Relative homological algebra in categories of representations of infinite quivers

Sergio Estrada*† Salahattin Özdemir‡§

September 2, 2010

Abstract

In the first part of this paper, we prove the existence of torsion free covers in the category of representations of quivers, $(Q, R\text{-}\mathcal{M}od)$, for a wide class of quivers included in the class of the so-called source injective representation quivers provided that any direct sum of torsion free and injective R-modules is injective. In the second part, we prove the existence of $\mathscr{F}_{cw}$-covers and $\mathscr{F}_{cw}^{\perp}$-envelopes for any quiver Q and any ring R with unity, where $\mathscr{F}_{cw}$ is the class of all "componentwise" flat representations of Q.

Key Words: cover; envelope; torsion free; flat; representations of a quiver.

2000 Mathematics Subject Classification: Primary 16G20; Secondary 16D90

1 Introduction

The aim of this paper is to continue with a program initiated in Enochs & Herzog (1999) and continued in Enochs et al. (2003a), Enochs et al. (2004b), Enochs & Estrada (2005a) and Enochs et al. (2009) to develop new techniques on the study of representations by modules over (possibly infinite) quivers. In contrast to the classical representation theory of quivers motivated by Gabriel's work (Gabriel (1972)), we do not assume that the base ring is an algebraically closed field and that all vector spaces involved are finite dimensional.

Techniques on representation theory of infinite quivers have recently proved to be very useful in leading to simplifications of proofs as well as the descriptions of objects in related categories. For instance, in Enochs & Estrada (2005b) it is shown that the category of quasi-coherent sheaves on an arbitrary scheme is equivalent to a category of representations of a quiver (with

certain modifications on the representations). And this point of view allows to introduce new versions of homological algebra in such categories (see Enochs & Estrada (2005b, §5) and Enochs et al. (2003b)). Infinite quivers also appear when considering the category of $\mathbb{Z}$-graded modules over the graded ring $R[x]$. This category is equivalent to the category of representations over R of the quiver $A_\infty^\infty \equiv \cdots \to \bullet \to \bullet \to \bullet \to \cdots$. And, in general, one can find less trivial example involving group rings $R[G]$ with the obvious grading.

Our goal on this paper is to introduce new classes in the category of representations of a (possibly infinite) quiver to compute (unique up to homotopy) resolutions which give rise to new versions of homological algebra on it.

The first of such versions turns to Enochs' proof on the existence of torsion free covers of modules over an integral domain (see Enochs (1963)) and its subsequent generalization by Teply and Golan in Teply (1969) and Golan & Teply (1973) to more general torsion theories in $R\text{-}\mathcal{M}od$. In the first part of the paper we prove that torsion free covers exist for a wide class of quivers included in the class of the so-called source injective representation quivers as introduced in Enochs et al. (2009). This important class of quivers includes all finite quivers with no oriented cycles, but also includes infinite line quivers:

$$A_\infty \equiv \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \, ,$$

$$A^\infty \equiv \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \, ,$$

$$A_\infty^\infty \equiv \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

and quiver of pure semisimple type as introduced by Drozdowski and Simson in Drozdowski & Simson (1979).

On the second part, we will focus on the existence of a version of relative homological algebra by using the class of componentwise flat representations in $(Q, R\text{-}\mathcal{M}od)$. Recently, it has been proved by Rump that flat covers do exist on each abelian locally finitely presented category (see Rump (2010)). Here by "flat" the author means Stenström's concept of flat object (Stenström (1968)) in terms of the Theory of Purity that one can always define in a locally finitely presented category (see Crawley-Boevey (1994)). We call such flat objects "categorical flat". For abelian locally finitely presented categories with enough projectives, this notion of "flatness" is equivalent to be direct limit of certain projective objects. As $(Q, R\text{-}\mathcal{M}od)$ is a locally finitely presented Grothendieck category with enough projectives we infer by using Rump's result that $(Q, R\text{-}\mathcal{M}od)$ admits "categorical flat" covers for any quiver Q and any associative ring R with unity. But there are categories in which there is a classical notion of flatness having nothing to do with respect to a Theory of Purity. This is the case of the notion of "flatness" in

categories of presheaves or quasi-coherent sheaves, where "flatness" is more related with a "componentwise" notion. Those categories may be viewed as certain categories of representations of quivers, so we devote the second part to prove the existence of "componentwise" flat covers for any quiver and any ring R with unity. In particular if X is a topological space, an easy modification of our techniques can prove the existence of a flat cover (in the algebraic geometrical sense) for any presheaf on X over $R\text{-}\mathcal{M}od$.

2   Preliminaries

All rings considered in this paper will be associative with identity and, unless otherwise specified, not necessarily commutative. The letter R will usually denote a ring. All R-modules are left unitary modules, and all torsion theories considered for $R\text{-}\mathcal{M}od$ are *hereditary*, that is, the torsion class is closed under submodules, or equivalently, the torsion free class is closed under injective envelopes, and *faithful* that is, R is torsion free. We refer to Enochs & Jenda (2000) and Assem et al. (2006) for any undefined notion on covers and envelopes or quivers used in the text.

A *quiver* is a directed graph whose edges are called arrows. As usual we denote a quiver by Q understanding that $Q = (V, E)$ where V is the set of vertices and E the set of arrows. An arrow of a quiver from a vertex v_1 to a vertex v_2 is denoted by $a : v_1 \to v_2$. In this case we write $s(a) = v_1$ the initial (starting) vertex and $t(a) = v_2$ the terminal (ending) vertex. A *path p* of a quiver Q is a sequence of arrows $a_n \cdots a_2 a_1$ with $t(a_i) = s(a_{i+1})$. Thus $s(p) = s(a_1)$ and $t(p) = t(a_n)$. Two paths p and q can be composed, getting another path qp (or pq) whenever $t(p) = s(q)$ $(t(q) = s(p))$.

A quiver Q may be thought as a category in which the objects are the vertices of Q and the morphisms are the paths of Q.

A representation by modules X of a given quiver Q is a functor $X : Q \longrightarrow R\text{-}\mathcal{M}od$. Such a representation is determined by giving a module $X(v)$ to each vertex v of Q and a homomorphism $X(a) : X(v_1) \to X(v_2)$ to each arrow $a : v_1 \to v_2$ of Q. A morphism η between two representations X and Y is a natural transformation, so it will be a family η_v such that $Y(a) \circ \eta_{v_1} = \eta_{v_2} \circ X(a)$ for any arrow $a : v_1 \to v_2$ of Q. Thus, the representations of a quiver Q by modules over a ring R form a category, denoted by $(Q, R\text{-}\mathcal{M}od)$.

For a given quiver Q and a ring R, the path ring of Q over R, denoted by RQ, is defined as the free left R-module, whose base are the paths p of Q, and where the multiplication is the obvious composition between two paths. This is a ring with enough idempotents, so in fact it is a ring with local units (see Wisbauer (1991, Ch.10, §49)). We denote by $RQ\text{-}\mathcal{M}od$ the category of unital RQ-modules (i.e. $_{RQ}M$ such that $RQM = M$). It is known that RQ is a projective generator of the category and that the categories $RQ\text{-}\mathcal{M}od$ and $(Q, R\text{-}\mathcal{M}od)$ are equivalent categories, so $(Q, R\text{-}\mathcal{M}od)$ is Grothendieck

category with enough projectives.

For a given quiver Q, one can define a family of projective generators from an adjoint situation as it is shown in Mitchell (1972). For every vertex $v \in V$ and the embedding morphism $\{v\} \subseteq Q$ the family $\{S_v(R) : v \in V\}$ is a family of projective generators of Q where the functor $S_v : R\text{-}\mathcal{M}od \longrightarrow (Q, R\text{-}\mathcal{M}od)$ is defined in Mitchell (1972, §28) as $S_v(M)(w) = \oplus_{Q(v,w)} M$ where $Q(v,w)$ is the set of paths of Q starting at v and ending at w. Then S_v is a left adjoint functor of the evaluation functor $T_v : (Q, R\text{-}\mathcal{M}od) \longrightarrow R\text{-}\mathcal{M}od$ given by $T_v(X) = X(v)$ for any representation $X \in (Q, R\text{-}\mathcal{M}od)$.

Let $\mathscr{F}$ be a class of objects in an abelian category $\mathscr{A}$. Recall from Enochs (1981) that, an $\mathscr{F}$-precover of an object C is a morphism $\psi : F \to C$ with $F \in \mathscr{F}$ such that $\text{Hom}(F', F) \to \text{Hom}(F', C) \to 0$ is exact for every $F' \in \mathscr{F}$. If, moreover, every morphism $f : F \to F$ such that $\psi \circ f = \psi$ is an automorphism, then ψ is said to be an $\mathscr{F}$-cover. $\mathscr{F}$-preenvelopes and $\mathscr{F}$-envelopes are defined dually.

Throughout this paper, by a representation of a quiver we will mean a representation by modules over a ring R.

During this paper we consider the following properties:

(**A**) Any direct sum of torsion free and injective modules is injective.

(**B**) For each vertex v of Q, the set $\{t(a) \mid s(a) = v\}$ is finite.

3   Torsion free covers in the category of representations relative to a torsion theory

Throughout this section Q will be a *source injective representation quiver* (see Enochs et al. (2009, Definition 2.2)), that is, for any ring R any *injective* representation X of $(Q, R\text{-}\mathcal{M}od)$ can be characterized in terms of the following conditions:

(i) $X(v)$ is injective R-module, for any vertex v of Q.

(ii) For any vertex v the morphism

$$X(v) \longrightarrow \prod_{s(a)=v} X(t(a))$$

induced by $X(v) \longrightarrow X(t(a))$ is a splitting epimorphism.

Example 3.1.   *(1)* Each quiver with a finite number of vertices and without oriented cycles is source injective.

(2) The infinite line quivers:

$$A_\infty \equiv \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \, ,$$

$$A^\infty \equiv \quad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots ,$$

$$A_\infty^\infty \equiv \quad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

are source injective representation quivers.

(3) Infinite barren trees are source injective representation quivers, where a tree T with root v is said to be *barren* if the number of vertices n_i of the i'th state of T is finite for every $i \in \mathbb{N}$ and the sequence of positive natural numbers $n_1, n_2, \ldots$ stabilizes (see Enochs et al. (2009, Corollaries 5.4-5.5)). For example the tree:
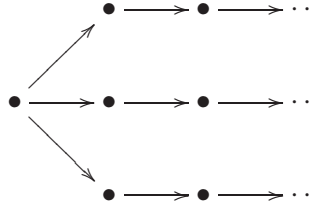
is barren.

(4) The quiver $\bullet \rightrightarrows \bullet$ is source injective, but does not satisfy (**B**).

Example 3.2. The n-loop, that is, a loop with n vertices, is not a source injective representation quiver. To see this, let v_i be a vertex and $a_i : v_i \to v_{i+1}$ be an arrow of the quiver for all $i = 1, 2, \ldots, n$ where $v_{n+1} = v_1$. Now consider the representation X defined as follows: $X(v_i) = E \times \cdots \times E$ (n times) where E is an injective R-module and $X(a_i)(x_1, \ldots x_n) = (x_n, x_1, \ldots, x_{n-1})$ where $x_i \in E$ for all $i = 1, \ldots n$. Then it is clear that X satisfies the conditions (i) and (ii) of the source injective representations quiver. But X is not an injective representation since it is not a divisible RQ-module. This is because, there is a nonzero element $(a_n a_{n-1} \cdots a_1 + a_1 a_n \cdots a_2 + \cdots + a_{n-1} a_{n-2} \cdots a_n) - 1$ of RQ such that

$$[(a_n a_{n-1} \cdots a_1 + a_1 a_n \cdots a_2 + \cdots + a_{n-1} a_{n-2} \cdots a_n) - 1] \cdot m = 0$$

for every element $m = (m_1, \ldots, m_n)$ where $m_i \in E \times \cdots \times E$ for all $i = 1, 2, \ldots, n$ (notice that if $x \in X$ then $x \in E \times \cdots \times E$).

We recall the definition of a torsion theory.

Definition 3.3. (Dickson, 1966) A *torsion theory* for an abelian category $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ such that

(1) $\operatorname{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$.

(2) If $\operatorname{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$.

(3) If $\mathrm{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

Here, $\mathcal{T}$ is called *torsion class* and its objects are *torsion*, while $\mathcal{F}$ is called *torsion free class* and its objects are *torsion free*. A torsion theory is called *hereditary* if the torsion class is closed under subobjects.

Now let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for $R\text{-}\mathcal{M}od$. Then we can define a torsion theory $(\mathcal{T}_{cw}, \mathcal{F}_{cw})$ for $(Q, R\text{-}\mathcal{M}od)$, by defining the torsion class such that $X \in \mathcal{T}_{cw}$ if and only if $X(v) \in \mathcal{T}$ for all $v \in V$. This is because $\mathcal{T}_{cw}$ is closed under quotient representations, direct sums and extensions (as so is $\mathcal{T}$) (see, for example, Stenström (1975, VI, Proposition 2.1)).

Remark 3.4. Since the torsion class $\mathcal{T}_{cw}$ is closed under subrepresentations, the torsion theory $(\mathcal{T}_{cw}, \mathcal{F}_{cw})$ is hereditary.

Proposition 3.5. *Let $X \in (Q, R\text{-}\mathcal{M}od)$. Then $X \in \mathcal{F}_{cw}$ if and only if $X(v) \in \mathcal{F}$ for all $v \in V$.*

Proof. ($\Rightarrow$) Let $X \in \mathcal{F}_{cw}$. Then for any $M \in \mathcal{T}$, we have

$$\mathrm{Hom}_R(M, T_v(X)) \cong \mathrm{Hom}_Q(S_v(M), X) = 0$$

since $S_v(M) \in \mathcal{T}_{cw}$ (as $\mathcal{T}$ is closed under direct sums). Thus $X(v) = T_v(X) \in \mathcal{F}$ for all $v \in V$.

($\Leftarrow$) Suppose that $X(v) \in \mathcal{F}$ for all $v \in V$. Let $A \in \mathcal{T}_{cw}$. If $\gamma : A \longrightarrow X$ is a morphism of representations, then we have module homomorphisms $\gamma_v : A(v) \to X(v)$ for all $v \in V$. Since $A(v) \in \mathcal{T}$, then $\mathrm{Hom}_R(A(v), X(v)) = 0$ and so $\gamma_v = 0$ for all $v \in V$. Thus $\gamma = 0$, that is, $\mathrm{Hom}_Q(A, X) = 0$. This means that $X \in \mathcal{F}_{cw}$. $\qquad\square$

Theorem 3.6. *Any representation of $\mathcal{F}_{cw}$ can be embedded in a torsion free and injective representation.*

Proof. Let $X \in \mathcal{F}_{cw}$ be any representation of Q. Since $(Q, R\text{-}\mathcal{M}od)$ has enough injectives and $(\mathcal{T}_{cw}, \mathcal{F}_{cw})$ is hereditary, then $\mathcal{F}_{cw}$ is closed under injective envelopes (see Dickson (1966, Theorem 2.9)). Thus X can be embedded in its torsion free injective envelope. $\qquad\square$

Lemma 3.7. *Let X, X', Y and Z be representations of Q. Then*

(i) *If X has an $\mathcal{F}_{cw}$-precover and $Z \subseteq X$, then Z also has an $\mathcal{F}_{cw}$-precover.*

(ii) *If X is injective, then $\psi : X' \longrightarrow X$ is an $\mathcal{F}_{cw}$-precover of X if and only if for every morphism $\phi : Y \longrightarrow X$ with $Y \in \mathcal{F}_{cw}$ and Y injective, there exists $f : Y \longrightarrow X'$ such that $\psi \circ f = \phi$.*

Proof. (i) Let $\psi : X' \longrightarrow X$ be an $\mathcal{F}_{cw}$-precover. Consider the morphism $\psi_1 : \psi^{-1}(Z) \longrightarrow Z$. Then $\psi^{-1}(Z) \in \mathcal{F}_{cw}$ since $\mathcal{F}_{cw}$ is closed under subrepresentations. Now for any morphism $\phi : Y \longrightarrow Z$ with $Y \in \mathcal{F}_{cw}$, there is a morphism $f : Y \longrightarrow X'$ such that $\psi f = \phi$. Therefore, $f(Y) \subset \psi^{-1}(Z)$ and so ϕ can be factored through ψ_1.

(ii) The condition is clearly necessary. Let $\phi_1 : Y_1 \longrightarrow X$ be a morphism with $Y_1 \in \mathcal{F}_{cw}$. Then by Theorem 3.6, Y_1 can be embedded in a representation $Y \in \mathcal{F}_{cw}$ which is injective. Now since X is injective, there is a morphism $\phi : Y \longrightarrow X$ such that $\phi \mid_{Y_1} = \phi_1$. So, by hypothesis, there exists a morphism $f : Y \longrightarrow X'$ such that $\psi f = \phi$. It follows that $(\psi f) \mid_{Y_1} = \phi \mid_{Y_1} = \phi_1$. $\qquad\square$

Lemma 3.8. *Let E be an R-module and let $\{E_i\}_{i\in I}$ be a direct family of submodules of E. If $\oplus_{i\in I} E_i$ is injective, then $\sum_{i\in I} E_i$ is injective.*

Proof. Let $\varphi : \oplus E_i \to \sum E_i$ and $\psi : \sum E_i \to \oplus E_i$ be homomorphisms of R-modules such that $\varphi\psi = id$. Now for any ideal A of R, any map $f : A \to \sum E_i$ can be extended to $\varphi \circ h : R \to \sum E_i$:

$$
\begin{array}{ccc}
A & \hookrightarrow & R \\
\downarrow{\scriptstyle f} & \swarrow{\scriptstyle \varphi h} & \\
\sum E_i & \nearrow{\scriptstyle h} & \\
\uparrow{\scriptstyle \varphi} \downarrow{\scriptstyle \psi} & & \\
\oplus E_i & &
\end{array}
$$

where $(\varphi \circ h) \mid_A = \varphi \circ (h \mid_A) = \varphi \circ \psi \circ f = f$. Thus $\sum_{i\in I} E_i$ is injective. $\quad\square$

Lemma 3.9. *Let E be in $(Q, R\text{-}\mathcal{M}od)$ and let $\{E_i\}_{i\in I}$ be a direct family of injective subrepresentations of E such that $E_i \in \mathcal{F}_{cw}, \forall i \in I$. If R satisfies ($\mathbf{A}$) and if Q satisfies ($\mathbf{B}$), then $\sum_{i\in I} E_i \in \mathcal{F}_{cw}$ and it is injective.*

Proof. Since each E_i is an injective representation such that $E_i \in \mathcal{F}_{cw}$, then $E_i(v)$ is an injective module such that $E_i(v) \in \mathcal{F}$, $\forall v \in V$ and $\forall i \in I$. So $\oplus_{i\in I} E_i(v)$ is also an injective module by hypothesis. By Lemma 3.8, $\sum_{i\in I} E_i(v)$ is also injective. Then the representation $\sum_{i\in I} E_i$ satisfies (i). Now taking the union of the splitting epimorphisms $E_i(v) \to \prod_{s(a)=v} E_i(t(a))$, we obtain the following splitting epimorphism:

$$
\left(\sum_{i\in I} E_i\right)(v) \longrightarrow \sum_{i\in I} \prod_{s(a)=v} E_i(t(a)) \cong \prod_{s(a)=v} \left(\sum_{i\in I} E_i\right)(t(a))
$$

where the isomorphism follows since the product is finite by hypothesis. This means $\sum_{i\in I} E_i$ is also satisfies (ii). Thus it is an injective representation

since Q is a source injective representation quiver. Finally, since $E_i(v) \in \mathcal{F}$ then $\sum_{i \in I} E_i(v) \in \mathcal{F}$ for all $v \in V$, and so $\sum_{i \in I} E_i \in \mathcal{F}_{cw}$. $\qquad \square$

Proposition 3.10. *Let Q be any quiver satisfying* (**B**). *Then R satisfies* (**A**) *if and only if any direct sum of injective representations of $\mathcal{F}_{cw}$ is injective.*

*Proof.* ($\Rightarrow$) The proof is the same as the proof of Lemma 3.9 by taking $\oplus E_i$ instead of $\sum E_i$.

($\Leftarrow$) The proof is immediate by considering the quiver $Q \equiv \cdot v$ which trivially satisfies (**B**). This is because $(Q, R\text{-}\mathcal{M}od) \cong R\text{-}\mathcal{M}od$ in this case. $\quad \square$

Note that, in the previous proposition, which will be useful in the proof of the following theorem, we cannot omit the fact that Q satisfy (**B**) as the following example shows.

Example 3.11. Consider the quiver

$$Q \equiv \qquad v_0 \longrightarrow v_2 \nearrow^{v_1} \searrow_{v_3}$$
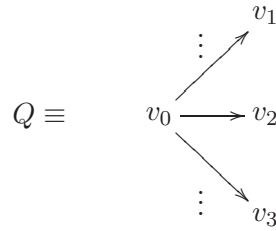
which, of course, does not satisfy (**B**) for the vertex v_0. For the ring of integers, $R = \mathbb{Z}$, consider the category $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$. Then the indecomposable injective and torsion free representations of $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$ (w.r.t usual torsion theory) are as follows:

$$E_0 \equiv \mathbb{Q} \longrightarrow 0, \qquad E_1 \equiv \mathbb{Q} \longrightarrow 0, \qquad E_2 \equiv \mathbb{Q} \longrightarrow \mathbb{Q} \cdots$$
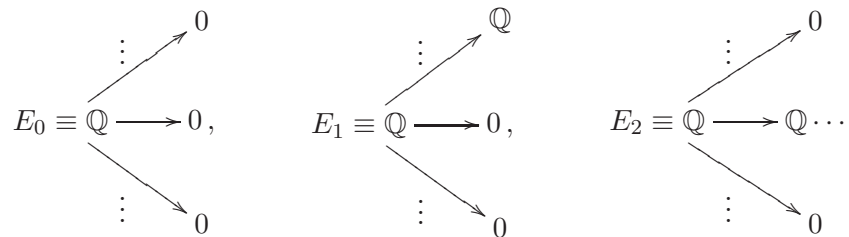
that is, for each $i \in \mathbb{N}$, the representation E_i has a module $\mathbb{Q}$ at the vertices v_0 and v_i, and 0 otherwise. Therefore, the direct sum of the representations

of E_i for $i \geq 1$ will be as follows:

$$
\begin{array}{ccc}
 & & \mathbb{Q} \\
 & \nearrow & \\
\vdots & & \\
\mathbb{Q}^{(\mathbb{N})} & \longrightarrow & \mathbb{Q} \\
\vdots & & \\
 & \searrow & \\
 & & \mathbb{Q}
\end{array}
$$
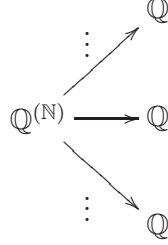
If we show that $\oplus_{i \geq 1} E_i$ is not an injective representation of $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$, then we will see that the statement of Proposition 3.10 does not hold for this Q (since $R = \mathbb{Z}$ satisfies $(\mathbf{A})$). Now suppose on the contrary that $\oplus_{i \geq 1} E_i$ is injective. Then since Q is source injective representation quiver as it is right rooted (see Enochs et al. (2009, Theorem 4.2)) we have (ii), that is,

$$
\bigoplus_{i \geq 1} E_i(v_0) \longrightarrow \prod_{s(a)=v_0} \bigoplus_{i \geq 1} E_i(t(a))
$$

is a splitting epimorphism, or equivalently, $\mathbb{Q}^{(\mathbb{N})} \longrightarrow \mathbb{Q}^{\mathbb{N}}$ is a splitting epimorphism. However, this is impossible since $\mathbb{Q}^{(\mathbb{N})}$ has a countable basis but $\mathbb{Q}^{\mathbb{N}}$ does not have it.

Now recall that a representation of a quiver Q is said to be *finitely generated* if it is finitely generated as an object of the category of representations of Q.

Theorem 3.12. *Let Q be any quiver satisfying $(\mathbf{B})$. If R satisfies $(\mathbf{A})$, then every injective representation of $\mathcal{F}_{cw}$ is the direct sum of indecomposable injective representations of $\mathcal{F}_{cw}$.*

Proof. Following the proof of Stenström (1975, Proposition 4.5) we argue as follows: let $E \in \mathcal{F}_{cw}$ be an injective representation of Q. Consider all independent families $(E_i)_{i \in I}$ of indecomposable torsion free and injective subrepresentations of E. Then by Zorn's lemma, there is a maximal such family $(E_i)_{i \in I}$. Since $\oplus_{i \in I} E_i \in \mathcal{F}_{cw}$ and it is injective (by Proposition 3.10), we can write $E = (\oplus E_i) \oplus E'$. To show that $E' = 0$ it is enough to show that every injective representation with $0 \neq E' \in \mathcal{F}_{cw}$ contains a non-zero indecomposable direct summand. Consider the set of all subrepresentations of E' such that:

$$
\Sigma = \{E'' \subset E' \mid E'' \in \mathcal{F}_{cw}, \text{ injective s.t. } C \nsubseteq E'' \text{ where } 0 \neq C \subset E' \text{ is f.g.}\}
$$

(In fact, we can take such a non-zero finitely generated representation C, since$(Q, R\text{-}\mathcal{M}od)$ is locally finitely generated). Now take $\overline{E} = \sum_{E'' \in \Omega} E''$ where Ω is a chain of Σ. Then by Lemma 3.9, $\overline{E} \in \mathcal{F}_{cw}$ and it is injective.

Clearly $C \nsubseteq \overline{E}$ since C is finitely generated (if $C \subset \overline{E}$ then $C \subset E''$ for some $E'' \in \Omega$ which is impossible). This shows that $\overline{E} \in \Sigma$ and in fact it is an upper bound of Ω. Then by Zorn's lemma Σ has a maximal element, say E''. Now we have $E' = E'' \oplus D$ where $0 \neq D$ is an indecomposable representation. For if $D = D' \oplus D''$ with $D' \neq 0$ and $D'' \neq 0$, then $(E''+D') \cap (E''+D'') = E''$ and so either $C \nsubseteq E''+D'$ or $C \nsubseteq E''+D''$ which contradicts the maximality of E'' in Σ. Hence, every non-zero E' contains an indecomposable direct summand, which completes the proof. $\qquad\square$

Proposition 3.13. *Let Q be any quiver satisfying (**B**). If R satisfies (**A**), then $(Q, R\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-precovers.*

Proof. Since the category $(Q, R\text{-}\mathcal{M}od)$ has enough injectives, it suffices to show that an *injective* representation X has an $\mathcal{F}_{cw}$-precover (by Lemma 3.7-(i)), and so we can take an injective representation $Y \in \mathcal{F}_{cw}$ (by Lemma 3.7-(ii)). Let $\{E_\mu \mid \mu \in \Lambda\}$ denote the set of representatives of indecomposable injective representations of $\mathcal{F}_{cw}$. Let $H_\mu = \mathrm{Hom}_Q(E_\mu, X)$ and then define $X' = \oplus_{\mu \in \Lambda} E_\mu^{(H_\mu)}$. So there is a morphism $\psi : X' \longrightarrow X$ such that $\psi \mid_{E_\mu} \in H_\mu$. Thus every morphism $\phi : Y \longrightarrow X$ with an injective representation $Y \in \mathcal{F}_{cw}$ factors through the canonical map $\psi : X' \longrightarrow X$, since $Y = \oplus_{\mu \in \Lambda'} E_\mu$ by Theorem 3.12 where $\Lambda' \subseteq \Lambda$. $\qquad\square$

To prove that $(Q, R\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-covers, we need the following lemmas by the same methods of proofs given in, for example, Xu (1996, Lemmas 1.3.6-1.3.7) for usual torsion theories for $R\text{-}\mathcal{M}od$.

Lemma 3.14. *Let Q be any quiver satisfying (**B**) and let R satisfy (**A**). If $\psi : X' \longrightarrow X$ is an $\mathcal{F}_{cw}$-precover of the representation X, then we can derive an $\mathcal{F}_{cw}$-precover $\phi : Y \longrightarrow X$ such that there is no non-trivial sub-representation $S \subseteq \ker(\phi)$ with $Y/S \in \mathcal{F}_{cw}$.*

Lemma 3.15. *Let Q be any quiver satisfying (**B**) and let R satisfy (**A**). If $\phi : Y \longrightarrow X$ is an $\mathcal{F}_{cw}$-precover of X with no non-trivial subrepresentation $S \subseteq Y$ such that $S \subseteq \ker(\phi)$ and $Y/S \in \mathcal{F}_{cw}$, then this $\mathcal{F}_{cw}$-precover is actually an $\mathcal{F}_{cw}$-cover of X.*

Theorem 3.16. *Let Q be any quiver satisfying (**B**) and let R satisfy (**A**). Then every representation in $(Q, R\text{-}\mathcal{M}od)$ has a unique, up to isomorphism, $\mathcal{F}_{cw}$-cover.*

Proof. The existence part of the proof follows by Lemmas 3.14 and 3.15, and the uniqueness part follows by Xu (1996, Theorem 1.2.6). $\qquad\square$

Example 3.17. Let R satisfy (**A**). Consider the quiver $Q \equiv \bullet \to \bullet$. For any module M, if we take the torsion free cover $\psi : T \to M$ of M (this is possible in the category of $R\text{-}\mathcal{M}od$, see Teply (1969)), then

$$
\begin{array}{ccc}
\overline{T} & \quad & \mathrm{Ker}\,\psi \xrightarrow{\ i\ } T \\
\downarrow{\scriptstyle \overline{\psi}} & & \downarrow \qquad\quad \downarrow{\scriptstyle \psi} \\
\overline{M} & & 0 \xrightarrow{\qquad} M
\end{array}
$$

is an $\mathcal{F}_{cw}$-cover of the representation $0 \to M$. In fact, if there is a morphism

$$
\begin{array}{ccc}
T_1 & \xrightarrow{\ \alpha\ } & T_2 \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
0 & \xrightarrow{\qquad} & M
\end{array}
$$

where $T_1 \to T_2 \in \mathcal{F}_{cw}$, then there exists $f : T_2 \to T$ such that $\psi f = \beta$ since ψ is torsion free precover, and so taking $g = f\alpha : T_1 \to \mathrm{Ker}\,\psi$ (it is well-defined since for any $x \in T_1, \psi f \alpha(x) = \beta\alpha(x) = 0$) we see that it is an $\mathcal{F}_{cw}$-precover. And if there is an endomorphism $\overline{f} = (f, g) : \overline{T} \to \overline{T}$ such that $\overline{\psi} \circ \overline{f} = \overline{\psi}$ then f is automorphism (since ψ is a torsion free cover), and so g is a monomorphism. To show that g is epic, take any $y \in \mathrm{Ker}\,\psi$. Then $y = f(x)$ for some $x \in T$ (since f is epic). Since $\psi(x) = \psi f(x) = 0$, $x \in \mathrm{Ker}\,\psi$ and thus $y = f(x) = g(x)$ implies that g is epic. Hence $\overline{f}$ is an automorphism, that is, $\overline{\psi}$ is an $\mathcal{F}_{cw}$-cover.

Remark 3.18. In Dunkum (2009), the question was raised whether the category $(A^{\infty}, R\text{-}\mathcal{M}od)$ admits torsion free covers, where $A^{\infty} \equiv \bullet \to \bullet \to \cdots$. By Theorem3.16, if R satisfies (**A**) then the category $(A_{\infty}, R\text{-}\mathcal{M}od)$ admits torsion free covers since A^{∞} satisfies (**B**).

4 Componentwise flat covers in the category of representations

Let $\mathscr{A}$ be a Grothendieck category. Recall that an object C of $\mathscr{A}$ is *finitely presented* if it is finitely generated and every epimorphism $B \to C$, where B is finitely generated, has a finitely generated kernel. $\mathscr{A}$ is said to be a *locally finitely presented* category if it has a family of finitely presented generators.

In Rump (2010), flat covers are shown to exist in locally finitely presented Grothendieck categories. Then the category $(Q, R\text{-}\mathcal{M}od)$ admits flat covers for any quiver Q since it is a locally finitely presented Grothendieck category. This is because $(Q, R\text{-}\mathcal{M}od)$ has a family of finitely generated projective (and so finitely presented) generators. Here by "flat" we mean *categorical flat* representations of Q, that is, $\varinjlim P_i$ where each P_i is a projective representation of Q.

Now we will define flat representations *componentwise* which are different from *categorical* flat representations.

Definition 4.1. Let Q be any quiver and let M be a representation of Q. We call M *componentwise flat* if $M(v)$ is a flat R-module for all $v \in V$.

This definition is not the categorical definition of flat representations, but it is the correct one when we consider $(Q, R\text{-}\mathcal{M}od)$ as the category of presheaves over a topological space. By $\mathscr{F}_{cw}$ we denote the class of all componentwise flat representations.

Also, let us define *pure subrepresentations* componentwise.

Definition 4.2. An exact sequence of left R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is *pure* if for every right R-module L, the induced sequence

$$0 \longrightarrow L \otimes_R A \longrightarrow L \otimes_R B \longrightarrow L \otimes_R C \longrightarrow 0$$

is exact. A submodule $A \subseteq B$ is *pure* if the induced sequence is pure.

Definition 4.3. Let M be a representation of Q. We call a subrepresentation $P \subseteq M$ *componentwise pure* if $P(v) \subseteq M(v)$ is pure submodule for all $v \in V$.

In the proof of the following lemma, we can consider the representation generated by an element "x". Let M be a representation of Q and let $x \in M$ (so $x \in M(v)$ for some $v \in V$). Since S_v is a left adjoint of T_v, we have

$$\text{Hom}_{(\{v\}, R\text{-}\mathcal{M}od)}(R, M(v)) \cong \text{Hom}_{(Q, R\text{-}\mathcal{M}od)}(S_v(R), M)$$

for all $v \in V$. So we have a unique morphism $\varphi : S_v(R) \longrightarrow M$ corresponds to the R-homomorphism $\varphi_x : R \to M(v)$ given by $\varphi_x(1) = x$. Thus $\text{Im}(\varphi)$ is the subrepresentation of M generated by x.

The cardinality of a representation M of a quiver Q is defined as

$$|M| = \left| \coprod_{v \in V} M(v) \right|.$$

Lemma 4.4. *Let $\aleph$ be an infinite cardinal such that $\aleph \geq \sup\{|R|, |V|, |E|\}$. Let M be a representation of Q. Then for each $x \in M$, there exists a componentwise pure subrepresentation P of M such that $|P| \leq \aleph$ and $x \in P$.*

Proof. Let $x \in M(v)$ with $v \in V$. Then consider the subrepresentation $M^0 \subseteq M$ generated by x. Then $|M^0| \leq \aleph$ since

$$|S_v(R)(w)| = |\oplus_{Q(v,w)} R| \leq |V| \cdot |E| \cdot |\mathbb{N}| \cdot |R| \leq \aleph \cdot \aleph_0 = \aleph.$$

Since $|M^0(v)| \leq \aleph$ for all $v \in V$, we can apply Xu (1996, Lemma 2.5.2), so there exist pure submodules $M^1(v)$ of $M(v)$ such that $|M^1(v)| \leq \aleph$ and $M^0(v) \subseteq M^1(v)$, $\forall v \in V$. Now consider the subrepresentation M^2 of M generated by $M^1(v)$ such that $M^1(v) \subseteq M^2(v)$ for all $v \in V$. Then $|M^2| = |\coprod_{v \in V} M^2(v)| = |V| \cdot |M^2(v)| \leq \aleph$ since $|M^2(v)| \leq \aleph$ as $|M^1(v)| \leq \aleph$ for all $v \in V$. So applying Xu (1996, Lemma 2.5.2) again, there exist pure submodules $M^3(v)$ of $M(v)$ such that $|M^3(v)| \leq \aleph$ and $M^2(v) \subseteq M^3(v)$ for all $v \in V$. Now consider the subrepresentation M^4 of M generated by $M^3(v)$ such that $M^3(v) \subseteq M^4(v)$. Then $|M^4| \leq \aleph$. So proceed by induction to find a chain of subrepresentations of M: $M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots$ such that $|M^n| \leq \aleph$ for every $n \in \mathbb{N}$ and $v \in V$. Therefore, by taking $P = \bigcup_{n < \omega} M^n$ we obtain a pure subrepresentation P of M which satisfies the hypothesis of the lemma. Indeed, P is componentwise pure subrepresentation of M, because for each $v \in V$, the set $\{n \in \mathbb{N} : M^n(v)$ is pure in $M(v)\}$ is cofinal, and the set $\{n \in \mathbb{N} : M^n$ is a subrepresentation of $M\}$ is also cofinal. Finally, it is clear that $|P| \leq \aleph_0 \cdot \aleph = \aleph$, and $x \in P$ since $x \in M^0(v)$. □

We recall that a chain of subobjects of a given object C of an abelian category, $\{C_\alpha : \alpha < \lambda\}$ where λ is an ordinal number, is said to be *continuous* provided that $C_\omega = \cup_{\alpha < \omega} C_\alpha$ for any limit ordinal $\omega < \lambda$.

Given a class $\mathscr{F}$ of objects in an abelian category $\mathscr{A}$, by $\mathscr{F}^\perp$ we denote the class of objects C of $\mathscr{A}$ such that $\text{Ext}^1(F, C) = 0$ for all $F \in \mathscr{F}$, and we say that $(\mathscr{F}, \mathscr{F}^\perp)$ is cogenerated by a set if there exists a set $T \subseteq \mathscr{F}$ such that $T^\perp = \mathscr{F}^\perp$ (see, for example, Enochs & Jenda (2000, Chp.7)).

Theorem 4.5. *The pair $(\mathscr{F}_{cw}, \mathscr{F}_{cw}^\perp)$ is cogenerated by a set.*

Proof. Let $F \in \mathscr{F}_{cw}$ and take any element $x_0 \in F$. Then by Lemma 4.4, there exists a componentwise pure subrepresentation $F_0 \subseteq F$ such that $x_0 \in F_0$ and $|F_0| \leq \aleph$ for a suitable cardinal number. Since a pure submodule of a flat module is flat, then $F_0 \in \mathscr{F}_{cw}$, and so $F/F_0 \in \mathscr{F}_{cw}$. Then take any element $x_1 \in F/F_0$ and find a componentwise pure (and so componentwise flat) subrepresentation $F_1/F_0 \subseteq F/F_0$ such that $x_1 \in F_1/F_0$ and $|F_1/F_0| \leq \aleph$. Since $F_0, F_1/F_0 \in \mathscr{F}_{cw}$, we have $F_1 \in \mathscr{F}_{cw}$ and so $F/F_1 \in \mathscr{F}_{cw}$. Now take $x_2 \in F/F_1$ and, since $\mathscr{F}_{cw}$ is closed under direct limits, proceed by transfinite induction to find, when α is a successor ordinal, subrepresentations $F_\alpha \subseteq F$ such that $F_\alpha/F_{\alpha-1} \in \mathscr{F}_{cw}$ (and so $F_\alpha \in \mathscr{F}_{cw}$) and that $|F_\alpha/F_{\alpha-1}| \leq \aleph$. When ω is a limit ordinal, define $F_\omega = \bigcup_{\alpha < \omega} F_\alpha$. So $F_\omega \in \mathscr{F}_{cw}$ and $|F_\omega| \leq \aleph$ for every ω. Now there exists an ordinal λ such that F is a direct union of the continuous chain $\{F_\alpha \mid \alpha < \lambda\}$ where by construction $F_0, F_{\alpha+1}/F_\alpha \in \mathscr{F}_{cw}$ and $|F_0| \leq \aleph, |F_{\alpha+1}/F_\alpha| \leq \aleph$. Thus if we choose a set T of representatives of all componentwise flat representations with cardinality less than or equal to $\aleph$, then by Eklof & Trlifaj (2001, Lemma 1), we see that the pair $(\mathscr{F}_{cw}, \mathscr{F}_{cw}^\perp)$ is cogenerated by T (note that Eklof & Trlifaj (2001,

Lemma 1) is for module categories, but the same arguments of the proof carry over general Grothendieck categories).

$\square$

Theorem 4.6. *For any quiver Q, any representation of Q has an $\mathscr{F}_{cw}$-cover and an $\mathscr{F}_{cw}^{\perp}$-envelope.*

Proof. It is clear that $\mathscr{F}_{cw}$ is closed under direct sums, extensions and well ordered direct limits (as so is the class of all flat modules). Moreover, $S_v(R)(w) = \oplus_{Q(v,w)} R$ is a projective (and so flat) module for all $w \in V$. Therefore $S_v(R) \in \mathscr{F}_{cw}$. Now, apply Theorem 2.6 in Enochs et al. (2004a) with Theorem 4.5 to obtain the result. $\square$

Recall that a commutative integral domain is called *Prüfer domain* if every finitely generated ideal is projective. Over such a domain a module is flat if and only if it is torsion free (see Rotman (1979) for the details). Combining this fact with the previous result, we have that

Theorem 4.7. *Let R be a Prüfer domain. Then every representation in $(Q, R\text{-}\mathcal{M}od)$ has an $\mathscr{F}_{cw}$-cover agreeing with its $\mathcal{F}_{cw}$-cover.*

Remark 4.8. In Example 3.11, since Q does not satisfy (**B**) we cannot use Theorem 3.16 to determine whether $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-covers. However, since $R = \mathbb{Z}$ is a Prüfer domain, $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-covers by Theorem 4.7.

5 Examples of comparing categorical flat covers with $\mathscr{F}_{cw}$-covers

In this section, we will provide some examples on the different kinds of covers studied throughout the paper.

The *categorical* flat representations are characterized (for rooted quivers) in Enochs et al. (2004b, Theorem 3.7) as follows: a representation F of a quiver Q is *flat* if and only if $F(v)$ is a flat module and the morphism $\oplus_{t(a)=v} F(s(a)) \to F(v)$ is a pure monomorphism for every vertex $v \in V$. In this case, as we pointed out at the beginning of Section 4, it is known that $(Q, R\text{-}\mathcal{M}od)$ admits categorical flat covers for any quiver Q. Moreover, we have proved in Theorem 4.6 that $(Q, R\text{-}\mathcal{M}od)$ also admits $\mathscr{F}_{cw}$-covers (i.e. componentwise flat covers). In this section, we will give some examples of categorical flat covers and of $\mathscr{F}_{cw}$-covers showing that these two kind of covers do not coincide in general.

Recall that a module C is called *cotorsion* if $\text{Ext}^1(F, C) = 0$ for any flat module F. Since every module has a flat cover (Bican et al. (2001)), every module has a cotorsion envelope by Xu (1996, Theorem 3.4.6).

Example 5.1. Let Q be the quiver $\bullet \to \bullet$. Let us take any module M and the flat cover $\varphi : F \to M$ of it. Then:

(1)

$$
\begin{array}{ccc}
0 & \longrightarrow & F \\
\downarrow & & \downarrow {\scriptstyle \varphi} \\
0 & \longrightarrow & M
\end{array}
$$

is a flat cover of the representation $0 \to M$. To see this, let

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\ \alpha\ } & F_2 \\
\downarrow & & \downarrow {\scriptstyle \beta} \\
0 & \longrightarrow & M
\end{array}
$$

be a morphism, where α is a pure monomorphism and F_1, F_2 are flat modules. Then F_2/F_1 is also a flat module. Since φ is a flat cover, there exists $\delta : F_2 \to F$ such that $\varphi\delta = \beta$. It is clear that $\varphi\delta\alpha = \beta\alpha = 0$, and so there exists a unique $h : F_1 \to \operatorname{Ker}\varphi$ such that $\delta\alpha = ih$, where $i : \operatorname{Ker}\varphi \to F$. From the short exact sequence $0 \to F_1 \to F_2 \to F_2/F_1 \to 0$, we obtain

$$\operatorname{Hom}(F_2, \operatorname{Ker}\varphi) \to \operatorname{Hom}(F_1, \operatorname{Ker}\varphi) \to 0,$$

since $\operatorname{Ext}^1(F_2/F_1, \operatorname{Ker}\varphi) = 0$ by Wakamutsu's lemma (see Xu (1996, Lemma 2.1.1)). So there is $z : F_2 \to \operatorname{Ker}\varphi$ such that $z\alpha = h$. Now if we consider $\delta - z : F_2 \to F$, then clearly $\varphi(\delta - z) = \beta$ and $(\delta - z)\alpha = 0$.

(2) If we take the flat cover $f : G \to \operatorname{Ker}\varphi$ of $\operatorname{Ker}\varphi$ then

$$
\begin{array}{ccc}
G & \xrightarrow{\ t\ } & F \\
\downarrow & & \downarrow {\scriptstyle \varphi} \\
0 & \longrightarrow & M
\end{array}
$$

is an $\mathscr{F}_{cw}$-cover of the representation $0 \to M$. In fact, if

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\ \alpha\ } & F_2 \\
\downarrow & & \downarrow {\scriptstyle \beta} \\
0 & \longrightarrow & M
\end{array}
$$

is a morphism where $F_1 \to F_2 \in \mathscr{F}_{cw}$, then clearly there exists $h : F_2 \to F$ such that $\varphi h = \beta$, since φ is a flat cover. Since $\varphi h\alpha = \beta\alpha = 0$, the map $h\alpha : F_1 \to \operatorname{Ker}\varphi$ is defined. Then there exists $h' : F_1 \to G$ such that $fh' = h\alpha$ since f is a flat cover, and so $h\alpha = th'$. This shows

that $\{0, \varphi\}$ is an $\mathscr{F}_{cw}$-precover. To see that it is a cover, suppose there is an endomorphism

$$
\begin{array}{ccc}
G & \xrightarrow{\ t\ } & F \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\
G & \xrightarrow{\ t\ } & F
\end{array}
$$

such that $0g = 0$ and $\varphi g' = \varphi$. Then clearly g' is an automorphism since φ is a flat cover. Now we show that g is also an automorphism. Since $\varphi g' i = 0$, there exists $\psi : \operatorname{Ker}\varphi \to \operatorname{Ker}\varphi$ where $i : \operatorname{Ker}\varphi \to F$. Actually, ψ is an automorphism (see the comment of g being an automorphism in Example 3.17), and so from the commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\ f\ } & \operatorname{Ker}\varphi \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
G & \xrightarrow{\ f\ } & \operatorname{Ker}\varphi
\end{array}
$$

we obtain that g is also an automorphism (by using the fact that f is a cover).

Remark 5.2. Note that in the previous example $0 \to F$ cannot be an $\mathscr{F}_{cw}$-precover of $0 \to M$. Because by (2), $G \to F$ is an $\mathscr{F}_{cw}$-cover of $0 \to M$ with $G \neq 0$ and it is known that covers are direct summand of precovers (see Xu (1996, Theorem 1.2.7)). So if $0 \to F$ were an $\mathscr{F}_{cw}$-precover of $0 \to M$, then we would have

$$(0 \to F) = (G \to F) \oplus (H_1 \to H_2) = (G \oplus H_1 \to F \oplus H_2)$$

for some representation $H_1 \to H_2$ of Q. This implies $0 = G \oplus H_1$ which contradicts the fact that $G \neq 0$.

Remark 5.3. Comparing with Example 3.17; $\operatorname{Ker}\varphi \to F$ is a torsion free cover but not an $\mathscr{F}_{cw}$-cover of $0 \to M$ (unless $\operatorname{Ker}\varphi$ is a flat module). Since the class of torsion free modules is closed under submodules, but the class of flat modules is not.

Example 5.4. Let Q be the quiver $\bullet \to \bullet$. Let us take any module M and the flat cover $\varphi : F \to M$ of it. Then,

$$
\begin{array}{ccc}
F & \xrightarrow{\ id\ } & F \\
{\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\
M & \xrightarrow{\ id\ } & M
\end{array}
$$

is both a (categorical) flat cover and an $\mathscr{F}_{cw}$-cover of the representation $M \xrightarrow{\;id\;} M$. In fact, if there is a morphism

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\;h\;} & F_2 \\
{\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle \psi_2} \\
M & \xrightarrow{\;id\;} & M
\end{array}
$$

where F_1, F_2 are flat modules and h is a pure monomorphism, then clearly there is $f : F_2 \to F$ such that $\varphi f = \psi_2$ (since φ is a flat cover). Taking $fh : F_1 \to F$, we see that $\varphi fh = \psi_2 h = \psi_1$. This means that $F \xrightarrow{\;id\;} F$ is a flat precover, and clearly it is a flat cover (since id_F is a pure monomorphism). Since we have not used the fact that h is pure, then $F \xrightarrow{\;id\;} F$ is also a $\mathscr{F}_{cw}$-cover of $M \xrightarrow{\;id\;} M$.

Example 5.5. Let Q be the quiver $\bullet \to \bullet \to \bullet$ and let M be a module. Let us take the flat cover $\varphi : F \to M$ of M. Then:

(1) If we take the cotorsion envelope $i : F \to C$ of F, then C will be a flat module by Xu (1996, Theorem 3.4.2)). Therefore, we have a (categorical) flat representation $\overline{F} \equiv F \xrightarrow{\;i\;} C \xrightarrow{\;k_1\;} C \times F$ where k_1 is canonical inclusion (since k_1 and i are pure monomorphisms). We show that

$$
\begin{array}{ccc}
\overline{F} & &
\end{array}
\qquad
\begin{array}{ccccc}
F & \xrightarrow{\;i\;} & C & \xrightarrow{\;k_1\;} & C \times F \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi p_2} \\
M & \longrightarrow & 0 & \longrightarrow & M
\end{array}
$$

with $\overline{\varphi} \downarrow$ and $\overline{M}$ on the left.

is a flat cover of the representation $\overline{M}$ of Q, where $p_2 : C \times F \to F$ is a projection. In fact, if there is a morphism

$$
\begin{array}{ccccc}
F_1 & \xrightarrow{\;\alpha\;} & F_2 & \xrightarrow{\;\beta\;} & F_3 \\
{\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle t_3} \\
M & \longrightarrow & 0 & \longrightarrow & M
\end{array}
$$

with F_1, F_2, F_3 are flat modules and α, β are pure monomorphisms, then clearly there exists $f : F_1 \to F$ such that $\varphi f = t_1$ since F is a flat cover of M. From the short exact sequence $0 \to F_1 \to F_2 \to F_2/F_1 \to 0$, we obtain that

$$
\mathrm{Hom}(F_2, C) \to \mathrm{Hom}(F_1, C) \to 0
$$

is exact, since $\mathrm{Ext}^1(F_2/F_1, C) = 0$ (as α is a pure monomorphism and C is cotorsion). So there exists $g : F_2 \to C$ such that $g\alpha = if$. Now, since F is a flat cover of M, there exists $\tau_2 : F_3 \to F$ such that $\varphi\tau_2 = t_3$. Moreover, from the short exact sequence

$$0 \to F_2 \to F_3 \to F_3/F_2 \to 0 \tag{5.1}$$

we obtain that

$$\mathrm{Hom}(F_3, C) \to \mathrm{Hom}(F_2, C) \to 0$$

is exact. Then there exists $\tau_1 : F_3 \to C$ such that $\tau_1\beta = g$. Since $\varphi\tau_2\beta = t_3\beta = 0$, there exists a unique $\gamma : F_2 \to \mathrm{Ker}\,\varphi$ such that $\gamma = \tau_2\beta$. Similarly, if we take $\mathrm{Ker}\,\varphi$ instead of C, by (5.1), there exists $z : F_3 \to \mathrm{Ker}\,\varphi$ such that $z\beta = \gamma$. Therefore, by defining $h : F_3 \to C \times F$ such that $h(x) = \big(\tau_1(x), (\tau_2 - z)(x)\big)$ for all $x \in F_3$, we see that $\varphi p_2 h = \varphi(\tau_2 - z) = t_3$, and moreover $h\beta = \big(\tau_1\beta, (\tau_2 - z)\beta\big) = (g, 0) = k_1 g$. Thus $\overline{\varphi} : \overline{F} \to \overline{M}$ is a flat precover. To see that it is a cover, suppose $s = \{f, g, h\} : \overline{F} \to \overline{F}$ is an endomorphism such that $\overline{\varphi} \circ s = \overline{\varphi}$. It is clear that f and g are automorphisms. For $h : C \times F \to C \times F$, we set $h_{ij} = \pi_i h e_j$ for $i, j = 1, 2$ where π_k is projection and e_k is injection. We can write h in a matrix form as

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Since $hk_1 = k_1 g$ then $h_{11} = g$ and $h_{21} = 0$, and since φ is a cover, $\varphi = \varphi h_{22}$ implies that h_{22} is an automorphism. Hence h is an automorphism and so is s, that is, $\overline{\varphi}$ is a cover.

(2) If we take the flat cover $f : G \to \mathrm{Ker}\,\varphi$ of $\mathrm{Ker}\,\varphi$, then it is immediate that

$$
\begin{array}{ccccc}
F & \xrightarrow{0} & G & \xrightarrow{t} & F \\
\downarrow{\varphi} & & \downarrow{0} & & \downarrow{\varphi} \\
M & \xrightarrow{0} & 0 & \xrightarrow{0} & M
\end{array}
$$

is an $\mathscr{F}_{cw}$-cover of the representation $M \to 0 \to M$.

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