ALTERNATIVE POLARIZATIONS OF BOREL FIXED IDEALS

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ABSTRACT. Let I be a monomial ideal of a polynomial ring S. *Polarization* is a classical technique constructing a squarefree monomial ideal $\mathsf{pol}(I) \subset \widetilde{S}$ which has (for example) the same $\mathbb{Z}$-graded Betti numbers as I. We show that if I is Borel fixed, it has an alternative polarization. While the standard polarization sends the monomial x^2yz^2 to $x_1x_2y_1z_1z_2$, our version sends it to $x_1x_2y_3z_4z_5$.

1. PREPARATION

We introduce the convention and notation used throughout the paper. Let $S = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. The i^{th} coordinate of $\mathbf{a} \in \mathbb{N}^n$ is denote by a_i (i.e., we change the font). For $\mathbf{a} \in \mathbb{N}^n$, $x^{\mathbf{a}}$ denotes the monomial $\prod_{i=1}^n x_i^{a_i} \in S$. For a monomial $\mathsf{m} := x^{\mathbf{a}}$, set $\deg(\mathsf{m}) := \sum_{i=1}^n a_i$ and $\deg_i(\mathsf{m}) := a_i$. For a monomial ideal $I \subset S$, $G(I)$ denotes the set of minimal (monomial) generators of I. We define the order $\succeq$ on $\mathbb{N}^n$ so that $\mathbf{a} \succeq \mathbf{b}$ if $a_i \geq b_i$ for all i. We refer [1, 2] for unexplained terminology.

Take $\mathbf{d} \in \mathbb{N}^n$ with $d_i \geq 1$ for all i, and set

$$\widetilde{S} := \Bbbk[\, x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d_i \,].$$

Note that

$$\Theta := \{x_{i,1} - x_{i,j} \mid 1 \leq i \leq n,\ 2 \leq j \leq d_i\,\} \subset \widetilde{S}$$

forms a regular sequence with $\widetilde{S}/(\Theta) \cong S$. (Throughout this paper, $\widetilde{S}$ and Θ are used in this meaning, while the value of $\mathbf{d} \in \mathbb{N}^n$ depends on the context.)

Definition 1.1. For a monomial ideal $I \subset S$, a *polarization* of I is a squarefree monomial ideal $J \subset \widetilde{S}$ satisfying the following conditions:

(i) $\widetilde{S}/J \otimes_{\widetilde{S}} \widetilde{S}/(\Theta) \cong S/I$,

(ii) Θ forms a $\widetilde{S}/J$-regular sequence.

The following is a well-known fact, and the proof is found in [6, Lemma 6.9].

Lemma 1.2 (c.f. [6, Lemma 6.9])**.** *Let I and J be monomial ideals of S and $\widetilde{S}$ respectively. Assume that the condition (i) of Definition 1.1 is satisfied. Then the condition (ii) is equivalent to the following.*

(ii') $\beta^{\widetilde{S}}_{i,j}(J) = \beta^S_{i,j}(I)$ *for all* i, j.

While the proof in [6] concerns only the case $\#\Theta = 1$, it works in the general case. If Θ does *not* form a $\widetilde{S}/J$-regular sequence, the relation between $\beta^{\widetilde{S}}_{i,j}(J)$ and $\beta^S_{i,j}(I)$ is not simple. So it is better to compare the Hilbert series of $\widetilde{S}/J$ with that of S/I (recall that the Hilbert series is determined by Betti numbers.)

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For a monomial $x^{\mathbf{a}}$ with $\mathbf{a} \preceq \mathbf{d}$, set

$$\mathsf{pol}(x^{\mathbf{a}}) := \prod_{1 \le i \le n} x_{i,1} x_{i,2} \cdots x_{i,a_i} \in \widetilde{S}.$$

Let $I = (x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_r}) \subset S$ be a monomial ideal with $\mathbf{a}_i \preceq \mathbf{d}$ for all i. Then it is well-known that

$$\mathsf{pol}(I) = (\, \mathsf{pol}(x^{\mathbf{a}_1}), \dots, \mathsf{pol}(x^{\mathbf{a}_r}) \,)$$

gives a polarization of I, which is called the *standard polarization* of I. (If the reader is nervous about the choice of $\mathbf{d}$, take it so that $x^{\mathbf{d}}$ is the least common multiple of the minimal generators of I.) While all monomial ideals have the standard polarizations, some have alternative ones.

Let d be a positive integer, and set

$$\widetilde{S} := \Bbbk[\, x_{i,j} \mid 1 \le i \le n,\ 1 \le j \le d \,]. \tag{1.1}$$

For a monomial $x^{\mathbf{a}} \in S$ with $\deg(x^{\mathbf{a}}) \le d$, set $b_i := \sum_{j=1}^{i} a_j$ for each $i \ge 0$ (here $b_0 = 0$), and

$$\mathsf{b\text{-}pol}(x^{\mathbf{a}}) := \prod_{\substack{1 \le i \le n \\ b_{i-1}+1 \le j \le b_i}} x_{i,j} \in \widetilde{S}.$$

If $a_i = 0$ then $b_{i-1} = b_i$ and $x_{i,j}$ does not divide $\mathsf{b\text{-}pol}(x^{\mathbf{a}})$ for all j. Let $I \subset S$ be a monomial ideal with $\deg(\mathsf{m}) \le d$ for all $\mathsf{m} \in G(I)$. Set

$$\mathsf{b\text{-}pol}(I) := (\, \mathsf{b\text{-}pol}(\mathsf{m}) \mid \mathsf{m} \in G(I) \,) \subset \widetilde{S}.$$

This ideal sometimes gives a polarization of I, and sometimes not. Note that the condition (i) of Definition 1.1 is always satisfied, and the problem is the condition (ii).

In the sequel, when we treat $\mathsf{b\text{-}pol}(I)$, we assume that $\widetilde{S}$ is the one in (1.1) and $\deg(\mathsf{m}) \le d$ for all $\mathsf{m} \in G(I)$.

Example 1.3. (1) For $I = (x^2, xy, xz, y^2, yz) \subset \Bbbk[x, y, z]$, we have

$$\mathsf{b\text{-}pol}(I) = (x_1x_2, x_1y_2, x_1z_2, y_1y_2, y_1z_2),$$

and it gives a polarization. In fact, since I is Borel fixed, we can use Theorem 2.3 below. Moreover, it is essentially different to the standard polarization

$$\mathsf{pol}(I) = (x_1x_2, x_1y_1, x_1z_1, y_1y_2, y_1z_1).$$

More precisely, $\mathsf{b\text{-}pol}(I)$ and $\mathsf{pol}(I)$ are different even after permutations of variables. In fact, if $\mathsf{m} \in G(\mathsf{pol}(I))$ is divided by y_i for some i, then y_1 divides m. However, y_1 does not divide $x_1y_2 \in G(\mathsf{b\text{-}pol}(I))$ and y_2 does not divide $y_1z_2 \in G(\mathsf{b\text{-}pol}(I))$. Of course, there are many similar examples (in the class of Borel fixed ideals).

(2) In general, $\mathsf{b\text{-}pol}(I)$ does not give a polarization. For example, if $I = (x^2y, xy^2, xz^2, y^2z)$, then $\mathsf{b\text{-}pol}(I) = (x_1x_2y_3, x_1y_2y_3, x_1z_2z_3, y_1y_2z_3)$ and it is not a polarization (to see this compare the Betti numbers of $\mathsf{b\text{-}pol}(I)$ with that of I using *Macaulay2*).

The polarization of the form $\mathsf{b\text{-}pol}(I)$ first appeared in Nagel and Reiner [6, Corollary 2.21]. Inspired by this, Lohne [4] undertakes a study of all possible polarizations of certain monomial ideals. He calls $\mathsf{b\text{-}pol}(I)$ the *box polarization*, since the ideals treated in [6] are constructed from combinatorial data described by several "boxes" (something like Young diagrams). While the name "box" is no longer natural in our case, we use the symbol $\mathsf{b\text{-}pol}$.

Definition 1.4. We say a polarization J of I is *faithful*, if Θ forms an $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$-regular sequence for all i.

If a polarization J of I is faithful, then we have

$$\operatorname{Ext}^i_S(S/I, S) \cong \operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}) \otimes_{\widetilde{S}} \widetilde{S}/(\Theta).$$

In fact, the long exact sequences of $\operatorname{Ext}^\bullet_{\widetilde{S}}(-, \widetilde{S})$ yield

$$\operatorname{Ext}^{i+(\#\Theta)}_S(\widetilde{S}/(J + (\Theta)), \widetilde{S}) \cong \operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}) \otimes_{\widetilde{S}} \widetilde{S}/(\Theta).$$

Since $\Theta \subset \widetilde{S}$ forms a regular sequence and $\widetilde{S}/(J + (\Theta)) \cong S/I$, we have

$$\operatorname{Ext}^{i+(\#\Theta)}_S(\widetilde{S}/(J + (\Theta)), \widetilde{S}) \cong \operatorname{Ext}^i_S(S/I, S).$$

Hence, if J is faithful, $\operatorname{Ext}^i_S(S/I, S)$ and $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$ have the same degree and Betti numbers. So S/I and $\widetilde{S}/J$ have the same arithmetic degree in this case.

Remark 1.5. For any I, the standard polarization is always faithful by [7, Corollary 4.10] (see also [9, Theorem 4.4]). If S/I is Cohen-Macaulay, then any polarization J of I is faithful. In fact, $\widetilde{S}/J$ is also Cohen-Macaulay in this case, and the only non-vanishing Ext-module is $\operatorname{Ext}^c_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S})$ (here, $c := \operatorname{codim} J$), which is the canonical module of $\widetilde{S}/J$. Hence the assertion is immediate from the well-known fact on canonical modules.

Example 1.6. For the ideal $I := (x^2y, x^2z, xyz, xz^2, y^3, y^2z, yz^2)$ of $S := \Bbbk[x, y, z]$, $J := \mathsf{b\text{-}pol}(I) \subset \widetilde{S}$ gives a polarization (to see this, compute the Betti numbers). However, $\deg \operatorname{Ext}^3_S(S/I, S) = 5$ and $\deg \operatorname{Ext}^3_{\widetilde{S}}(\widetilde{S}/J, \widetilde{S}) = 4$. Hence J is not faithful.

Let M be a finitely generated S-module. We say M is *sequentially Cohen-Macaulay* if $\operatorname{Ext}^{n-i}_S(M, S)$ is either a Cohen-Macaulay module of dimension i or the 0 module for all i. The original definition is given by the existence of a certain filtration (see [8, III, Definition 2.9]), however it is equivalent to the above one by [8, III, Theorem 2.11].

Lemma 1.7. *Let M be a sequentially Cohen-Macaulay S-module, and $y \in S$ a non-zero divisor of M. Then M/yM is a sequentially Cohen-Macaulay module with $\operatorname{Ext}^{i+1}_S(M/yM, S) \cong \operatorname{Ext}^i_S(M, S)/y \cdot \operatorname{Ext}^i_S(M, S)$ for all i. Moreover, we have*

$$\operatorname{Ass}(M/yM) = \{\, \mathfrak{p} \mid \mathfrak{p} \textit{ is a minimal prime of } \mathfrak{p}' + (y) \textit{ for some } \mathfrak{p}' \in \operatorname{Ass}(M) \,\}.$$

To prove this lemma, recall the following basic properties of a finitely generated module N over S (c.f. [1, Theorem 8.1.1]).

(1) $\dim_S \operatorname{Ext}^i_S(N, S) \leq n - i$ for all i.

(2) For a prime ideal $\mathfrak{p} \subset S$ of codimension c, $\mathfrak{p} \in \operatorname{Ass}(N)$ if and only if $\mathfrak{p}$ is an associated (equivalently, minimal) prime of $\operatorname{Ext}^c_S(N, S)$.

Proof. Since y is a non-zero divisor of $\operatorname{Ext}^i_S(M, S)$ for all i by the above remark, the first assertion is clear. To see the second assertion, let $\mathfrak{p} \subset S$ be a prime ideal of codimension c. Then we have;

$$\begin{aligned}
\mathfrak{p} \in \operatorname{Ass}(M/yM) &\iff \mathfrak{p}S_\mathfrak{p} \in \operatorname{Ass}_{S_\mathfrak{p}}(\operatorname{Ext}^c_S(M/yM, S) \otimes_S S_\mathfrak{p}) \\
&\iff \dim_{S_\mathfrak{p}}(\operatorname{Ext}^{c-1}_S(M, S) \otimes_S S_\mathfrak{p}) = n - c + 1 \text{ and } y \in \mathfrak{p} \\
&\iff \exists\, \mathfrak{p}' \in \operatorname{Ass}(\operatorname{Ext}^{c-1}_S(M, S)) \text{ with } \operatorname{codim} \mathfrak{p}' = c - 1, \\
&\qquad \mathfrak{p}' \subset \mathfrak{p} \text{ and } y \in \mathfrak{p} \\
&\iff \exists\, \mathfrak{p}' \in \operatorname{Ass}(M) \text{ with } \operatorname{codim} \mathfrak{p}' = c - 1,\ \mathfrak{p}' \subset \mathfrak{p} \text{ and } y \in \mathfrak{p}.
\end{aligned}$$

□

Lemma 1.8. *Let J be a polarization of I. If $\widetilde{S}/J$ is sequentially Cohen-Macaulay, then so is S/I, and J is faithful.*

Proof. Follows from the first assertion of Lemma 1.7. □

Remark 1.9. Even if S/I is sequentially Cohen-Macaulay, a polarization J is not necessarily faithful. In fact, S/I of Example 1.6 is sequentially Cohen-Macaulay.

Let M be an S-module, and let

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r = M$$

be a prime filtration, that is, there is a prime ideal $\mathfrak{p}_i$ such that $M_i/M_{i-1} \cong S/\mathfrak{p}_i$ for each $1 \le i \le r$. Herzog and Popescu ([3]) call the filtration $\mathcal{F}$ is *pretty clean* if $i < j$ and $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ imply $\mathfrak{p}_i = \mathfrak{p}_j$. By [3, Theorem 4.1 and Corollary 3.4], if M admits a pretty clean filtration $\mathcal{F}$ then M is sequentially Cohen-Macaulay and $\operatorname{Ass} M = \{\, \mathfrak{p}_i \mid 1 \le i \le r \,\}$.

2. Main Results

We say a monomial ideal I is *Borel fixed*, if $\mathsf{m} \in I$, $x_i | \mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$. Of course, if $\operatorname{char}(\Bbbk) > 0$, this terminology is unnatural. However, we use it for simplicity. Borel fixed ideals play an important role in Gröbner basis theory, since they appear as the "generic initial ideals" of homogeneous ideals (c.f. [2, §15.9]).

For a monomial $\mathsf{m} \in S$, set

$$\nu(\mathsf{m}) := \max\{\, i \mid x_i \text{ divides } \mathsf{m} \,\}.$$

If I is Borel fixed, it is well know that

$$\operatorname{depth} S/I = n - \max\{\, \nu(\mathsf{m}) \mid \mathsf{m} \in G(I) \,\}$$

(c.f. [2, Corollary 15.25]).

Lemma 2.1. *If I is a Borel fixed ideal (with* $\deg(\mathsf{m}) \le d$ *for all* $\mathsf{m} \in G(I)$*), then*

$$\mathsf{b\text{-}pol}(I) = (\, \mathsf{b\text{-}pol}(\mathsf{m}) \mid \mathsf{m} \in I \textit{ with } \deg(\mathsf{m}) \le d \,).$$

Proof. Since the inclusion "$\subseteq$" is clear, it suffices to show the converse. To prove this by a contradiction, assume that there is some $\mathsf{m} \in I$ with $\deg(\mathsf{m}) \leq d$ and $\mathsf{b\text{-}pol}(\mathsf{m}) \notin \mathsf{b\text{-}pol}(I)$. Take m so that it has the smallest degree among these monomials. It is clear that $\mathsf{m} \notin G(I)$. Hence there is some i with $x_i | \mathsf{m}$ and $\mathsf{m}' := \mathsf{m}/x_i \in I$. Set $l := \nu(\mathsf{m})$. Since I is Borel fixed, we have $\mathsf{m}'' := \mathsf{m}/x_l = (x_i/x_l) \cdot \mathsf{m}' \in I$. Since $\deg(\mathsf{m}'') < \deg(\mathsf{m}) =: e$, we have $\mathsf{b\text{-}pol}(\mathsf{m}'') \in \mathsf{b\text{-}pol}(I)$. Hence $\mathsf{b\text{-}pol}(\mathsf{m}) = x_{l,e} \cdot \mathsf{b\text{-}pol}(\mathsf{m}'') \in \mathsf{b\text{-}pol}(I)$. This is a contradiction. □

As shown in [3, Proposition 5.2], the quotient S/I of a Borel fixed ideal I has a pretty clean filtration. The next result states that the same is true for $J := \mathsf{b\text{-}pol}(I)$. Moreover, since J is a radical ideal, $\widetilde{S}/J$ actually admits a clean filtration by [3, Corollary 3.5]. Hence the simplicial complex associated with J is non-pure shellable.

Theorem 2.2. *Let I be a Borel fixed ideal, and set $J := \mathsf{b\text{-}pol}(I)$. Then $\widetilde{S}/J$ has a pretty clean filtration, in particular, $\widetilde{S}/J$ is sequentially Cohen-Macaulay.*

Proof. Set $l := n - \operatorname{depth} S/I$. Then $\{\, \mathsf{m} \in G(I) \mid \nu(\mathsf{m}) = l \,\}$ is non-empty. Let m be the maximum element of this set with respect to the lexicographic order. If $\mathsf{m} = x_l$, then I (resp. J) is a prime ideal $(x_1, \dots, x_l)$ (resp. $(x_{1,1}, x_{2,1} \dots, x_{l,1})$) and there is nothing to prove. So we may assume that $\mathsf{m} \neq x_l$, and set $\mathsf{m}_1 := \mathsf{m}/x_l$. Since $\mathsf{m} \in G(I)$, we have $\mathsf{m}_1 \notin I$.

Claim 1. The ideal $I_1 := I + (\mathsf{m}_1)$ is Borel fixed.

Proof of Claim 1. It suffices to show that $x_i | \mathsf{m}_1$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m}_1 \in I$. Note that $\mathsf{m}' := x_l \cdot (x_j/x_i) \cdot \mathsf{m}_1 = (x_j/x_i) \cdot \mathsf{m} \in I$ and $\mathsf{m}' \succ \mathsf{m}$ with respect to the lexicographic order. From our choice of m, we have $\mathsf{m}' \notin G(I)$. Hence there is some k such that $x_k | \mathsf{m}'$ and $\mathsf{m}'/x_k \in I$. Since I is Borel fixed $(x_j/x_i) \cdot \mathsf{m}_1 = \mathsf{m}'/x_l = (x_k/x_l) \cdot (\mathsf{m}'/x_k) \in I$. □

If $\mathsf{m}_1 = \prod_{i=1}^{l} x_i^{a_i}$, then

$$\mathsf{n} := \mathsf{b\text{-}pol}(\mathsf{m}_1) = \prod_{\substack{1 \leq i \leq l \\ b_{i-1}+1 \leq j \leq b_i}} x_{i,j},$$

where $b_i := \sum_{j=1}^{i} a_j$ for each $i \geq 0$ (here $b_0 = 0$). Note that $b_l = \deg(\mathsf{n}) =: e$, and there is a nondecreasing sequence $\{\alpha_j\}_{1 \leq j \leq e}$ with

$$\mathsf{n} = \prod_{j=1}^{e} x_{\alpha_j, j}.$$

Claim 2. With the above notation, we have $J : \mathsf{n} = (x_{i,b_i+1} \mid 1 \leq i \leq l)$.

Proof of Claim 2. First we prove $x_{i,b_i+1} \cdot \mathsf{n} \in J$ for $1 \leq i \leq l$. Note that $x_i \cdot \mathsf{m}_1 = (x_i/x_l) \cdot \mathsf{m} \in I$. If $\nu(\mathsf{m}_1) \leq i$, then we have $x_{i,b_i+1} \cdot \mathsf{n} = \mathsf{b\text{-}pol}(x_i \cdot \mathsf{m}_1) \in J$. Hence we may assume that $\nu(\mathsf{m}_1) > i$, and we can take $k := \min\{\, j \mid a_j > 0,\, j > i \,\}$. Since $\mathsf{m}' := (x_i/x_k) \cdot \mathsf{m}_1$ is in I by Claim 1, we have $x_{i,b_i+1} \cdot \mathsf{n} = x_{k,b_i+1} \cdot \mathsf{b\text{-}pol}(\mathsf{m}') \in J$.

Next we prove $J : \mathsf{n} \subseteq (x_{i,b_i+1} \mid 1 \leq i \leq l)$. Assume that a monomial $\mathsf{n}' \in \widetilde{S}$ satisfies $\mathsf{n}' \cdot \mathsf{n} \in J$. Then there is a monomial $\mathsf{m}'' \in G(I)$ such that $\mathsf{b\text{-}pol}(\mathsf{m}'')$ divides

$\mathsf{n}' \cdot \mathsf{n}$. If $\mathsf{n}' \notin (x_{i,b_i+1} \mid 1 \le i \le l)$, then $\mathsf{b\text{-}pol}(\mathsf{m}'') \notin (x_{i,b_i+1} \mid 1 \le i \le l)$ also, and $\mathsf{b\text{-}pol}(\mathsf{m}'') = \prod x_{\beta_j,j}$ with $\alpha_j \le \beta_j$ for all $1 \le j \le e$, and $\beta_j \le l$ for all j. Moreover, since $\mathsf{n}' \cdot \mathsf{n}$ can not be divided by $x_{l,e+1}$, we have $\deg(\mathsf{m}'') \le e$. It means that if $\mathsf{m}'' = \prod x_i^{c_i}$ then $b_i \,(= \sum_{j=1}^{i} a_j) \ge \sum_{j=1}^{i} c_j$ for all i. Since I is Borel fixed, $\mathsf{m}'' \in I$ implies $\mathsf{m}_1 \in I$. This is a contradiction. □

Set $J_1 := J + (\mathsf{n})$ and $\mathfrak{p} := (x_{i,b_i+1} \mid 1 \le i \le l)$. Then $J_1/J \cong (\widetilde{S}/\mathfrak{p})(-e)$.

Note that $\mathsf{b\text{-}pol}(I_1) = J_1$. If I_1 is not a prime ideal, applying the above argument to I_1, we get a Borel fixed ideal $I_2 \,(\supset I_1)$ such that $\mathsf{b\text{-}pol}(I_2)/J_1$ satisfies the similar property to J_1/J. Repeating this procedure, we have a sequence of Borel fixed ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_t$$

of S such that $J_i := \mathsf{b\text{-}pol}(I_i)$ satisfies $J_i/J_{i-1} \cong \widetilde{S}/\mathfrak{p}_i$ up to degree shifting for all $i \ge 1$. Here $\mathfrak{p}_i \subset \widetilde{S}$ is a prime ideal of the form $(\, x_{j,c_{i,j}} \mid 1 \le j \le l_i)$ for some $l_i, c_{i,j} \in \mathbb{N}$. By the noetherian property of S, the procedure eventually terminates, that is, I_t will become a prime ideal. In this case, $J_t = \mathsf{b\text{-}pol}(I_t)$ is also a prime ideal, and we have a prime filtration

$$0 \subset J_1/J \subset J_2/J \subset \cdots \subset J_t/J \subset \widetilde{S}/J$$

This is a pretty clean filtration. In fact, $\operatorname{depth} S/I_1 \ge \operatorname{depth} S/I$ by construction. Similarly, $\operatorname{depth} S/I_j \ge \operatorname{depth} S/I_i$ holds for $j \ge i$. On the other hand, we have $\operatorname{codim} \mathfrak{p}_i = l_i = n - \operatorname{depth} S/I_i$. Hence $\operatorname{codim} \mathfrak{p}_j \le \operatorname{codim} \mathfrak{p}_i$ for $j \ge i$. □

By the proof of the above result, we see that any associated prime of J is of the form $(\, x_{i,c_i} \mid 1 \le i \le m)$ for some $m, c_i \in \mathbb{N}$.

Theorem 2.3. *If $I \subset S$ is a Borel fixed ideal, then $J := \mathsf{b\text{-}pol}(I)$ gives a polarization of I, which is faithful.*

Proof. To see that J is a polarization, it suffices to show that if a subset Θ' of Θ forms a $\widetilde{S}/J$-regular sequence then so dose $\Theta' \cup \{\, x_{i,1} - x_{i,j} \,\}$ for $x_{i,1} - x_{i,j} \in \Theta \setminus \Theta'$. Since any $\mathfrak{p} \in \operatorname{Ass} \widetilde{S}/J$ is a monomial prime ideal, $\mathfrak{p} + (\Theta'')$ is a prime ideal for all $\Theta'' \subset \Theta'$. Since $\widetilde{S}/J$ is sequentially Cohen-Macaulay, so is $\widetilde{S}/(J + (\Theta'))$ and

$$\operatorname{Ass}_S(\widetilde{S}/(J + (\Theta'))) = \{\, \mathfrak{p} + (\Theta') \mid \mathfrak{p} \in \operatorname{Ass} \widetilde{S}/J \,\}$$

by Lemma 1.7. Recall that all $\mathfrak{p} \in \operatorname{Ass}(\widetilde{S}/J)$ is of the form $(x_{k,c_k} \mid 1 \le k \le m)$. Hence $x_{i,1} - x_{i,j}$ is $\widetilde{S}/(J + (\Theta'))$-regular.

The last assertion follows from Lemma 1.8. □

Remark 2.4. S. Murai told us that Theorem 2.3 can be shown by his result [5, Proposition 1.9]. In fact, from a nondecreasing sequence $\{a_i\}_{i\in\mathbb{N}}$ of integers, he defined the operator α^a acting on the set of monomials of variables $x_1, x_2, \ldots$. If we take $a_i = (i-1) \cdot n$ for each i, this operator corresponds to our $\mathsf{b\text{-}pol}$. To see this, assign our variable $x_{i,j}$ to his $x_{(j-1)n+i}$. Since the operator α^a preserves Betti numbers by [5, Proposition 1.9], it gives a polarization by Lemma 1.2. (For general $\{a_i\}_{i\in\mathbb{N}}$, α^a has no relation to polarization. Our choice of $\{a_i\}_{i\in\mathbb{N}}$ makes this operator "polarization-like".) However, this proof does not give a pretty clean filtration (i.e., the non-pure shellability of the associated simplicial complex) of

$\widetilde{S}/\operatorname{b-pol}(I)$. Moreover, the following generalization of Theorem 2.3 can not be proved in this way.

Theorem 2.5. *Let A be a subset of $\{1, 2, \dots, n\}$. For a monomial $\mathsf{m} = x^{\mathbf{a}} \in S$, set $\mathsf{m}_A := \prod_{i \in A} x_i^{a_i}$, $\mathsf{m}_{-A} := \prod_{i \notin A} x_i^{a_i}$ and*

$$\operatorname{b-pol}_A(\mathsf{m}) := \operatorname{b-pol}(\mathsf{m}_A) \cdot \operatorname{pol}(\mathsf{m}_{-A}) \in \widetilde{S}$$

(we set $\widetilde{S} := \Bbbk[\, x_{i,j} \mid 1 \le i \le n, 1 \le j \le d]$, where $d := \max\{\deg(\mathsf{m}) \mid \mathsf{m} \in G(I)\,\}$). If I is Borel fixed,

$$\operatorname{b-pol}_A(I) := (\, \operatorname{b-pol}_A(\mathsf{m}) \mid \mathsf{m} \in G(I)\,)$$

is a faithful polarization of I.

By the above theorem, Borel fixed ideals have many alternative polarizations (if n is large).

Lemma 2.6. *In the situation of Theorem 2.5, we have*

$$J = (\, \operatorname{b-pol}_A(\mathsf{m}) \mid \mathsf{m} \in I \text{ with } \deg(\mathsf{m}) \le d\,).$$

Clearly, this is a generalization of Lemma 2.1.

Proof. It suffices to prove "$\supseteq$". For a contradiction, assume that there is some $\mathsf{m} = x^{\mathbf{a}} \in I$ with $\deg(\mathsf{m}) \le d$ and $\operatorname{b-pol}_A(\mathsf{m}) \notin J$. Since $\mathsf{m} \notin G(I)$, there is some i with $x_i | \mathsf{m}$ and $\mathsf{m}' := \mathsf{m}/x_i \in I$. If $i \notin A$, then it is easy to see that $\operatorname{b-pol}_A(\mathsf{m}) = x_{i,a_i} \cdot \operatorname{b-pol}_A(\mathsf{m}') \in J$. Hence we have $i \in A$. Set $\nu_A(\mathsf{m}) := \max\{i \in A \mid a_i > 0\}$. If we replace $\nu(A)$ by $\nu_A(\mathsf{m})$, the last part of the proof of Lemma 2.1 works verbatim, except that $\operatorname{b-pol}_A(\mathsf{m}) = x_{\nu_A(\mathsf{m}),f} \cdot \operatorname{b-pol}_A(\mathsf{m}'')$ with $f := \sum_{i \in A} a_i$. □

Proof of Theorem 2.5. We will show that the corresponding statement (and the proof) of Theorem 2.2 holds for $J := \operatorname{b-pol}_A(I)$, that is, $\widetilde{S}/J$ has a pretty clean filtration and any associated prime of J is of the form $(\, x_{i,c_i} \mid 1 \le i \le m)$. If this fact is given, then the present theorem follows from the same argument as Theorem 2.3.

To prove the above fact, take the same $\mathsf{m} \in \widetilde{S}$ as the proof of Theorem 2.2 (here $\nu(\mathsf{m}) = l := n - \operatorname{depth} S/I$, and $\nu_A(\mathsf{m})$ is *not* used). Clearly, as shown in Claim 1, $I + (\mathsf{m}_1)$ is Borel fixed.

For the statement corresponding to Claim 2, we need modification. Under the assumption that $\mathsf{m} \neq x_l$, set $\mathsf{m}_1 := \mathsf{m}/x_l = \prod_{i=1}^{n} x^{a_i}$ and $\mathsf{n} = \operatorname{b-pol}_A(\mathsf{m}_1)$. For each $i \in A$, set

$$b_i := \sum_{j \in A,\, j \le i} a_j.$$

In the rest of the proof, we will show that $J : \mathsf{n} = \mathfrak{p}$ where

$$\mathfrak{p} := (x_{i,b_i+1} \mid i \in A, i \le l) + (x_{i,a_i+1} \mid i \notin A, i \le l).$$

Note that $x_i \cdot \mathsf{m}_1 = (x_i/x_l) \cdot \mathsf{m} \in I$ for $i \le l$. If $i \notin A$, then we have $x_{i,a_i+1} \cdot \mathsf{n} = \operatorname{b-pol}_A(x_i \cdot \mathsf{m}_1) \in \operatorname{b-pol}_A(I)$. If $i \in A$, then we can show that $x_{i,b_i+1} \cdot \mathsf{n} \in \operatorname{b-pol}_A(I)$ by a similar argument to Claim 2, while we have to replace $\min\{\, j \mid a_j > 0,\, j > i\,\}$ by $\min\{\, j \in A \mid a_j > 0,\, j > i\,\}$. Hence we have $J : \mathsf{n} \supset \mathfrak{p}$.

To prove the converse, assume that a monomial $\mathsf{n}' \in \widetilde{S}$ satisfies $\mathsf{n}' \cdot \mathsf{n} \in J$. Then there is a monomial $\mathsf{m}'' \in G(I)$ such that $\operatorname{b-pol}_A(\mathsf{m}'')$ divides $\mathsf{n}' \cdot \mathsf{n}$. If $\mathsf{n}' \notin$

$(x_{i,a_i+1} \mid i \notin A,\, i \leq l)$, then $\mathsf{b\text{-}pol}_A(\mathsf{m}'') \notin (x_{i,a_i+1} \mid 1 \leq i \leq l)$ also. It means that $\deg_i(\mathsf{m}'') \leq a_i = \deg_i(\mathsf{m}_1)$ for all $i \notin A$. Now concentrating our attention on the variables x_i with $i \in A$ and $i \leq l$, we can use the proof of Claim 2 (almost) verbatim, and we see that the assumption $\mathsf{n}' \notin \mathfrak{p}$ yields a contradiction. Hence the analogy of Theorem 2.2 holds for $\mathsf{b\text{-}pol}_A(I)$. □

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