Characterizations of processes with stationary and independent increments under G-expectation

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Abstract

Our purpose is to investigate properties for processes with stationary and independent increments under G-expectation. As applications, we prove the martingale characterization to G-Brownian motion and present a decomposition for generalized G-Brownian motion.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_t\}$ be a continuous process in $(\Omega, \mathcal{F}, P)$ with stationary, independent increments and finite variation. It's known that there exists some constant c such that $A_t = ct$. However, it's not the case in the G-expectation space. A counterexample is $\{\langle B\rangle_t\}$, the quadratic variation process for G-Brownian motion $\{B_t\}$. The process $\{\langle B\rangle_t\}$ is very important in the theory of G-expectation, which shows, in many aspects, the difference between Probability space and G-expectation space. However, up to now, few properties of this process are known. For example, we don't know whether $\{\frac{d}{ds}\langle B\rangle_s\}$ belongs to $M_G^1(0,T)$.

In this article, we shall prove that if $A_t = \int_0^t h_s ds$ (respectively $A_t = \int_0^t h_s d\langle B\rangle_s$) is a process with stationary, independent increments and $h \in M_G^1(0,T)$ (respectively $h \in M_G^{1,+}(0,T)$), there exists some constant c such that $h \equiv c$. As applications, we prove the following conclusions (Question 1 and 3 are put forward by Prof. Shige Peng in private communications):

1. $\{\frac{d}{ds}\langle B\rangle_s\} \notin M_G^1(0,T)$.

2.(Martingale characterization)

A symmetric G-martingale $\{M_t\}$ is a G-Brownian motion if and only if its quadratic variation process $\{\langle M\rangle_t\}$ has stationary and independent increments;

A symmetric G-martingale $\{M_t\}$ is a G-Brownian motion if and only if its quadratic variation process $\langle M\rangle_t = c\langle B\rangle_t$ for some $c \geq 0$.

The sufficiency of the second part is implied by that of the first part, but the necessity is not trivial.

3. *Let $\{X_t\}$ be a generalized G-Brownian motion with zero mean, then we have the following decomposition:*

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a (symmetric) G-Brownian motion, and $\{L_t\}$ is a negative, decreasing G-martingale with stationary and independent increments.

This article is organized as follows: In section 2, we recall some basic notions and results of G-expectation and the related space of random variables. In section 3, we give characterizations to processes with stationary and independent increments. In section 4, as applications, we prove the martingale characterization to G-Brownian motion and present a decomposition for generalized G-Brownian motion.

2 Preliminary

We recall some basic notions and results of G-expectation and the related space of random variables. More details of this section can be found in [P06, P07, P08, P10].

Definition 2.1 Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Definition 2.2 Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, $\forall \varphi \in C_{l,Lip}(R^n)$, where $C_{l,Lip}(R^n)$ is the space of real continuous functions defined on R^n such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \forall x, y \in R^n,$$

where k depends only on φ.

Definition 2.3 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}(\cdot)$, denoted by $Y \perp X$, if for each test function $\varphi \in C_{l,Lip}(R^m \times R^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}]$.

Definition 2.4 (G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normal distributed if for each $a, b \in R$ we have

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(A) := \frac{1}{2}\hat{E}[(AX, X)] : S_d \to R,$$

where S_d denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot) : S_d \to R$ is a monotonic, sublinear mapping on S_d and $G(A) = \frac{1}{2}\hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} Tr(\gamma A).$$

If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta Tr(A - B)$ for any $A \geq B$, we call the G-normal distribution is non-degenerate, which is the case we consider throughout this article.

Definition 2.5 i) Let $\Omega_T = C_0([0, T]; R^d)$ with the supremum norm, $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, ..., B_{t_n}) | \forall n \geq 1, t_1, ..., t_n \in [0, T], \forall \varphi \in C_{l,Lip}(R^{d \times n})\}$, G-expectation is a sublinear expectation defined by

$$\hat{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$

$$= \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \cdots, \xi_n$ are identically distributed d-dimensional G-normal distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$ such that ξ_{i+1} is independent to $(\xi_1, \cdots, \xi_i)$ for each $i = 1, \cdots, m$. $(\Omega_T, \mathcal{H}_T^0, \hat{E})$ is called a G-expectation space.

ii) For $t \in [0, T]$ and $\xi = \varphi(B_{t_1}, ..., B_{t_n}) \in \mathcal{H}_T^0$, the conditional expectation defined by(there is no loss of generality, we assume $t = t_i$)

$$\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$
$$= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{E}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

Define $\|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then $\forall t \in [0, T]$, $\hat{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}_T^0$ with norm $\|\cdot\|_{1,G}$ and therefore can be extended continuously to the completion $L_G^1(\Omega_T)$ of $\mathcal{H}_T^0$ under norm $\|\cdot\|_{1,G}$.

Definition 2.6 Let $\{X_t\}$ be a d-dimensional process defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ such that

(i) $X_0 = 0$;

(ii) $\{X_t\}$ is a process with stationary and independent increments;

(iii) $\lim_{t\to 0} \hat{E}[|X_t|^3]t^{-1} = 0$. Then $\{X_t\}$ is called a generalized G-Brownian motion.

If in addition $\hat{E}(X_t) = \hat{E}(-X_t) = 0$, $\{X_t\}$ is called a (symmetric) G-Brownian motion.

Let $H_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For each $\eta \in H_G^0(0, T)$, let $\|\eta\|_{H_G^p} = \{\hat{E}(\int_0^T |\eta_s|^2 ds)^{p/2}\}^{1/p}$ and denote $H_G^p(0, T)$ the completion of $H_G^0(0, T)$ under norm $\|\cdot\|_{H_G^p}$.

Definition 2.7 For each $\eta \in H_G^0(0, T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

By B-D-G inequality, the mapping $I : H_G^0(0,T) \to L_G^p(\Omega_T)$ is continuous under $\|\cdot\|_{H_G^p}$ and thus can be continuously extended to $H_G^p(0,T)$.

Definition 2.8 A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G-martingale if $\hat{E}_s(M_t) = M_s$ for any $s \le t$. If $\{M_t\}$ and $\{-M_t\}$ are both G-martingale, we call $\{M_t\}$ symmetric G-martingale.

Definition 2.9 We say that $\{X_t\}$ on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ is a process with independent increments if for any $0 < t < T$ and $s_0 \le \cdots s_m \le t \le t_0 \le \cdots t_n \le T$, $(X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}) \perp (B_{s_1} - B_{s_0}, \cdots, B_{s_m} - B_{s_{m-1}})$.

3 Characterization of processes with stationary and independent increments

In the sequel, we only consider 1-dimensional G-Brownian motion with $\overline{\sigma}^2 \ge \underline{\sigma}^2 > 0$.

Lemma 3.1 For $\zeta \in M_G^1(0,T)$ and $\varepsilon > 0$, let

$$\zeta_t^\varepsilon = \frac{1}{\varepsilon} \int_{(t-\varepsilon)^+}^t \zeta_s ds$$

and

$$\zeta_t^{\varepsilon,0} = \sum_{k=1}^{k_\varepsilon - 1} \frac{1}{\varepsilon} \int_{(k-1)\varepsilon}^{k\varepsilon} \zeta_s ds 1_{]k\varepsilon,(k+1)\varepsilon]}(t),$$

where $t \in [0,T]$, $k_\varepsilon \varepsilon \le T < (k_\varepsilon + 1)\varepsilon$. Then as $\varepsilon \to 0$

$$\|\zeta^\varepsilon - \zeta\|_{M_G^1(0,T)} \to 0$$

and

$$\|\zeta^{\varepsilon,0} - \zeta\|_{M_G^1(0,T)} \to 0.$$

Proof. The proofs to the two cases are similar. Here we only prove the second case. Our proof starts with the observation that for any $\zeta, \zeta' \in M_G^1(0,T)$

$$\|\zeta^{\varepsilon,0} - \zeta'^{\varepsilon,0}\|_{M_G^1(0,T)} \le \|\zeta - \zeta'\|_{M_G^1(0,T)}. \tag{3.0.1}$$

In fact,

$$\begin{aligned}
& \|\zeta^{\varepsilon,0} - \zeta'^{\varepsilon,0}\|_{M_G^1(0,T)} \\
= & \hat{E}[\int_0^T |\zeta^{\varepsilon,0} - \zeta'^{\varepsilon,0}|]ds \\
= & \hat{E}[\Sigma_{k=1}^{k_\varepsilon - 1}| \int_{(k-1)\varepsilon}^{k\varepsilon} \zeta_s ds - \int_{(k-1)\varepsilon}^{k\varepsilon} \zeta'_s ds|] \\
\leq & \hat{E}[\Sigma_{k=1}^{k_\varepsilon - 1} \int_{(k-1)\varepsilon}^{k\varepsilon} |\zeta_s - \zeta'|_s ds] \\
\leq & \|\zeta - \zeta'\|_{M_G^1(0,T)}.
\end{aligned}$$

By the definition of space $M_G^1(0,T)$, we know that for each $\zeta \in M_G^1(0,T)$, there exists a sequence of processes $\{\zeta^n\}$ with

$$\zeta_t^n = \Sigma_{k=0}^{m_n - 1} \xi_{t_k}^n 1_{]t_k, t_{k+1}]}(t)$$

and $\xi_{t_k}^n \in L_{ip}(\Omega_{t_k})$ such that

$$\|\zeta - \zeta^n\|_{M_G^1(0,T)} \to 0 \text{ as } n \to \infty. \tag{3.0.2}$$

It is easily seen that for each n,

$$\|\zeta^{n;\varepsilon,0} - \zeta^n\|_{M_G^1(0,T)} \to 0 \text{ as } \varepsilon \to 0. \tag{3.0.3}$$

Thus we get

$$\begin{aligned}
& \|\zeta^{\varepsilon,0} - \zeta\|_{M_G^1(0,T)} \\
\leq & \|\zeta^{\varepsilon,0} - \zeta^{n;\varepsilon,0}\|_{M_G^1(0,T)} + \|\zeta^n - \zeta^{n;\varepsilon,0}\|_{M_G^1(0,T)} + \|\zeta^n - \zeta\|_{M_G^1(0,T)} \\
\leq & 2\|\zeta^n - \zeta\|_{M_G^1(0,T)} + \|\zeta^n - \zeta^{n;\varepsilon,0}\|_{M_G^1(0,T)}.
\end{aligned}$$

The second inequality follows from (3.0.1). Combining (3.0.2) and (3.0.3), first letting $\varepsilon \to 0$, then letting $n \to \infty$, we have

$$\|\zeta^{\varepsilon,0} - \zeta\|_{M_G^1(0,T)} \to 0 \text{ as } \varepsilon \to 0.$$

□

Theorem 3.2 Let $A_t = \int_0^t h_s ds$ with $h \in M_G^1(0,T)$ be a process with stationary and independent increments, then we have $h \equiv c$ for some constant c.

Proof. Let $\overline{c} := \hat{E}(A_T)/T \geq -\hat{E}(-A_T)/T =: \underline{c}$. Then we have

$$
\begin{aligned}
&\|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\
= \;& \hat{E}[\sum_{k=0}^{2n-1} \int_{kT/(2n)}^{(k+1)T/(2n)} |h_s - h_s^{T/(2n),0}| ds] \\
\geq \;& \hat{E}[\sum_{k=0}^{n-1} \int_{2kT/(2n)}^{(2k+1)T/(2n)} |h_s - h_s^{T/(2n),0}| ds] \\
\geq \;& \hat{E}[\sum_{k=1}^{n-1} \int_{2kT/(2n)}^{(2k+1)T/(2n)} (h_s - h_s^{T/(2n),0}) ds] \\
= \;& \hat{E}[\sum_{k=1}^{n-1} (\int_{2kT/(2n)}^{(2k+1)T/(2n)} h_s ds - \int_{(2k-1)T/(2n)}^{2kT/(2n)} h_s ds)] \\
= \;& \hat{E}\sum_{k=1}^{n-1} [(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)T/2n})] \\
= \;& \sum_{k=1}^{n-1} \hat{E}[(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)T/2n})] \\
= \;& \sum_{k=1}^{n-1} (\overline{c} - \underline{c}) T/(2n) \\
= \;& (\overline{c} - \underline{c})(n-1)T/(2n).
\end{aligned}
$$

So by Lemma 3.1, letting $n \to \infty$, we have $\overline{c} = \underline{c}$. Furthermore, we note that $M_t := A_t - \overline{c}t$ is a G-martingale. In fact, for $t > s$, we see

$$
\begin{aligned}
&\hat{E}_s(M_t) \\
= \;& \hat{E}_s(M_t - M_s) + M_s \\
= \;& \hat{E}(M_t - M_s) + M_s \\
= \;& M_s.
\end{aligned}
$$

So $\{M_t\}$ is a symmetric G-martingale with finite variation, from which we conclude that $M_t \equiv 0$, hence that $A_t = \overline{c}t$.□

Corollary 3.3 Assume $\overline{\sigma} > \underline{\sigma} > 0$. $\{\frac{d}{ds}\langle B\rangle_s\} \notin M_G^1(0,T)$.

Proof. The proof is straightforward from Theorem 3.2. □

Corollary 3.4 There is no symmetric G-martingale $\{M_t\}$ which is a standard Brownian motion under G-expectation(i.e. $\langle M\rangle_t = t$).

Proof. Let $\{M_t\}$ be a symmetric G-martingale. If $\{M_t\}$ is also a standard Brownian motion, by Theorem 4.8 in [Song10a] or Corollary 5.2 in [Song10b], there exists $\{h_s\} \in M_G^2(0,T)$ such that

$$M_t = \int_0^t h_s dB_s$$

and

$$\int_0^t h_s^2 d\langle B\rangle_s = t.$$

Thus we have $\frac{d}{ds}\langle B\rangle_s = h_s^{-2} \in M_G^1(0,T)$, which contradicts the conclusion of Corollary 3.3. $\square$

Lemma 3.5 Let $A_t = \int_0^t h_s ds$ with $h \in M_G^1(0,T)$ be a process with independent increments. Then A_t is symmetric for each $t \in [0,T]$.

Proof. By arguments similar to that in the proof of Theorem 3.2, we have

$$\begin{aligned}
& \|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\
= \;& \hat{E}[\sum_{k=0}^{2n-1} \int_{kT/(2n)}^{(k+1)T/(2n)} |h_s - h_s^{T/(2n),0}| ds] \\
\geq \;& \hat{E}[\sum_{k=0}^{n-1} \int_{2kT/(2n)}^{(2k+1)T/(2n)} |h_s - h_s^{T/(2n),0}| ds] \\
\geq \;& \hat{E}[\sum_{k=0}^{n-1} \int_{2kT/(2n)}^{(2k+1)T/(2n)} (h_s - h_s^{T/(2n),0}) ds] \\
= \;& \hat{E}[\sum_{k=0}^{n-1} (\int_{2kT/(2n)}^{(2k+1)T/(2n)} h_s ds - \int_{(2k-1)^+T/(2n)}^{2kT/(2n)} h_s ds)] \\
= \;& \hat{E}\sum_{k=0}^{n-1} [(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)^+T/2n})] \\
= \;& \sum_{k=0}^{n-1} \{\hat{E}(A_{(2k+1)T/2n} - A_{2kT/2n}) + \hat{E}[-(A_{2kT/2n} - A_{(2k-1)^+T/2n})]\}.
\end{aligned}$$

The right side of the first inequality is only the sum of the odd terms. Summing up the even terms only, we have

$$\begin{aligned}
& \|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\
\geq \;& \sum_{k=0}^{n-1} \{\hat{E}(A_{(2k+2)T/2n} - A_{(2k+1)T/2n}) + \hat{E}[-(A_{(2k+1)T/2n} - A_{2kT/2n})]\}.
\end{aligned}$$

Combining the above inequalities, we have

$$
\begin{aligned}
& 2\|h-h^{T/(2n),0}\|_{M_G^1(0,T)} \\
\geq\ & \sum_{k=0}^{2n-1}\{\hat{E}(A_{(k+1)T/2n}-A_{(kT/2n)})+\hat{E}[-(A_{(k+1)T/2n}-A_{kT/2n})]\} \\
\geq\ & \hat{E}\sum_{k=0}^{2n-1}(A_{(k+1)T/2n}-A_{(kT/2n)})+\hat{E}\sum_{k=0}^{2n-1}[-(A_{(k+1)T/2n}-A_{kT/2n})] \\
=\ & \hat{E}(A_T)+\hat{E}(-A_T),
\end{aligned}
$$

Thus by Lemma 3.1, letting $n\to\infty$, we have $\hat{E}(A_T)+\hat{E}(-A_T)=0$, which means that A_T is symmetric. □

Theorem 3.6 Let $A_t=\int_0^t h_s d\langle B\rangle_s$ be a process with stationary, independent increments and $h\in M_G^{1,+}(0,T)$. Then there exists a constant $c\geq 0$ such that $A_t=c\langle B\rangle_t$.

Proof. Noting that $K_t:=\int_0^t h_s ds=\int_0^t(\frac{d\langle B\rangle_s}{ds})^{-1}dA_s$ is a process with independent increments, we know that K_T is symmetric by Lemma 3.5.

Let $M_t=\int_0^t h_s d\langle B\rangle_s-\int_0^t 2G(h_s)ds$ and $N_t=\int_0^t h_s d\langle B\rangle_s-\lambda^2 t$, where $\lambda^2=\hat{E}(A_T)/T$. We know that both $\{M_t\}$ and $\{N_t\}$ are G-martingale with finite variation. Let $L_t=\hat{E}_t(\lambda^2 T-\overline{\sigma}^2 K_T)$. Then $\{L_t\}$ is a symmetric G-martingale. By the symmetry of $\{L_t\}$ we have

$$M_t=\hat{E}_t(M_T)=\hat{E}_t(L_T+N_T)=L_t+N_t.$$

By uniqueness of G-martingale decomposition, we get $L\equiv 0$ and $h\equiv\frac{\lambda^2}{\overline{\sigma}^2}$. □

4 Characterization of G-Brownian motion

Theorem 4.1(Martingale characterization of G-Brownian motion)

Let $\{M_t\}$ be a symmetric G-martingale with $M_T\in L_G^2(\Omega_T)$ and $\{\langle M\rangle_t\}$ a process with stationary and independent increments. Then $\{M_t\}$ is a G-Brownian motion;

Let $\{M_t\}$ be a G-Brownian motion on $(\Omega_T, L_G^1(\Omega_T),\hat{E})$. Then there exists a positive constant c such that $\langle M\rangle_t=c\langle B\rangle_t$.

Proof. By Theorem 4.8 in [Song10a] or Corollary 5.2 in [Song10b], there exists $h\in M_G^2(0,T)$ such that $M_t=\int_0^t h_s dB_s$. So $\langle M\rangle_t=\int_0^t h_s^2 d\langle B\rangle_s$. By Theorem 3.6, there exists some constant $c\geq 0$ such that $h^2\equiv c^2$. Thus by the representation of G-expectation given in [DHP08], $\{M_t\}$ is a G-Brownian motion with M_t distributed as $N(0, [c^2\underline{\sigma}^2 t, c^2\overline{\sigma}^2 t])$.

On the other hand, if $\{M_t\}$ is a G-Brownian motion on $(\Omega_T, L_G^1(\Omega_T)$, then $\{M_t\}$ is a symmetric G-martingale. By the above arguments, we have $\langle M\rangle_t = c\langle B\rangle_t$ for some positive constant c. $\square$

Let $\mathcal{H} = \{a|\ a(t) = \Sigma_{k=0}^{n-1} a_{t_k} 1_{]t_k, t_{k+1}]}(t)\}$ and $H = \{a \in \mathcal{H} | \lambda[a = 0] = 0\}$, where λ is the Lebesgue measure.

In the following Lemma, we say that $\{X_t\}$ on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ is a process with independent increments in the following sense, which is somewhat different from Definition 2.9.

For any $0 < t < T$ and $s_0 \le \cdots s_m \le t \le t_0 \le \cdots t_n \le T$,

$$(X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}) \perp (X_{s_1} - X_{s_0}, \cdots, X_{s_m} - X_{s_{m-1}}).$$

Lemma 4.2 Let $\{L_t\}$ be a process with absolutely continuous paths. Assume $\underline{c}t \le L_t \le \overline{c}t$ for real numbers $\underline{c} \le \overline{c}$. Let $C(a) = \overline{c}a^+ - \underline{c}a^-$ for any $a \in R$. If

$$\hat{E}(\int_0^T a(s) dL_s) = \int_0^T C(a(s)) ds,\ \forall\ a \in \mathcal{H},$$

we have that $\{L_t\}$ is a process with stationary and independent increments such that $\underline{c}t = -\hat{E}(-L_t) \le \hat{E}(L_t) = \overline{c}t$.

Proof. It suffices to prove the Lemma for the case $\underline{c} < \overline{c}$. For any $a \in H$, let

$$\theta_s^a = \overline{c} 1_{[a(s) \ge 0]} + \underline{c} 1_{[a(s) < 0]}.$$

By assumption,

$$\hat{E}(\int_0^T a(s) dL_s) = \int_0^T a(s) \theta_s^a ds.$$

On the other hand, by the representation theorem for G-expectation given in [DHP08] or [HP09], there exists some weak compact subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that

$$\hat{E}(\xi) = \max_{P \in \mathcal{P}} E_P(\xi),\ \forall \xi \in L_G^1(\Omega_T),$$

which means that there exists $P_a \in \mathcal{P}$ such that

$$E_{P_a}(\int_0^T a(s) dL_s) = \int_0^T a(s) \theta_s^a ds.$$

By the assumption for L_t, we have $P_a\{L_t = \int_0^t \theta_s^a ds, \forall t\} = 1$. From this we have

$$\hat{E}[\varphi(L_{t_1} - L_{t_0}, \cdots, L_{t_n} - L_{t_{n-1}})] \ge \varphi(\int_{t_0}^{t_1} \theta_s^a ds, \cdots, \int_{t_{n-1}}^{t_n} \theta_s^a ds)$$

for any $\varphi \in C_b(R^n)$ and $n \in N$. Consequently,

$$\begin{aligned} & \hat{E}[\varphi(L_{t_1} - L_{t_0}, \cdots, L_{t_n} - L_{t_{n-1}})] \\ \geq \ & \sup_{a \in H} \varphi(\int_{t_0}^{t_1} \theta_s^a ds, \cdots, \int_{t_{n-1}}^{t_n} \theta_s^a ds) \\ = \ & \sup_{c_1, \cdots, c_n \in [\underline{c}, \overline{c}]} \varphi(c_1(t_1 - t_0), \cdots, c_n(t_n - t_{n-1})). \end{aligned}$$

The converse inequality is obvious. Thus $\{L_t\}$ is a process with stationary and independent increments such that $\underline{c}t = -\hat{E}(-L_t) \leq \hat{E}(L_t) = \overline{c}t$. □

Lemma 4.3 Let $\{L_t\}$ be a G-martingale with finite variation and $L_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$. Then $\{L_t\}$ is decreasing. Particularly, $L_t \leq L_0 = \hat{E}(L_T)$.

Proof. By Theorem 4.5 in [Song10a], we know $\{L_t\}$ has the following decomposition

$$L_t = \hat{E}(L_T) + M_t + K_t,$$

where $\{M_t\}$ is a symmetric G-martingale and $\{K_t\}$ is a negative, decreasing G-martingale. Since both $\{L_t\}$ and $\{K_t\}$ are processes with finite variation, we get $M_t \equiv 0$. Therefore, we have $L_t = \hat{E}(L_T) + K_t \leq \hat{E}(L_T) = L_0$. □

Theorem 4.4 Let $\{X_t\}$ be a generalized G-Brownian motion with zero mean. Then we have the following decomposition:

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a symmetric G-Brownian motion, and $\{L_t\}$ is a negative, decreasing G-martingale with stationary and independent increments.

Proof. Clearly $\{X_t\}$ is a G-martingale. By Theorem 4.5 in [Song10a], we have the following decomposition

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a symmetric G-martingale, and $\{L_t\}$ is a negative, decreasing G-martingale. Since $\{X_t\}$ is a process with independent increments, it is easily seen that $\{M_t\}$ and $\{L_t\}$ are both processes with independent increments by the uniqueness of G-martingale decomposition.

In the sequel, we first prove that $\{L_t\}$ is a process with stationary increments. Noting that $\hat{E}(-L_t) = \hat{E}(-X_t) = ct$ for some positive constant c since $\{X_t\}$ is a process with stationary and independent increments, we claim that $-L_t - ct$ is a G-martingale. To prove this, it suffices to show that for any $t > s$, $\hat{E}_s[-(L_t - L_s)] = c(t-s)$. In fact,

$$\begin{aligned}
& \hat{E}_s[-(L_t - L_s)] \\
= & \hat{E}_s[-(X_t - M_t - X_s + M_s)] \\
= & \hat{E}_s[-(X_t - X_s)] \\
= & \hat{E}[-(X_t - X_s)] \\
= & c(t-s).
\end{aligned}$$

Combining this with Lemma 4.3, we have $-L_t - ct \leq 0$. On the other hand, for any $a \in \mathcal{H}$

$$\begin{aligned}
& \hat{E}[\int_0^T a(s)dL_s] \\
= & \hat{E}[\int_0^T a(s)dX_s] \\
= & \hat{E}[\Sigma_{k=0}^{n-1} a_{t_k}(X_{t_{k+1}} - X_{t_k})] \\
= & \Sigma_{k=0}^{n-1} c a_{t_k}^-(t_{k+1} - t_k) \\
= & \int_0^T ca^-(s)ds = \int_0^T C(a(s))ds,
\end{aligned}$$

where $C(a(s))$ is defined as in Lemma 4.2 with $\overline{c} = 0, \underline{c} = -c$. By Lemma 4.2, $\{L_t\}$ is a process with stationary increments.

Now we are in a position to show that $\{M_t\}$ is a (symmetric) G-Brownian motion. To this end, by Theorem 4.1, it suffices to prove that $\{\langle M\rangle_t\}$ is a process with stationary and independent increments. For $n \in N$, let

$$X_t^n = \sum_{k=0}^{2^n-1} X_{\frac{kT}{2^n}} 1_{]\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]}(t)$$

and

$$\Omega_t^n(X) = \sum_{k=0}^{2^n-1} (X_{\frac{kt}{2^n}} - X_{\frac{kt}{2^n}})^2.$$

Observing that $\Omega_t^n(X) = X_t^2 - 2\int_0^t X_s^n dX_s$, we have

$$\begin{aligned}
& |\Omega_t^n(X) - \Omega_t^{m+n}(X)| \\
\leq & 2(|\int_0^t (X_s^n - X_s^{m+n})dM_s| + |\int_0^t (X_s^n - X_s^{m+n})dL_s|) \\
= & 2(I + II).
\end{aligned}$$

for any $n, m \in N$. It's easy to check that

$$\hat{E}(II) \leq c \int_0^t \hat{E}(|X_s^n - X_s^{m+n}|)ds \to 0 \text{ as } m, n \to \infty.$$

Noting that

$$\begin{aligned} I &= \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} (X_{\frac{i}{2^n}+\frac{j}{2^{n+m}}} - X_{\frac{i}{2^n}})(M_{\frac{i}{2^n}+\frac{j+1}{2^{n+m}}} - M_{\frac{i}{2^n}+\frac{j}{2^{n+m}}}) \\ &= \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} I_i^j, \end{aligned}$$

we get

$$\hat{E}(I^2) \leq \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} \hat{E}[(I_i^j)^2].$$

It's easily seen that $\hat{E}[(I_i^j)^2] \leq C\frac{j}{2^{2(n+m)}}$ for some constant C, hence that $\hat{E}(I^2) \to 0$, and finally that $\hat{E}(|\Omega_t^n(X) - \Omega_t^{m+n}(X)|) \to 0$ as $m, n \to \infty$. Then

$$\langle X \rangle_t := \lim_{L_G^1(\Omega_T), n\to\infty} \Omega_t^n$$

is a process with stationary and independent increments. Noting that $\langle M \rangle_t = \langle X \rangle_t$, $\langle M \rangle_t$ is also a process with stationary and independent increments. □

References

[DHP08] Denis, L., Hu, M. and Peng S. *Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes.* arXiv:0802.1240v1 [math.PR] 9 Feb, 2008

[HP09] Hu, Mingshang and Peng, Shige (2009) *On representation theorem of Gexpectations and paths of G-Brownian motion.* Acta Math Appl Sinica English Series, 25(3): 1-8.

[P06] Peng, S. (2006) *G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type*, preprint (pdf-file available in arXiv:math.PR/0601035v1 3Jan 2006), to appear in Proceedings of the 2005 Abel Symposium.

[P08] Peng, S. (2008) *Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation*, in Stochastic Processes and their Applications, 118(12), 2223-2253.

[P07] Peng, S. (2007) *G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty*, Preprint: arXiv:0711.2834v1 [math.PR] 19 Nov 2007.

[P09] Peng, S. (2009) *Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations*, Science in China Series A: Mathematics, 52, No.7, 1391-1411, (www.springerlink.com).

[P10] Peng, S. (2010) *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, arXiv:1002.4546v1 [math.PR] 24 Feb 2010.

[Song10a] Song, Y.(2010) *Some properties on G-evaluation and its applications to G-martingale decomposition. Preprint* arXiv:1001.2802v1 [math.PR] 17 Jan 2010.

[Song10b] Song, Y.(2010) *Properties of hitting times for G-martingale. Preprint.* arXiv:1001.4907v1 [math.PR] 27 Jan 2010.

[XZ09] Xu J, Zhang B. (2009) *Martingale characterization of G-Brownian motion.* Stochastic Processes Appl., 119(1): 232-248.