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Traveling Wave Solutions of WBK Shallow Water Equations by Differential Transform Method

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Abstract

By using differential transform method (DTM) to coupled Whitham–Broer–Kaup (WBK), we find the explicit traveling wave solutions of WBK equations in the form of a convergent polynomial series. In addition two examples the special case of WBK equations namely modified Boussinesq (MB) and approximate long wave (ALW) equations are discussed in details and compared with previous solutions. The obtained results demonstrate the reliability of the algorithm and the DTM is an attractive method in solving the systems of nonlinear differential equations.

Keywords: Differential transform method (DTM), Whitham–Broer–Kaup equations, Approximate long wave equation, Boussinesq equation

1 Introduction

To describe the propagation of shallow water, many well known completely integral models are introduced, such as Boussinesq equation, KP equation, KdV equation and WBK equation. We consider the WBK equations, which have been studied by Whitham [1], Broer [2] and Kaup [3]. Eq. (1) is a model for water waves where the field of horizontal velocity is represented by $u = u(x,t)$ and $v = v(x,t)$ is the height that deviate from equilibrium position of liquid, and α, β are constants, which are represented in different diffusion powers [4].

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$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\beta\frac{\partial^2 u}{\partial x^2}=0, \tag{1}$$
$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(uv)+\alpha\frac{\partial^3 u}{\partial x^3}-\beta\frac{\partial^2 v}{\partial x^2}=0,$$

The exact solutions of $u=u(x,t)$ and $v=v(x,t)$ are given by [5]

$$\begin{aligned} u(x,t) &= \lambda-2k(\alpha+\beta^2)^{0.5}\coth\left(k(x-\lambda t+x_0)\right), \\ v(x,t) &= -2k^2\left(\alpha+\beta^2+\beta(\alpha+\beta^2)^{0.5}\right)\operatorname{csch}^2\left(k(x-\lambda t+x_0)\right), \end{aligned} \tag{2}$$

where λ, k and x_0 are arbitrary constants. Above system is a very good model to describe dispersive waves. If $\alpha=1$ and $\beta=0$, then the system represents the modified Boussinesq (MB) equations [5]. If $\alpha=0$ and $\beta\neq 0$, then the system represents the classical long wave equations that describe shallow water wave with dispersion [4]. Ablowitz [6] studied inverse transformation solution for the special case of the WBK. Xie et al. [5] applied the hyperbolic function method to the WBK equations and found some new solitary wave solutions. El-Sayed and Kaya [7] by the Adomian decomposition method (ADM), Rafei and Daniali [8] by the variational iteration method (VIM), Rashidi et al. by the homotopy perturbation method (HPM) [9] and Homotopy Analysis Method (HAM) [10] obtained explicit traveling wave solutions of the Whitham–Broer–Kaup equations.

Most scientific problems and phenomena are modeled by nonlinear ordinary or partial differential equations. Some of them are solved using numerical methods and some are solved using analytic methods of perturbation [11]. Although with the advancement of the symbolic computation software such as MATHEMATICA, MAPLE and so on approximate analytic methods for nonlinear problems have been adopted by many researchers. Among these are the HPM [12, 13], homotopy analysis method (HAM) [14, 15, 16] and the DTM [17]. The concept of DTM was first introduced by Zhou [17] in 1986 and it was used to solve both linear and nonlinear initial value problems in electric circuit analysis. The DTM is a semi analytical-numerical technique that formulizes Taylor series in a very different manner. In this method, we applied certain transformation rules hence the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations and the solution of these algebraic equations gives the desired solution of the problem. Chen and Ho [18] developed this method for partial differential equations and obtained closed form series solutions for linear and nonlinear initial value problems and Ayaz [19] applied it to the system of differential equations. Rashidi and Erfani [20] used the DTM to solve Burgers' and nonlinear heat transfer equations and compared the DTM with the HAM.

2 Basic idea of differential transform method

Consider a function of two variable $w(x,y)$ be analytic in the domain Ω and let $(x,y)=(x_0,y_0)$ in this domain. The function $w(x,y)$ is then represented by one series whose centre at located at $w(x_0, y_0)$. The differential transform of the function is the form

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)}, \tag{3}$$

where $w(x,y)$ is the original function and $W(k,h)$ is the transformed function. Then its inverse transform is defined as

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)(x-x_0)^k (y-y_0)^h. \tag{4}$$

The relations Eq. (3) and Eq. (4) imply that

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)} (x-x_0)^k (y-y_0)^h, \tag{5}$$

In a real application and when (x_0, y_0) are taken as $(0,0)$, then the function $w(x,y)$ is expressed by a finite series and Eq. (4) can be written as

$$w(x,y) \cong \sum_{k=0}^{m}\sum_{h=0}^{n} W(k,h) x^k y^h. \tag{6}$$

Table 1
The operations for the two-dimensional differential transform method**.**

Original function	Transformed function
$w(x,y) = u(x,y) \pm v(x,y)$	$W(k,h) = U(k,h) \pm V(k,h)$
$w(x,y) = \lambda u(x,y)$	$W(k,h) = \lambda U(k,h)$, (λ is a constant)
$w(x,y) = \frac{\partial u(x,y)}{\partial x}$	$W(k,h) = (k+1)U(k+1,h)$
$w(x,y) = \frac{\partial^{r+s} u(x,y)}{\partial x^r \partial y^s}$	$W(k,h) = (k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)U(k+r,h+s)$
$w(x,y) = \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial y}$	$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(h-s+1)U(k-r+1,s)V(r,h-s+1)$
$w(x,y) = u(x,y)\frac{\partial v(x,y)}{\partial x}$	$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,s)V(k-r+1,h-s)$

3 Application

Consider the Whitham–Broer–Kaup (WBK) equations Eq. (1), with the initial conditions [5]

$$\begin{aligned} u(x,0) &= \lambda - 2k(\alpha+\beta^2)^{0.5}\coth\left(k(x+x_0)\right), \\ v(x,0) &= -2k^2\left(\alpha+\beta^2+\beta(\alpha+\beta^2)^{0.5}\right)\operatorname{csch}^2\left(k(x+x_0)\right). \end{aligned} \tag{7}$$

Taking the two-dimensional transform of Eq. (1) by using the related definitions in Table1, we have

$$
\begin{aligned}
&(j+1)U(i,j+1)+\sum_{r=0}^{i}\sum_{s=0}^{j}(i-r+1)U(r,s)U(i-r+1,j-s)+(i+1)V(i+1,j)\\
&\qquad +\beta(i+1)(i+2)U(i+2,j)=0,\\
&(j+1)V(i,j+1)+\sum_{r=0}^{i}\sum_{s=0}^{j}(i-r+1)U(r,s)V(i-r+1,j-s)\\
&\qquad +\sum_{r=0}^{i}\sum_{s=0}^{j}(i-r+1)V(r,s)U(i-r+1,j-s)\\
&\qquad +\alpha(i+1)(i+2)(i+3)U(i+3,j)-\beta(i+1)(i+2)V(i+2,j)=0.
\end{aligned}
\tag{8}
$$

By applying the initial conditions Eq. (7) into Eq. (4), the initial transformation coefficients are thus determined by

$$
\begin{aligned}
&\sum_{r=0}^{\infty}U(k,0)x^r=\lambda-2k\sqrt{\alpha+\beta^2}\coth\left(k\,x_0\right)-2k^2\sqrt{\alpha+\beta^2}\operatorname{csch}^2\left(k\,x_0\right)x\\
&\qquad -2k^3\sqrt{\alpha+\beta^2}\coth\left(k\,x_0\right)\operatorname{csch}^2\left(k\,x_0\right)x^2+\cdots,\\
&\sum_{r=0}^{\infty}V(k,0)x^r=-2k^2\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\operatorname{csch}^2\left(k\,x_0\right)\\
&\qquad +4k^3\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\coth\left(k\,x_0\right)\operatorname{csch}^2\left(k\,x_0\right)x\\
&\qquad -2k^4\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\left(2+\cosh\left(2k\,x_0\right)\right)\operatorname{csch}^4\left(k\,x_0\right)x^2.
\end{aligned}
\tag{9}
$$

Hence from Eq. (9)

$$
\begin{aligned}
&U(0,0)=\lambda-2k\sqrt{\alpha+\beta^2}\coth\left(k\,x_0\right),\ U(1,0)=-2k^2\sqrt{\alpha+\beta^2}\operatorname{csch}^2\left(k\,x_0\right),\\
&U(2,0)=-2k^3\sqrt{\alpha+\beta^2}\coth\left(k\,x_0\right)\operatorname{csch}^2\left(k\,x_0\right),\\
&V(0,0)=-2k^2\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\operatorname{csch}^2\left(k\,x_0\right),\\
&V(1,0)=+4k^3\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\coth\left(k\,x_0\right)\operatorname{csch}^2\left(k\,x_0\right),\\
&V(2,0)=-2k^4\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\left(2+\cosh\left(2k\,x_0\right)\right)\operatorname{csch}^4\left(k\,x_0\right).
\end{aligned}
\tag{10}
$$

Substituting Eq. (10) in Eq. (8), and by recursive method we can calculate another values of $U(k,h)$ and $V(k,h)$. Hence, substituting all $U(k,h)$ and $V(k,h)$ into Eq. (6), we have series solution as below

$$
\begin{aligned}
&u(x,t) = U(0,0) + U(1,0)x + U(0,1)t + U(1,1)\,xt + \cdots U(m,n)x^m t^n \;\Rightarrow \\
&u(x,t) = \lambda - 2k\sqrt{\alpha+\beta^2}\coth\left(k\,x_0\right) + 2k^2\sqrt{\alpha+\beta^2}\operatorname{csch}^2\left(k\,x_0\right)x - \\
&\qquad 2k^2\lambda\sqrt{\alpha+\beta^2}\operatorname{csch}^2\left(k\,x_0\right)^2 t \\
&\qquad + 4k^3\lambda\sqrt{\alpha+\beta^2}\coth\left(k\,x_0\right)\operatorname{csch}^2\left(k\,x_0\right)xt + \cdots,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&v(x,t) = V(0,0) + V(1,0)\,x + V(0,1)\,t + V(1,1)\,xt + \cdots V(m,n)x^m t^n \;\Rightarrow \\
&v(x,t) = -2k^2\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\operatorname{csch}^2\left(k\,x_0\right) + \\
&\qquad 4k^3\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\coth\left(k\,x_0\right)\operatorname{csch}^2\left(k\,x_0\right)x + \\
&\qquad 4k^4\lambda\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)\left(2+\cosh\left(2k\,x_0\right)\right)\operatorname{csch}^4\left(k\,x_0\right)xt + \cdots.
\end{aligned} \tag{12}
$$

Our approximation has one more interesting property, if we expand exact solutions Eq. (2) using Taylor's expansion about $(0,0)$, we have the series same as the our approximation Eq. (11) and Eq. (12).

4. Numerical experiments and discussion

In this section, we obtain numerical solutions of the WBK equations. In order to verify the efficiency of the proposed method in comparison with the ADM, VIM, HPM and HAM, we report the absolute errors for the DTM $(m = 10, n = 10)$, the DTM $(m = 20, n = 20)$, ADM [7] (5-term approximate solution), VIM [8] (4-term approximate solution), HPM [9] (2-term approximate solution) and HAM [10] (8-term approximate solution), in the following cases: "*Case 1.* The WBK equations Eq. (1), for α=1.5 and β=1.5, in Table 2; *Case 2.* The modified Boussinesq (MB) equations [5], reduced of the WBK equations for α=1 and β=0, in Table 3; *Case 3.* The approximate long wave (ALW) equations in shallow water [4], reduced of the WBK equations for α=0 and β=0.5, in Table 4". The results clearly show that even the DTM $(10,10)$ is the most accurate method of all the others method. Note that the DTM is easier to calculate than HAM, ADM, VIM and HPM because in the DTM we have iterative procedure where do not need to solve any differential equations or integrate equations. In the other methods we must in each iterate solve differential equations or integrate equations. In Fig. 1 we show the results obtained by DTM $(15,15)$, in comparison with the exact solutions Eq. (2), for various parameter λ when $k = 0.1$, $x_0 = 12$, $\alpha = 1.5$ and $\beta = 1.5$ (WBK equation). From Fig. 1, it can be concluded that our results are good agreement with exact solutions Eq. (2). It is also evident that when more terms for the DTM are computed the numerical results get much closer to the exact solutions.

4. Conclusions

In this paper, the DTM has been applied to the coupled Whitham–Broer–Kaup problem. The results for three numerical examples in Tables 2-4 showed the validity and accuracy of this procedure. From Tables 2-4 it is obvious that the DTM is the most accurate method of all the others method.

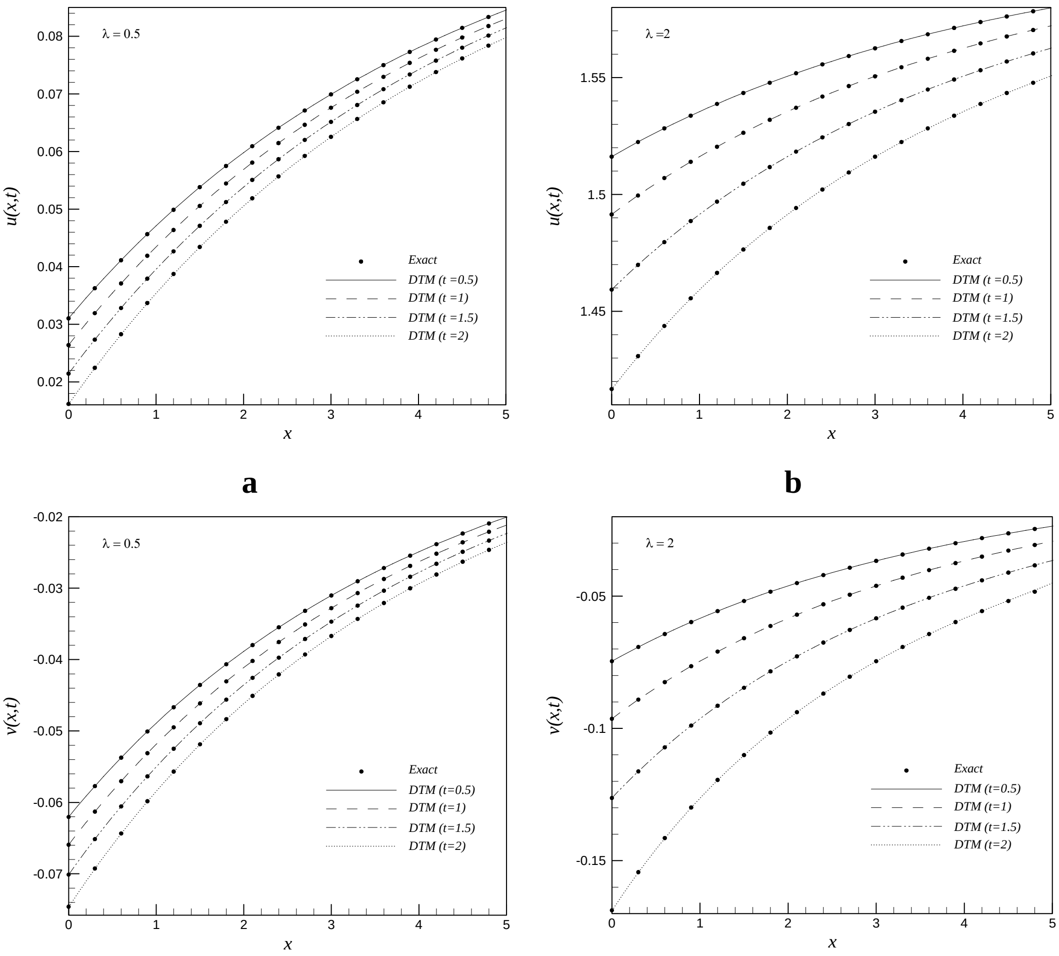

Fig. 1. The results obtained by DTM $(15,15)$, in comparison with the exact solutions for various parameter λ when $k = 0.1, x_0 = 12, \alpha = 1.5$ and β=1.5. (a), (c) $\lambda = 0.5$; (b),(d) $\lambda = 2$.

Table 2
The absolute errors for the case of $k = 0.1, \lambda = 0.005, x_0 = 10, \alpha = 1.5$ and β=1.5.

x	t	DTM(10,10)	DTM(20,20)	HAM [10]	VIM [8]	ADM [7]	HPM [9]
				$\lvert u_{Exact} - u_{Approximat} \rvert$			
0.1	0.1	1.1102E−16	1.1102E−16	8.4109E−08	1.2303E−04	1.0489E–04	8.9190E−10
	0.5	1.1102E−16	1.1102E−16	2.4944E−08	6.1687E−04	8.8831E–05	2.2302E–08
0.5	0.1	1.7208E−15	5.5511E−17	8.4345 E−08	1.1094E−04	2.7952E–03	7.8779E−10
	0.5	1.8319E−15	0.0000	2.2294 E−07	5.5623E−04	2.3618E–03	1.9699E–08
				$\lvert v_{Exact} - v_{Approximat} \rvert$			
0.1	0.1	1.3878 E−17	1.3878 E−17	2.6499E−06	1.1043E−04	6.4142E–03	9.6607E−10
	0.5	4.1633 E−17	4.1633 E−17	1.2946E−05	5.5407E−04	5.6151E–03	2.4158E–08
0.5	0.1	7.3552 E−15	2.7756 E−17	2.6396E−06	9.7538E−05	3.7519E–02	8.2720E−10
	0.5	7.4524 E−15	2.7756 E−17	1.2935E−05	4.8934E−04	2 .2624E–03	2.0685E–08

Table 3
The absolute errors for the case of $k = 0.1, \lambda = 0.005, x_0 = 10, \alpha = 1$ and β=0.

x	t	DTM(10,10)	DTM(20,20)	HAM [10]	VIM [8]	ADM [7]	HPM [9]
				$\lvert u_{Exact} - u_{Approximat} \rvert$			
0.1	0.1	5.5511E−17	5.5511E−17	2.5947E−07	6.3527E−05	8.1630E−07	4.6057E−10
	0.5	0.0000	0.0000	2.5097E−06	3.1855E−04	7.1608E−07	1.1517E−08
0.5	0.1	9.9920 E−16	0.0000	2.3362E−07	5.7287E−05	2.0341E−05	4.0681E−10
	0.5	8.8818 E−16	5.5511 E−17	2.2555E−06	2.8724E−04	1.7853E−05	1.0172E−08
				$\lvert v_{Exact} - v_{Approximat} \rvert$			
0.1	0.1	5.2042 E−18	5.2042 E−18	1.5483 E−06	1.65942E−05	5.8868E−05	1.4517E−10
	0.5	8.6736 E−18	8.6736 E−18	7.7433 E−06	8.32598E−05	5.2717E−05	3.6302E−09
0.5	0.1	1.1085 E−15	3.4694 E−18	1.3716E−06	1.46569E−05	2.9916E−04	1.2430E−10
	0.5	1.1206 E−15	5.2041 E−18	6.8600E−06	7.35317E−05	2.6787E−04	3.1083E−09

Table 4
The absolute errors for the case of $k = 0.1, \lambda = 0.005, x_0 = 10, \alpha = 0$ and β=0.5.

x	t	DTM(10,10)	DTM(20,20)	HAM [10]	VIM [8]	ADM [7]	HPM [9]
				$\lvert u_{Exact} - u_{Approximat} \rvert$			
0.1	0.1	2.7756 E−17	2.7756 E−17	1.2980E−07	3.17634E−05	8.0299E−06	2.3029E−10
	0.5	0.0000	0.0000	1.2621E−06	1.59274E−04	6.7992E−06	5.7584E−09
0.5	0.1	4.9960 E−16	0.0000	1.1686E−07	2.86433E−05	2.0619E−04	2.0341E−10
	0.5	4.4409 E−16	2.7756 E−17	1.1338E−06	1.43620E−04	1.7451E−04	5.0862E−09
				$\lvert v_{Exact} - v_{Approximat} \rvert$			
0.1	0.1	2.6021 E−18	2.6021 E−18	7.7417E−07	8.29712E−06	4.8190E−04	7.2585E−11
	0.5	4.3368 E−18	4.3368 E−18	3.8722E−06	4.16299E−05	4.2222E−04	1.8151E−09
0.5	0.1	5.5424 E−16	1.7347 E−18	6.8585E−07	7.32847E−06	2.5440E−03	6.2151E−11
	0.5	5.6032 E−16	2.6020 E−18	3.4304E−06	3.67658E−05	2.2258E−03	1.5541E−09

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