rho}f_{\tau\rho} + u_{00}f_{\tau 0} = 0$. Hence, as an algebraic equation, we have

$$u_{00} + \sum_{\rho=1}^{n} u_{0\rho}\xi_{\tau\rho} = 0 \tag{4.6}$$

under the constraint $u_{00}^{(h)} = z_\tau$. As a consequence, $z_\tau = -(\sum_{\rho=1}^{n} u_{0\rho}\xi_{\tau\rho})^{(h)}$. The theorem is proved. $\square$

In the proof of Theorem 4.22, some equations are valid in the algebraic case only. To avoid confusion, we introduce the following notations:

$$\begin{aligned}
{}^a\mathbb{P}_0^{(0)} &= {}^a\mathbb{P}_0 := u_{00} + u_{01}y_1 + \cdots + u_{0n}y_n\\
{}^a\mathbb{P}_0^{(1)} &= {}^a\mathbb{P}_0' := u_{00}' + u_{01}'y_1 + u_{01}y_1' + \cdots + u_{0n}'y_n + u_{0n}y_n'\\
&\cdots\\
{}^a\mathbb{P}_0^{(s)} &:= u_{00}^{(s)} + \textstyle\sum_{j=1}^{n}\sum_{k=0}^{s}\binom{s}{k}u_{0j}^{(k)}y_j^{(s-k)}
\end{aligned}$$

which are considered to be algebraic polynomials in $\mathcal{F}(u_{00}, \ldots, u_{0n}, \ldots, u_{00}^{(s)}, \ldots, u_{0n}^{(s)})[y_1, \ldots, y_n, \ldots, y_1^{(s)}, \ldots, y_n^{(s)}]$, and $u_{ij}^{(k)}, y_i^{(j)}$ are treated as algebraic indeterminates. A point $(\eta_1, \ldots, \eta_n)$ is said to be lying on ${}^a\mathbb{P}_0^{(k)}$ if regarded as an algebraic point, $(\eta_1, \ldots, \eta_n, \ldots, \eta_1^{(k)}, \ldots, \eta_n^{(k)})$ is a zero of ${}^a\mathbb{P}_0^{(k)}$. As a consequence of (4.6) in the proof of Theorem 4.22, we have

Corollary 4.23. $(\xi_{\tau1}, \ldots, \xi_{\tau n}, \ldots, \xi_{\tau1}^{(h-1)}, \ldots, \xi_{\tau n}^{(h-1)})$ $(\tau = 1, \ldots, g)$ are common zeros of ${}^a\mathbb{P}_0 = 0$, ${}^a\mathbb{P}_0' = 0, \ldots,$ ${}^a\mathbb{P}_0^{(h-1)} = 0$ under the constraint $u_{00}^{(h)} = z_\tau$, where z_τ is defined in (4.5).

Example 4.24. Continue from Example 4.4 again. In this example, $F(u) = u_1^2(u_0')^2 - 2u_1u_1'u_0u_0' + (u_1')^2u_0^2 + 4u_1^3u_0$, so $g = 2$. And $F(u) = u_1^2(u_0' + \xi_{11}u_1' + 2\sqrt{-1}\sqrt{u_0u_1})(u_0' + \xi_{21}u_1' - 2\sqrt{-1}\sqrt{u_0u_1}) = u_1^2(u_0 + \xi_{11}u_1)'(u_0 + \xi_{21}u_1)'$ where $\xi_{11} = -u_0/u_1$ with u_0, u_1 satisfying $u_0' = \frac{u_0}{u_1}u_1' - 2\sqrt{-1}\sqrt{u_0u_1}$, $\xi_{21} = -u_0/u_1$ with u_0, u_1 satisfying $u_0' = \frac{u_0}{u_1}u_1' + 2\sqrt{-1}\sqrt{u_0u_1}$. Note that both ξ_{i1} $(i = 1, 2)$ satisfy ${}^a\mathbb{P}_0 = u_0 + u_1\xi_{i1} = 0$, but ${}^a\mathbb{P}_0^{(1)} = u_0' + u_1'\xi_{i1} + u_1\xi_{i1}' \neq 0$. Instead, from the Chow form, we have $\xi_{i1}'^2 + 4u_0/u_1 = 0$.

Lemma 4.25. *In equation (4.4), $A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ is free of $u_{0i}^{(h)}$ $(i = 1, \ldots, n)$.*

Proof: Since f is homogenous in the indeterminates u_{0i} and its derivatives up to the order h, we have

$$\sum_{\rho=1}^{n} u_{0\rho}^{(h)} \frac{\partial f}{\partial u_{0\rho}^{(h)}} + u_{00}^{(h)} \frac{\partial f}{\partial u_{00}^{(h)}} + \sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}} = rf, r \in \mathbb{N}$$

In this equation, let $u_{00}^{(h)} = z_\tau$, we obtain

$$\sum_{\rho=1}^{n} u_{0\rho}^{(h)} f_{\tau\rho} + z_\tau f_{\tau 0} + \sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)} \overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}} = 0.$$

Consequently,

$$z_\tau = -\sum_{\rho=1}^{n} u_{0\rho}^{(h)} \xi_{\tau\rho} - \sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)} \overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}} \Big/ f_{\tau 0}.$$

Henceforth,

$$f(\mathbf{u}; u_{00}, u_{10}, \ldots, u_{d0}) = A \prod_{\tau=1}^{g} \Big(u_{00}^{(h)} + \sum_{\rho=1}^{n} u_{0\rho}^{(h)} \xi_{\tau\rho} + \sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)} \overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}} \Big/ f_{\tau 0} \Big) \tag{4.7}$$

where $\overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}}$ means replacing $u_{00}^{(h)}$ by z_τ in $\frac{\partial f}{\partial u_{0\rho}^{(k)}}$.

We claim that $\xi_{\tau\rho}$ and $\sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)} \overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}}/f_{\tau 0}$ are algebraically independent of $u_{0i}^{(h)}\,(i = 1, \ldots, n)$. Firstly, since ξ_ρ is algebraically independent of $u_{0i}^{(h)}(i = 1, \ldots, n)$, and

$$\xi_\rho = \frac{\partial f}{\partial u_{0\rho}^{(h)}} \Big/ \frac{\partial f}{\partial u_{00}^{(h)}} \Big|_{(u_{00},\ldots,u_{d0})=(\zeta_0,\ldots,\zeta_d)} = \overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}} \Big/ \frac{\partial f}{\partial \zeta_0^{(h)}},$$

we have

$$\frac{\partial \xi_\rho}{\partial u_{0i}^{(h)}} = \frac{\frac{\partial}{\partial u_{0i}^{(h)}}\left(\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}\right)\frac{\partial f}{\partial \zeta_0^{(h)}} - \frac{\partial}{\partial u_{0i}^{(h)}}\left(\frac{\partial f}{\partial \zeta_0^{(h)}}\right)\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}}{(\frac{\partial f}{\partial \zeta_0^{(h)}})^2} = 0,$$

or equivalently

$$\frac{\partial}{\partial u_{0i}^{(h)}}\left(\frac{\partial f}{\partial u_{0\rho}^{(h)}}\right)\frac{\partial f}{\partial u_{00}^{(h)}} - \frac{\partial}{\partial u_{0i}^{(h)}}\left(\frac{\partial f}{\partial u_{00}^{(h)}}\right)\frac{\partial f}{\partial u_{0\rho}^{(h)}} \in \mathrm{sat}(f).$$

Set $u_{00}^{(h)} = z_\tau$, we have

$$\frac{\partial}{\partial u_{0i}^{(h)}}\Big(f_{\tau\rho}\Big) f_{\tau 0} - \frac{\partial}{\partial u_{0i}^{(h)}}\Big(f_{\tau 0}\Big) f_{\tau\rho} = 0.$$

Thus,

$$\frac{\partial \xi_{\tau\rho}}{\partial u_{0i}^{(h)}} = \frac{\partial (f_{\tau\rho}/f_{\tau 0})}{\partial u_{0i}^{(h)}} = 0. \tag{4.8}$$

Secondly, set $(u_{00}, \ldots, u_{d0}) = (\zeta_0, \ldots, \zeta_d)$ in the equation

$$\sum_{\rho=1}^{n} u_{0\rho}^{(h)} \frac{\partial f}{\partial u_{0\rho}^{(h)}} + u_{00}^{(h)} \frac{\partial f}{\partial u_{00}^{(h)}} + \sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)} \frac{\partial f}{\partial u_{0\rho}^{(k)}} = rf, r \in \mathbb{N}.$$

We have

$$\sum_{\rho=1}^{n} u_{0\rho}^{(h)}\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}+\zeta_0^{(h)}\frac{\partial f}{\partial \zeta_0^{(h)}}+\overline{\sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)}\frac{\partial f}{\partial u_{0\rho}^{(k)}}}=0.$$

By Theorem 4.13,

$$\sum_{\rho=1}^{n} u_{0\rho}^{(h)}\xi_\rho+\zeta_0^{(h)}+\overline{\sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)}\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\Big/\frac{\partial f}{\partial \zeta_0^{(h)}}=0.$$

Then,

$$\frac{\partial}{\partial u_{0i}^{(h)}}\left(\overline{\sum_{k=0}^{h-1}\sum_{\rho=0}^{n} u_{0\rho}^{(k)}\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\Big/\frac{\partial f}{\partial \zeta_0^{(h)}}\right)=-\xi_i-(-\xi_i)=0$$

$$=\frac{\frac{\partial}{\partial u_{0i}^{(h)}}\left(\overline{\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\right)\frac{\partial f}{\partial\zeta_0^{(h)}}-\frac{\partial}{\partial u_{0i}^{(h)}}\left(\frac{\partial f}{\partial\zeta_0^{(h)}}\right)\left(\overline{\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\right)}{\left(\frac{\partial f}{\partial\zeta_0^{(h)}}\right)^2}.$$

Thus, we have $\frac{\partial}{\partial u_{0i}^{(h)}}\left(\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\frac{\partial f}{\partial u_{0\rho}^{(k)}}\right)\frac{\partial f}{\partial u_{00}^{(h)}}-\frac{\partial}{\partial u_{0i}^{(h)}}\left(\frac{\partial f}{\partial u_{00}^{(h)}}\right)\left(\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\frac{\partial f}{\partial u_{0\rho}^{(k)}}\right)\in \mathrm{sat}(f)$. Equivalently,

$$\frac{\partial\left(\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\Big/f_{\tau 0}\right)}{\partial u_{0i}^{(h)}}=0. \tag{4.9}$$

From (4.8) and (4.9), $\xi_{\tau\rho}$ and $\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\Big/f_{\tau 0}$ are algebraically independent of $u_{0i}^{(h)}$ $(i=1,\ldots,n)$. Then as symmetric functions in $\xi_{\tau1},\ldots,\xi_{\tau n}$, $\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\Big/f_{\tau 0}$ are rational functions in the set of indeterminates $\{u_{ik},\ldots,u_{ik}^{(h)}, u_{0k},\ldots,u_{0k}^{(h-1)}: i=1,\ldots,d;\ k=0,\ldots,n\}$ only. Therefore, $\prod_{\tau=1}^{g}(u_{00}^{(h)}-z_\tau)=\frac{\phi}{\psi}$ where ψ is free of $u_{0i}^{(h)}$ $(i=1,\ldots,n)$ and $\gcd(\phi,\psi)=1$. Thus $A\phi=f\psi$. Since f is irreducible, we conclude that $A=\psi$ is free of $u_{0i}^{(h)}$ $(i=1,\ldots,n)$. □

Theorem 4.26. *The quantities $\xi_{\tau1},\ldots,\xi_{\tau n}$ in (4.4) are unique and (4.7) is a factorization of F as a form in $u_{00}^{(h)},\ldots,u_{0n}^{(h)}$ in an extension field of $\mathcal{F}(u_{ik},\ldots,u_{ik}^{(h)}, u_{0k},\ldots,u_{0k}^{(h-1)}: i=1,\ldots,d;\ k=0,\ldots,n)$.*

Proof: From Lemma 4.25, equations (4.8) and (4.9), we can see that $A(\mathbf{u}_0,\ldots,\mathbf{u}_d)$, $\xi_{\tau j}$, and $\sum_{k=0}^{h-1}\sum_{\rho=0}^{n}u_{0\rho}^{(k)}\overline{\frac{\partial f}{\partial u_{0\rho}^{(k)}}}\Big/f_{\tau 0}$ are free of $u_{0i}^{(h)}(i=1,\ldots,n)$. Then, (4.7) is a factorization of the Chow form $F(\mathbf{u}_0,\ldots,\mathbf{u}_n)$ in the polynomial ring $\mathcal{F}(u_{ik},\ldots,u_{ik}^{(h)}, u_{0k},\ldots,u_{0k}^{(h-1)}: i=1,\ldots,d;\ k=0,\ldots,n)[u_{00}^{(h)},\ldots,u_{0n}^{(h)}]$. Thus, the factorization (4.7) must be unique and hence $\xi_{\tau i}$. □

Definition 4.27. By Lemma 4.18 and Theorem 4.26, the number g in (4.4) is an invariant of V under linear transformations, which is called the *leading differential degree* of V.

Theorem 4.28. *The points* $(\xi_{\tau 1},\ldots,\xi_{\tau n})\,(\tau=1,\ldots,g)$ *in* (3) *are generic points of the variety* V*, and satisfy the equations*

$$u_{\sigma 0}+\sum_{\rho=1}^{n}u_{\sigma\rho}y_\rho=0\,(\sigma=1,\ldots,d).$$

Proof: Suppose $\phi(y_1,\ldots,y_n)\in\mathcal{F}\{\mathbb{Y}\}$ is any differential polynomial vanishing on V. Then $\phi(\xi_1,\ldots,\xi_n)=0$. From Theorem 4.13, $\xi_\rho=\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}\Big/\frac{\partial f}{\partial\zeta_0^{(h)}}$, we have

$$\phi\Big(\overline{\frac{\partial f}{\partial u_{01}^{(h)}}}\Big/\frac{\partial f}{\partial\zeta_0^{(h)}},\ldots,\overline{\frac{\partial f}{\partial u_{0n}^{(h)}}}\Big/\frac{\partial f}{\partial\zeta_0^{(h)}}\Big)=0$$

Hence, $\phi(\frac{\partial f}{\partial u_{01}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}},\ldots,\frac{\partial f}{\partial u_{0n}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}})$ vanishes for $(u_{00},\ldots,u_{d0})=(\zeta_0,\ldots,\zeta_d)$. Then there exists an $m\in\mathbb{N}$, such that $(\frac{\partial f}{\partial u_{00}^{(h)}})^m\phi(\frac{\partial f}{\partial u_{01}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}},\ldots,\frac{\partial f}{\partial u_{0n}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}})\in\mathrm{sat}(f)\bigcap\mathcal{F}\{\mathbf{u};u_{00},\ \ldots,u_{d0}\}$. Since $\mathrm{I}_f*\mathrm{S}_f$ does not vanish at $u_{00}^{(h)}=z_\tau$, when regarding f and polynomials in $\mathrm{sat}(f)$ as differential polynomials in u_{00}, we have $(f_{\tau 0})^m\phi(\xi_{\tau 1},\ldots,\xi_{\tau n})=0$, thus $\phi(\xi_{\tau 1},\ldots,\xi_{\tau n})=0$, which means that $(\xi_{\tau 1},\ldots,\xi_{\tau n})\in V$.

Conversely, for any $p\in\mathcal{F}\{\mathbb{Y}\}$ such that $p(\xi_{\tau 1},\ldots,\xi_{\tau n})=0$, when we regard the polynomial $(\frac{\partial f}{\partial u_{00}^{(h)}})^l p(\frac{\partial f}{\partial u_{01}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}},\ldots,\frac{\partial f}{\partial u_{0n}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}})$ $(l\in\mathbb{N})$ as a differential polynomial in u_{00}, it vanishes at $u_{00}^{(h)}=z_\tau$. Hence, $(\frac{\partial f}{\partial u_{00}^{(h)}})^l p(\frac{\partial f}{\partial u_{01}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}},\ldots,\frac{\partial f}{\partial u_{0n}^{(h)}}\Big/\frac{\partial f}{\partial u_{00}^{(h)}})$ $\in\mathrm{sat}(f)$ in the differential ring $\mathcal{F}\langle\mathbf{u}_1,\ldots,\mathbf{u}_d,u_{01},\cdots,u_{0n}\rangle\{u_{00}\}$, and thus $p(\xi_1,\ldots,\xi_n)=0$. So $(\xi_{\tau 1},\ldots,\xi_{\tau n})$ is a generic point of V.

Since $\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}+\frac{\partial f}{\partial\zeta_0^{(h)}}(-\xi_\rho)=0$, we have $\sum_{\rho=1}^n u_{\sigma\rho}\overline{\frac{\partial f}{\partial u_{0\rho}^{(h)}}}+\zeta_\sigma\frac{\partial f}{\partial\zeta_0^{(h)}}=0$. Thus, $\sum_{\rho=0}^n u_{\sigma\rho}\frac{\partial f}{\partial u_{0\rho}^{(h)}}$ vanishes at $(u_{00},\ldots,u_{d0})=(\zeta_0,\ldots,\zeta_d)$. In the case $\sigma\neq 0$, $\sum_{\rho=0}^n u_{\sigma\rho}\frac{\partial f}{\partial u_{0\rho}^{(h)}}=0$. Consequently, $u_{\sigma 0}+\sum_{\rho=1}^n u_{\sigma\rho}\xi_{\tau\rho}=0(\sigma=1,\ldots,d)$. □

To end this subsection, we will prove a result which gives the geometrical meaning of the leading differential degree.

Suppose $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$ is the differential Chow form of V which is of dimension d, order g, and leading differential degree g. Theorem 4.28 and Corollary 4.23 show that $(\xi_{\tau 1},\ldots,\xi_{\tau n})(\tau=1,\ldots,g)$ are intersection points of V and $\mathbb{P}_i=0(i=1,\ldots,d)$ as well as ${}^a\mathbb{P}_0^{(k)}=0\,(k=0,\ldots,h-1)$. In the next theorem, we will prove the converse of this result, that is, $(\xi_{\tau 1},\ldots,\xi_{\tau n})\,(\tau=1,\ldots,g)$ are the only elements in V which are also on $\mathbb{P}_i=0\,(i=1,\ldots,d)$ as well as ${}^a\mathbb{P}_0^{(k)}=0\,(k=0,\ldots,h-1)$. Intuitively, we use $\mathbb{P}_i=0\,(i=1,\ldots,d)$ to decrease the dimension of V to zero and use ${}^a\mathbb{P}_0^{(k)}=0\,(k=0,\ldots,h-1)$ to determine the h arbitrary constants in the solutions of the zero dimensional differential variety.

Theorem 4.29. $(\xi_{\tau 1},\ldots,\xi_{\tau n})\,(\tau=1,\ldots,g)$ *defined in (4.4) are the only elements of* V *which also lie on* $\mathbb{P}_1,\ldots,\mathbb{P}_d$ *as well as on* ${}^a\mathbb{P}_0,{}^a\mathbb{P}_0',\ldots,{}^a\mathbb{P}_0^{(h-1)}$.

Proof: Firstly, from Theorem 4.28 and Corollary 4.23, $(\xi_{\tau 1},\ldots,\xi_{\tau n})\,(\tau=1,\ldots,g)$ are solutions of $\mathbb{I}(V)$ and $\mathbb{P}_1,\ldots,\mathbb{P}_d$ which also lie on ${}^a\mathbb{P}_0,{}^a\mathbb{P}_0',\ldots,{}^a\mathbb{P}_0^{(h-1)}$. It suffices to show that the number of solutions of $\mathbb{I}(V)$ and $\mathbb{P}_1,\ldots,\mathbb{P}_d$ which also lie on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0', \ldots, {}^a\mathbb{P}_0^{(h-1)}$ does not exceed g.

Let $\mathcal{J}=[\mathbb{I}(V),\mathbb{P}_1,\ldots,\mathbb{P}_d]$. By Theorem 3.14, $\mathcal{J}$ is a differential ideal of dimension zero and order h. Let $\mathcal{J}^{<h>}=\mathcal{J}\cap\mathcal{F}(\mathbb{U}_d)[y_1,\ldots,y_n,\ldots,y_1^{(h)},\ldots,y_n^{(h)}]$, where $\mathbb{U}_d=\{u_{ij}^{(k)}: i=1,\ldots,d; j=0,\ldots,n; k\geq 0\}$. Since $\mathcal{J}$ is of dimension zero and order h, its differential dimension polynomial is of the form $\omega(t)=h$, for $t\geq h$. So $\mathcal{J}^{<h>}$ is an algebraic prime ideal of dimension h.

Let $\mathcal{J}_0=(\mathcal{J}^{<h>}, {}^a\mathbb{P}_0, {}^a\mathbb{P}_0', \ldots, {}^a\mathbb{P}_0^{(h-1)})$ in the polynomial ring $\mathcal{F}(\mathbb{U}_d\cup\{u_{0j}^{(l)}; j=1,\ldots,n;\, l=0,\ldots,h-1\})[y_1,\ldots,y_n,\ldots,y_1^{(h)},\ldots,y_n^{(h)},u_{00},\ldots,u_{00}^{(h-1)}]$. It is clear that $\mathcal{J}_0$ is a prime ideal of dimension h. If we can prove that $\{u_{00},\ldots,u_{00}^{(h-1)}\}$ is a parametric set of $\mathcal{J}_0$, then it is clear that $\mathcal{J}_1=(\mathcal{J}_0)$ is a prime ideal of dimension zero in $\mathcal{F}(\mathbb{U}^*)[y_1,\ldots,y_n,\ldots,y_1^{(h)},\ldots,y_n^{(h)}]$, where $\mathbb{U}^*=\mathbb{U}_d\cup\{u_{0j}^{(l)}; j=0,1,\ldots,n; l=0,\ldots,h-1\}$. So it needs to prove that $\{u_{00},\ldots,\, u_{00}^{(h-1)}\}$ is a parametric set of $\mathcal{J}_0$. Suppose the contrary, then there exists a nonzero polynomial involving only $\{u_{00},\ldots,u_{00}^{(h-1)}\}$ as well as the other u, which belongs to $\mathcal{J}_0$. Such a polynomial also belongs to $[\mathcal{J},\mathbb{P}_0]\in\mathcal{F}\langle u_{ij},u_{0l}: i=1,\ldots,d; j=0,\ldots,n; l=1,\ldots,n\rangle\{y_1,\ldots,y_n,u_{00}\}$. From the proof of Theorem 4.11, $\{F, S_Fy_1-\frac{\partial F}{\partial u_{01}^{(s)}},\ \ldots,\ S_Fy_n-\frac{\partial F}{\partial u_{0n}^{(s)}}\}$ is a characteristic set of $[\mathcal{J},\mathbb{P}_0]$ w.r.t. the elimination ranking $u_{00}\prec y_1\prec\ldots\prec y_n$. So this polynomial can be reduced to zero by $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$. But $\mathrm{ord}(F,u_{00})=h$, a contradiction. So we have proved that $\mathcal{J}_1$ is a prime ideal of dimension zero.

It is clear that $\mathcal{J}_2=(\mathcal{J}_1, {}^a\mathbb{P}_0^{(h)})\in\mathcal{F}(\mathbb{U}^*,u_{0j}^{(h)}: j=1,\ldots,n)[y_1,\ldots,y_n,\ldots,y_1^{(h)},\ldots,y_n^{(h)},\, u_{00}^{(h)}]$ is a prime ideal of dimension zero. Then, there exists an irreducible polynomial involving only $u_{ij}^{(k)}$ and $u_{00}^{(h)}$. Similarly as above, it also belongs to $[\mathcal{J},\mathbb{P}_0]$, thus it can be divisible by F. Since F is irreducible, it differs from F only by a factor in $\mathcal{F}$. Thus, $F=f(\mathbf{u};u_{00},u_{10},\ldots,u_{d0})\in\mathcal{J}_2$.

Let $(\xi_1,\ldots,\xi_n)$ ba a generic point of V and $\zeta_i=-\sum_{j=1}^n u_{ij}\xi_j\ (i=0,\ldots,d)$. Then the differential ideal $[\mathbb{I}(V),\mathbb{P}_1,\ldots,\mathbb{P}_d,\mathbb{P}_0]$ in $\mathcal{F}\langle\mathbf{u}\rangle\{y_1,\ldots,y_n,u_{00},\ldots,u_{d0}\}$ has a generic point $(\xi_1,\ldots,\xi_n,\zeta_0,\ldots,\zeta_d)$. Since $f(\mathbf{u};\zeta_0,\ldots,\zeta_d)=0$, differentiate both sides of this identity w.r.t. $u_{0j}^{(k)}$, we have the following identities

$$\overline{\frac{\partial f}{\partial u_{0j}^{(k)}}}+\sum_{l=k}^{h}\frac{\partial f}{\partial\zeta_0^{(l)}}\left(-\binom{l}{k}\xi_j^{(l-k)}\right)=0,\qquad (j=1,\ldots,n;k=0,\ldots,h),$$

where $\overline{\frac{\partial f}{\partial u_{0j}^{(k)}}}$ and $\frac{\partial f}{\partial\zeta_0^{(l)}}$ are respectively obtained by substituting ζ_i to u_{i0} in $\frac{\partial f}{\partial u_{0j}^{(k)}}$ and $\frac{\partial f}{\partial u_{00}^{(l)}}$. Let $g_{jk}=\binom{h}{k}\frac{\partial f}{\partial u_{00}^{(h)}}y_j^{(k)}+\sum_{l=1}^k\binom{h-l}{k-l}\frac{\partial f}{\partial u_{00}^{(h-l)}}y_j^{(k-l)}-\frac{\partial f}{\partial u_{0j}^{(h-k)}}\ (j=1,\ldots,n;k=0,\ldots,h)$. Then $g_{jk}\in[\mathbb{I}(V),\mathbb{P}_1,\ldots,\mathbb{P}_d,\mathbb{P}_0]\subset[\mathcal{J},\mathbb{P}_0]$, for g_{jk} vanishes at $(\xi_1,\ldots,\xi_n,\zeta_0,\ldots,\zeta_d)$. Denote the algebraic ideal $[\mathcal{J},\mathbb{P}_0]\cap\mathcal{F}(\mathbb{U}_d,u_{0j}^{(l)}: j=1,\ldots,n; l=0,\ldots,h)[y_1,\ldots,y_n,\ldots,y_1^{(h)},\ldots,y_n^{(h)},u_{00},\ \ldots,u_{00}^{(h)}]$ by $[\mathcal{J},\mathbb{P}_0]^{<h>}$. It is clear that $g_{jk}\in[\mathcal{J},\mathbb{P}_0]^{<h>}$. We will show that $[\mathcal{J},\mathbb{P}_0]^{<h>}=(\mathcal{J}^{<h>}, {}^a\mathbb{P}_0,\ldots, {}^a\mathbb{P}_0^{(h)})$, which implies that $g_{jk}\in\mathcal{J}_2$. Let $\eta=(\eta_1,\ldots,\eta_n)$ be a generic point of $\mathcal{J}$. Then $(\eta_1,\ldots,\eta_n,-\sum_{j=1}^n u_{0j}\eta_j)$ is a generic point of $[\mathcal{J},\mathbb{P}_0]$. Thus, $(\eta_1,\ldots,\eta_n,\ldots,\eta_1^{(h)},\ldots,\eta_n^{(h)},-\sum_{j=1}^n u_{0j}\eta_j,\ldots,-\sum_{j=1}^n(u_{0j}\eta_j)^{(h)})$ is a generic point of $[\mathcal{J},\mathbb{P}_0]^{<h>}$. Of course, it is also a generic point of $(\mathcal{J}^{<h>}, {}^a\mathbb{P}_0,\ldots, {}^a\mathbb{P}_0^{(h)})$. So the two ideals are identical. Thus, g_{jk} belongs to $\mathcal{J}_2$. Note that the coefficient of $y_j^{(k)}$ in g_{jk} is $\binom{h}{k}S_f = \binom{h}{k}\frac{\partial f}{\partial u_{00}^{(h)}}$. So $\mathbb{V}(\mathcal{J}_2) \subseteq \mathbb{V}(f(\mathbb{U}_0, u_{00}^{(h)}), g_{jk} : j = 1, \ldots, n; k = 0, \ldots, h)$ and the latter algebraic variety of consists exactly g elements. Thus, $|\mathbb{V}(\mathcal{J}_1)| = |\mathbb{V}(\mathcal{J}_2)| \leq g$, which completes the proof. □

Example 4.30. Let V be the general component of $P = y'^2 - 4y = 0$. As in Theorem 4.29, we introduce the equation ${}^a\mathbb{P}_0 = u_0 + u_1 y$ which intersects V at two points: $y = -u_0/u_1$ and $y' = \pm 2\sqrt{-u_0/u_1}$. The condition $y' = \pm 2\sqrt{-u_0/u_1}$ is equivalent to the vanishing of the Chow form or the condition $u_0' = \frac{u_0}{u_1}u_1' \mp 2\sqrt{-1}\sqrt{u_0 u_1}$ given in Example 4.24. Using the terminologies of Theorem 4.29, we have $\xi_{11} = -u_0/u_1$ with u_0, u_1 satisfying $u_0' = \frac{u_0}{u_1}u_1' - 2\sqrt{-1}\sqrt{u_0 u_1}$ and $\xi_{21} = -u_0/u_1$ with u_0, u_1 satisfying $u_0' = \frac{u_0}{u_1}u_1' + 2\sqrt{-1}\sqrt{u_0 u_1}$.

With Theorems 4.22, 4.26, 4.28, and 4.29, we proved the second and third statements of Theorem 1.2.

4.5. **Relations between the differential Chow form and the variety.** In the algebraic case, we can obtain the defining equations of a variety from its Chow form. But in the differential case, this is not valid. Now we proceed as follows to obtain a weaker result.

Lemma 4.31. *Let V be an irreducible differential variety of dimension $d > 0$ and $(0, 0, \ldots, 0) \notin V$. Then, the intersection of V with a generic prime passing through $(0, \ldots, 0)$ is either empty or unmixed of dimension $d - 1$. Moreover, in the case $d > 1$, it is exactly unmixed of dimension $d - 1$.*

Proof: Let $\mathcal{P} = \mathbb{I}(V)$ be the prime differential ideal corresponding to V. A generic prime passing through $(0, \ldots, 0)$ is $u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$ where the u_i are differential indeterminates. Since $(0, 0, \ldots, 0) \notin V$, we have

$$\begin{aligned}
& V \cap \mathbb{V}(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n) \\
= \; & \mathbb{V}(\mathcal{P}, u_1 y_1 + u_2 y_2 + \cdots + u_n y_n) \\
= \; & \bigcup_{i=1}^{n} \mathbb{V}([\mathcal{P}, u_1 y_1 + u_2 y_2 + \cdots + u_n y_n]/y_i) \\
= \; & \bigcup_{i=1}^{n} \mathbb{V}([\mathcal{P}, u_1 y_1 + u_2 y_2 + \cdots + u_n y_n] : y_i^{\infty})
\end{aligned}$$

Suppose a generic point of V is $(\xi_1, \ldots, \xi_n)$. Since $(0, 0, \ldots, 0) \notin V$, there exists at least one $i \in \{1, \ldots, n\}$, such that $\xi_i \neq 0$. Of course, $\xi_i = 0$ means $\mathbb{V}([\mathcal{P}, u_1 y_1 + u_2 y_2 + \cdots + u_n y_n] : y_i^{\infty}) = \emptyset$. So we need only to consider the case when $\xi_i \neq 0$. Without loss of generality, we suppose $\xi_1 \neq 0$.

Let

$$\mathcal{Q} = [\mathcal{P}, u_1 y_1 + u_2 y_2 + \cdots + u_n y_n] : y_1^{\infty} \subseteq \mathcal{F}\langle u_1, \ldots, u_n \rangle \{y_1, \ldots, y_n\}$$

and

$$\mathcal{Q}_0 = [\mathcal{P}, u_1 y_1 + u_2 y_2 + \cdots + u_n y_n] : y_1^{\infty} \subseteq \mathcal{F}\langle u_2, \ldots, u_n \rangle \{y_1, \ldots, y_n, u_1\}$$

A generic point of $\mathcal{Q}_0$ is $(\xi_1, \ldots, \xi_n, -\frac{u_2\xi_2 + \cdots + u_n\xi_n}{\xi_1})$ and $\dim(\mathcal{Q}_0) = d$. Now we discuss in three cases.

Case 1) $\mathcal{P}\cap\mathcal{F}\{y_1\}\neq\{0\}$, that is, ξ_1 is differentially algebraic over $\mathcal{F}$. We have

$$\dim V = d = \mathrm{d.tr.deg}\,\mathcal{F}\langle\xi_1,\ldots,\xi_n\rangle/\mathcal{F} = \mathrm{d.tr.deg}\,\mathcal{F}\langle\xi_1\rangle\langle\xi_2,\ldots,\xi_n\rangle/\mathcal{F}\langle\xi_1\rangle.$$

Suppose $\xi_2,\ldots,\xi_{d+1}$ are differentially independent over $\mathcal{F}\langle\xi_1\rangle$.

Firstly, $\mathcal{Q}_0\cap\mathcal{F}\langle u_2,\ldots,u_n\rangle\{u_1\}=\{0\}$. For if not, we have a nonzero differential polynomial $h(u_2,\ldots,u_n,u_1)\in\mathcal{F}\{u_2,\ldots,u_n,u_1\}$ such that $h(u_2,\ldots,u_n, -\frac{u_2\xi_2+\cdots+u_n\xi_n}{\xi_1})=0$. For a fixed i between 2 and n, if we specialize u_i to -1, $u_j\,(j\neq i)$ to 0, then by Theorem 2.14, ξ_i/ξ_1 is differentially algebraic over $\mathcal{F}$. So each $\xi_i\,(i=1,\ldots,n)$ is differentially algebraic over $\mathcal{F}$, which contradicts to the fact $d>0$.

Secondly, since $y_2,\ldots,y_{d+1}$ is a parametric set of $\mathcal{P}$, it is also a parametric set for $\mathcal{Q}_0$. So $y_2,\ldots,y_{d+1},u_1$ are differentially dependent modulo $\mathcal{Q}_0$. Since $\mathcal{Q}_0\cap\mathcal{F}\langle u_2,\ldots,u_n\rangle\{u_1\}=\{0\}$, we know that $y_2,\ldots,y_{d+1}$ are differentially dependent modulo $\mathcal{Q}$. Using the fact that each remaining y_i and $y_2,\ldots,y_{d+1}$ are differentially dependent modulo Q, we obtain $\dim(\mathcal{Q})\leq d-1$. We claim that $\dim(\mathcal{Q})=d-1$ by proving that $y_2,\ldots,y_d$ are differentially independent modulo $\mathcal{Q}$. For if not, there exists $0\neq h(y_2,\ldots,y_d,u_1)\in\mathcal{Q}_0$ such that $h(\xi_2,\ldots,\xi_d,-\frac{u_2\xi_2+\cdots+u_n\xi_n}{\xi_1})=0$. By Theorem 2.14, we can specialize u_{d+1} to -1, the other u_i to zero, and conclude that $\xi_2,\ldots,\xi_d,\frac{\xi_{d+1}}{\xi_1}$ are differentially dependent over $\mathcal{F}$. Since $\xi_2,\ldots,\xi_d$ are differentially independent over $\mathcal{F}$, ξ_{d+1} is differentially algebraic over $\mathcal{F}\langle\xi_1,\ldots,\xi_d\rangle$, which is a contradiction. Thus $\dim\mathcal{Q}=d-1$.

Case 2) $d\geq 2$ and ξ_1 is differentially transcendental over $\mathcal{F}$. In this case, we suppose a differential transcendence basis is $\xi_1,\ldots,\xi_d$.

Firstly, $\mathcal{Q}_0\cap\mathcal{F}\langle u_2,\ldots,u_n\rangle\{u_1\}=\{0\}$. For if not, as proceeded in the preceding case, we conclude that ξ_i/ξ_1 is differentially algebraic over $\mathcal{F}$, that is, ξ_i,ξ_1 are differentially algebraic over $\mathcal{F}$, which contradicts to the fact $d>1$. So $\mathcal{Q}$ is a nontrivial prime differential ideal.

Secondly, $\dim(\mathcal{Q})=d-1$, for on the one hand from the fact that $y_1,y_2,\ldots,y_d,u_1$ are differentially dependent modulo $\mathcal{Q}_0$, we have $\dim(\mathcal{Q})\leq d-1$, and on the other hand, from the fact that $y_2,\ldots,y_d,u_1$ are differentially independent modulo $\mathcal{Q}_0$, it comes $\dim(\mathcal{Q})\geq d-1$.

Case 3) $d=1$ and ξ_1 is differentially transcendental over $\mathcal{F}$. If $\mathcal{Q}_0\cap\mathcal{F}\langle u_2,\ldots,u_n\rangle\{u_1\}\neq\{0\}$, the intersection is empty. If $\mathcal{Q}\neq[1]$, similar to case 2, we can easily prove that the intersection is of dimension zero.

So for each $i\in\{1,\ldots,n\}$ such that $\xi_i\neq 0$, we can show that $\mathbb{V}([\mathcal{P},u_1y_1+u_2y_2+\cdots+u_ny_n]:y_i^\infty)$ is either empty or of dimension $d-1$ similarly as the above steps for the case $i=1$. And if $d>1$, it is exactly of dimension $d-1$. Thus the theorem is proved. $\square$

The following result gives an equivalent condition for a point to be in a variety.

Theorem 4.32. *Let V be a differential variety of dimension d. Then $\overline{x}\in V$ if and only if $d+1$ independent generic primes $\mathbb{P}_0,\mathbb{P}_1,\ldots,\mathbb{P}_d$ passing through $\overline{x}$ meet V.*

Proof: The necessity of the condition is obviously true. We now consider the sufficiency. We adjoin the coordinates of $\overline{x}$ to $\mathcal{F}$, and denote $\overline{\mathcal{F}}$ to be the differential field thus obtained. Regarded as a variety over $\overline{\mathcal{F}}$, V is the sum of a finite number of irreducible varieties $\overline{V}_i$, which are of dimension d [29, p.51]. Suppose $\overline{x}\notin V$, and therefore does not lie in any component of V. We now prove that any $d+1$ independent generic primes passing through $\overline{x}$ do not meet $\overline{V}_i$. Without loss of generality, suppose $\overline{x} = (0, 0, \ldots, 0)$. Then a generic prime passing through $\overline{x}$ is $s_1y_1 + \cdots + s_ny_n$ with s_i differential indeterminates. We proceed by induction on d.

If $d = 0$, then for $(a_1, \ldots, a_n) \in V$, each a_i is differentially algebraic over $\mathcal{F}$. If $V \cap \mathbb{V}(s_1y_1 + \cdots + s_ny_n) \neq \emptyset$, then there exists a $(a_1, \ldots, a_n) \in V$ such that $s_1a_1 + \cdots + s_na_n = 0$. Since the s_i are differentially independent over $\mathcal{F}$. Thus $(a_1, \ldots, a_n) = (0, \ldots, 0)$, a contradiction to the fact that $\overline{x} \notin V$. Thus the theorem is proved when $d = 0$.

We therefore assume the truth of the theorem for varieties of dimension less than d, and consider a variety V of dimension d. Let $\mathbb{P}_0, \ldots, \mathbb{P}_d$ be $d+1$ independent generic primes passing through $\overline{x}$. The equation $\mathbb{P}_d$ can be written as $s_1y_1 + \cdots + s_ny_n = 0$ with s_i differential indeterminates. From Lemma 4.31, $\mathbb{P}_d = 0$ meets V in a variety $\mathbb{W}$ of dimension less than d. By the hypothesis of the induction, $\mathbb{P}_0, \ldots, \mathbb{P}_{d-1}$ do not meet $\mathbb{W}$, it follows that V does not meet $\mathbb{P}_0, \ldots, \mathbb{P}_d$. Therefore the theorem is proved. $\square$

The following result proves the fourth statement of Theorem 1.2.

Theorem 4.33. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ be the differential Chow form of V and $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$. Suppose that $\mathbf{u}_i (i = 0, \ldots, d)$ specialize to sets $\mathbf{v}_i$ of specific elements in an extension field of $\mathcal{F}$ and $\overline{\mathbb{P}}_i\,(i = 0, \ldots, d)$ are obtained by substituting $\mathbf{u}_i$ by $\mathbf{v}_i$ in $\mathbb{P}_i$. If $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, d)$ meet V, then $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$. Furthermore, if $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$ and $S_F(\mathbf{v}_0, \ldots, \mathbf{v}_d) \neq 0$, then the $d+1$ primes $\overline{\mathbb{P}}_i = 0$ $(i = 0, \ldots, d)$ meet V.*

Proof: Let $\mathcal{I} = \mathbb{I}(V) \subseteq \mathcal{F}\{\mathbb{Y}\}$, $\mathbb{I}_{\zeta,\xi} = [\mathcal{I}, \mathbb{P}_0, \ldots, \mathbb{P}_d] \subseteq \mathcal{F}\langle\mathbf{u}\rangle\{y_1, \ldots, y_n, u_{00}, \ldots, u_{d0}\}$, and $\mathcal{I}_1 = [\mathbb{I}_{\zeta,\xi}] \subseteq \mathcal{F}\langle\mathbf{u}_0, \ldots, \mathbf{u}_d\rangle\{y_1, \ldots, y_n\}$. By Lemma 4.10, $\{F, \frac{\partial F}{\partial u_{00}^{(h)}}y_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \ldots, \frac{\partial F}{\partial u_{00}^{(h)}}y_n - \frac{\partial F}{\partial u_{0n}^{(h)}}\}$ is a characteristic set of $\mathbb{I}_{\zeta,\xi}$ w.r.t. the elimination ranking $u_{d0} \prec \cdots \prec u_{00} \prec y_1 \prec \cdots \prec y_n$. Since F is irreducible, $\mathbb{I}_{\zeta,\xi} = [F, S_Fy_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \ldots, S_Fy_n - \frac{\partial F}{\partial u_{0n}^{(h)}}] : S_F^\infty$ with $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$.

When $\mathbf{u}_i$ specializes to $\mathbf{v}_i$, $\mathcal{I}_1$ becomes an ideal in $\mathcal{F}\langle\mathbf{v}_0, \ldots, \mathbf{v}_n\rangle\{\mathbb{Y}\}$. If $\overline{\mathbb{P}}_0, \ldots, \overline{\mathbb{P}}_d$ meet V, then $\overline{\mathcal{I}_1} = [\mathcal{I}, \overline{\mathbb{P}}_0, \ldots, \overline{\mathbb{P}}_d] \neq [1]$ which implies $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$ since $F \in \mathbb{I}_{\zeta,\xi}$.

If $S_F(\mathbf{v}_0, \ldots, \mathbf{v}_d) \neq 0$ and $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$, then let $\overline{y}_i = (\frac{\partial F}{\partial u_{0i}^{(h)}}(\mathbf{v}_0, \ldots, \mathbf{v}_d))\big/(S_F(\mathbf{v}_0, \ldots, \mathbf{v}_d))(i = 1, \ldots, n)$. We claim that $(\overline{y}_1, \ldots, \overline{y}_n)$ lies in V and the $d+1$ primes $\overline{\mathbb{P}}_0, \ldots, \overline{\mathbb{P}}_d$, which implies that $\overline{\mathbb{P}}_0, \ldots, \overline{\mathbb{P}}_d$ meet V.

Firstly, let p be any polynomial in $\mathcal{I}$. Then $p \in \mathbb{I}_{\zeta,\xi}$, so there exists an integer m such that $S_F^m p \in [F, S_Fy_1 - \frac{\partial F}{\partial u_{01}^{(h)}}, \ldots, S_Fy_n - \frac{\partial F}{\partial u_{0n}^{(h)}}]$. If we specialize $u_{ij} \to v_{ij}, u_{i0} \to v_{i0}$ and let $y_i = \overline{y}_i$, then we have $S_F^m(\mathbf{v}_0, \ldots, \mathbf{v}_d)p(\overline{y}_1, \ldots, \overline{y}_n) = 0$, so $p(\overline{y}_1, \ldots, \overline{y}_n) = 0$. That is, $(\overline{y}_1, \ldots, \overline{y}_n) \in V$. Secondly, since $\mathbb{P}_i \in \mathbb{I}_{\zeta,\xi}$, similarly as in the above, it follows that $(\overline{y}_1, \ldots, \overline{y}_n)$ lies in $\overline{\mathbb{P}}_i$. So $\overline{\mathbb{P}}_0, \ldots, \overline{\mathbb{P}}_d$ meet V. $\square$

Similar to the algebraic case [16, p.22], we can show that a generic differential prime passing through a given point $x = (x_1, x_2, \ldots, x_n)$ is of the form $a_0 + a_1y_1 + \cdots + a_ny_n = 0$ with $a_i = \sum_{j=0}^n s_{ij}x_j (i = 0, 1, \ldots, n)$, where $x_0 = 1$ and $S = (s_{ij})$ is an $(n+1) \times (n+1)$ skew-symmetric matrix with $s_{ij} (i < j)$ independent differential indeterminates. That is,

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = S \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

For convenience, we denote such a prime by Sx and say a generic prime passing through a point x is of the form Sx.

Now we write $\mathbf{u}_i = (u_{i0}, u_{i1}, \ldots, u_{in})^T = S^i Y$ where $Y = (1, y_1, \ldots, y_n)^T$ and the S^i are skew-symmetric matrices with $s^i_{jk} (j < k)$ independent differential indeterminates. Substitute the $\mathbf{u}_i$ in $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ by these equations. Then we get a differential polynomial involving $s^i_{jk} (j < k)$ and the y_l. Regarding this differential polynomial as a differential polynomial in $s^i_{jk} (j < k)$, then we have $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = F(S^0 Y, S^1 Y, \ldots, S^d Y) = \sum g_\phi(y_1, \ldots, y_n)\phi(s^i_{jk})$ where $\phi(s^i_{jk})$ are different differential monomials. In this way, we get a finite number of differential polynomials $g_\phi(y_1, \ldots, y_n)$ over $\mathcal{F}$, which we denote the set by $\mathcal{P}$. Similarly, in this way, we will get another set $\mathcal{D}$ of differential polynomials from the polynomial $S_F(u_0, \ldots, u_d)$.

Theorem 4.34. *Let V be an irreducible differential variety with dimension d and $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ its differential Chow form. Then $V \setminus \mathbb{V}(\mathcal{D}) = \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$, where $\mathcal{P}, \mathcal{D}$ are the differential polynomial sets obtained from $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ and $S_F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ as above.*

Proof: On the one hand, for any $\overline{x} \in V$, from Theorem 4.32, any $d+1$ generic primes passing through $\overline{x}$ meet V. So $S^0\overline{x}, S^1\overline{x}, \ldots, S^d\overline{x}$ meet V. By the proof of Theorem 4.33, $F(S^0\overline{x}, S^1\overline{x}, \ldots, S^d\overline{x}) = 0$. Since $s^i_{jk} (j < k)$ are differential indeterminates, $\overline{x} \in \mathbb{V}(\mathcal{P})$. So $V \setminus \mathbb{V}(\mathcal{D}) \subseteq \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D})$.

On the other hand, for any $\overline{x} \in \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D})$, since any $d+1$ generic primes passing through $\overline{x}$ are of the form $S^0\overline{x}, S^1\overline{x}, \ldots, S^d\overline{x}$ with the S^i indeterminate skew-symmetric matrices, we have $F(S^0\overline{x}, S^1\overline{x}, \ldots, S^d\overline{x}) = 0$ and $S_F(S^0\overline{x}, S^1\overline{x}, \ldots, S^d\overline{x}) \neq 0$. From Theorem 4.33, $S^0\overline{x}, S^1\overline{x}, \ldots, S^d\overline{x}$ meet V. Thus from Theorem 4.32, $\overline{x} \in V$. Thus $V \setminus \mathbb{V}(\mathcal{D}) = \mathbb{V}(\mathcal{P}) \setminus \mathbb{V}(\mathcal{D})$.

Now, we show that $V \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$. Suppose the contrary, i.e. $V \subset \mathbb{V}(\mathcal{D})$, in particular, its generic point $(\xi_1, \ldots, \xi_n) \in \mathbb{V}(\mathcal{D})$. Thus, $S_F(S^0\xi, S^1\xi, \ldots, S^d\xi) = 0$, where $\xi = (1, \xi_1, \ldots, \xi_n)$. Recall that $s^i_{jk} (j < k; i = 0, 1, \ldots, d)$ are independent differential indeterminates over $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle$. Now we consider a differential endomorphism ϕ of $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle\{s^i_{jk}\ (j < k; i = 0, 1, \ldots, d)\}$ over $\mathcal{F}\langle\xi_1, \ldots, \xi_n\rangle$ satisfying $\phi(s^i_{0k}) = -s^i_{0k}$ and $\phi(s^i_{jk}) = 0\,(j < k; j = 1, \ldots, n)$. It is clear that $\phi(S_F(S^0\xi, \ldots, S^d\xi)) = S_F(s^i_{0k}; -\sum_{k=1}^n s^0_{0k}\xi_k, \ldots, -\sum_{k=1}^n s^d_{0k}\xi_k) = 0$. If we denote s^i_{0k} by u_{ik}, we have $S_F(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$, thus $S_F \in \mathrm{sat}(F)$, which is a contradiction. So $V \setminus \mathbb{V}(\mathcal{D}) \neq \emptyset$. □

Since V is an irreducible differential variety, $V \cap \mathbb{V}(\mathcal{D})$ is a subset of V with lower dimension than that of V or with the same dimension but of lower order. Thus, $V \setminus \mathbb{V}(\mathcal{D})$ contains almost all points of V.

Example 4.35. Continue from Example 4.4. In this example, $F(\mathbf{u}_0) = u_1^2(u_0')^2 - 2u_1u_1'u_0u_0' + (u_1')^2u_0^2 + 4u_1^3u_0$ and $S_F(\mathbf{u}_0) = 2u_1^2u_0' - 2u_1u_1'u_0$. Following the steps as above, we obtain $\mathcal{P} = \{(y')^2 - 4y\}$ and $\mathcal{D} = \{y'\}$.

5. Differential Chow variety

In Theorem 1.2, we have listed four properties for the differential Chow form. In this section, we are going to prove that these properties are also the sufficient conditions for a differential polynomial $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ to be the Chow form for a differential variety. Based on these sufficient conditions, we can define the differential Chow quasi-variety for certain classes of differential varieties in the sense that a point in the differential Chow quasi-variety represents a differential variety in the class. In other words, we give a parametrization of all differential varieties in the class. Obviously, this is an extension of the concept of Chow variety in algebraic case [13, 16].

5.1. Sufficient conditions for a polynomial to be a differential Chow form.

The following result gives sufficient conditions for a differential polynomial to be the Chow form of an irreducible variety. From Theorem 1.2, they are also necessary conditions.

Theorem 5.1. *Let $F(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d)$ be an irreducible differential polynomial in $\mathcal{F}\{\mathbf{u}_0,\mathbf{u}_1,\ \dots,\mathbf{u}_d\}$ where $\mathbf{u}_i=(u_{i0},u_{i1},\dots,u_{in})\,(i=0,\dots,d)$. If F satisfies the following conditions, then it is the Chow form for an irreducible differential variety of dimension d and order h.*

1. $F(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d)$ is differentially homogenous of the same degree in each $\mathbf{u}_i$ and $\mathrm{ord}(F,u_{ij})=h$ for all u_{ij} occurring in F.

2. $F(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d)$ can be factored uniquely into the following form

$$\begin{aligned} F(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d) &= A(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d)\prod_{\tau=1}^{g}(u_{00}^{(h)}+\sum_{\rho=1}^{n}u_{0\rho}^{(h)}\xi_{\tau\rho}+t_\tau)\\ &= A(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d)\prod_{\tau=1}^{g}(u_{00}+\sum_{\rho=1}^{n}u_{0\rho}\xi_{\tau\rho})^{(h)}\end{aligned}$$

where $g=\deg(F,u_{00}^{(h)})$ and $\xi_{\tau\rho}$ are in an extension field of $\mathcal{F}$. The first " $=$ " is obtained by factoring $F(\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d)$ as an algebraic polynomial in the variables $u_{00}^{(h)},u_{01}^{(h)},\dots,u_{0n}^{(h)}$, while the second one is a differential expression by defining derivatives of $\xi_{\tau\rho}$ to be

$$\xi_{\tau\rho}^{(m)}=(\delta_u\xi_{\tau\rho}^{(m-1)})|_{u_{00}^{(h)}=-\sum_{\rho=1}^{n}u_{0\rho}^{(h)}\xi_{\tau\rho}-t_\tau}\ (m\geq 1)$$

recursively, where δ_u is the natural derivation over $\mathcal{F}\langle\mathbf{u}_0,\mathbf{u}_1,\dots,\mathbf{u}_d\rangle$.

3. $\Xi_\tau=(\xi_{\tau1},\dots,\xi_{\tau n})\,(\tau=1,\dots,g)$ are on the differential primes $\mathbb{P}_\sigma=0\,(\sigma=1,\dots,d)$ as well as on the algebraic primes ${}^a\mathbb{P}_0^{(\delta)}=0\,(\delta=0,\dots,h-1)$.

4. For each τ, if $v_{i0}+v_{i1}\xi_{\tau1}+\cdots+v_{in}\xi_{\tau n}=0$ $(i=0,\dots,d)$, then $F(\mathbf{v}_0,\dots,\mathbf{v}_d)=0$, where $\mathbf{v}_i=(v_{i0},v_{i1},\dots,v_{in})$ and v_{ij} are from an extension field of $\mathcal{F}$. It is equivalent to say that if $S^0,\dots,S^d$ are $(n+1)\times(n+1)$ skew-symmetric matrices, each having independent differential indeterminates above its principle diagonal, then $F(S^0\xi_\tau,\dots,S^d\xi_\tau)=0$, where $\xi_\tau=(1,\xi_{\tau1},\dots,\xi_{\tau n})$.

Before proving the theorem, we need several lemmas.

Lemma 5.2. [15, p.11, Theorem 1] *Let $\mathcal{R}$ and $\mathcal{S}^*$ be two rings and $\mathcal{R}$ isomorphic to a subring $\mathcal{S}$ of $\mathcal{S}^*$. Then there exists an extension ring $\mathcal{R}^*$ of $\mathcal{R}$ which is isomorphic with $\mathcal{S}^*$, this isomorphism including that between $\mathcal{R}$ and $\mathcal{S}$.*

Lemma 5.3. *Let V be an irreducible differential variety of dimension $d>0$ over $\mathcal{F}$ and $\mathbb{P}:=u_0+u_1y_1+\cdots+u_ny_n$ a generic prime. Then every generic point of $\mathbb{V}\big(\mathbb{I}(V),\mathbb{P}\big)$ over $\mathcal{F}\langle u_0,\ldots,u_n\rangle$ is a generic point of V over $\mathcal{F}$.*

Proof: By Theorem 3.7, $[\mathbb{I}(V),\mathbb{P}]$ is a prime differential ideal of dimension $d-1$. Let η be a generic point of $\mathbb{V}\big(\mathbb{I}(V),\mathbb{P}\big)$. Then for any differential polynomial p in $\mathbb{I}(V)$, we have $p(\eta)=0$. On the other hand, for any differential polynomial $p\in\mathcal{F}\{y_1,\ldots,y_n\}$ such that $p(\eta)=0$, we have $p\in[\mathbb{I}(V),\mathbb{P}]$. Then $p\equiv\sum_i h_i\mathbb{P}^{(i)} \bmod \mathbb{I}(V)$. Substituting u_0 by $-u_1y_1-\cdots-u_ny_n$ in the above equality, we have $p\equiv 0 \bmod \mathbb{I}(V)$. Hence η is a generic point of V. □

In the next result, we will show that the following stronger version of condition 4) from Theorem 5.1 is also valid.

Lemma 5.4. *Consider F in Theorem 5.1 as an algebraic polynomial $f(u_{\sigma j}^{(k)},u_{0j}^{(l)}$, $u_{00}^{(h)},\ldots,u_{0n}^{(h)})$ in $u_{ij}^{(k)}$. If $v_{i0k}+\sum_{j=1}^n\sum_{m=0}^k\binom{k}{m}v_{ijm}\xi_{\tau j}^{(k-m)}=0\,(i=0,\ldots,d;k=0,\ldots,h)$, then $f(v_{\sigma jk},v_{0jl},v_{00h},\,\ldots,v_{0nh})=0$, where the v are in some extension field of $\mathcal{F}$.*

Proof: Regard $\mathcal{Q}=[v_{i0}+\sum_{j=1}^n v_{ij}\xi_{\tau j}\,:\,i=0,\ldots,d]$ as a differential ideal in $\mathcal{F}\langle\xi_{\tau j}\,:\,j=1,\ldots,n\rangle\{v_{i0},\ldots,v_{in}\,:\,i=0,\ldots,d\}$, where v_{ij} are differential indeterminates. From condition 4) of Theorem 5.1, $F(\mathbf{v}_0,\ldots,\mathbf{v}_d)|_{\mathbb{V}(\mathcal{Q})}\equiv 0$. It is clear that $\mathcal{Q}$ is a prime ideal and $\{v_{i0}+\sum_{j=1}^n v_{ij}\xi_{\tau j}\,:\,i=0,\ldots,d\}$ is its characteristic set with v_{i0} as leaders. By the differential Nullstellensatz, $F(\mathbf{v}_0,\ldots,\mathbf{v}_d)\in\mathcal{Q}$. From condition 1) of Theorem 5.1, $\mathrm{ord}(F,v_{i0})=h$. Then $F(\mathbf{v}_0,\ldots,\mathbf{v}_d)\in(v_{i0}+\sum_{j=1}^n v_{ij}\xi_{\tau j},\ldots,v_{i0}^{(h)}+\sum_{j=1}^n v_{ij}^{(h)}\xi_{\tau j}+\sum_{j=1}^n\sum_{m=0}^{h-1}\binom{h}{m}v_{ij}^{(m)}\xi_{\tau j}^{(h-m)}\,:\,i=0,\ldots,d)$. In this algebraic relation, we can change $v_{ij}^{(k)}$ to algebraic indeterminates v_{ijk} and regard F as an algebraic polynomial. Then $f(v_{\sigma jk},v_{0jl},v_{00h},\ldots,v_{0nh})\in(v_{i00}+\sum_{j=1}^n v_{ij0}\xi_{\tau j},\ldots,v_{i0h}+\sum_{j=1}^n v_{ijh}\xi_{\tau j}+\sum_{j=1}^n\sum_{m=0}^{h-1}\binom{h}{m}v_{ijm}\xi_{\tau j}^{(h-m)}\,:\,i=0,\ldots,d)$, which shows that lemma is valid. □

Proof of Theorem 5.1. Let V_τ $(\tau=1,\ldots,g)$ be the irreducible differential variety over $\mathcal{F}$ with $(\xi_{\tau_1},\ldots,\xi_{\tau n})$ as its generic point over $\mathcal{F}$. We will show later that all the varieties V_τ are the same.

Firstly, we claim that the generic points of V_τ which lie on $\mathbb{P}_1,\ldots,\mathbb{P}_d$ as well as on ${}^a\mathbb{P}_0,{}^a\,\mathbb{P}_0',\ldots,{}^a\mathbb{P}_0^{(h-1)}$ are included in $\{(\xi_{\tau 1},\ldots,\xi_{\tau n})\,:\,\tau=1,\ldots,g\}$. Without loss of generality, we consider V_1. It suffices to show that if $(\eta_{10},\ldots,\eta_{n0},\ldots,\eta_{1h},\ldots,\eta_{nh})$ is a generic point of the algebraic ideal $\mathbb{I}(V_1)^{<h>}=\mathbb{I}(V_1)\cap\mathcal{F}[y_1,\ldots,y_n,\ldots,y_1^{(h)},\ldots,y_n^{(h)}]$ which lies on ${}^a\mathbb{P}_\sigma^{(k)},{}^a\,\mathbb{P}_0^{(l)}$ $(\sigma=1,\ldots,d;k=0,\ldots,h;l=0,\ldots,h-1)$, then there must exist some τ such that $\eta_{j0}=\xi_{\tau j}$ for $j=1,\ldots,n$. Similar to the proof of Lemma 5.4, rewrite F as an algebraic polynomial $f(u_{\sigma j}^{(k)},u_{0j}^{(l)},\,u_{00}^{(h)},\ldots,\,u_{0n}^{(h)})$ and consider the condition 2) as a pure algebraic factorization. Let $(\eta_{10},\ldots,\eta_{n0},\ldots,\eta_{1h},\ldots,\eta_{nh})$ be such a generic point of $\mathbb{I}(V_1)^{<h>}$ other than $(\xi_{11},\ldots,\xi_{1n},\ldots,\xi_{11}^{(h)},\ldots,\xi_{1n}^{(h)})$. Then we have the following isomorphism $\mathcal{F}(\eta_{10},\ldots,\eta_{n0},\ldots,\eta_{1h},\ldots,\eta_{nh})\cong\mathcal{F}(\xi_{11},\ldots,\xi_{1n},\,\ldots,\xi_{11}^{(h)},\,\ldots,\xi_{1n}^{(h)})$ which maps η_{jk} to $\xi_{1j}^{(k)}$ for j from 1 to n, k from 0 to h and leaving elements of $\mathcal{F}$ unchanged. By Lemma 5.2, this isomorphism can be extended to an isomorphism

$$\mathcal{F}(\eta_{10},\ldots,\eta_{n0},\ldots,\eta_{1h},\ldots,\eta_{nh},u_{\sigma j}^{(k)},u_{0j}^{(l)})\cong\mathcal{F}(\xi_{11},\ldots,\xi_{1n},\ldots,\xi_{11}^{(h)},\ldots,\xi_{1n}^{(h)},w_{\sigma jk},w_{0jl})$$

where $\sigma = 1,\dots,d; j = 0,\dots,n; k = 0,\dots,h; l = 0,\dots,h-1$, and $u_{\sigma j}^{(k)}$ and $u_{0j}^{(l)}$ map to $w_{\sigma jk}$ and w_{0jl} respectively. Since $(\eta_{10},\dots,\eta_{n0},\dots,\eta_{1h},\dots,\eta_{nh})$ lies on ${}^a\mathbb{P}_\sigma^{(k)}, {}^a\mathbb{P}_0^{(l)}$, the relation $u_{\sigma 0}^{(k)} + \sum_{j=1}^n \sum_{m=0}^k \binom{k}{m} u_{\sigma j}^{(m)}\ \eta_{j,k-m} = 0$ implies that $w_{\sigma 0k} + \sum_{j=1}^n \sum_{m=0}^k \binom{k}{m} w_{\sigma jm}\xi_{1j}^{(k-m)} = 0\,(\sigma = 1,\dots,d)$ and the relation $u_{00}^{(l)} + \sum_{j=1}^n \sum_{m=0}^l \binom{l}{m} u_{0j}^{(m)}\ \eta_{j,l-m} = 0$ implies that $w_{00l} + \sum_{j=1}^n \sum_{m=0}^l \binom{l}{m} w_{0jm}\ \xi_{1j}^{(l-m)} = 0\,(l = 0,\dots,h-1)$. Furthermore, if $w_{00h} + \sum_{i=1}^n w_{0ih}\xi_{1i}^{(h)} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} w_{0jm}\xi_{1j}^{(h-m)} = 0$ is valid, then from Lemma 5.4, $f(w_{\sigma jk}, w_{0jl}, w_{00h},\dots,w_{0nh}) = 0$. Then, by the Hilbert Nullstellensatz, when regarded as a polynomial in the algebraic indeterminates $u_{00}^{(h)},\dots,u_{0n}^{(h)}$, $f(w_{\sigma jk}, w_{0jl}, u_{00}^{(h)},\dots, u_{0n}^{(h)}) \in (u_{00}^{(h)} + u_{01}^{(h)}\xi_{11} + \cdots + u_{0n}^{(h)}\xi_{1n} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m}\ w_{0jm}\xi_{1j}^{(h-m)})$. Reversing the above isomorphism, we have $f(u_{\sigma j}^{(k)}, u_{0j}^{(l)}, u_{00}^{(h)},\ \dots, u_{0n}^{(h)})$ is divisible by $u_{00}^{(h)} + u_{01}^{(h)}\eta_{10} + \cdots + u_{0n}^{(h)}\eta_{n0} + \sum_{j=1}^n \sum_{m=0}^{h-1} \binom{h}{m} u_{0j}^{(m)}\ \eta_{j,h-m}$. From condition 2), we have $f(u_{\sigma j}^{(k)}, u_{0j}^{(l)}, u_{00}^{(h)},\dots,u_{0n}^{(h)}) = A\prod_{\tau=1}^g \big(u_{00}^{(h)} + u_{01}^{(h)}\xi_{\tau 1} + \cdots + u_{0n}^{(h)}\ \xi_{\tau n} + \sum_{j=1}^n \sum_{k=0}^{h-1} \binom{h}{k} u_{0j}^{(k)}\xi_{\tau j}^{(h-k)}\big)$. Thus, there exists some τ such that $\eta_{j0} = \xi_{\tau j}\,(j = 1,\dots,n)$, which completes the proof of the claim.

Denote the dimension and order of V_τ by d_τ and h_τ respectively. We claim that $d_\tau = d$ and $h_\tau = h$. Since V_τ meets $\mathbb{P}_1,\dots,\mathbb{P}_d$ and $(\xi_{\tau 1},\dots,\xi_{\tau n})$ are such points in their intersection variety, by Theorem 3.7, $d_\tau \ge d$. If $d_\tau > d$, then V_τ meets $\mathbb{P}_1,\dots,\mathbb{P}_d,\mathbb{P}_0$. Let $(\eta_1,\dots,\eta_n)$ be a generic point of $\mathbb{V}(\mathbb{I}(V_\tau),\mathbb{P}_1,\dots,\mathbb{P}_d,\mathbb{P}_0)$. Then by Lemma 5.3, $(\eta_1,\dots,\eta_n)$ is also a generic point of V_τ. Since $(\eta_1,\dots,\eta_n)$ lies on $\mathbb{P}_0$, it also lies on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0',\dots, {}^a\mathbb{P}_0^{(h-1)}$. From the above claim, there exists some τ such that $(\eta_1,\dots,\eta_n) = (\xi_{\tau 1},\dots,\xi_{\tau n})$. Thus, $(\xi_{\tau 1},\dots,\xi_{\tau n})$ lies on $\mathbb{P}_0$, which implies that $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ is a zero differential polynomial, which is a contradiction. So $d_\tau = d$.

It remains to show that $h_\tau = h$. We first prove $h_\tau \ge h$. Suppose the contrary, then $h_\tau \le h-1$. Similar to the proof of Theorem 4.29, we can prove that $\mathbb{V}\big([\mathbb{I}(V_\tau),\mathbb{P}_1,\dots,\mathbb{P}_d]^{<h>}, {}^a\mathbb{P}_0,\dots,\ {}^a\mathbb{P}_0^{(h-1)}\big) = \emptyset$. But $(\xi_{\tau_1},\dots,\xi_{\tau n})$ is an element of $\mathbb{V}(\mathbb{I}(V_\tau),\mathbb{P}_1,\dots,\mathbb{P}_d)$ which also lies on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0',\dots,\ {}^a\mathbb{P}_0^{(h-1)}$, which is a contradiction. Now suppose that $h_\tau > h$, then $h_\tau - 1 \ge h$. From Theorems 4.29 and 4.28, every point of V_τ which lies both on $\mathbb{P}_1,\dots,\mathbb{P}_d$ and on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0',\dots, {}^a\mathbb{P}_0^{(h_\tau-1)}$ is a generic point of V_τ. But the generic points of V_τ which lie on $\mathbb{P}_1,\dots,\mathbb{P}_d$ as well as on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0',\dots, {}^a\mathbb{P}_0^{(h-1)}$ are included in $\{(\xi_{\tau 1},\dots,\xi_{\tau n}) : \tau = 1,\dots,g\}$. So some $(\xi_{\tau 1},\dots,\xi_{\tau n})$ lies on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0',\dots, {}^a\mathbb{P}_0^{(h_\tau-1)}$. Since $h_\tau - 1 \ge h$, we have $(\xi_{\tau 1},\dots,\xi_{\tau n})$ lies on ${}^a\mathbb{P}_0^{(h)}$, which implies $F(\mathbf{u}_0,\dots,\mathbf{u}_d) \equiv 0$, a contradiction. Thus, we have proved that $d_\tau = d$ and $h_\tau = h$.

Since the solutions of V_τ and $\mathbb{P}_1,\dots,\mathbb{P}_d$ which also lie on ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0',\dots, {}^a\mathbb{P}_0^{(h-1)}$ are generic points of V_τ and these are therefore contained in $\{(\xi_{\tau 1},\dots,\xi_{\tau n}) : \tau = 1,\dots,g\}$. Hence, the differential Chow form of V_τ is of the form

$$F_\tau(\mathbf{u}_0,\dots,\mathbf{u}_d) = A_\tau \prod_{\rho=1}^g \big((u_{00} + u_{01}\xi_{\tau 1} + \cdots + u_{0n}\xi_{\tau n})^{(h)}\big)^{l_{\tau\rho}},$$

where $l_{\tau\rho} = 1$ or 0 according to whether $(\xi_{\tau 1},\dots,\xi_{\tau n})$ is in V_τ. Since both F_τ and F are irreducible, they differ at most by a factor in $\mathcal{F}$. Therefore, $V_\tau\,(\tau = 1,\dots,g)$ are the same variety, and $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ is their differential Chow form. $\square$

In order to define Chow varieties in the next subsection, we will introduce the concept of order-unmixed varieties. A variety V is called *order-unmixed* if all its components have the same dimension and order. Let V be an order-unmixed differential variety with dimension d and order h and $V = \bigcup_{i=1}^{l} V_i$ its minimal irreducible decomposition with $F_i(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ the differential Chow form of V_i. A differential Chow form of V is defined to be

$$F(\mathbf{u}_0, \ldots, \mathbf{u}_d) = \prod_{i=1}^{l} F_i(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)^{s_i} \tag{5.1}$$

with s_i arbitrary nonnegative integers. Associated with (5.1), we introduce the concept of *multiplicative differential variety* $\mathbf{V} = \sum_{i=1}^{l} s_i \mathbf{V}_i$, where s_i is called the multiplicity of $\mathbf{V}_i$ in $\mathbf{V}$. Recall that, we defined the differential degree m and leading differential degree g for an irreducible differential variety V in Definitions 4.17 and 4.27 respectively. Let g_i and m_i be the leading differential degree and differential degree of V_i respectively. Then the leading differential degree and differential degree of $\mathbf{V}$ is defined to be $\sum_{i=1}^{l} s_i g_i$ and $\sum_{i=1}^{l} s_i m_i$ respectively. Given a differential polynomial $G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ with $\mathrm{ord}(G, u_{00}) = h$, it may be reducible over $\mathcal{F}$ such that some of its irreducible factors are free of $u_{00}^{(h)}$. In that case, if the product of all such factors is L, then we define the primitive part of G w.r.t. $u_{00}^{(h)}$ to be G/L. Otherwise, its primitive part w.r.t. $u_{00}^{(h)}$ by convention is defined to be itself. Then we have

Theorem 5.5. *Let $F(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ be a differential polynomial and $\widetilde{F}$ the primitive part of F with respect to the variable $u_{00}^{(h)}$. If F satisfies the four conditions in Theorem 5.1, then $\widetilde{F}$ is the Chow form for an order-unmixed multiplicative differential variety of dimension d and order h.*

Proof: By definition, $F = B\widetilde{F}$, where $\mathrm{ord}(B, \mathbf{u}_0) < h$. Since F is differentially homogenous in $\mathbf{u}_i$ for each i, $\widetilde{F}$ is differentially homogenous in each $\mathbf{u}_i$ too. And since B is free of $u_{00}^{(h)}$, i.e. B divides A, then $\widetilde{F}$ satisfies conditions 2) and 3), and moreover the $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ in the factorization are the same as that of F. By the proof of Theorem 5.1, we have $\mathbb{I}(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ is of dimension d and order h over $\mathcal{F}$. Then similarly as the proof of Lemma 4.1 and Theorem 4.11, we conclude that $\mathbb{I}(-\sum_{j=1}^{n} v_{0j}\xi_{\tau j}, \ldots, -\sum_{j=1}^{n} v_{dj}\xi_{\tau j})$ is of dimension d over $\mathcal{F}\langle v_{ij} : i = 0, \ldots, d; j = 1, \ldots, n\rangle$ and its relative order w.r.t. any parametric set is h, where $v_{ij}\,(i = 0, \ldots, d; j = 1, \ldots, n)$ are differential indeterminates over $\mathcal{F}\langle \xi_{\tau 1}, \ldots, \xi_{\tau n}\rangle$. In particular, $\mathrm{tr.deg}\,\mathcal{F}\langle \zeta_0, \zeta_1, \ldots, \zeta_d\rangle/\mathcal{F}\langle \zeta_1, \ldots, \zeta_d\rangle = h$, where $\zeta_i = -\sum_{j=1}^{n} v_{ij}\xi_{\tau j}$. Thus $B(-\sum_{j=1}^{n} v_{0j}\xi_{\tau j}, \ldots, -\sum_{j=1}^{n} v_{dj}\xi_{\tau j}) \neq 0$. But $F(-\sum_{j=1}^{n} v_{0j}\xi_{\tau j}, \ldots, -\sum_{j=1}^{n} v_{dj}\xi_{\tau j}) = 0$, so $\widetilde{F}(-\sum_{j=1}^{n} v_{0j}\xi_{\tau j}, \ldots, -\sum_{j=1}^{n} v_{dj}\xi_{\tau j}) = 0$. It follows that $\widetilde{F}(S^0\xi_\tau, \ldots, S^d\xi_\tau) = 0$, for if we suppose the contrary, then $B(S^0\xi_\tau, \ldots, S^d\xi_\tau) = 0$. But if we specialize $s_{jk}^{i}(j < k, j > 0)$ to 0 and $s_{0k}^{i}\,(k > 0)$ to $-v_{ik}$, then $B(-\sum_{j=1}^{n} v_{0j}\xi_{\tau j}, \ldots, -\sum_{j=1}^{n} v_{dj}\xi_{\tau j}) = 0$, which is a contradiction. Thus, $\widetilde{F}$ satisfies condition 4).

Now we claim that $\widetilde{F}$ is the Chow form of some variety. Let $V_\tau = \mathbb{I}(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ over $\mathcal{F}$. Following the steps in the proof of Theorem 5.1 exactly, we arrive at the conclusion that the differential Chow form of V_τ is of the form

$$F_\tau(\mathbf{u}_0,\dots,\mathbf{u}_d) = A_\tau \prod_{\rho=1}^{g} \left((u_{00}+u_{01}\xi_{\tau 1}+\cdots+u_{0n}\xi_{\tau n})^{(h)}\right)^{l_{\tau\rho}},$$

where $l_{\tau\rho} = 1$ or 0 according to whether $(\xi_{\tau 1},\dots,\xi_{\tau n})$ is in V_τ. Since each ξ_τ is in at least one of the varieties V_i, the differential Chow form of $\bigcup_{\tau=1}^{g} V_\tau$ is of the form $G(\mathbf{u}_0,\dots,\mathbf{u}_d) = \prod_{\tau=1}^{g}(F_\tau(\mathbf{u}_0,\dots,\mathbf{u}_d))^{s_\tau} = C(\mathbf{u}_0,\dots,\mathbf{u}_d)\prod_{\rho=1}^{g}\big((u_{00} + u_{01}\xi_{\tau 1}+\cdots+u_{0n}\xi_{\tau n})^{(h)}\big)^{\eta_{\tau\rho}}$ with $\eta_{\tau\rho}>0$. Since $\widetilde{F}$ and G have the same factors $(u_{00}+u_{01}\xi_{\tau 1}+\cdots+u_{0n}\xi_{\tau n})^{(h)}$ and the primitive factor of $\widetilde{F}$ w.r.t. $u_{00}^{(h)}$ is itself, thus we can find $\eta_{\tau\rho}$ such that $\widetilde{F}=G$ which completes the proof. □

5.2. **Differential Chow quasi-variety.** An irreducible variety V in the n dimensional space with dimension d, order h, leading differential degree g, and differential degree m is said to be of index (n,d,h,g,m). In this section, we will define the differential Chow quasi-variety in certain cases such that each point in this variety represents a differential variety with a given index (n,d,h,g,m).

For a given index (n,d,h,g,m), a differential polynomial $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ which has unknown coefficients $a_\lambda\,(\lambda=0,\dots,D)$ and satisfies the following two conditions is referred to as a *differential polynomial with index* (n,d,h,g,m).

1) F is a homogenous polynomial of the same degree m in each set of indeterminates $\mathbf{u}_i=(u_{i0},u_{i1},\dots,u_{in})\,(i=0,\dots,d)$ and their derivatives. Furthermore, for each u_{ij}, $\mathrm{ord}(F,u_{ij})$ is either h or $-\infty$. In particular, $\mathrm{ord}(F,u_{00})=h$.

2) As a polynomial in $u_{00}^{(h)},u_{01}^{(h)},\dots,u_{0n}^{(h)}$, its total degree is g. In particular, $\deg(F,u_{00}^{(h)})=g$.

We want to determine the necessary and sufficient conditions imposed on $a_\lambda\,(\lambda = 0,\dots,D)$ in order that F is the differential Chow form for a differential variety of index (n,d,h,g,m). Proceeding in this way, if the necessary and sufficient conditions given in Theorem 5.1 can be expressed by some differential polynomials in a_λ, then the variety defined by them is called the differential Chow (quasi)-variety. More precisely, we have the following definition.

Definition 5.6. Let $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ be a differential polynomial with differential indeterminates $a_i\,(i=0,\dots,D)$ as coefficients and with index (n,d,h,g,m). A quasi-variety $\mathbb{CV}$ in the variables a_i is called the *Chow quasi-variety* with index (n,d,h,g,m) if a point $\bar{a}_i$ is in $\mathbb{CV}$ if and only if $\tilde{F}$ is the Chow form for an order-unmixed differential variety with index (n,d,h,g,m_1) with $m_1\le m$, where $\tilde{F}$ is obtained from F by first replacing a_i by $\bar{a}_i$ and then taking the primitive part with respect to the variable $u_{00}^{(h)}$.

In the case $h=0$, since Theorems 1.2 and 5.1 become their algebraic counterparts, we can obtain the equations for the algebraic Chow variety in the same way as in [16, p.56-57]. So in the following, we only consider the case $h>0$. For $h>0$, the case $g=1$ is relatively simple. The following result shows how to determine the defining equations for the differential Chow quasi-variety with index (n,d,h,g,m) in the case of $g=1$.

Theorem 5.7. *Let $F(\mathbf{u}_0,\dots,\mathbf{u}_d)$ be a differential polynomial with differential indeterminates $a_\nu\,(i=0,\dots,D)$ as coefficients and with index (n,d,h,g,m) with $g=1$.*

Let I_F be the initial of F w.r.t. the elimination ranking $u_{00} \succ u_{i,j}$ and $a_0, \ldots, a_I$ the coefficients of I_F. Then we can find a set of differentially homogeneous polynomials

$$R_\omega(a_0, \ldots, a_D)\,(\omega = 1, \ldots, \upsilon)$$

in a_ν such that $\mathbb{V}(R_\omega : \omega = 1, \ldots, \upsilon) \setminus \mathbb{V}(a_0, \ldots, a_I)$ is the Chow quasi-variety of index (n, d, h, g, m) with $g = 1$.

Proof: In order for F to be a differential Chow form, by Theorem 1.2, F must be differentially homogeneous in each $\mathbf{u}_i$. Let λ be a differential indeterminate. For each i, replacing $\mathbf{u}_i$ by $\lambda\mathbf{u}_i$ in F, we should have

$$F(\mathbf{u}_0, \ldots, \mathbf{u}_{i-1}, \lambda\mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_d) = \lambda^m F(\mathbf{u}_0, \ldots, \mathbf{u}_d).$$

Comparing the coefficients of the power products of λ, u_{ij} and their derivatives, we obtain a system of linearly homogenous equations $R_\omega(a_0, \ldots, a_D) = 0$, ($\omega = 1, \ldots, e_1$) in a_ν, which are the conditions for F to be differentially homogeneous and with degree m in each $\mathbf{u}_i$. So by Gaussian elimination in linear algebra, we can obtain a basis for the solution space of $R_\omega = 0\,(\omega = 1, \ldots, e_1)$. More precisely, if the coefficient matrix of this linear equations is of rank r, then r of $\{a_0, \ldots, a_D\}$ are the linear combinations of the other $D+1-r$ of a_ν. Now substitute these r relations into F and denote the new polynomial by F_1. That is, F_1 is a differentially homogenous polynomial in each $\mathbf{u}_i$, which only involves $D+1-r$ independent coefficients a_ν.

Since $g = 1$, F_1 can be written in the form

$$F_1(\mathbf{u}_0, \ldots, \mathbf{u}_d) = A_0 u_{00}^{(h)} + A_1 u_{01}^{(h)} + \cdots + A_n u_{0n}^{(h)} + B,$$

where A_i and B are free of $u_{0k}^{(h)}$. Denote $-(A_1 u_{01}^{(h)} + \cdots + A_n u_{0n}^{(h)} + B)/A_0$ by z. Then $u_{00}^{(h)} = z$ is the solution of F_1 as an algebraic polynomial in $u_{00}^{(h)}$. Let $\xi_j = \frac{\partial F_1}{\partial u_{0j}^{(h)}} \Big/ \frac{\partial F_1}{\partial u_{00}^{(h)}} \Big|_{u_{00}^{(h)}=z} = A_j/A_0\big|_{u_{00}^{(h)}=z}$ for $j = 1, \ldots, n$. Proceeding as in the proof of Theorem 4.22, we define the derivatives of ξ_j to be $\xi_j^{(k)} = (\delta_u \xi_j^{(k-1)})\big|_{u_{00}^{(h)}=z}$ where δ_u is referred to be the natural derivation operator in $\mathcal{F}\langle\mathbf{u}_0, \ldots, \mathbf{u}_d\rangle$. It is easy to see that this definition is well defined. Since F_1 is differentially homogeneous in $\mathbf{u}_0$, by Theorem 4.15, for $r \neq 0$

$$\sum_{j=0}^{n} \sum_{k \geq 0} \binom{k+r}{r} u_{0j}^{(k)} \frac{\partial F_1}{\partial u_{0j}^{(k+r)}} = 0.$$

In the case $r = h$, we have $\sum_{j=0}^n u_{0j} \frac{\partial F_1}{\partial u_{0j}^{(h)}} = 0$. Set $u_{00}^{(h)} = z$ in the identity $\sum_{j=0}^n u_{0j} \frac{\partial F_1}{\partial u_{0j}^{(h)}} = 0$, then we have $u_{00} + \sum_{j=1}^n u_{0j}\xi_j = 0$ with $u_{00}^{(h)} = z$. So $(\xi_1, \ldots, \xi_n, \ldots, \xi_1^{(h-1)}, \ldots, \xi_n^{(h-1)})$ is a solution of ${}^a\mathbb{P}_0, {}^a\mathbb{P}_0', \ldots, {}^a\mathbb{P}_0^{(h-1)}$ and $z = -(\sum_{j=1}^n u_{0j}\xi_j)^{(h)}$. So $F_1(\mathbf{u}_0, \ldots, \mathbf{u}_d) = A_0(u_{00} + \sum_{j=1}^n u_{0j}\xi_j)^{(h)}$. As a consequence, with these ξ_i, the second condition and the second part of the third condition in Theorem 5.1 are satisfied.

In order for F_1 to be the differential Chow form for some differential variety, by Theorem 5.1, $(\xi_1, \ldots, \xi_n)$ should satisfy $\mathbb{P}_\sigma = 0\,(\sigma = 1, \ldots, d)$ and $F_1(S^0\xi, \ldots, S^d\xi) = 0$ where S^i are $(n+1) \times (n+1)$-skew symmetric matrices with elements independent indeterminates and $\xi = (1, \xi_1, \ldots, \xi_n)^T$.

Firstly, setting $y_j = A_j/A_0$ in $\mathbb{P}_\sigma = 0$, we get $u_{\sigma 0}A_0 + \sum_{j=1}^n u_{\sigma j}A_j = 0$. Then we obtain some equations in a_ν by equating to zero the coefficients of the various differential products of $\mathbf{u}_0, \ldots, \mathbf{u}_d$. This gives polynomials $R_\omega(a_0, \ldots, a_D)\,(\omega = e_1+1, \ldots, e_2)$.

Secondly, we obtain some differential equations $\chi_\tau(a_\nu, y_1, \ldots, y_d)$ by equating to zero the coefficients of all differential products of the independent indeterminates $s^i_{jk}\,(j>k)$ in $F_1(S^0Y, \ldots, S^dY) = 0$ with $Y = (1, y_1, \ldots, y_n)^T$. Then setting $y_j^{(k)} = \xi_j^{(k)}$ in the above χ_τ and clearing denominators, we obtain polynomial equations p_μ in $u_{ij}^{(k)}$ and a_ν. Equating to zero the coefficients of the power products of the $u_{ij}^{(k)}$ in p_μ, we finally obtain differential polynomials in a_ν: $R_\omega(a_0, \ldots, a_D)\,(\omega = e_2+1, \ldots, \upsilon)$. We then obtain the defining equations $R_\omega = 0$ for the Chow variety.

We now show that all the R_ω are homogenous polynomials. We have known for $\nu = 1, \ldots, e_1$, R_ω are linearly homogenous polynomials. Since F_1 as well as A_i are linearly homogenous in $a_0, \ldots, a_D$, $R_\omega(a_0, \ldots, a_D)\,(\omega = e_1+1, \ldots, e_2)$ are linearly homogenous polynomials. To show $R_\omega(a_0, \ldots, a_D)\,(\omega = e_2+1, \ldots, \upsilon)$ are differentially homogenous polynomials, it suffices to show that as a rational function, $F_1(S^0\xi, \ldots, S^d\xi)$ is differentially homogenous in a_ν. Indeed, ξ_j and their derivatives are differentially homogenous rational functions in a_ν. And since F_1 is linearly homogenous in a_ν, for any differential indeterminate λ, $F_1(S^0\xi, \ldots, S^d\xi)(\lambda a_0, \ldots, \lambda a_D) = \lambda^m F_1(S^0\xi, \ldots, S^d\xi)(a_0, \ldots, a_D)$. Clearing denominators, we get differentially homogenous polynomials in $a_0, \ldots, a_D$, which implies that $R_\omega(a_0, \ldots, a_D)\,(\omega = e_2+1, \ldots, \upsilon)$ are differentially homogeneous polynomials.

Let $a_0, \ldots, a_I$ be the coefficients of I_F. Then we claim that the quasi-projective variety $\mathbb{CV} = \mathbb{V}(R_\omega : \omega = 1, \ldots, \upsilon) \setminus \mathbb{V}(a_0, \ldots, a_I)$ is the differential Chow quasi-variety. Indeed, for every element $(\bar{a_0}, \ldots, \bar{a_D})$ in $\mathbb{CV}$, following the proof of this theorem, $\overline{F}$ with coefficients $\bar{a_\nu}$ satisfies the four conditions in Theorem 5.1. And since $g = 1$, its primitive part must be irreducible and satisfies the four conditions too, which consequently must be the differential Chow form for some irreducible variety with index $(n, d, h, 1, m_1)$ with $m_1 \leq m$. $\square$

The following example illustrates the procedure to compute the Chow quasi-variety in the case of $g = 1$.

Example 5.8. We consider a differential polynomial which has 16 terms and has index $(2,1,1,1,2)$ to illustrate the proof of Theorem 5.7: $F = a_1u_{12}^2u_{01}u_{00}' + a_2u_{11}u_{12}u_{02}u_{00}' + a_3u_{01}u_{02}u_{12}u_{10}' + a_4u_{02}^2u_{11}u_{10}' + a_5u_{12}^2u_{00}u_{01}' + a_6u_{10}u_{12}u_{02}u_{01}' + a_7u_{00}u_{02}u_{12}u_{11}' + a_8u_{02}^2u_{10}u_{11}' + a_9u_{10}u_{01}u_{12}u_{02}' + a_{10}u_{00}u_{02}u_{11}u_{12}' + a_{11}u_{11}u_{12}u_{00}u_{02}' + a_{12}u_{01}u_{02}u_{10}u_{12}' + a_{13}u_{00}u_{11}^2u_{02} + a_{14}u_{00}\,u_{01}u_{11}u_{12} + a_{15}u_{01}u_{10}u_{02}u_{11} + a_{16}u_{10}u_{01}^2u_{12}$. We will derive the conditions about the coefficients a_ν under which F is a differential Chow form. Firstly, in order for F to be differentially homogenous, we have $R_1 = a_5+a_1, R_2 = a_8+a_4, R_3 = a_9+a_6, R_4 = a_{10}+a_7, R_5 = a_{11}+a_2, R_6 = a_{12}+a_3$. Replacing $a_5, a_8, a_9, a_{10}, a_{11}, a_{12}$ by $-a_1, -a_4, -a_6, -a_7, -a_2, -a_3$ respectively in F to obtain F_1.

For such an F_1, $A_0 = a_1u_{12}^2u_{01} + a_2u_{11}u_{12}u_{02}$, $A_1 = -a_1u_{12}^2u_{00} + a_6u_{10}u_{12}u_{02}$, $A_2 = -a_6u_{10}u_{01}u_{12} - a_2u_{11}u_{12}\,u_{00}$, and $B = a_3u_{01}u_{02}u_{12}u_{10}' + a_4u_{02}^2u_{11}u_{10}' + a_7u_{00}u_{02}u_{12}u_{11}' - a_4u_{02}^2u_{10}u_{11}' - a_7u_{00}u_{02}u_{11}u_{12}' - a_3u_{01}u_{02}u_{10}u_{12}' + a_{13}u_{00}u_{11}^2u_{02} + a_{14}u_{00}\,u_{01}u_{11}u_{12} + a_{15}u_{01}u_{10}\,u_{02}u_{11} + a_{16}u_{10}u_{01}^2u_{12}$. Then $z = (F_1 - A_0u_{00}')/A_0$, $\xi_1 = A_1/A_0|_{u_{00}'=z}$, and $\xi_2 = A_2/A_0|_{u_{00}'=z}$. To confirm that $u_{10} + u_{11}\xi_1 + u_{12}\xi_2 = 0$, we must have $R_7 = a_6 - a_1 = 0, R_8 = a_2 + a_1 = 0$.

In order to satisfy the fourth condition of Theorem 5.1, we obtain a set of differential polynomial equations $R_\omega(a_0, \ldots, a_{16}) = 0$ which have more complicated forms. By simplifying them with $R_7 = 0$ and $R_8 = 0$, we obtain $R_9 = a_7a_1(a_{15} + a_{16}), R_{10} = a_7a_1(a_{14} + a_{16}), R_{11} = a_7a_1(a_{13} - a_{16}), R_{12} = a_1(a_4 - a_7), R_{13} = a_1(a_3 + a_7)$, $R_{14} = a_1a_{16}(a_1 - a_7)$, $R_{15} = a_{15}a_1^2 + a_1a_7a_{16}$, $R_{16} = a_{14}a_1^2 + a_1a_7a_{16}$, $R_{17} = a_7a_1(a_1 - a_7)$, $R_{18} = a_1^2a_{13} - a_1a_7a_{16}$, $R_{19} = a_1^3 - a_7^2a_1$. Thus the Chow quasi-variety is $\mathbb{V}(R_1, \ldots, R_{19})/\mathbb{V}(a_1, a_2) = \mathbb{V}(a_2 + a_1, a_3 + a_1, a_4 - a_1, a_5 + a_1, a_6 - a_1, a_7 - a_1, a_8 + a_1, a_9 + a_1, a_{10} + a_1, a_{11} - a_1, a_{12} - a_1, a_{14} + a_{13}, a_{15} + a_{13}, a_{16} - a_{13})/\mathbb{V}(a_1)$. From Example 4.3, it is easy to check that each point of this quasi-variety is the coefficients of the Chow form for $\mathbb{V}(a_1y_1' + a_{13}y_2)$ for some $a_1, a_{13} \in \mathcal{F}$. Note that a_{13} could be zero and the result is still valid.

We are unable to prove the existence of the Chow quasi-variety in the case of $g > 1$. The main difficulty is how to do elimination for a mixed system consisting of both differential and algebraic equations. In our case, conditions 1, 2, and the second part of condition 3 of Theorem 5.1 generate algebraic equations in the coefficients of F and ξ_{ij}, while the first part of condition 3 and condition 4 of Theorem 5.1 generate differential equations. And we need to eliminate variables ξ_{ij} from these equations.

The following example shows that the Chow quasi-variety can be easily defined in a very special case.

Example 5.9. If $n = 1$ and $d = 0$, then every irreducible differentially homogeneous polynomial in $\mathbf{u}_0 = (u_{00}, u_{01})$ is the differential Chow form for some irreducible differential variety.

Proof: Let $F(\mathbf{u}_0) = F(u_{00}, u_{01})$ be an irreducible differentially homogenous polynomial with degree m and order h. Then $F(-\frac{u_{00}}{u_{01}}, -1) = (-\frac{1}{u_{01}})^m F(u_{00}, u_{01})$. Let $g(-\frac{u_{00}}{u_{01}}) = F(-\frac{u_{00}}{u_{01}}, -1)$. It is easy to show that $g(y)$ is an irreducible polynomial. By Example 4.3, the Chow form of the differential prime ideal $\text{sat}(g(y))$ is $(-u_{01})^m g(-\frac{u_{00}}{u_{01}}) = (-u_{01})^m F(-\frac{u_{00}}{u_{01}}, -1) = F(u_{00}, u_{01})$, and the result is proved. $\square$

6. Generalized differential Chow form and differential resultant

We mentioned that the differential Chow form can be obtained by intersecting the variety with generic differential primes. In this section, we show that when intersecting an irreducible differential variety of dimension d by $d+1$ generic differential primals, we can obtain the generalized Chow form which has similar properties to the Chow form. As a direct consequence, we can define the differential resultant and give some new properties for it.

6.1. Generalized differential Chow form.

Let V be an irreducible differential variety with dimension d and order h, $(\xi_1, \ldots, \xi_n)$ a generic point of V, and

$$\mathbb{P}_i = u_{i0} + \sum_{j=1}^{n}\sum_{k=0}^{s_i} u_{ijk}y_j^{(k)} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)} \\ 1 < |\alpha| \leq m_i}} u_{i\alpha}(\mathbb{Y}^{(s_i)})^{\alpha},\ (i = 0, \ldots, d) \tag{6.1}$$

a generic differential polynomial of order $s_i \geq 0$ and degree $m_i \geq 1$, where $u_{ijk}, u_{i\alpha}$ ($i = 0, \ldots, d; j = 1, \ldots, n; k = 0, \ldots, s_i; \alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)}, 1 < |\alpha| \leq m_i$) are differential indeterminates and $(\mathbb{Y}^{(s_i)})^{\alpha}$ is a monomial in $\mathcal{F}[y_1, \ldots, y_n, y_1', \ldots, y_n', \ldots, y_1^{(s_i)}, \ldots, y_n^{(s_i)}]$ with exponent vector $\alpha = (\alpha_{10}, \ldots, \alpha_{n0}, \alpha_{11}, \ldots, \alpha_{n1}, \ldots, \alpha_{1s_i}, \ldots, \alpha_{ns_i})$, i.e.

$(\mathbb{Y}^{(s_i)})^\alpha = \prod_{j=1}^{n}\prod_{k=0}^{s_i}(y_j^{(k)})^{\alpha_{jk}}$ and $|\alpha| = \sum_{j=1}^{n}\sum_{k=0}^{s_i}\alpha_{jk}$. For convenience in the rest of the paper, we denote the nonlinear part of each $\mathbb{P}_i$ by f_i, that is,

$$\mathbb{P}_i = u_{i0} + \sum_{j=1}^{n}\sum_{k=0}^{s_i} u_{ijk}y_j^{(k)} + f_i\,(i=0,\ldots,d)$$

Denote $\mathbf{u}$ to be the set consisting of all the u_{ijk} and $u_{i\alpha}$ for $i=0,\ldots,d$. We define $d+1$ elements $\zeta_0,\zeta_1,\ldots,\zeta_d$ of $\mathcal{F}\langle\mathbf{u},\xi_1,\ldots,\xi_n\rangle$:

$$\zeta_i = -\sum_{j=1}^{n}\sum_{k=0}^{s_i} u_{ijk}\xi_j^{(k)} - f_i(\xi_1,\ldots,\xi_n)\,(i=0,\ldots,d) \tag{6.2}$$

Similar to the proof of Lemma 4.1, we can prove that any set of d elements of $\zeta_0,\ldots,\zeta_d$ is a differential transcendence basis of $\mathcal{F}\langle\mathbf{u}\rangle\langle\zeta_0,\ldots,\zeta_d\rangle$ over $\mathcal{F}\langle\mathbf{u}\rangle$. We thus have

Lemma 6.1. $\text{d.tr.deg}\,\mathcal{F}\langle\mathbf{u}\rangle\langle\zeta_0,\ldots,\zeta_d\rangle/\mathcal{F}\langle\mathbf{u}\rangle = d$

Let $\mathbb{I}_\zeta$ be the prime differential ideal in $\mathcal{R} = \mathcal{F}\langle\mathbf{u}\rangle\{z_0,\ldots,z_d\}$ having $\zeta = (\zeta_0,\ldots,\zeta_d)$ as a generic point. By Lemma 6.1, the dimension of $\mathbb{I}_\zeta$ is d. Then, the characteristic set of $\mathbb{I}_\zeta$ w.r.t. any ranking consists of an irreducible differential polynomial $g(z_0,\ldots,z_d)$ in $\mathcal{R}$ and $\mathbb{I}_\zeta = \text{sat}(g)$. Since the coefficients of $g(z_0,\ldots,z_d)$ are elements in $\mathcal{F}\langle\mathbf{u}\rangle$, without loss of generality, we assume that $g(\mathbf{u};z_0,\ldots,z_d)$ is irreducible in $\mathcal{F}\{\mathbf{u};z_0,\ldots,z_d\}$. We shall subsequently replace $z_0,\ldots,z_d$ by $u_{00},\ldots,u_{d0}$, and obtain

$$G(\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_d) = g(\mathbf{u};u_{00},\ldots,u_{d0}), \tag{6.3}$$

where $\mathbf{u}_i = (u_{i0},\ldots,u_{ijk},\ldots,u_{i\alpha},\ldots)$ is the sequence of the coefficients of $\mathbb{P}_i$.

Definition 6.2. The differential polynomial defined in (6.3) is called the *generalized Chow form* of V or the prime ideal $\mathbb{I}(V)$.

Similar to Lemma 4.9, we can prove that two generalized Chow forms for a differential variety can only differ by a factor in $\mathcal{F}$. Similar to Theorem 4.16, we can prove that the generalized Chow form is a differentially homogeneous polynomial in each set of indeterminates $\mathbf{u}_i$, but in this case the homogeneous degree for distinct $\mathbf{u}_i$ may be distinct. The *order of the generalized Chow form w.r.t.* $\mathbf{u}_i$, denoted by $\text{ord}(G,\mathbf{u}_i)$, is defined to be $\max_{u\in\mathbf{u}_i}\{\text{ord}(g,u))\}$. Now we will consider the order of the generalized Chow form.

Theorem 6.3. *Let $\mathcal{I}$ be a prime differential ideal with dimension d and order h, and $G(\mathbf{u}_0,\mathbf{u}_1,\ \ldots,\mathbf{u}_d) = g(\mathbf{u};u_{00},u_{10},\ldots,u_{d0})$ its generalized Chow form. Then for a fixed i between 0 and d, $\text{ord}(g,u_{i0}) = h+s-s_i$ with $s=\sum_{l=0}^{d}s_l$. Moreover, $\text{ord}(G,\mathbf{u}_i) = h+s-s_i$.*

Proof: Use the notations as above in this section. Let $\mathcal{I}_d = [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_{i-1},\mathbb{P}_{i+1},\ldots,\mathbb{P}_d] \subset \mathcal{F}\langle\mathbf{u}_0,\ldots,\mathbf{u}_{i-1},\mathbf{u}_{i+1},\ldots,\mathbf{u}_d\rangle\{y_1,\ldots,y_n\}$. By Theorem 3.13, $\mathcal{I}_d$ is a prime ideal with dimension 0 and order $h+s_0+\cdots+s_{i-1}+s_{i+1}+\cdots+s_d = h+s-s_i$, where $s=\sum_{l=0}^{d}s_l$.

Let $\mathbb{I}_{\zeta,\xi} = [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_d] \subset \mathcal{F}\langle\mathbf{u}\rangle\{u_{00},\ldots,u_{d0},y_1,\ldots,y_n\}$ and $\mathcal{I}_1 = [\mathcal{I},\mathbb{P}_0,\ldots,\mathbb{P}_d] = [\mathcal{I}_d,\mathbb{P}_0] \subset \mathcal{F}\langle\widehat{\mathbf{u}}\setminus\{u_{i0}\}\rangle\{u_{i0},y_1,\ \ldots,y_n\}$, where $\widehat{\mathbf{u}} = \mathbf{u}_0\cup\cdots\cup\mathbf{u}_d$. Denote $\text{ord}(G,u_{i0})$ by h_1. Similar to the proof of Lemma 4.10, we can show that

$\mathcal{A} = g(\mathbf{u}; u_{00}, \ldots, u_{d0}),\ \frac{\partial g}{\partial u_{i0}^{(h_1)}} y_1 - \frac{\partial g}{\partial u_{i10}^{(h_1)}}, \ldots, \frac{\partial g}{\partial u_{i0}^{(h_1)}} y_n - \frac{\partial g}{\partial u_{in0}^{(h_1)}}$ is a characteristic set of $\mathbb{I}_{\zeta,\xi}$ w.r.t. the elimination ranking $u_{00} \prec \cdots u_{i-1,0} \prec u_{i+1,0} \prec u_{d0} \prec u_{i0} \prec y_1 \prec \cdots \prec y_n$. Clearly, $\mathcal{I}_1$ is the differential ideal generated by $\mathbb{I}_{\zeta,\xi}$ in $\mathcal{F}\langle \widehat{\mathbf{u}} \setminus \{u_{i0}\}\rangle\{u_{i0}, y_1, \ldots, y_n\}$. Since $\{u_{00}, \ldots, u_{i-1,0}, u_{i+1,0}, \ldots, u_{d0}\}$ is a parametric set of $\mathbb{I}_{\zeta,\xi}$, $\mathcal{A}$ is also a characteristic set of $\mathcal{I}_1$ w.r.t. the elimination ranking $u_{i0} \prec y_1 \prec \cdots \prec y_n$. Since $\dim(\mathcal{I}_1) = 0$, from Corollary 2.9, we have $\mathrm{ord}(\mathcal{I}_1) = \mathrm{ord}(\mathcal{A}) = \mathrm{ord}(g, u_{i0})$.

On the other hand, let $\mathcal{I}_1^{(l)} = [\mathcal{I}_d, u_{i0}^{(l)} + \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + f_i] \subset \mathcal{F}\langle \widehat{\mathbf{u}} \setminus \{u_{i0}\}\rangle\{u_{i0}, y_1, \ldots, y_n\}$ $(l = 1, \ldots, s_i)$. Since $\dim(\mathcal{I}_1^{(l)}) = 0$, u_{i0} is a leading variable of $\mathcal{I}_1^{(l)}$ for any ranking. Thus, by Lemma 3.11, we have $\mathrm{ord}(\mathcal{I}_1^{(l+1)}) = \mathrm{ord}(\mathcal{I}_1^{(l)}) + 1$, which follows that $\mathrm{ord}(\mathcal{I}_1^{(s_i)}) = \mathrm{ord}(\mathcal{I}_1) + s_i$. And it is easy to see that $\mathrm{ord}(\mathcal{I}_1^{(s_i)}) = \mathrm{ord}(\mathcal{I}_d) + s_i$. Indeed, let $\mathcal{A}$ be a characteristic set of $\mathcal{I}_d$ w.r.t. some orderly ranking $\mathscr{R}$, and let t be the pseudo remainder of $u_{i0}^{(s_i)} + \sum_{j=1}^n \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} + f_i$ w.r.t. $\mathcal{A}$ under the ranking $\mathscr{R}$. Clearly, $\mathrm{ord}(t, u_{i0}) = s_i$. It is obvious that for some orderly ranking, $\{\mathcal{A}, t\}$ is a characteristic set of $\mathcal{I}_1^{(s_i)}$ with $\mathrm{ld}(\mathcal{A})$ and $u_{i0}^{(s_i)}$ as its leaders, so $\mathrm{ord}(\mathcal{I}_1^{(s_i)}) = \mathrm{ord}(\mathcal{I}_d) + s_i$. Thus, $\mathrm{ord}(\mathcal{I}_1) = \mathrm{ord}(\mathcal{I}_d) = h + s - s_i$, and consequently, $\mathrm{ord}(g, u_{i0}) = h + s - s_i$.

It remains to show that $\mathrm{ord}(g, u_{ijk})$ $(j = 1, \ldots, n; k = 0, 1, \ldots, s_i)$ and $\mathrm{ord}(g, u_{i\alpha})$ cannot exceed $\mathrm{ord}(g, u_{i0})$. If $\mathrm{ord}(g, u_{ijk}) = l > \mathrm{ord}(g, u_{i0})$, then differentiate the identity $g(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$ w.r.t. $u_{ijk}^{(l)}$, we have $\frac{\partial g}{\partial u_{ijk}^{(l)}}(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$. Thus, $\frac{\partial g}{\partial u_{ijk}^{(l)}}$ can be divisible by g, a contradiction. So $\mathrm{ord}(g, u_{ijk}) \leq \mathrm{ord}(g, u_{i0})$. Similarly, we can prove that $\mathrm{ord}(g, u_{i\alpha}) \leq \mathrm{ord}(g, u_{i0})$. Thus, $\mathrm{ord}(G, \mathbf{u}_i) = \mathrm{ord}(g, u_{i0})$. $\square$

In the following, we consider the factorization of the generalized Chow form. Denote $h + s - s_i$ by h_i $(i = 0, \ldots, d)$ where $s = \sum_{l=0}^d s_l$. Now consider G as a polynomial in $u_{00}^{(h_0)}$ with coefficients in $\mathcal{F}_0 = \mathcal{F}(\cup_{l=0}^d \widehat{\mathbf{u}}_l^{(h_l)} \setminus \{u_{00}^{(h_0)}\})$, where $\widehat{\mathbf{u}}_l^{(h_l)} = \{u^{(i)} : u \in \mathbf{u}_l, i = 0, \ldots, h_l\}$. Then, in an algebraic extension field of $\mathcal{F}_0$, we have

$$g = A \prod_{\tau=1}^{t_0} (u_{00}^{(h_0)} - z_\tau)$$

where $t_0 = \deg(g, u_{00}^{(h_0)})$. Let $\xi_{\tau\rho k} = g_{\tau\rho k}/g_{\tau 0}$ $(\rho = 1, \ldots, n;\ k = 0, \ldots, s_0)$ and $\xi_{\tau\alpha} = g_{\tau\alpha}/g_{\tau 0}$, where $g_{\tau\rho k} = \frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}\Big|_{u_{00}^{(h_0)} = z_\tau}$, $g_{\tau\alpha} = \frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}\Big|_{u_{00}^{(h_0)} = z_\tau}$ and $g_{\tau 0} = \frac{\partial g}{\partial u_{00}^{(h_0)}}\Big|_{u_{00}^{(h_0)} = z_\tau}$. Similarly as in Section 4.4, we can uniquely define the derivatives of z_τ and $\xi_{\tau\rho 0}$ to make them elements in a differential field. From $g(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$, if we differentiate this equality w.r.t. $u_{0\rho k}^{(h_0)}$, then we have

$$\overline{\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}} + \frac{\partial g}{\partial \zeta_0^{(h_0)}}(-\xi_\rho^{(k)}) = 0 \tag{6.4}$$

And if we differentiate $g(\mathbf{u}; \zeta_0, \ldots, \zeta_d) = 0$ w.r.t. $u_{0\alpha}^{(h_0)}$, then

$$\overline{\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}} + \frac{\partial g}{\partial \zeta_0^{(h_0)}}(-(\xi^{(s_0)})^\alpha) = 0 \tag{6.5}$$

where $(\xi^{(s_0)})^\alpha = (\mathbb{Y}^{(s_0)})^\alpha|_{(y_1,\ldots,y_n)=(\xi_1,\ldots,\xi_n)}$. And in the above equations, $\overline{\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}}$, $\overline{\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}}$ and $\frac{\partial g}{\partial \zeta_0^{(h_0)}}$ represent $\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}$, $\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}$ and $\frac{\partial g}{\partial u_{00}^{(h_0)}}$ when substituting u_{i0} by ζ_i. For each $\rho = 1,\ldots,n$ and $k = 0,\ldots,s_0$, multiplying the equations in (6.4) by $u_{0\rho k}$, and for $\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)}$, $1 < |\alpha| \leq m_0$, multiplying the equations in (6.5) by $u_{0\alpha}$, then adding all of the equations obtained together, we have

$$\zeta_0 \frac{\partial g}{\partial \zeta_0^{(h_0)}} + \sum_{\rho=1}^{n}\sum_{k=0}^{s_0} u_{0\rho k}\overline{\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1<|\alpha|\leq m_0}} u_{0\alpha}\overline{\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}} = 0.$$

Thus, the polynomial $u_{00}\frac{\partial g}{\partial u_{00}^{(h_0)}} + \sum\limits_{\rho=1}^{n}\sum\limits_{k=0}^{s_0} u_{0\rho k}\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}} + \sum\limits_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1<|\alpha|\leq m_0}} u_{0\alpha}\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}$ vanishes at $(u_{00},\ldots,u_{d0}) = (\zeta_0,\ldots,\zeta_d)$. Since it is at most of the same order as g, it must be divisible by g. And since it has the same degree as g, there exists some $a \in \mathcal{F}$ such that

$$u_{00}\frac{\partial g}{\partial u_{00}^{(h_0)}} + \sum_{\rho=1}^{n}\sum_{k=0}^{s_0} u_{0\rho k}\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1<|\alpha|\leq m_0}} u_{0\alpha}\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}} = ag.$$

Setting $u_{00}^{(h_0)} = z_\tau$ in both sides of the above equation, we have

$$u_{00}g_{\tau 0} + \sum_{\rho=1}^{n}\sum_{k=0}^{s_0} u_{0\rho k}g_{\tau\rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1<|\alpha|\leq m_0}} u_{0\alpha}g_{\tau\alpha} = 0.$$

Or,

$$u_{00} + \sum_{\rho=1}^{n}\sum_{k=0}^{s_0} u_{0\rho k}\xi_{\tau\rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1<|\alpha|\leq m_0}} u_{0\alpha}\xi_{\tau\alpha} = 0.$$

Then, we have

$$\begin{aligned}&(u_{00} + \sum_{\rho=1}^{n}\sum_{k=0}^{s_0} u_{0\rho k}\xi_{\tau\rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1<|\alpha|\leq m_0}} u_{0\alpha}\xi_{\tau\alpha})^{(h_0)} \\ &= z_\tau + (\sum_{\rho=1}^{n}\sum_{k=0}^{s_0} u_{0\rho k}\xi_{\tau\rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1<|\alpha|\leq m_0}} u_{0\alpha}\xi_{\tau\alpha})^{(h_0)} = 0.\end{aligned}$$

We have the following theorem

Theorem 6.4. *Let $G(\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_d)$ be the generalized Chow form of a differential variety of dimension d and order h. Then, there exist $\xi_{\tau\rho}(\rho = 1,\ldots,n;\tau =$*

$1, \ldots, t_0)$ *in an extension field of* $\mathcal{F}_0$ *such that*

$$G(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{t_0} \mathbb{P}_0(\xi_{\tau 1}, \ldots, \xi_{\tau n})^{(h_0)} \tag{6.6}$$

where $A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ *is in* $\mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d\}$ *and* $t_0 = \deg(g, u_{00}^{(h_0)})$.

Proof: From what we have proved,

$$G(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) = A(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d) \prod_{\tau=1}^{t_0} \Big(u_{00} + \sum_{\rho=1}^{n} \sum_{k=0}^{s_0} u_{0\rho k}\xi_{\tau\rho k} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_0+1)} \\ 1 < |\alpha| \leq m_0}} u_{0\alpha}\xi_{\tau\alpha}\Big)^{(h_0)}.$$

Denote $\xi_{\tau\rho 0}$ by $\xi_{\tau\rho}$. To complete the proof, it remains to show that $\xi_{\tau\rho k} = (\xi_{\tau\rho 0})^{(k)} (k = 1, \ldots, s_0)$ and $\xi_{\tau\alpha} = \prod_{\rho=1}^{n} \prod_{j=0}^{s_0} \Big((\xi_{\tau\rho 0})^{(j)}\Big)^{\alpha_{\rho j}}$. From equation (6.4) and equation (6.5) , we have $\xi_\rho^{(k)} = \overline{\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}} \Big/ \frac{\partial g}{\partial \zeta_0^{(h_0)}}$ $(k = 0, \ldots, s_0)$ and $(\xi^{(s_0)})^\alpha = \overline{\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}} \Big/ \frac{\partial g}{\partial \zeta_0^{(h_0)}}$. So we have the equalities: $\left(\overline{\frac{\partial g}{\partial u_{0\rho 0}^{(h_0)}}} \Big/ \frac{\partial g}{\partial \zeta_0^{(h_0)}}\right)^{(k)} = \overline{\frac{\partial g}{\partial u_{0\rho k}^{(h_0)}}} \Big/ \frac{\partial g}{\partial \zeta_0^{(h_0)}}$ and $\overline{\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}}} \Big/ \frac{\partial g}{\partial \zeta_0^{(h_0)}} = \prod_{\rho=1}^{n} \prod_{j=0}^{s_0} \left(\left(\overline{\frac{\partial g}{\partial u_{0\rho 0}^{(h_0)}}} \Big/ \frac{\partial g}{\partial \zeta_0^{(h_0)}}\right)^{(j)}\right)^{\alpha_{\rho j}}$. Thus, $\left(\frac{\partial g}{\partial u_{0\rho 0}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}}\right)^{(k)} - \frac{\partial g}{\partial u_{0\rho k}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}}$ and $\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}} - \prod_{\rho=1}^{n} \prod_{j=0}^{s_0} \left(\left(\frac{\partial g}{\partial u_{0\rho 0}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}}\right)^{(j)}\right)^{\alpha_{\rho j}}$ vanish at $(u_{00}, \ldots, u_{d0}) = (\zeta_0, \ldots, \zeta_d)$. Similarly as in the proof of Theorem 4.28, we can see $\left(\frac{\partial g}{\partial u_{0\rho 0}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}}\right)^{(k)} - \frac{\partial g}{\partial u_{0\rho k}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}}$ and $\frac{\partial g}{\partial u_{0\alpha}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}} - \prod_{\rho=1}^{n} \prod_{j=0}^{s_0} \left(\left(\frac{\partial g}{\partial u_{0\rho 0}^{(h_0)}} \Big/ \frac{\partial g}{\partial u_{00}^{(h_0)}}\right)^{(j)}\right)^{\alpha_{\rho j}}$ vanish at $u_{00}^{(h_0+j)} = z_\tau^{(j)} (j \geq 0)$. Thus, $\xi_{\tau\rho 0}^{(k)} = \xi_{\tau\rho k}$ and $\xi_{\tau\alpha} - \prod_{\rho=1}^{n} \prod_{j=0}^{s_0} \Big((\xi_{\tau\rho 0})^{(j)}\Big)^{\alpha_{\rho j}} = 0$. The proof is completed. □

Theorem 6.5. *The points* $(\xi_{\tau 1}, \ldots, \xi_{\tau n}) (\tau = 1, \ldots, t_0)$ *in* (10) *are generic points of the variety* V*, and satisfy the equations*

$$\mathbb{P}_\sigma(y_1, \ldots, y_n) = u_{\sigma 0} + \sum_{\rho=1}^{n} \sum_{k=0}^{s_\sigma} u_{\sigma\rho k} y_\rho^{(k)} + f_\sigma = 0 \, (\sigma = 1, \ldots, d)$$

Proof: The proof is similar to that of Theorem 4.28. □

Theorem 6.6. *Let* $G(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ *be the generalized Chow form of* V *and* $S_G = \frac{\partial G}{\partial u_{00}^{(h_0)}}$ *with* $\mathrm{ord}(G, u_{00}) = h_0$*. Suppose that* $\mathbf{u}_i (i = 0, \ldots, d)$ *specialize to sets* $\mathbf{v}_i$ *of specific elements in an extension field of* $\mathcal{F}$ *and* $\overline{\mathbb{P}}_i \, (i = 0, \ldots, d)$ *are obtained by substituting* $\mathbf{u}_i$ *by* $\mathbf{v}_i$ *in* $\mathbb{P}_i$*. If* $\overline{\mathbb{P}}_i = 0 (i = 0, \ldots, d)$ *meet* V*, then* $G(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$*. Furthermore, if* $G(\mathbf{v}_0, \ldots, \mathbf{v}_d) = 0$ *and* $S_G(\mathbf{v}_0, \ldots, \mathbf{v}_d) \neq 0$*, then the* $d + 1$ *differential hypersurfaces* $\overline{\mathbb{P}}_i = 0$ $(i = 0, \ldots, d)$ *meet* V*.*

Proof: The proof is similar to that of Theorem 4.33. □

6.2. **Differential resultant of multivariate differential polynomials.** As an application of the generalized Chow form, we can define the differential resultant of $n + 1$ generic differential polynomials in n variables. Let $\mathcal{I} = [0]$ be the ideal generated by 0 in $\mathcal{F}\{\mathbb{Y}\}$. Then $\dim(\mathcal{I}) = n$. Let $G(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ be the generalized Chow form for $\mathcal{I}$. Then we will define $G(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ to be the differential resultant for the $n+1$ generic differential polynomials given in (6.1).

Definition 6.7. The *differential resultant* for the $n+1$ generic differential polynomials $\mathbb{P}_i$ in (6.1) is defined to be the generalized Chow form of $\mathcal{I} = [0]$ associated with these $\mathbb{P}_i$, and will be denoted by $R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = G(\mathbf{u}_0, \ldots, \mathbf{u}_n)$.

Theorem 6.8. *Let $R(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ be the differential resultant of the $n+1$ differential polynomials $\mathbb{P}_0, \ldots, \mathbb{P}_n$ given in (6.1) with $\mathrm{ord}(\mathbb{P}_i) = s_i$ and $\deg(\mathbb{P}_i) = m_i$, where $\mathbf{u}_i = (u_{i0}, \ldots, u_{ijk}, \ldots, u_{i\alpha}, \ldots)\,(i = 0, \ldots, n)$. Denote $s = \sum_{i=0}^n s_i$, $\mathbf{u} = \cup_{i=0}^n \mathbf{u}_i \backslash \{u_{00}, \ldots, u_{n0}\}$ and $D = \max\{m_0, m_1, \ldots, m_n\}$. Then there exist $h_{jk} \in \mathcal{F}\langle\mathbf{u}\rangle[y_1, \ldots, y_n, \ldots, y_1^{(s)}, \ldots, y_n^{(s)}]$ such that*

$$R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{j=0}^{n} \sum_{k=0}^{s-s_j} h_{jk} \delta^k \mathbb{P}_j.$$

Moreover, the degree of h_{jk} is bounded by $(sn+n)^2 D^{sn+n} + D(sn+n)$.

Proof: Let $\mathcal{J}$ be the ideal generated by $\mathbb{P}_0, \ldots, \mathbb{P}_n$ in $\mathcal{F}\{\mathbf{u}_0, \ldots, \mathbf{u}_n, y_1, \ldots, y_n\}$. Let $\mathscr{R}$ be the elimination ranking $\mathbf{u} \prec y_n \prec \cdots \prec y_1 \prec u_{n0} \prec \cdots \prec u_{00}$ with arbitrary ranking endowed on $\Theta(\mathbf{u}) = (\theta u : u \in \mathbf{u}; \theta \in \Theta)$. Clearly, $\mathcal{J}$ is a prime ideal with $\mathbb{P}_0, \ldots, \mathbb{P}_n$ as its characteristic set w.r.t. $\mathscr{R}$. Thus, $\mathbf{u} \cup \{y_1, \ldots, y_n\}$ is a parametric set of $\mathcal{J}$. From the definition of R, $R \in \mathcal{J}$. In $R(\mathbf{u}_0, \ldots, \mathbf{u}_n) = G(\mathbf{u}_0, \ldots, \mathbf{u}_n) = g(\mathbf{u}; u_{00}, \ldots, u_{n0})$, let $u_{i0}\,(i = 0, \ldots, n)$ be replaced respectively by

$$\mathbb{P}_i - \sum_{j=1}^{n} \sum_{k=0}^{s_i} u_{ijk} y_j^{(k)} - \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)} \\ 1 < |\alpha| \leq m_i}} u_{i\alpha} (\mathbb{Y}^{(s_i)})^{\alpha},\ (i = 0, \ldots, d),$$

and let R be expanded as a polynomial in $\mathbb{P}_0, \ldots, \mathbb{P}_n$ and their derivatives. The term not involving $\mathbb{P}_0, \ldots, \mathbb{P}_n$ or their derivatives will be a differential polynomial only in $\mathbf{u} \cup \{y_1, \ldots, y_n\}$ which also belong to $\mathcal{J}$. Since $\mathcal{J} \bigcap \mathcal{F}\{\mathbf{u}, y_1, \ldots, y_n\} = \{0\}$, such term will be identically zero. So R is a linear combinations of $\mathbb{P}_0, \ldots, \mathbb{P}_n$ and some of their derivatives. Since $\mathrm{ord}(R, u_{i0}) = s - s_i$, the above expansion for R involving $\mathbb{P}_i$ only up to the order $s - s_i$ and the coefficients in the linear combination are polynomials in $\mathcal{F}\{\mathbf{u}\}[y_1, \ldots, y_n, \ldots, y_1^{(s)}, \ldots, y_n^{(s)}]$. Thus, $R \in (\delta^{s-s_0}\mathbb{P}_0, \ldots, \delta\mathbb{P}_0, \mathbb{P}_0, \ldots, \delta^{s-s_n}\mathbb{P}_n, \ldots, \delta\mathbb{P}_n, \mathbb{P}_n) \subseteq \mathcal{F}\langle\mathbf{u}\rangle[y_1, \ldots, y_n, \ldots, y_1^{(s)}, \ldots, y_n^{(s)}]$, which implies that $(\delta^{s-s_0}\mathbb{P}_0, \ldots, \delta\mathbb{P}_0, \mathbb{P}_0, \ldots, \delta^{s-s_n}\mathbb{P}_n, \ldots, \delta\mathbb{P}_n, \mathbb{P}_n)$ in $\mathcal{F}\langle\mathbf{u}\rangle[y_1, \ldots, y_n, \ldots, y_1^{(s)}, \ldots, y_n^{(s)}]$ is the unit ideal. By [28, Theorem 1], there exist $A_{jk} \in \mathcal{F}\langle\mathbf{u}\rangle[y_1, \ldots, y_n, \ldots, y_1^{(s)}, \ldots, y_n^{(s)}]$ with $\deg(A_{jk}) \leq (sn+n)^2 D^{sn+n} + D(sn+n)$ such that

$$1 = \sum_{j=0}^{n} \sum_{k=0}^{s-s_j} A_{jk} \delta^k \mathbb{P}_j,$$

where $D = \max\{m_0, m_1, \ldots, m_n\}$. If we multiply the above equation by R and denote $A_{jk}R$ by h_{jk}, we complete the proof. $\square$

As a consequence of the above five theorems proved in this section, the properties of the differential resultant listed in Theorem 1.3 are proved.

7. Conclusion

In this paper, an intersection theory for generic differential polynomials is presented by giving the explicit formulas for the dimension and order of the intersection of an irreducible differential variety with a generic differential hypersurface. As a consequence, we show that the differential dimension conjecture is true for generic differential polynomials.

The Chow form for an irreducible differential variety is defined. Most of the properties of the algebraic Chow form have been extended to its differential counterpart. In particular, we introduce the concept of Chow quasi-variety for a special class of differential varieties. Furthermore, the generalized Chow form for an irreducible differential variety is also defined and its properties are proved. As an application of the generalized differential Chow form, we can give a rigorous definition for the differential resultant and establish its properties which are similar to that of the Sylvester resultant of two univariate polynomials.

As we mentioned in Section 5, the theory of differential Chow quasi-varieties is not fully developed and the main difficulty is to develop an elimination theory for mixed systems with both algebraic and differential equations. We mentioned in Section 1 that the algebraic Chow form has many important applications. It is very interesting to see whether some of these applications can be extended to the differential case.

In this paper, we only consider Chow forms for affine differential varieties. It is not difficult to extend the results in this paper to differentially projective varieties. In thus a case, the properties of Chow form is almost the same except in Theorem 4.33 where the condition $S_F \neq 0$ is not needed. And the Chow quasi-variety becomes Chow variety, just as the algebraic counterpart in [16, 13].

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