EXOTIC SMOOTH STRUCTURES ON TOPOLOGICAL FIBRE BUNDLES

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ABSTRACT. We use a variation of Hatcher's construction to construct virtually all stable exotic smooth structures on compact smooth manifold bundles whose fibers have sufficiently large odd dimension (at least twice the base dimension plus 3). We show that, rationally stably, such smooth structures are classified by a cohomology class in the total space and the relative higher Igusa-Klein (IK) torsion is equal to the push-down of that cohomology class. This answers the question, in the relative case, of which cohomology classes can occur as relative higher torsion classes.

CONTENTS

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0. Introduction and outline

Higher analogues of Reidemeister torsion and Ray-Singer analytic torsion were developed by J. Wagoner, J.R. Klein, M. Bismut, J. Lott, W. Dwyer, M. Weiss, E.B. Williams, W. Dorabiala, B. Badzioch, the authors of this paper and many others. ([26], [25], [24], [5], [9], [4], [11], [12],[13], [21], [3], [2]).

The purpose of this work is to determine to what extent higher Reidemeister torsion distinguishes between different smooth structures on the same (smoothable) topological manifold bundle and to determine which cohomology classes occur as higher torsion classes. Since the higher torsion is a sequence of real cohomology classes which are "stable", it can only detect the torsion-free part of the group of stable smooth structures on topological bundles. Following Dwyer, Weiss and Williams we eschew classical smoothing theory by assuming that we are given a fixed linearization of the vertical tangent microbundle of a topological manifold bundle. We also assume that there exists at least one smoothing. With these points in mind, we give a complete answer to these two questions in the two main theorems of this paper.

The first main result is the Realization Theorem (or Arc de Triomph Theorem 3.1.1) which says that the Arc de Triomph construction gives virtually all stable tangential smoothings of a compact topological manifold bundle which admits at least one smoothing. The second main result relates the cohomological relative smooth structure class arising from the theory of Dwyer-Weiss-Williams, with the higher Igusa-Klein (IK) torsion. This is Theorem 3.0.8 which states that the relative higher IK torsion is the push-down of the relative smooth structure class.

This paper discusses in detail how these (the Arc de Triomph construction and the relative smooth structure class) are defined and what are their basic properties. The two main theorems are difficult to state precisely and their proofs are intertwined. The main corollaries however are easier to state and show how this paper is an extension of earlier work of the first author [11] in which a large class of examples of bundles with nontrivial higher torsion was constructed using Hatcher's example. The following theorem is a reformulation of Corollary 3.0.9.

Theorem 0.0.1. *Let $p : M \to B$ be a smooth manifold bundle whose base B, fiber X and total space M are closed oriented smooth manifold. Suppose that $\dim X = N$ is odd and at least $2 \dim B + 3$. Let $\theta \in H^{4k}(B; \mathbb{R})$ be a cohomology class whose Poincaré dual is the push-down of an integral homology class in M. Then there exists another smooth bundle $p' : M' \to B$ which is fiberwise tangentially homeomorphic to p so that the relative torsion $\tau^{IK}(M', M)$ is a nonzero multiple of θ*

Fiberwise tangentially homeomorphic (A.3.3) means that there is a homeomorphism $f : M \to M'$ over B covered by a vector bundle isomorphism between the vertical tangent bundles $T^v f : T^v M \to T^v M'$ which is compatible in a certain sense with the topologically defined map of Euclidean bundles $E^v f : E^v M \to E^v M'$ induced by f.

There are two constructions which produce the examples in the theorem above. We call these the *immersed Hatcher handle* construction and the *Arc de Triomph* (AdT) construction. The first construction is geometrically easier and the second has a better algebraic description.

The group of all stable smooth structures on a smooth bundle is given by a theorem coming from Dwyer-Weiss-Williams smoothing theory (Appendix A) which has the following especially nice formulation in the case when the base and fiber are closed oriented manifolds (B.2.2).

Theorem 0.0.2. *Let $\widetilde{\mathcal{S}}^s_B(M)$ be the space of all stable smooth bundles $M' \to B$ which are fiberwise tangentially homeomorphic to M over B. Then $\pi_0 \widetilde{\mathcal{S}}^s_B(M)$ is a finitely generated abelian group and*

$$\pi_0 \widetilde{\mathcal{S}}^s_B(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H^{N+4k}(M; \mathbb{R})$$

where N is the fiber dimension.

The element in the cohomology/homology group corresponding to $M' \to B$ is called the *(relative) smooth structure class* and will be denoted

$$\Theta(M',M) \in \bigoplus_{k>0} H^{N+4k}(M;\mathbb{R}) \cong \bigoplus_{k>0} H_{\dim B - 4k}(M;\mathbb{R})$$

The two main theorem of this paper can now be stated.

Theorem 0.0.3 (AdT Theorem 3.1.1). *When $N \geq 2\dim B + 3$ is odd, the relative smooth structure classes of the smooth bundles $M' \to B$ given by the AdT construction span the vector space $\bigoplus_{k>0} H^{N+4k}(M;\mathbb{R})$.*

Remark 0.0.4. The group $\pi_0 \widetilde{\mathcal{S}}_B^s(M)$ is finitely generated and the stable smooth structures given by the AdT construction form a subgroup of full rank, i.e., a subgroup of finite index. So, the AdT construction gives virtually all stable smooth structures on $M \to B$.

Proposition 0.0.5 (Lemma 3.1.8). *The AdT construction and the immersed Hatcher construction are equivalent in the sense that they produce the same set of smooth bundles.*

Theorem 0.0.6 (Theorem 3.0.8, Corollary 3.0.11). *Given a smooth bundle $M' \to B$ which is fiberwise tangentially homeomorphic to $M \to B$, the relative higher torsion class $\tau^{IK}(M',M)$ is equal to the image of the relative smooth structure class $\Theta(M',M)$ under the push-down map:*

$$p_* : \bigoplus_{k>0} H^{N+4k}(M;\mathbb{R}) \to \bigoplus_{k>0} H^{4k}(B;\mathbb{R})$$

In short: $\tau^{IK} = p_* \circ \Theta$.

This theorem can be interpreted as saying that the relative higher IK-torsion is proportional to the relative higher DWW-torsion if we define the latter to be the push-down of the relative smooth structure class. This agrees with the recent theorem of Badzioch, Dorabiala, Klein and Williams [2] but the two results do not imply each other since the absolute higher torsion (DWW or IK) is not always defined.

When the fibers are closed even dimensional manifolds, the theorem above still holds by Corollary 3.0.11. However, the relative higher torsion class $\tau^{IK}(M',M)$ is equal to zero in that case:

$$\tau^{IK}(M',M) = \tau^{IK}(M') - \tau^{IK}(M) = 0$$

since $\tau^{IK}(M)$ depends only on the vertical tangent bundle of M over B by [22]. This leads to the following conjecture.

Conjecture 0.0.7 (Rigidity conjecture). *The stable smooth structure class vanishes when the fiber is a closed oriented even dimensional manifold:*

$$\Theta(M',M) = 0$$

In other words, rationally stably, there are no exotic smooth structures on manifold bundles with closed oriented even dimensional fibers.

Theorem 0.0.6 implies that $\Theta(M',M)$ must lie in the kernel of the push-down map p_* in the closed even dimensional fiber case since the higher relative torsion is zero.

0.1. **Basic definitions.** In Appendix A we explain the Dwyer-Weiss-Williams smoothing theory. This is an expanded version of the handwritten notes by Bruce Williams [28] which explains this version of their result which is not contained in the final published version of their paper [9]. (It comes from the introduction of the earlier unpublished version.) In Appendix B we add to this theory the elementary homotopy theory calculation which simplifies the result in the case when base and fiber are closed and oriented. We also extend the result to the "stratified" case. The

stratified version is a formal consequence of the standard version. Both follow from the special case when the base is a disk and the bundle is trivial. But in this case the base has a boundary.

The model that we will use is due to Hatcher. He constructed a disk bundle with an exotic smooth structure. So, we also need to consider the case when the fiber X has a boundary. The subbundle of M with fiber ∂X is called the vertical or relative boundary $\partial^{\mathsf{v}} M$.

0.1.1. *Stable smoothing theory.* Stabilization of a bundle simply means replacing a smooth bundle $M \to B$ with a linear disk bundle $D(\eta) \to B$ where $D(\eta)$ is the disk bundle of a vector bundle η over M. For example $M \times D^k \to B$ is a stabilization of M. Stabilization has the property that it does not see the boundary in the sense that, we can fix the boundary or not and it makes no difference after stabilization (Proposition A.4.5). So, we use whichever is more convenient. We use the version in which the boundary has a variable smooth structure to compute the stabilization and we use the version in which the boundary is fixed to give explicit constructions of smooth structures on M in low dimensions.

From now on we let $q = \dim B$ and $N = \dim X$. Suppose that ∂B is a union of two smooth $q-1$ manifolds $\partial_0 B, \partial_1 B$ which meet along their common boundary. Let $\widetilde{\mathcal{S}}^{\partial}_{B,\partial_0}(M)$ be the space of all smooth bundles $M' \to B$ which are tangentially homeomorphic to M relative to

$$\partial_0 M = M_{\partial_0 B} \cup \partial^{\mathsf{v}} M$$

in the sense that the fiberwise tangential homeomorphism $M \to M'$ is smooth on $\partial_0 M$.

We define this to be a (fiberwise) **exotic smooth structure** on M over $(B, \partial_0 B)$ since M' represents a new smooth structure on the underlying topological manifold bundle $M \to B$ which agrees with the given structure on $M_{\partial_0 B}$ and on the fiberwise boundary $\partial^{\mathsf{v}} M$.

Then DWW smoothing theory tells us that $\widetilde{\mathcal{S}}^{s}_{B,\partial_0 B}(M)$ is homotopy equivalent to the space of section $\Gamma_{B,\partial_0 B} \mathcal{H}^{\%}_B(M)$ of the bundle $\mathcal{H}^{\%}_B(M)$ over B whose fibers are $\mathcal{H}^{\%}(X)$ where $\mathcal{H}^{\%}(X) = \Omega^{\infty}(X_+ \wedge \mathcal{H}(*))$ is the homology theory whose coefficient spectrum is the space $\mathcal{H}(*)$ of stable h-cobordisms of a point. $\Gamma_{B,\partial_0 B}$ means sections over B which are fixed over $\partial_0 B$.

Theorem 0.1.1 (Dwyer-Weiss-Williams, Theorem A.5.14). $\widetilde{\mathcal{S}}^{s}_{B,\partial_0 B}(M) \simeq \Gamma_{B,\partial_0 B} \mathcal{H}^{\%}_B(M)$

An elementary homotopy calculation shows that

Proposition 0.1.2 (Corollary B.2.2). $\pi_0 \Gamma_{B,\partial_0 B} \mathcal{H}^{\%}_B(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}; \mathbb{R})$.

Here $M_{\partial_1 B}$ is the restriction of M to $\partial_1 B$. By Poincaré duality we have:

$$\bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}) \cong \bigoplus_{k>0} H^{N+4k}(M, \partial_0 M)$$

where we take homology with coefficients in $\mathbb{R}$

In the special case when $X = D^N$ is a disk, we denote the total space by $M = E \simeq B$ and we can use the fundamental theorem of Waldhausen [27] that $\mathcal{H}(*)$ is rationally homotopy equivalent to BO and the calculation of higher torsion to see that we have a sequence of isomorphisms:

$$\pi_0 \widetilde{\mathcal{S}}^{s}_{B,\partial_0 B}(E) \otimes \mathbb{R} \xrightarrow{\gamma_E} \pi_0 \Gamma_{B,\partial_0 B} \mathcal{H}^{\%}_B(E) \otimes \mathbb{R} \xrightarrow{\theta_E} \bigoplus_{k>0} H^{N+4k}(E, \partial_0 E) \xrightarrow{p_*} \bigoplus_{k>0} H^{4k}(B, \partial_0 B)$$

The isomorphism θ_E is normalized so that this composition is equal to the higher relative IK-torsion invariant τ^{IK} (Proposition B.2.3).

0.1.2. *Stratified smoothing theorem.* We show that the isomorphism given by the DWW smoothing theory is natural with respect to immersions in the following sense. Suppose that L is a compact q manifold with boundary $\partial L = \partial_0 L \cup \partial_1 L$ and $\lambda : L \to B$ is a codimension-0 immersion with image disjoint from $\partial_0 B$ so that $\lambda^{-1}(\partial B) = \partial_1 L$. Let $\tilde{\lambda} : L \to M$ be an embedding over λ and let E be a

D^N bundle diffeomorphic to a neighborhood of the image of $\tilde{\lambda}$. Then we have a smooth embedding of bundles:

$$D(\tilde{\lambda}) : E \to M$$

An exotic smooth structure E' on E relative to $\partial_0 E$ gives an exotic smooth structure on M simply by

$$M' = (M - E) \cup E'$$

since $\partial_0 E$ is the intersection of E with the closure of its complement in M. This defines a mapping

$$D(\tilde{\lambda})_* : \widetilde{\mathcal{S}}^s_{L,\partial_0 L}(E) \to \widetilde{\mathcal{S}}^s_{B,\partial_0 B}(M)$$

We show, as an easy extension of the naturality of the DWW construction that:

Theorem 0.1.3 (stratified deformation theorem)**.** *The following diagram commutes where the vertical arrows are induced by the embedding* $D(\tilde{\lambda}) : (E, E_{\partial_1 L}) \to (M, M_{\partial_1 B})$

$$
\begin{array}{ccccc}
\pi_0 \widetilde{\mathcal{S}}^s_{L,\partial_0}(E) & \xrightarrow[\cong]{\gamma_E} & \pi_0 \Gamma_{L,\partial_0} \mathcal{H}^{\%}_L(E) & \xrightarrow{\theta_E} & \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M) & \xrightarrow[\cong]{\gamma_M} & \pi_0 \Gamma_{B,\partial_0} \mathcal{H}^{\%}_B(M) & \xrightarrow{\theta_M} & \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1})
\end{array}
$$

Proof. The left hand square commutes by Corollary A.5.19 and the right hand square commutes by Corollary B.4.2. $\qquad\qquad\square$

The composition of the horizontal arrows is by definition the homological smooth structure map Θ. So, we can abbreviate this diagram:

$$
\begin{array}{ccc}
\pi_0 \widetilde{\mathcal{S}}^s_{L,\partial_0}(E) & \xrightarrow{\Theta} & \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1}) \\
\downarrow{\scriptstyle D(\tilde{\lambda})_*} & & \downarrow{\scriptstyle D(\tilde{\lambda})_*} \\
\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M) & \xrightarrow{\Theta} & \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1})
\end{array}
$$

0.2. **Outline of the proofs.** The statements and proofs of the main theorems are contained in a single diagram which contains the above diagram in its middle.

The Arc de Triomph construction can be expressed as a homomorphism

$$AdT : SD^{G/O}_{B,\partial_0}(M) \to \pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M)$$

where $SD^{G/O}_{B,\partial_0}(M)$ is the group of all equivalence classes of input data for the Arc de Triomph construction. This factors through the quotient group $\overline{SD}^{G/O}_{B,\partial_0}(M)$ shown in the diagram above.

The AdT Theorem 3.1.1 says that $\overline{AdT}$ is rationally surjective, in other words, the cokernel is a finite group. The proposition (Lemma 3.1.8) that every AdT construction is equivalent to

an immersed Hatcher construction is formulated more precisely in the statement that we have a surjective map

$$\sum \Sigma_{\tilde{\lambda}} : \bigoplus G(L, \partial_0 L) \twoheadrightarrow \overline{SD}_{B,\partial_0}^{G/O}(M)$$

Here $G(L, \partial_0 L)$ is just the group of all homotopy classes of maps $L/\partial_0 L \to G/O$.

The other main theorem:

$$\tau^{\mathrm{IK}} = p_* \circ \Theta$$

is the commutativity of the lower right curved triangle. This follows from the AdT Theorem and the commutativity of the corresponding upper right curved triangle which is Proposition B.2.3.

0.2.1. *Hatcher's example and its variations.* The basic construction is due to Hatcher. Hatcher's construction starts with an n-plane bundle ξ over a q-manifold L which is trivial over $\partial_0 L$ and which has the property that the associated sphere bundle is fiber homotopically trivial. In other words, ξ gives a mapping

$$\xi : L/\partial_0 L \to G/O$$

We use the notation $G(L, \partial_0 L)$ to denote the set of homotopy classes of pointed maps $L/\partial_0 L \to G/O$ giving the input data for this construction. Using this data, Hatcher constructed a disk bundle $E^{n,m}(\xi)$ over B which is homeomorphic to $B \times D^{n+m}$ but not fiberwise diffeomorphic to any linear disk bundle. We show (Theorem 1.2.2) that this construction can be realized in fiber dimension $n + m \geq 2q + 3$.

Hatcher's disk bundle can be used to construct what we call "Hatcher handles" (subsection 2.3). These are two thickenings of Hatcher's disk bundle which we call "negative" and "positive" suspension of $E^{n,m}(\xi, \eta)$ and denote $A^{n,m}(\xi, \eta)$ and $B^{n,m}(\xi, \eta)$. We show that the positive Hatcher handles $B^{n,m}(\xi, \eta)$ can be attached along the top $M \times 1$ of a manifold bundle $M \times I$ over B along an embedding $\tilde{\lambda} : L \to M$ which lies over a codimension 0 immersion $\lambda : L \to B$. We call this the *immersed Hatcher construction.*

The attaching map for the negative Hatcher handle $A^{n,m}(\xi, \eta)$ can deformed to be on top of the positive Hatcher handle $B^{n,m}(\xi, \eta)$ in such a way that they cancel as shown in Figure 2.1. We call this the *Arc de Triomph* construction. This construction has as input data a pair (Σ, ψ) where Σ is a smooth oriented q-manifold embedded in M with the property that the projection $\Sigma \to B$ has only fold singularities. The mapping $\psi : \Sigma \to G/O$ gives the data for positive and negative Hatcher handles to be attached along Σ_+, Σ_- which are the subsets of Σ along which the projection $p : \Sigma \to B$ is orientation preserving or reversing, respectively. The group of deformation classes of such input data we denote $SD_{B,\partial_0}^{G/O}(M)$. The Arc de Triomph construction thus gives a map

$$AdT : SD_{B,\partial_0}^{G/O}(M) \to \pi_0 \widetilde{\mathcal{S}}_{B,\partial_0}^s(M)$$

which we show to be additive (Proposition 3.1.4). One of the main theorems is that this map is rationally surjective, i.e., its cokernel is finite.

To prove this we use the computation of the homotopy type of the space of generalized Morse functions [18] which implies that there is a fiberwise generalized morse function $f : M \to I$ whose singular set $\Sigma(f)$ together with is vector bundle data ξ given by the second derivative of f gives an element of $SD_{B,\partial_0}^{G/O}(M)$ which maps onto a spanning subset of the real homology group

$$\bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}) \cong \pi_0 SD_{B,\partial_0 B}^{G/O}(M) \otimes \mathbb{R}$$

In the main diagram, this is expressed by saying that the curved mapping $(-1)^n 2\widetilde{ch}$ from $\overline{SD}_{B,\partial_0 B}^{G/O}(M)$ to $\bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B})$ maps onto a spanning subset. We know from the Appendices that Θ is an isomorphism. So, it suffices to show that $\Theta \circ AdT = (-1)^n 2\widetilde{ch}$, i.e. that the lower left curved triangle commutes.

0.2.2. *Higher IK torsion.* In order to show the commutativity of the lower part of the main diagram, we use the computation of higher IK torsion on the upper part of the diagram and the fact that the middle of the diagram commutes by way of the stratified smoothing theorem and the related homotopy calculation as proved in the Appendix. The stratified deformation lemma 3.2.1 is used to prove that every AdT construction can be deformed into an immersed Hatcher construction. This is used to pass to the upper part of the main diagram. Here, the bundle E is a disk bundle. So, E is homotopy equivalent to its base L and the higher torsion is equal to the smooth structure class

$$\Theta : \pi_0 \widetilde{\mathcal{S}}^s_{L,\partial_0 L}(E) \otimes \mathbb{R} \to \bigoplus_{k>0} H_{q-4k}(L, \partial_1 L)$$

by definition of the latter.

By passing down to M, we see that the image in the cohomology of M of the smooth structure class of E maps to the image in the cohomology of B of the higher torsion invariant. By commutativity of the main diagram this implies that the smooth structure class in general maps to the higher IK torsion. This is the second main theorem.

1. Hatcher's example

Hatcher's famous construction gives smooth disk bundles over S^{4k} which are homeomorphic but not diffeomorphic to $S^{4k} \times D^n$. The exact statement is given below.

1.1. **Homotopy theory.** John Klein helped us to find the lowest dimension in which this part of the construction works.

Suppose that B is a compact smooth q-manifold and $\partial B = \partial_0 B \cup \partial_1 B$ as before. Let

$$f : B/\partial_0 B \to G/O$$

be a continuous map, i.e., f is a continuous mapping on B which sends $\partial_0 B$ to the basepoint of G/O, the fiber of $BO \to BG$. This classifies a stable vector bundle over B which is trivial over $\partial_0 B$ and trivial over B as a spherical fibration. Take $n > q$. Then $BO_n \to BO$ is $q+1$-connected and therefore this stable vector bundle is given by a unique oriented n-plane bundle ξ over B which is trivial over $\partial_0 B$.

Remark 1.1.1. Since G/O is rationally homotopy equivalent to BO, the Chern characters of all real vector bundles ξ obtained in this way will span the vector space

$$\bigoplus_{0<k\leq q/4} H^{4k}(B, \partial_0 B; \mathbb{R}).$$

Recall that G_n is the topological monoid of all unpointed self-homotopy equivalences of S^{n-1}. Taking unreduced suspension we get a mapping $G_n \to F_n$ where $F_n \subset \Omega^n S^n$ is the union of the degree ± 1 components. It follows from a theorem of Haefliger [15] that (F_n, G_n) is $2n-3$ connected.

Furthermore, the components of $\Omega^n S^n$ are all homotopy equivalent and $\pi_k G_n \cong \pi_k F_n$ is stable and thus finite for $k \leq n - 2$. (This also follows from the EHP sequence.) Therefore,

$$[B/\partial_0 B, BG_n] \cong [B/\partial_0 B, BG]$$

for $n > q$. So, the composition

$$B/\partial_0 B \xrightarrow{\xi} BO_n \to BG_n$$

is null homotopic for $n > q$. This implies that the sphere bundle $S^{n-1}(\xi)$ associated to ξ is fiberwise homotopy equivalent to the trivial bundle:

$$g : S^{n-1}(\xi) \simeq S^{n-1} \times B$$

and this trivialization agrees with the given trivialization over $\partial_0 B$.

Take the fiberwise mapping cone of g. This gives a fibration over B whose fibers are contractible n-dimensional cell complexes which are homeomorphic to the standard n-disk over $\partial_0 B$. When we thicken this up we will get an exotic smooth structure on a trivial disk bundle over B.

Remark 1.1.2. For any space X recall [1, 17] that $J(X)$ is the group of stable vector bundles over X modulo the equivalence relation that $\xi \sim \eta$ if the sphere bundles over ξ and η are fiberwise homotopy equivalent. The group operation is fiberwise join which corresponds to direct sum of underlying bundles. If ξ is any vector bundle over X then $J(\xi)$ denotes its image in $J(X)$. If X is a finite complex then it is well known that $J(X)$ is a finite group. (See, e.g. [17].) The above argument shows that if $J(\xi)$ is trivial in $J(B/\partial_0 B)$ and $\dim \xi > \dim B$ then the sphere bundle of ξ is fiberwise homotopically trivial.

1.2. **Thickening.** We have a family of finite cell complexes over B which we want to thicken to get a manifold bundle. If we embed this fibration in $D^N \times B$ and take a "regular neighborhood" we will get a smooth N disk bundle over B which is homeomorphic but not diffeomorphic to $D^N \times B$.

We start by thickening the trivial sphere bundle $S^{n-1} \times B$ to get $S^{n-1} \times I \times D^m \times B$. This is the trivial bundle over B with fiber $S^{n-1} \times I \times D^m$. We also need this to be embedded in a trivial disk bundle $D^n \times D^m \times B$ in a standard way. We can take the obvious embedding

$$f : S^{n-1} \times I \times D^m \hookrightarrow D^n \times D^m$$

given by $f(x, y, z) = \left(\frac{1}{2}(1 + y)x, z \right)$. Note that $S^{n-1} \times 0 \times D^m$ is mapped into the sides of the "donut hole" which is the closure of the complement of the image of f in $D^n \times D^m$.

We attach an n-handle $D^n(\xi) \oplus D^m(\eta)$ to this (with η necessarily being a complementary bundle to ξ) to fill in the donut hole and create a smooth (after rounding corners) bundle over B with fiber

$$S^{n-1} \times I \times D^m \cup D^n \times D^m \cong D^{n+m}$$

The data needed to attach such a handle embedded in $D^n \times D^m \times B$ is a smooth embedding of pairs

$$D(j) : (D^n(\xi), S^{n-1}(\xi)) \oplus D^m(\eta) \to (D^n, S^{n-1}) \times D^m \times B$$

where D^m represents the hemisphere in the boundary of D^{m+1}.

$$D(j)(D^n(\xi) \times 0)$$
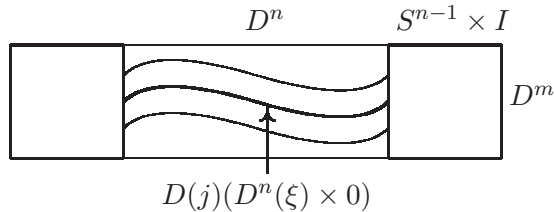
This embedding $D(j)$ is essentially given by its restriction to the core $D^n(\xi) \times 0$.

Lemma 1.2.1. *If $m > n > q$ then there is a smooth fiberwise embedding of pairs:*

$$j : (D^n(\xi), S^{n-1}(\xi)) \to (D^n, S^{n-1}) \times D^m \times B$$

over B which is the standard embedding over $\partial_0 B$ and which is transverse to $S^{n-1} \times D^m$. Furthermore, if $m \geq q + 3$ then this fiberwise embedding will be unique up to fiberwise isotopy.

Proof. When $q = 0$, this holds by transversality. So suppose $q > 0$. We use [20, Thm 6.5] which says that the inclusion

$$\mathrm{Emb}((D^n, S^{n-1}), (W^{n+m}, \partial_0 W)) \to \mathrm{Map}((D^n, S^{n-1}), (W^{n+m}, \partial_0 W))$$

of the smooth embedding space into the mapping space is c-connected where

$$c = m - n - 1 + \min(s, n, m - 2, n + m - 4)$$

and s is the connectivity of the pair $(W, \partial_0 W)$. In our case $s = n-1$. So the condition $m > n > q > 0$ implies that $c \geq q$ giving the existence part of the lemma and if $m \geq q + 3$ then either $m \geq n + 2$ or $n \geq q + 2$ and we get $c > q$ which implies the uniqueness part. $\square$

The embedding j gives an m-dimensional normal bundle η for ξ and a smooth codimension 0 embedding

$$D(j) : D^n(\xi) \oplus D^m(\eta) \to D^n \times D^m \times B$$

Restricting this to $\partial D^n(\xi) \oplus D^m(\eta)$ we get a fiberwise embedding

$$S(j) : S^{n-1}(\xi) \oplus D^m(\eta) \to S^{n-1} \times D^m \times B$$

We can use $S(j)$ to construct a smooth bundle (with corners rounded):

$$E^{n,m}(\xi) = D^n(\xi) \oplus D^m(\eta) \cup_{S(j)} S^{n-1} \times I \times D^m \times B.$$

We can also use $D(j)$ to embed this in the trivial disk bundle of the same dimension:

$$F(j) = D(j) \cup f : E^{n,m}(\xi) \hookrightarrow D^n \times D^m \times B$$

This is **Hatcher's example**. Since $m > q$, the m-plane bundle η is the stable complement to ξ and is thus uniquely determined. If $m \geq q + 3$ then, up to fiberwise diffeomorphism, $E(\xi)$ is independent of the choice of j. Finally, we note the crucial point that the bundle $E(\xi)$ is canonically diffeomorphic to the trivial bundle over $\partial_0 B$. Summarizing the construction above and the easy calculation of the higher torsion of this bundle we get the following well known theorem.

Theorem 1.2.2. *Suppose that B is a smooth q-manifold and $m > n > q$. Suppose that ξ is an n-plane bundle over B which is trivial over $\partial_0 B \subset \partial B$ so that $J(\xi) = 0 \in J(B/\partial_0 B)$. Then Hatcher's construction gives a smooth bundle $E^{n,m}(\xi)$ over B with fiber D^{n+m}. This bundle is fiberwise diffeomorphic to the trivial bundle over $\partial_0 B$ and fiberwise homeomorphic to the trivial bundle over B with fiber D^{n+m}. Furthermore its higher IK-torsion in degree $4k$ is given by*

$$\tau_{2k}^{IK}(E^{n,m}(\xi)) = (-1)^{k+n} \zeta(2k+1) \tfrac{1}{2} ch_{4k}(\xi \otimes \mathbb{C}) \in H^{4k}(B, \partial_0 B; \mathbb{R})$$

where ζ is the Riemann zeta function.

Remark 1.2.3. If we use the **normalized chern character**

$$\widetilde{ch}_{4k}(\xi) = (-1)^k \zeta(2k+1) \tfrac{1}{2} ch_{4k}(\xi \otimes \mathbb{C})$$

then the statement of the theorem simplifies to

$$\tau_{2k}^{IK}(E^{n,m}(\xi)) = (-1)^n \widetilde{ch}_{4k}(\xi)$$

Proof. The higher torsion is calculated using the Framing Principle from [21, 23]. Here we use the version in [22] which says that, given a smooth handlebody structure on the fibers with handles attached in the same order for each fiber, the axiomatic higher torsion is defined and equal to a linear combination of the suitably normalized chern characters of the bundles giving the core and cocores of the handle. For IK torsion the coefficient for the cocore is zero and the coefficient for the core is $(-1)^k \zeta(2k+1)$ which is what we are using.

The bundle is topologically trivial by the Alexander trick. (The topological group of homeomorphism of the disk D^{n+m} which are the identity on the southern hemisphere is contractible.) $\qquad \square$

Take $q = 4k, n = 4k+1, m \geq 4k+2, B = S^{4k}$ and using the well known fact that the order of the image of the J-homomorphism $J : \pi_{4k-1}O \to \pi_{4k-1}^s$, which we denote a_k, is the denominator of $B_k/4k$ where B_k is the k-th Bernoulli number [1], we get the following.

Corollary 1.2.4. *For any $k > 0, N \geq 8k+3$ Hatcher's construction gives a smooth N-disk bundle over S^{4k} which is tangentially homeomorphic to $D^N \times S^{4k}$ but has higher torsion invariant $\tau_{2k} \in H^{4k}(S^{4k}; \mathbb{R})$ equal to $\zeta(2k+1)a_k$ times the generator of $H^{4k}(S^{4k}; \mathbb{Z})$ for k odd and half of that number when k is even. In both cases this gives a nontrivial element of $\pi_{4k-1}Diff(D^N)/O_N \otimes \mathbb{R}$.*

Proof. It follows from Bott periodicity ([6], [17, 18.9]) that the chern character of the stable complex vector bundle over S^{2k} corresponding to a generator of $\pi_{2k}BU = \mathbb{Z}$ is equal to a generator of $H^{2k}(S^{2k}; \mathbb{Z})$. Also, the homotopy fiber sequence $BO \to BU \to \Omega^6 BO$ given by the inclusion map $O \to U$ implies that the generator of $\pi_{4k}BO$ maps to the generator of $\pi_{4k}BU$ for k even and to twice the generator when k is odd. The generator of the kernel of the J-homomorphism is a_k times this element. By the theorem above, the higher torsion of this exotic bundle is given by multiplying this element by $\frac{1}{2}\zeta(2k+1)$ giving the formula in the corollary up to sign. $\qquad \square$

2. Variations of Hatcher's construction

We need several variations and extensions of Hatcher's construction in order to construct a full rank subgroup of the group of all possible tangential smooth on a smooth manifold bundle with sufficiently large odd dimensional fibers. The idea is to construct "positive" and "negative" "suspensions" of Hatcher's basic construction which will cancel. We call this the "Arc de Triomph" construction due to the appearance of the figures used to explain the construction. Since the stabilization of bundles with even dimensional fibers includes bundles whose fiber dimensions are arbitrarily large and odd, this construction also produces "all" stable tangential smooth structures on bundles with even dimensional fibers.

2.1. **Arc de Triomph: basic construction.** There are two "suspensions" of $E^{n,m}$ to one higher dimension. We will see that their union is trivial:
$$E^{n,m+1}(\xi) \cup E^{n+1,m}(\xi) \cong D^{n+m+1} \times B$$
This is in keeping with the calculation of their higher torsions:
$$\tau_{2k}^{\text{IK}}(E^{n,m+1}(\xi)) + \tau_{2k}^{\text{IK}}(E^{n+1,m}(\xi)) = (-1)^n \widetilde{ch}_{4k}(\xi) + (-1)^{n+1} \widetilde{ch}_{4k}(\xi) = 0$$
and the Additivity Axiom from [22] which says that the higher torsion of a union is the sum of torsions of the pieces.

The **positive suspension** of $E^{n,m}(\xi)$ is defined simply as the product (with corners rounded):
$$\sigma_+ E^{n,m}(\xi) = E^{n,m}(\xi) \times I$$
An examination of the definitions shows that this is the same as $E^{n,m+1}(\xi)$.

The **negative suspension** of $E^{n,m}(\xi)$ uses the embedding $F(j) : E^{n,m} \hookrightarrow D^n \times D^m \times B$ and is defined as follows.
$$\sigma_- E^{n,m}(\xi) = D^n \times D^m \times [-1,0] \times B \cup_{F(j)\times 0} E^{n,m}(\xi) \times I \cup_{F(j)\times 1} D^n \times D^m \times [1,2] \times B$$

This is a subbundle of $D^n \times D^m \times [-1, 2] \times B$. We claim that $\sigma_- E^{n,m}(\xi)$ is a model for $E^{n+1,m}(\xi)$ over B in the sense that the construction of $E^{n+1,m}(\xi)$, which may not be unique, could give $\sigma_0 E^{n,m}(\xi)$. Lemma 1.2.1 then tells us that we have uniqueness after stabilizing just once:

$$\sigma_- E^{n,m}(\xi) \times I \cong E^{n+1,m}(\xi) \times I = E^{n+1,m+1}(\xi)$$

since $m + 1 \geq q + 3$. To verify this claim note that $\sigma_- E^{n,m}(\xi)$ contains the trivial bundle over B with fiber

$$F = D^n \times D^m \times [-1, 0] \cup S^{n-1} \times I \times D^m \times [0, 1] \cup D^n \times D^m \times [1, 2]$$

which is diffeomorphic to $S^n \times D^{m+1}$ after its corners are rounded. On this is attached the $n + 1$ handle $D^n(\xi) \oplus D^m(\eta) \times I$ which is which is equivalent to $D^{n+1}(\xi) \oplus D^m(\eta)$ after corners are rounded. Since $D^{n+1}(\xi)$ is the core of this handle, the result is $E^{n+1,m}(\xi)$.

When we take the union of the positive and negative suspensions of $E^{n,m}(\xi)$, they cancel. This will follow from the following lemma which does not require proof.

Lemma 2.1.1. *Suppose that E_0, E_1 are compact smooth manifold bundles over B with the same fiber dimension. Let $f : E_0 \to E_1$ be a smooth embedding over B. Then*

$$E_0 \times [0, 1] \cup_{f \times 1} E_1 \times [1, 2]$$

is fiberwise diffeomorphic to $E_1 \times I$ after rounding off corners.

Remark 2.1.2. The example that we have in mind is

$$E^{n,m}(\xi) \times [0, 1] \cup_{F(j) \times 1} D^n \times D^m \times [1, 2] \times B \cong D^n \times D^m \times I \times B$$

We denote the construction on the left by $V^{n,m}(\xi)$.

Next we use another trivial lemma:

Lemma 2.1.3. *Suppose that $\partial^v E_1 = \partial_0 E_1 \cup \partial_1 E_1$ where $\partial_i E_1$ are smooth manifold bundles over B with the same fiberwise boundary. Let $f, g : \partial_0 E_1 \to \partial^v E_0$ be smooth embeddings over B which are fiberwise isotopic. Then $E_0 \cup_f E_1$ and $E_0 \cup_g E_1$ are fiberwise diffeomorphic over B after rounding off the corners.*

In our example, $\partial_0 E_0$ will be a disk bundle. So, we need the following well-known lemma.

Lemma 2.1.4. *Suppose that D is a smooth disk bundle over B and E is any smooth manifold bundle over B. Let $f, g : D \to E$ be smooth embeddings over B which agree over $\partial_0 B$. Then f, g are isotopic if and only if they are fiberwise tangentially homotopic fixing $\partial_0 B$.*

Proof. Necessity of the condition is clear. To prove necessity, we triangulate the base and construct the isotopy over the simplices one at a time. Thus it suffices to consider the case $N = D^q$ and $\partial_0 B = S^{q-1}$. In this case both bundles are trivial so f, g are maps

$$f, g : D^q \to \mathrm{Emb}(D^N, F)$$

which agree on S^{q-1} and whose 1-jets are homotopic fixing the boundary. Here F is the fiber of $E \to B$. But the derivative at the origin map from the embedding space $\mathrm{Emb}(D^N, F)$ to the space of N-frames in F is a homotopy equivalence since embeddings on disks can be uniformly contracted to the middle and made isotopic to the exponential map on the N-disk in the tangent bundle of F given by the N-frame. Therefore an isotopy of the 1-jet is locally equivalent to an isotopy of the embedding. $\square$

We use the last two lemmas for

$$(E_1, \partial_0 E_1) = (E^{n,m}(\xi) \times [0, 1] \cup_{F(j) \times 1} D^n \times D^m \times [1, 2] \times B, E^{n,m}(\xi) \times 0)$$

and $E_0 = M \times [-1, 0]$ with

$$M = E^{n,m}(\xi) \cup_{h \times id_B} D^n \times D^m \times B$$

where h is an orientation reversing diffeomorphism

$$\partial(D^n \times D^m) \supset D_0^{n+m-1} \xrightarrow{h} D_1^{n+m-1} \subset S^{n-1} \times 1 \times D^m$$

between a fixed $n + m - 1$ disk on the boundary of $D^n \times D^m$ and a fixed $n + m - 1$ disk in $S^{n-1} \times 1 \times D^m$ (the outside surface of the donut). The pasting map h needs to be orientation reversing in order for orientations of the two pieces to agree. It is a special case of the first trivial Lemma 2.1.1 that M is fiberwise diffeomorphic to $E^{n,m}(\xi)$. Note that h is unique up to isotopy assuming that $n \geq 2$.

In our example of Lemma 2.1.3, the mapping $f : \partial_0 E_1 \to \partial^\vee E_0$ is simply the inclusion map

$$f : E^{n,m}(\xi) \times 0 \subseteq M \times 0 \subseteq \partial^\vee E_0$$

and $g : \partial_0 E_1 \to \partial^\vee E_0$ is the embedding:

$$g : E^{n,m}(\xi) \times 0 \xrightarrow{F(j)} D^n \times D^m \times B \subseteq M \times 0 \subseteq \partial^\vee E_0$$

Since f, g are tangentially homotopic maps on disk bundles, they are isotopic. But, when we attach E_1 on top of $D^n \times D^m \times B \times [-1, 0]$ using the map $F(j)$ we get exactly the negative suspension $\sigma_- E^{n,m}(\xi)$. So, we have a diffeomorphism which preserves all the corner sets:

$$M \times [-1, 0] \cup_g E_1 = \sigma_- E^{n,m}(\xi) \cup_{h \times id_B} \sigma_+ E^{n,m}(\xi)$$

and

$$M \times [-1, 0] \cup_f E_1 = V^{n,m}(\xi) \cup_{h \times id_B} D^n \times D^m \times B \times [-1, 0] \cong D^{n+m+1} \times B.$$

where $V^{n,m}(\xi)$ is given in Remark 2.1.2. Since h is unique up to isotopy, any two choices of h will produce fiberwise diffeomorphic bundles. So we get the following. (See Figure 2.1. The notation $E_1 = A^{n,m}(\xi, \eta)$ is from subsection 2.3.)

Proposition 2.1.5 (basic cancellation lemma). *The oriented union of the positive and negative suspensions of $E^{n,m}(\xi)$ glued together along fixed $n + m$ disk bundles in the fixed parts of their boundary is fiberwise diffeomorphic to the trivial $n + m + 1$ disk bundle over B:*

$$\sigma_- E^{n,m}(\xi) \cup_{h \times id_B} \sigma_+ E^{n,m}(\xi) \cong D^{n+m+1} \times B.$$

2.2. **Twisted version.** Remark 1.1.1 and the main theorem (Corollary B.2.2) of Appendix B show that, rationally stably, all exotic smooth structures on trivial disk bundles are given by Hatcher's example. Now we consider nontrivial disk bundles.

Stably, it is easy to construct exotic smooth structures on nontrivial linear disk bundles. If we start with any vector bundle ξ_0 over B which is trivial over $\partial_0 B$, we can take the associated disk bundle $D^N(\xi_0)$. The fiberwise product

$$D^N(\xi_0) \oplus E^{n,m}(\xi)$$

with corners rounded is a smooth disk bundle fiberwise homeomorphic to $D^N(\xi_0) \times D^{n+m}$ with the same higher torsion as $E^{n,m}(\xi)$ since IK torsion has the property that it is invariant under passage to linear disk bundles.

Corollary 2.2.1. *Given any linear disk bundle $D^N(\xi_0)$ over B which is trivial over $\partial_0 B$, the collection of all stable smooth structures on $D^N(\xi_0)$ given by Hatcher's construction spans the vector space*

$$\pi_0 \widetilde{\mathcal{S}}_{B,\partial_0}^s(D^N(\xi_0)) \otimes \mathbb{R} \cong \bigoplus_{k>0} H^{4k}(B, \partial_0 B; \mathbb{R})$$

12

2

$A^{n,m}(\xi,\eta) = E_1$

negative Hatcher handle

1

0

$D^n \times D^m(\eta) \times I$

$V^{n,m}(\xi,\eta) \times I$

$D^n(\xi) \times D^m(\eta_0) \times I$

move

positive suspension

E_1

is attached by g

negative suspension

$V^{n,m}(\xi,\eta)$

Arc de Triomph

E_1

is attached by f

giving trivial
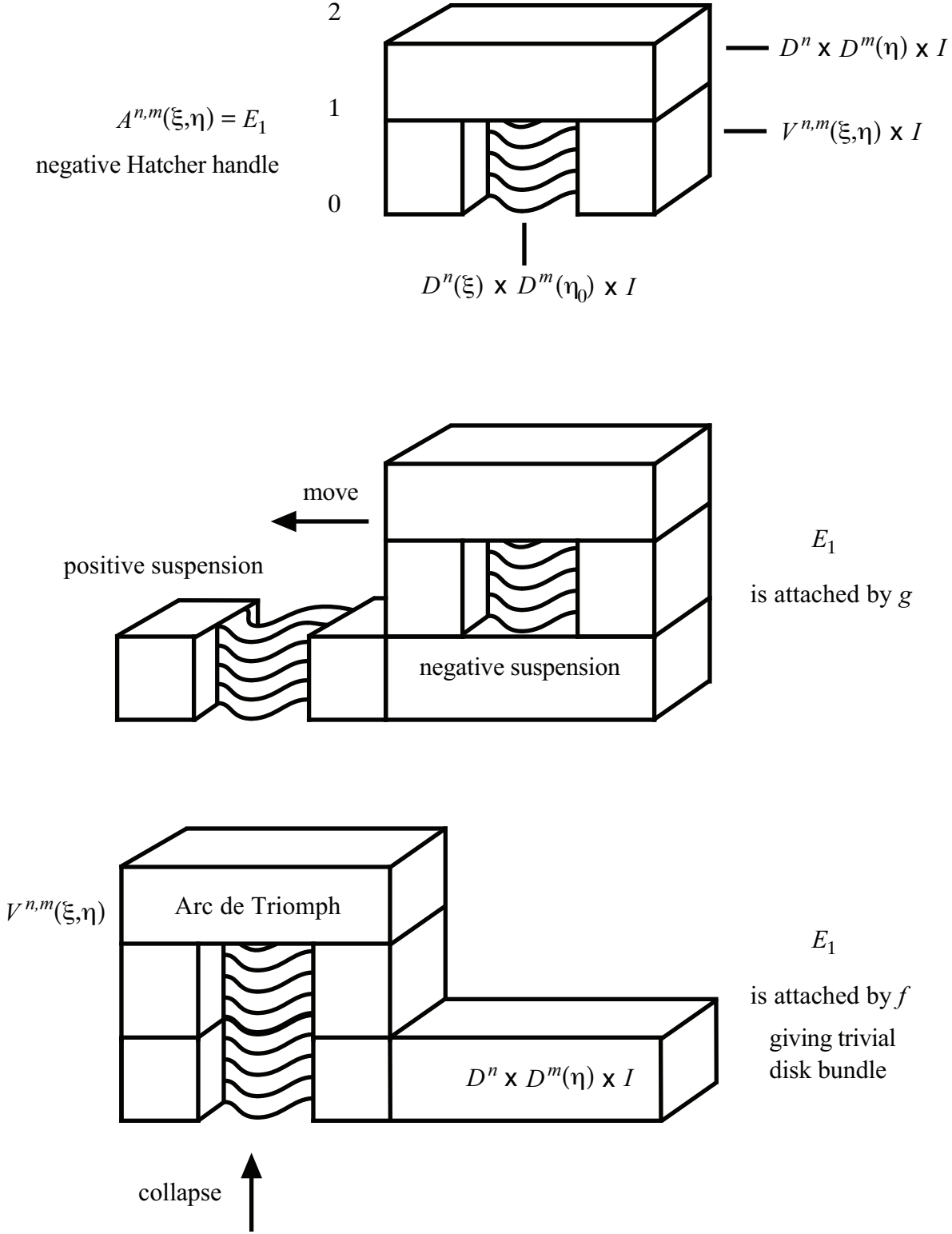disk bundle

$D^n \times D^m(\eta) \times I$

collapse

FIGURE 1. Positive and negative Hatcher handles are cancelled using Arc de Triomph

Now we give the unstable version of the last corollary and use it to define "Hatcher handles". Suppose that $(B, \partial_0 B)$ is a manifold pair as before with $\dim B = q$. Let ξ, η be vector bundles over B of dimension n, m so that ξ is trivial over $\partial_0 B$ and $J(\xi) = 0 \in J(B/\partial_0 B)$. As in Lemma 1.2.1 we have the following.

Lemma 2.2.2. *If $m > n > q$ then there is a smooth fiberwise embedding of pairs:*
$$j : (D^n(\xi), S^{n-1}(\xi)) \to (D^n, S^{n-1}) \times D^m(\eta)$$
over B which is a standard linear embedding over $\partial_0 B$ and which is transverse to $S^{n-1} \times D^m(\eta)$. Furthermore, if $m \geq q + 3$ then this fiberwise embedding is unique up to fiberwise isotopy.

Let η_0 be the unique m-plane bundle over B so that $\xi \oplus \eta_0 \cong \epsilon^n \oplus \eta$ where ϵ^n is the trivial n-plane bundle over B. Then the embedding given by the lemma thickens to a codimension 0 fiberwise embedding
$$(D(j), S(j)) : (D^n(\xi), S^{n-1}(\xi)) \oplus D^m(\eta_0) \hookrightarrow (D^n, S^{n-1}) \times D^m(\eta)$$
which is a standard linear embedding over $\partial_0 B$. Let $E^{n,m}(\xi, \eta)$ denote the $n + m$ disk bundle over B given by
$$E^{n,m}(\xi, \eta) = D^n(\xi) \oplus D^m(\eta_0) \cup_{S(j)} S^{n-1} \times I \times D^m(\eta)$$
with corners rounded. Up to fiberwise diffeomorphism, this is independent of the choice of g if $m \geq q + 3$. As before we have a fiberwise embedding $F(j) : E^{n,m}(\xi, \eta) \hookrightarrow D^n \times D^m(\eta)$ and we can define the positive and negative suspensions of $E^{n,m}(\xi)$ to be
$$\sigma_+ E^{n,m}(\xi, \eta) = E^{n,m}(\xi, \eta) \times I$$
which is fiberwise diffeomorphic to $E^{n,m+1}(\xi, \eta)$ after corners are rounded and
$$\sigma_- E^{n,m}(\xi, \eta) = D^n \times D^m(\eta) \times [-1, 0] \cup_{F(j) \times 0} E^{n,m}(\xi, \eta) \times I \cup_{F(j) \times 1} D^n \times D^m(\eta) \times [1, 2]$$
which is a model for $E^{n+1,m}(\xi, \eta)$. As before, the Framing Principle implies that the higher IK torsion of this bundle is the normalized Chern character of ξ:

Theorem 2.2.3. *$E^{n,m}(\xi, \eta)$ is a smooth $n + m$ disk bundle over B which is fiberwise diffeomorphic to the linear disk bundle $D^{n+m}(\eta)$ over $\partial_0 B$ and fiberwise homeomorphic to $D^{n+m}(\eta)$ over B. Furthermore,*
$$\tau_{2k}^{IK}(E^{n,m}(\xi, \eta)) = (-1)^n \widetilde{ch}_{4k}(\xi) \in H^{4k}(B, \partial_0 B; \mathbb{R})$$

Remark 2.2.4. This theorem can be stated as the commutativity of the following diagram:

$$
\begin{array}{ccc}
 & G(B, \partial_0 B) & \\
 {\scriptstyle E^n(-,\eta)} \swarrow & & \searrow {\scriptstyle (-1)^n \widetilde{ch}} \\
\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(D(\eta)) & \xrightarrow{\quad \tau^{IK} \quad} & \bigoplus_{k>0} H^{4k}(B, \partial_0 B; \mathbb{R})
\end{array}
$$

where $G(B, \partial_0 B)$ is the group of all homotopy classes of pointed maps $\xi : B/\partial_0 B \to G/O$. Here $E^n(-, \eta)$ is the map which sends ξ to the direct limit of $E^{n,m}(\xi, \eta)$ as m goes to ∞.

Since the torsion of a linear disk bundle is trivial, the torsion of the disk bundle $E^{n,m}(\xi, \eta)$ is equal to the torsion of the h-cobordism bundle given by deleting a neighborhood of a section. The fiberwise boundary of $E^{n,m}(\xi, \eta)$ is a smooth $n + m - 1$ dimensional sphere bundle over B which is fiberwise tangentially homeomorphic to the linear sphere bundle $S^{n+m-1}(\eta)$.

Corollary 2.2.5. *Suppose that $n + m - 1$ is odd. Then the vertical boundary $\partial^v E^{n,m}(\xi, \eta)$ of this disk bundle is a smooth sphere bundle which is fiberwise tangentially homeomorphic to the linear sphere bundle $S^{m+n-1}(\eta)$ and fiberwise diffeomorphic to this bundle over $\partial_0 B$ and the difference torsion is twice the normalized chern character of ξ:*
$$\tau_{2k}^{IK}(\partial^v E^{n,m}(\xi, \eta), S^{n+m-1}(\eta)) = (-1)^n 2\widetilde{ch}_{4k}(\xi) \in H^{4k}(B, \partial_0 B; \mathbb{R})$$

In particular, assuming that ξ is rationally nontrivial, this gives an exotic smooth structure on $S^{n+m-1}(\eta)$.

Proof. All axiomatic higher torsion invariants satisfy the equation

$$\tau_{2k}(E) = \tfrac{1}{2}\tau_{2k}(\partial^v E) + \tfrac{1}{2}\tau_{2k}(DE)$$

where $\partial^v E$ is the fiberwise boundary of E and DE is the fiberwise double of E and the double of $E^{n,m}(\xi,\eta)$, having closed even dimensional manifold fibers, has the same higher torsion as the linear sphere bundle $S^{n+m}(\eta)$:

$$\tau_{2k}^{\mathrm{IK}}(\partial^v E^{n,m}(\xi,\eta)) = 2\tau_{2k}(E^{n,m}(\xi,\eta)) - \tau_{2k}^{\mathrm{IK}}(S^{n+m}(\eta)) = (-1)^n 2\widetilde{ch}_{4k}(\xi) - 2\widetilde{ch}_{4k}(\eta)$$

$$\tau_{2k}^{\mathrm{IK}}(S^{n+m-1}) = -2\widetilde{ch}_{4k}(\eta)$$

The difference $(-1)^n 2\widetilde{ch}_{4k}(\xi)$ is the relative torsion. $\qquad\square$

2.3. Hatcher handles.

Suppose that $p : M \to B$ is a smooth manifold bundle whose fiber dimension is $N = n + m$ where $m > n > q$. Let $s : B \to M$ be a smooth section of p with image in the fiberwise interior of M. Since $m = N - n > q + 1$, the space of n frames in $\mathbb{R}^N$ is $q + 1$-connected. So there exists a smooth fiberwise embedding $f : D^n \times B \to M$ equal to s along the zero section section and f is uniquely determined up to isotopy by s. Let η be the vertical normal bundle to the image of f in M. This is the unique m plane bundle over B which is stably isomorphic to the pull back along s of the vertical tangent bundle of M. Then f extends to a fiberwise embedding

$$D(s) : D^n \times D^m(\eta) \hookrightarrow M$$

whose image is a tubular neighborhood of the image of the section s and $D(s)$ is determined up to isotopy by s. We will use this embedding $D(s)$ to attach positively and negatively suspended Hatcher disk bundles to the top $M \times 1$ of the bundle $M \times I \to B$. We call these *positive* and *negative Hatcher handles.*

2.3.1. *Positive Hatcher handles.*

Let $h_0 : D_0^n \hookrightarrow S^{n-1} \times I$ be a fixed smooth embedding where $D_0^n = D^n$ is a copy of the standard n-disk. Taking the product with $D^m(\eta)$ we get a fiberwise embedding of $D_0^n \times D^m(\eta)$ into $E^{n,m}(\xi,\eta)$:

$$h = h_0 \times id_{D^m(\eta)} : D_0^n \times D^m(\eta) \hookrightarrow S^{n-1} \times I \times D^m(\eta) \subseteq E^{n,m}(\xi,\eta)$$

Using these two maps we can attach $\sigma_+ E^{n,m}(\xi,\eta)$ to $M \times I$ to get

$$E_+^{n,m}(M,s,\xi) = M \times [0,1] \cup_{D(s) \times 1} D_0^n \times D^m(\eta) \times [1,2] \cup_{h \times 2} E^{n,m}(\xi,\eta) \times [2,3]$$

We can write this more succinctly as

$$E_+^{n,m}(M,s,\xi) = M \times I \cup_{D(s)} B^{n,m}(\xi,\eta)$$

where $B^{n,m}(\xi,\eta)$ is the **positive Hatcher handle** given by

$$B^{n,m}(\xi,\eta) = D_0^n \times D^m(\eta) \times I \cup_{h \times 1} E^{n,m}(\xi,\eta) \times [1,2]$$

attached on its base $\partial_0 B^{n,m}(\xi,\eta) = D_0^n \times D^m(\eta) \times 0$. Since the bundle pair $(B^{n,m}(\xi,\eta), \partial_0)$ is fiberwise homeomorphic to the disk bundle pair $D^n \times D^m(\eta) \times (I, 0)$, the bundle $E_+^{n,m}(M,s,\xi)$ is fiberwise homeomorphic to the bundle $M \times I$. However, $E_+^{n,m}(M,s,\xi)$ is a smooth bundle (when corners are rounded) whose fibers are h-cobordisms.

Theorem 2.3.1. *Let T be a closed fiberwise tubular neighborhood of $s(B)$ in M. Then there is a fiberwise homeomorphism $M \times I \to E_+^{n,m}(M,s,\xi)$ which is the identity (and thus a diffeomorphism) on $M \times 0$ and a diffeomorphism on the closure of $(M - T) \times I$. Furthermore the difference torsion is the same as the IK-torsion of $E^{n,m}(\xi,\eta)$:*

$$\tau(E_+^{n,m}(M,s,\xi), M \times I) = \tau(E^{n,m}(\xi,\eta)) = (-1)^n \widetilde{ch}_{4k}(\xi) \in H^{4k}(B, \partial_0 B; \mathbb{R})$$
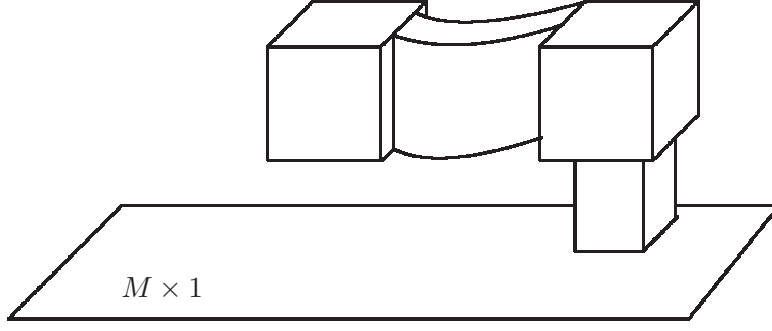
FIGURE 2. (Positive Hatcher handle) The positive suspension $\sigma_+ E^{n,m}(\xi,\eta)$ is attached to the top $M \times 1$ of $M \times I$ by the "stem" $D_0^n \times D^m(\eta) \times I$.

Remark 2.3.2. This theorem can be viewed as the commutativity of the diagram:

$$
\begin{array}{c}
G(B, \partial_0 B) \\
\end{array}
$$

$$
\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(D(\eta)) \xrightarrow{\ s_* \ } \pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M) \xrightarrow{\ \tau^{\mathrm{IK}} \ } \bigoplus_{k>0} H^{4k}(B, \partial_0 B; \mathbb{R})
$$

with arrows $E^n(-,\eta)$, $E_+^n(M,s,-)$, $(-1)^n \widetilde{ch}$, and τ^{IK}.

Let $M' = \partial_1 E_+^{n,m}(M,s,\xi)$ be the top boundary of the h-cobordism bundle $E_+^{n,m}(M,s,\xi)$.

Corollary 2.3.3. M' *is fiberwise tangentially homeomorphic to M and, if the fiber dimension $N = n + m$ of M' is odd, then the relative IK torsion is equal to twice the normalized chern character of ξ:*

$$
\tau(M', M) = (-1)^n 2 \widetilde{ch}_{4k}(\xi) \in H^{4k}(B, \partial_0 B; \mathbb{R})
$$

2.3.2. *Negative Hatcher handles.* Attaching the negative suspension of $E^{n,m}(\xi,\eta)$ to the top of $M \times I$ is very similar but easier since the attaching map will be simply the map $D(s)$ from $D^n \times D^m(\eta) \subseteq \partial^\vee \sigma_- E^{n,m}(\xi,\eta)$ into $M \times 1 \subseteq \partial^\vee M \times I$. However,

$$
M \times I \cup_{D(s) \times 1} D^n \times D^m(\eta) \times [1,2] \cong M \times I
$$

since we can pull $D^n \times D^m(\eta) \times I$ into $M \times I$. Therefore, $M \times I \cup \sigma_- E^{n,m}(\xi,\eta)$ can more economically be described by attaching the base $\partial_0 A^{n,m}(\xi,\eta) = E^{n,m}(\xi,\eta) \times 0$ of the **negative Hatcher handle**

$$
A^{n,m}(\xi,\eta) = E^{n,m}(\xi,\eta) \times I \cup_{F(j) \times 1} D^n \times D^m(\eta) \times [1,2]
$$

to the top of $M \times I$ using the composite map

$$
E^{n,m}(\xi,\eta) \xrightarrow{\ F(j) \ } D^n \times D^m(\eta) \xrightarrow{\ D(s) \ } M
$$

The negative Hatcher handle is shown in Figure 3 and also in the top figure in Figure 2.1 where $A^{n,m}(\xi,\eta) = E_1$. The argument above shows that $M \times I \cup \sigma_- E^{n,m}(\xi,\eta)$ is fiberwise diffeomorphic to:

$$
E_-^{n,m}(M,s,\xi) = M \times I \cup_{D(s)F(j)} A^{n,m}(\xi,\eta)
$$

with higher difference torsion given by

$$
\tau(E_-^{n,m}(M,s,\xi), M \times I) = -\tau(E^{n,m}(\xi,\eta)) = (-1)^{n+1} \widetilde{ch}_{4k}(\xi) \in H^{4k}(B, \partial_0 B; \mathbb{R})
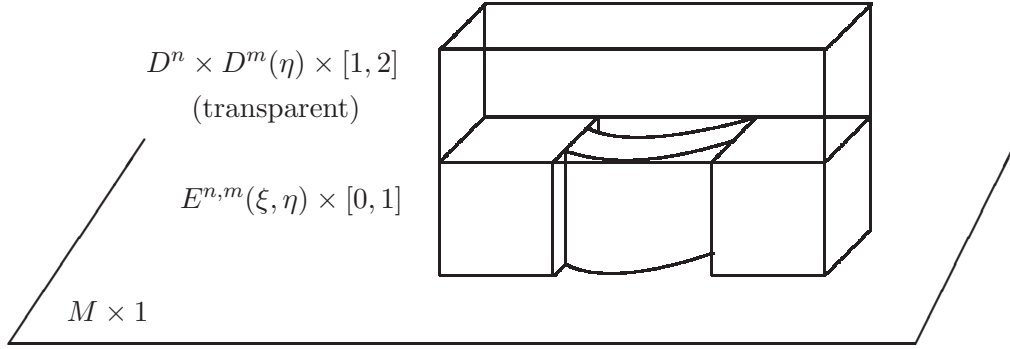$$

16

FIGURE 3. (Negative Hatcher handle) $A^{n,m}(\xi, \eta)$ is attached to the top $M \times 1$ of $M \times I$ along its base $E^{n,m}(\xi, \eta) \times 0$.

2.3.3. *Cancellation of Hatcher handles.* We will take the "union" of the two constructions given above and attach both positive and negative Hatcher handles along the same section $s : B \to M$ and show that they cancel. As before, we have a smooth embedding

$$D(s) : D^n \times D^m(\eta) \to M$$

whose image is a tubular neighborhood of $s(B)$. inside this disk bundle we create two smaller isomorphic disk bundles using embedding:

$$j_+, j_- : D^n \times D^m(\eta) \to D^n \times D^m(\eta)$$

given by $j_+(x, y) = (\frac{1}{3}(x + e_n), y)$ where e_n is the last unit vector of D^n and $j_-(x, y) = (\frac{1}{3}(x - e_n), y)$. Since they are less than half as wide, these two embeddings are disjoint. Suppose that $E^{n,m}(\xi, \eta)$ is a Hatcher disk bundle as in the construction above. We first attach the positive Hatcher handle $B^{n,m}(\xi, \eta)$ along its base $\partial_0 B^{n,m}(\xi, \eta) = D^n \times D^m(\eta) \times 0$ to the top $M \times 1$ of $M \times I$ using the fiberwise embedding $D(s) \circ j_-$. Next we attach the negative Hatcher handle $A^{n,m}(\xi, \eta)$ to the top of $M \times I$ along its base $\partial_0 A^{n,m}(\xi, \eta) = E^{n,m}(\xi, \eta)$ using the composite map

$$E^{n,m}(\xi, \eta) \xrightarrow{F(j)} D^n \times D^m(\eta) \xrightarrow{j_+} D^n \times D^m(\eta) \xrightarrow{D(s)} M$$

Let T be the image of $D(s)$ with corners rounded. Thus T is a D^{n+m}-bundle over B. Let $S = \partial' T$ be the fiberwise boundary of T. This is a sphere bundle over B. After attaching the positive and negative Hatcher handles to the top of $M \times I$ we get a new bundle

$$W = M \times I \cup_{D(s) \circ j_-} B^{n,m}(\xi, \eta) \cup_{D(s) \circ j_+ \circ F(j)} A^{n,m}(\xi, \eta)$$

Note that since $B^{n,m}(\xi, \eta)$ and $A^{n,m}(\xi, \eta)$ are both attached in the interior of T, this new bundle is the union of $C \times I$ and $T \times I \cup B \cup A$ where C is the closure of $M - T$ and A, B denote the Hatcher handles.

Proposition 2.3.4 (second cancellation lemma)**.** *W is fiberwise diffeomorphic to $M \times I$ after rounding corners and this diffeomorphism is the identity on $C \times I$ and on $M \times 0$.*

Proof. The argument is almost the same as in Proposition 2.1.5. Since $\partial_0 A^{n,m}(\xi, \eta) = E^{n,m}(\xi, \eta)$ is a disk bundle attached using the same tangential data as $B^{n,m}(\xi, \eta)$, there is an isotopy of the attaching map $D(s) \circ j_+ \circ F(j)$ of the negative Hatcher handle $A^{n,m}(\xi, \eta)$ to the mapping

$$E^{n,m}(\xi, \eta) \to E^{n,m}(\xi, \eta) \times 1 \subset (E^{n,m}(\xi, \eta) \cup D_0^n \times D^m(\eta)) \times I = B^{n,m}(\xi, \eta)$$

17

placing $A^{n,m}(\xi, \eta)$ onto the top sides $E^{n,m}(\xi, \eta) \times 1$ of the positive Hatcher handle $B^{n,m}(\xi, \eta) = E^{n,m}(\xi, \eta) \cup D_0^n \times D^m(\eta) \times I$. After moving the attaching map, $A^{n,m}(\xi, \eta)$ is attached on top of $E^{n,m}(\xi, \eta) \times I$ and their union is

$$V^{n,m}(\xi, \eta) = E^{n,m}(\xi, \eta) \times I \cup A^{n,m}(\xi, \eta) = E^{n,m}(\xi, \eta) \times [0, 2] \cup D^n \times D^m(\eta) \cong A^{n,m}(\xi, \eta)$$

which is attached on $M \times 1$ along the image of $D(s) \circ j_-$ by the "stem" $D_0^n \times D^m(\eta)$. By Lemma 2.1.1, $V^{n,m}(\xi, \eta) \cup D_0^n \times D^m(\eta)$ is fiberwise diffeomorphic to $D^n \times D^m(\eta) \cup D_0^n \times D^m(\eta)$. This is a linear disk bundle and, therefore, attaching this to the top of $T \times I$ gives a bundle X diffeomorphism of $T \times I$ fixing $S \times I$. This sequence of deformations and diffeomorphisms gives a differomorphism $T \times I \cup B \cup A \cong T \times I$ which is the identity on $S \times I$ and therefore, can be pasted with $C \times I$ to give a fiberwise diffeomorphism $W = C \times I \cup T \times I \cup B \cup A \cong M \times I$ as claimed. $\qquad\square$

2.4. **Immersed Hatcher handles.** Since "Hatcher handles" are attached in a neighborhood of one point, they can be attached at several points at the same time.

Let L be a q manifold with boundary $\partial L = \partial_0 L \cup \partial_1 L$ where $\partial_0 L, \partial_1 L$ are $q - 1$ manifolds meeting along their common boundary. Let $\lambda : L \to B$ be an immersion so that $\lambda^{-1}(\partial_1 B) = \partial_1 L$ and let $\tilde{\lambda} : L \to M$ be an embedding over λ. Suppose as before that $m > n > q$ and let

$$D(\tilde{\lambda}) : D^n \times D^m(\eta) \hookrightarrow M$$

be a smooth embedding over $\lambda : L \to B$ where η is the pull-back along $\tilde{\lambda} : L \to M$ of the stable vertical tangent bundle of M.

Let ξ be an n-plane bundle over L which is trivial over $\partial_1 L$ so that $J(\xi) = 0 \in J(L/\partial_1 L)$ and let η_0 be the unique m-plane bundle over L so that $\xi \oplus \eta_0 \cong \eta$. We define $W = E_+^{n,m}(M, \tilde{\lambda}, \xi)$ to be the smooth h-cobordism bundle over B so that $\partial_0 W = M$ given by

$$E_+^{n,m}(M, \tilde{\lambda}, \xi) = M \times I \cup_{D(\tilde{\lambda})} B^{n,m}(\xi, \eta)$$

where $B^{n,m}(\xi, \eta)$ is the positive Hatcher handle parametrized by L. This Hatcher handle needs to be "tapered off" along $\partial_0 L$. A more precise description of this is as follows.

First we note that the smooth disk bundle over L given by

$$E_L^{n,m+1}(\xi, \eta) = D^n \times D^m(\eta) \times I \cup B^{n,m}(\xi, \eta)$$

is fiberwise diffeomorphic to $D^n \times D^m \times I$ over $\partial_0 L$. We choose such a diffeomorphism. Let T be the image of $D(\tilde{\lambda}) : D^n \times D^m(\eta) \to M$. So $T \times I \subseteq M \times I$ is fiberwise diffeomorphic to $D^n \times D^m(\eta) \times I$. The smooth h-cobordism bundle $E_+^{n,m}(M, \tilde{\lambda}, \xi)$ can then be given by:

$$E_+^{n,m}(M, \tilde{\lambda}, \xi) = (M - T) \times I \cup E_L^{n,m+1}(\xi, \eta)$$

Theorem 2.4.1 (torsion of immersed Hatcher handle)**.** *The higher IK difference torsion of this bundle with respect to $M \times I$ is the image under the mapping*

$$\lambda_* : H^{4k}(L, \partial_0 L) \cong H_{q-4k}(L, \partial_1 L) \to H_{q-4k}(B, \partial_1 B) \cong H^{4k}(B, \partial_0 B)$$

of the normalized chern character of ξ:

$$\tau_{2k}^{IK}(E_+^{n,m}(M, \tilde{\lambda}, \xi), M \times I) = \lambda_* \left((-1)^n \widetilde{ch}_{4k}(\xi) \right) \in H^{4k}(B, \partial_0 B; \mathbb{R})$$

Remark 2.4.2. This theorem can be viewed as the commutativity of the diagram:

$$
\begin{array}{ccccc}
 & \xrightarrow{\quad (-1)^n \widetilde{ch} \quad} & & & \\
G(L, \partial_0 L) & \xrightarrow{\quad E_L^n(-, \eta) \quad} & \pi_0 \widetilde{\mathcal{S}}_{L, \partial_0}^s(D(\eta)) \xrightarrow{\quad \tau^{\mathrm{IK}} \quad} & \bigoplus_{k > 0} H^{4k}(L, \partial_0 L; \mathbb{R}) \\
 & {\scriptstyle E_+^n(M, \tilde{\lambda}, -)} \searrow & \Big\downarrow {\scriptstyle D(\tilde{\lambda})_*} & \Big\downarrow {\scriptstyle \lambda_*} \\
 & & \pi_0 \widetilde{\mathcal{S}}_{B, \partial_0}^s(M) \xrightarrow{\quad \tau^{\mathrm{IK}} \quad} & \bigoplus_{k > 0} H^{4k}(B, \partial_0 B; \mathbb{R})
\end{array}
$$

The commutativity of the upper curved triangle is Theorem 2.2.3.

To prove this, we need to recall the precise statement of the Framing Principle from [23]. Suppose that $W \to B$ is a smooth h-cobordism bundle with fiberwise boundary equal to

$$\partial^v W = M \cup \partial^v M \times I \cup M_1$$

and $f : W \to I$ is a fiberwise generalized Morse function equal to 0 on M and 1 on M_1 and equal to projection to I on $\partial^v M \times I$. Suppose that the fiberwise singular set $\Sigma(f)$ of f does not meet $W_{\partial_0 B}$. In particular, $W_{\partial_0 B} \cong M_{\partial_0 B} \times I$. We are in the restricted case when the birth death points of f are **framed** in the sense that the negative eigenspace bundle of $D^2 f$ is trivial over the birth-death points. This implies that, over the set $\Sigma_i(f)$ of Morse points of f of index i, the negative eigenspace bundle of $D^2 f$ is trivial along $\partial_0 \Sigma_i(f)$ which is equal to the set of birth-death points to which $\Sigma_i(f)$ converges. The Framing Principle was proved in this restricted case in [21].

In general, the negative eigenspace bundle is a well defined stable vector bundle $\xi = \xi(f)$ on the entire singular set $\Sigma(f)$. It is defined as follows. At each index i critical point x of f let $\xi(x) = \xi_i(x) \oplus \epsilon^{N-i}$ where ϵ^{N-i} is the trivial bundle with dimension $N - i$ and $N = n + m + 1$ is the dimension of the fiber of $W \to B$. This defines an N-plane bundle over $\Sigma_i(f)$. At each cubic point we identify the positive cubic direction with the positive first coordinate direction in ϵ^{N-i}. This has the effect of pasting together these N-plane bundles over $\Sigma_i(f)$ and $\Sigma_{i+1}(f)$ along their common boundary for each i. The result is an N-plane bundle over all of $\Sigma(f)$.

The projection mapping $p : (\Sigma(f), \partial \Sigma(f)) \to (B, \partial_1 B)$ induces a map in cohomology using Poincaré duality assuming that B is oriented. (If B is not oriented then just replace it with the disk bundle of the orientation line bundle.)

$$p_*^\Sigma : H^*(\Sigma(f)) \cong H_{q-*}(\Sigma(f), \partial \Sigma(f)) \to H_{q-*}(B, \partial_1 B) \cong H^*(B, \partial_0 B)$$

Similarly, for each index i we have the push-down operator:

$$p_* : H^*(\Sigma_i(f), \partial_0 \Sigma_i(f)) \cong H_{q-*}(\Sigma_i(f), \partial_1 \Sigma_i(f)) \to H_{q-*}(B, \partial_1 B) \cong H^*(B, \partial_0 B)$$

where $\partial_1 \Sigma_i(f) = \Sigma_i(f) \cap \partial \Sigma(f)$ and $\partial_0 \Sigma_i(f)$ is the set of birth-death points in the closure of $\Sigma_i(f)$. We use the orientation for $\Sigma_i(f)$ which agrees with the orientation of B and we take the orientation of $\Sigma(f)$ which agrees with the orientation of $\Sigma_i(f)$ for i even. As a result of these sign conventions we have the following observation.

Lemma 2.4.3. *In the restricted case when the birth-death points of f are framed, then the image under p_*^Σ of the Chern character of $\xi(f)$ is equal to the the alternating sum of images under the push-down operators:*

$$p_* : H^*(\Sigma_i(f), \partial_0 \Sigma_i(f)) \to H^{4k}(B, \partial_0 B)$$

of the Chern character of $\xi_i = \xi | \Sigma_i(f)$:

$$p_*^\Sigma(ch_{4k}(\xi \otimes \mathbb{C}) = \sum_i (-1)^i p_*(ch_{4k}(\xi_i \otimes \mathbb{C}) \in H^*(B, \partial_0 B)$$

19

Theorem 2.4.4 (Relative Framing Principle). *Suppose that the manifold B and the stable bundle $\xi = \xi(f)$ and both oriented. Then the degree $4k$ higher relative IK torsion invariant $\tau_{2k}^{IK}(W, M) \in H^{4k}(B, \partial_0 B; \mathbb{R})$ is given by the higher torsion of the family of acyclic chain complexes $C(f)$ given by f plus the push down of the normalized chern character of ξ:*

$$\tau_{2k}^{IK}(W, M) = \tau_{2k}(C(f)) + p_*^{\Sigma}(\widetilde{ch}_{4k}(\xi)) \in H^{4k}(B, \partial_0 B)$$

Proof. The published version of the Framing Principle [23] assumes that $\partial_0 B$ is empty. However, the relative case follows easily from the absolute case in the present setting where we have an h-cobordism bundle W. Just take the base $\partial_0 W = M$ and embed it into the boundary of a very large dimensional trivial disk bundle $B \times D^N$. Let ν_M be the vertical normal bundle of M in $B \times S^{N-1}$ and let ν_W be the extension of ν_M to W. Then we have a new bundle:

$$\Delta = B \times D^N \cup D(\nu_W)$$

over B. Since $D(\nu_W)$ is an h cobordism bundle, this is a smooth N-disk bundle over B (after rounding off corners). By additivity and invariance after passing to linear disk bundles, we have:

$$\tau^{IK}(W, M) = \tau^{IK}(D(\nu_W, \nu_M) = \tau^{IK}(\Delta, B \times D^N) = \tau^{IK}(\Delta)$$

But, Δ is a disk bundle over B which is trivial over $\partial_0 B$. So, we can collapse $\partial_0 B$ to a point to get a new bundle $\overline{\Delta}$ over $B/\partial_0 B$. The Framing Principle for $\overline{\Delta} \to B/\partial_0 B$ is then equivalent to the relative Framing Principle for (W, M).

To do this more precisely, we do the same trick as before, removing a tube $T = D(\nu_M) \times I$ in a collar neighborhood of $B \times S^{N-1}$ and replace it with W. The new fiberwise Morse function will be equal to the distance squared from the origin in $B \times D^N - T$ and equal to f (rescaled to match) on W. Now we collapse the bundle over $\partial_0 B$. By construction, the fiberwise generalized Morse function will factor through this quotient bundle and the original Framing Principle applies. $\square$

Proof of Theorem 2.4.1. We will start with a fiberwise oriented Morse function on the bundle $E_L^{n,m}(\xi, \eta) \to L$ and then modify it to give a fiberwise oriented generalized Morse function which is framed on the birth-death set.

The bundle $E_L = E_L^{n,m}(\xi, \eta)$ is obtained from $D^n \times D^m(\eta) \times I$ by attaching two handle with cores of dimension $n - 1$ and n. This means it has a fiberwise Morse function $f : E_L \to I$ which is equal to the projection map to I in a neighborhood of the bottom $D^n \times D^m(\eta) \times 0$ and sides $\partial(D^n \times D^m(\eta)) \times I$. Furthermore f will have two critical points over every point $t \in L$. These critical points x_t, y_t have index $n - 1$ and n respectively. The vertical tangent bundle of E_L splits as $\epsilon^{n-1} \oplus (\eta \oplus \epsilon^1)$ along the section x_t of E_L where the trivial $n - 1$ plane bundle ϵ^{n-1} is the negative eigenspace of $D^2 f_t$ along x_t. The vertical tangent bundle of E_L along y_t splits as $\xi \oplus (\eta_0 \oplus \epsilon^1)$ where the homotopically trivial ξ is the negative eigenspace bundle.

Along $\partial_0 L$, the bundle ξ is trivial and the handle corresponding to y_t is in cancelling position with the handle corresponding to x_t since they are both standard linear handle along $\partial_0 L$ by construction. This implies that these critical points can be cancelled along a birth-death set of index $n - 1$. Since the negative eigenspace bundle ξ is trivial along this set, this is a framed birth-death set. The new singular set $\Sigma(f)$ is now a q-manifold with boundary lying over $\partial_1 L$. It has a framed birth-death set and Morse sets in two indices $\Sigma_n(f)$ and $\Sigma_{n-1}(f)$. The descending bundles are $\xi_{n-1} = \epsilon^{n-1}$ and $\xi_n = \xi$. These are oriented bundle since they are homotopically trivial. Also the cellular chain complex is trivial at every point. Therefore, by the Framing Principle, the higher relative IK torsion of $E_L^{n,m}(\xi, \eta)$ is

$$\tau_{2k}^{IK}(E_L^{n,m}(\xi, \eta), D^n \times D^m(\eta) \times I) = (-1)^n \widetilde{ch}_{4k}(\xi) \in H^{4k}(L, \partial_0 L; \mathbb{R})$$

From this fiberwise oriented generalized Morse function we can construct a fiberwise oriented generalized Morse function F on $E_+^{n,m}(M, \tilde{\lambda}, \xi) = (M - T) \times I \cup E_L$ by taking projection to I on

the first piece $(M - T) \times I$ and f on the second piece E_L. The singular set of F is the image under $D(\tilde{\lambda})$ of the singular set of f. Consider the following commuting diagram.

$$\Sigma_n(f) \xrightarrow{\subset} \Sigma(f) \xrightarrow{D(\tilde{\lambda})} \Sigma(F)$$

This implies that the image of the push-down of the chern character of ξ along the map p is equal to the image of the chern character of ξ under λ. So, by the relative Framing Principle, we have

$$\tau_{4k}^{\mathrm{IK}}(E_+^{n,m}(M, \tilde{\lambda}, \xi), M) = (-1)^n p_*(\widetilde{ch}_{4k}(\xi)) = (-1)^n \lambda_*(\widetilde{ch}_{4k}(\xi))$$

as claimed. $\qquad\square$

3. Main Theorems

There are two main theorems in this paper. The first concerns the set of possible higher torsion invariants of exotic smooth structures on smooth manifold bundles.

The second theorem is that, rationally stably, the immersed Hatcher construction gives all possible exotic smooth structures on smooth manifold bundles with odd dimensional fibers. This is a combination of the following two theorems. First recall from Appendix B that

$$\pi_0 \widetilde{\mathcal{S}}_{B,\partial_0}^s(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}; \mathbb{R})$$

and the image of an exotic smooth structure M' on M is denoted

$$\Theta_M(M') = \Theta(M', M) \in \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}; \mathbb{R})$$

and we call it the **rational exotic structure class**.

Theorem 3.0.5. *When the fiber dimension is odd, the rational exotic structure class $\Theta(M', M)$ given by the immersed Hatcher construction $E_+^{n,m}(M, \tilde{\lambda}, \xi)$ is the image of the Poincaré dual of twice the normalized chern character of ξ under the map in homology induced by the embedding $\tilde{\lambda} : (L, \partial_1 L) \to (M, M_{\partial_1 B})$*

$$\tilde{\lambda}_* : \bigoplus_{k>0} H^{4k}(L, \partial_0 L) \cong \bigoplus_{k>0} H_{q-4k}(L, \partial_1 L) \to \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B})$$

Thus:

$$\Theta(M', M) = (-1)^n \tilde{\lambda}_*(2\widetilde{ch}(\xi))$$

where $\widetilde{ch}(\xi) = \sum_{k>0} \widetilde{ch}_{4k}(\xi)$ as defined in Remark 1.2.3.

Remark 3.0.6. By definition of the normalized chern character, the exotic structure class $\Theta(M', M)$ lies in the image of

$$\bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}; \zeta(2k+1)\mathbb{Q})$$

In particular, $\Theta(M', M)$ is a scalar multiple of an integral class in every degree.

Proof. The proof will show the commutativity of the following diagram which is a slightly stronger statement:

$$
\begin{array}{c}
G(L,\partial_0 L) \xrightarrow{\ top\,E_L^n(-,\eta)\ } \pi_0 \widetilde{\mathcal{S}}^s_{L,\partial_0}(E) \xrightarrow{\quad\quad\tau^{\mathrm{IK}}\quad\quad} \bigoplus_{k>0} H_{q-4k}(L,\partial L)
\end{array}
$$

(with the top curved arrow $(-1)^n 2\widetilde{ch}$, the diagonal $top\,E_+^n(M,\tilde\lambda,-)$, vertical maps $D(\tilde\lambda)_*$, λ_*, diagonal $\tilde\lambda_*$, and bottom row $\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M) \xrightarrow{\Theta} \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}) \xrightarrow{p_*} \bigoplus_{k>0} H_{q-4k}(B,\partial_1 B)$)

The middle portion can be expanded into the following diagram where $E = D^n \times D^m(\eta)$ is the disk bundle over L which is diffeomorphic to a tubular neighborhood of the image of $\tilde\lambda : L \to M$.

$$
\begin{array}{c}
\pi_0 \widetilde{\mathcal{S}}^s_{L,\partial_0}(E) \xrightarrow{\gamma_E} \pi_0 \Gamma_{L,\partial_0} \mathcal{H}_L^\%(E) \xrightarrow{\theta_E} \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1}) \xrightarrow{\cong} \bigoplus_{k>0} H_{q-4k}(L,\partial_1 L)
\end{array}
$$

(top curved arrow τ^{IK}; left vertical $D(\tilde\lambda)_*$; bottom row $\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M) \xrightarrow{\gamma_M} \pi_0 \Gamma_{B,\partial_0}\mathcal{H}_B^\%(M) \xrightarrow{\theta_M} \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1})$; diagonal $\tilde\lambda_*$; bottom curved arrow Θ_M)

The straight horizontal maps in the second diagram are isomorphisms of vector spaces after tensoring with $\mathbb{R}$ and the vertical maps are all induced by $\lambda : L \to B$, $\tilde\lambda : L \to M$ and $D(\tilde\lambda) : E \to M$. The left hand square commutes by the stratified smoothing theorem (Corollary A.5.19). The square in the middle commutes by the stratified homotopy calculation (Corollary B.4.2). The triangle on the right commutes since it comes from a commuting diagram of spaces. The composition of the top three arrows is equal to τ^{IK} by normalization of θ_E (Proposition B.2.3). The bottom curved triangle commutes by definition: $\Theta_M := \theta_M \circ \gamma_M$. Therefore, the second diagram commutes. So, the middle quadrilateral in the first diagram commutes.

If we look at the top of the immersed Hatcher handle we get an element

$$
top(E_+^{n,m}(M,\tilde\lambda,\xi)) \in \widetilde{\mathcal{S}}^s_{B,\partial_0}(M)
$$

which, by construction is the image of the Hatcher disk bundle $top(E_+^{n,m}(E,0,\xi)) \in \widetilde{\mathcal{S}}^s_{L,\partial_0}(E)$ under the stratified map $\widetilde{\mathcal{S}}^s_{L,\partial_0}(E) \to \widetilde{\mathcal{S}}^s_{B,\partial_0}(M)$. By Corollary 2.2.5 and Proposition B.2.3 the composition of the horizontal mapping on the top row of the first diagram gives the higher IK torsion invariant of the top E' of the h-cobordism bundle $E_+^{n,m}(E,0,\xi)$

$$
\tau^{\mathrm{IK}}(E') = (-1)^n 2\widetilde{ch}(\xi)
$$

since the bottom of E is a linear disk bundle over L. The theorem follows. $\qquad\square$

Proposition 3.0.7. *The vector space $\bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}; \mathbb{R})$ is spanned by the images of the possible maps*

$$
G(L,\partial_0 L) \to \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}; \mathbb{R})
$$

given by $\tilde\lambda_* \circ (-1)^n 2\widetilde{ch} = \Theta_M \circ top\,E_+^n(M,\tilde\lambda,-)$ in the theorem above.

This proposition is proved below using the Arc de Triomph construction.

Theorem 3.0.8. *When the fiber dimension N of $M \to B$ is odd and B is oriented, the higher IK relative torsion of an exotic smooth structure M' on M over $(B,\partial_0 B)$ is given by*

$$
\tau^{\mathrm{IK}}(M',M) = p_* \Theta(M',M).
$$

the image of the rational exotic smooth structure class $\Theta(M', M)$ under the push-down mapping p_* as indicated in the following commuting diagram.

$$\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M) \xrightarrow[\Theta]{} \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}) \xrightarrow[p_*]{} \bigoplus_{k>0} H_{q-4k}(B, \partial_1 B)$$

$$\tau^{IK}$$

Proof. The map p_* is $\mathbb{R}$-linear, and Theorem 3.0.5 and Proposition 3.0.7 above say that the immersed Hatcher construction gives generators for $\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B})$ and p_* sends these generators to their higher relative IK-torsion. The theorem follows. □

We have the following immediate corollary.

Corollary 3.0.9. *If M is a smooth bundle over B and both fiber and base are oriented manifolds with odd fiber dimension $N \geq 2q + 3$ then the possible values of the higher IK-relative torsion $\tau^{IK}(M', M)$ for M' an exotic smooth structure on M which agrees with M over $\partial_0 B$ will span the image of the push-down map*

$$p_* : \bigoplus_{k>0} H^{N+4k}(M, \partial_0 M; \mathbb{R}) \to \bigoplus_{k>0} H^{4k}(B, \partial_0 B; \mathbb{R})$$

where $\partial_0 M = M_{\partial_0 B} \cup \partial^v M$.

Remark 3.0.10. By Remark 3.0.6, we recover the theorem first proved in [22] that the higher IK-torsion lies in the image of

$$\bigoplus_{k>0} H^{4k}(B, \partial_0 B; \zeta(2k+1)\mathbb{Q})$$

Corollary 3.0.11. *Theorem 3.0.8 also holds in the case when the fiber dimension is even.*

Proof. If $M, M' \to B$ have even dimensional fibers then $M' \times I, M \times I \to B$ have odd dimensional fibers and we have:

$$\tau^{IK}(M', M) = \tau^{IK}(M' \times I, M \times I) = p_* \Theta(M' \times I, M \times I) = p_* \Theta(M', M)$$

by Theorem 3.0.8 since both τ^{IK} and Θ are stable invariants. □

3.1. **Arc de Triomph 2.** Proposition 3.0.7 follows from the Arc de Triomph construction and the stratified deformation lemma 3.2.1. The Arc de Triomph construction is an extension of the Hatcher construction which rationally stably produces all exotic smooth structures on a compact manifold bundle. The stratified deformation lemma shows that each AdT construction can be deformed into an immersed Hatcher construction.

We explained the basic construction in subsection 2.1. It only remains to describe the full construction and prove the following theorem.

Theorem 3.1.1 (Arc de Triomph Theorem)**.** *The AdT construction gives virtually all stable exotic smooth structures on a compact manifold bundle with odd dimensional fibers.*

Remark 3.1.2. If $M \to B$ is a smooth bundle whose fibers are even dimensional, the AdT construction rationally stably produces all exotic smooth structures on $M \times I \to B$. By definition these are stable smooth structures on $M \to B$. So, the theorem implies that *the AdT construction produces virtually all stable smooth structures on all compact manifold bundles.*

3.1.1. *AdT construction.* The Arc de Triomph construction goes as follows. Suppose that $M \to B$ is a smooth manifold bundle over a compact oriented q-manifold B with odd fiber dimension $N = n + m$ where $m > n > q$. Suppose $\partial B = \partial_0 B \cup \partial_1 B$ where $\partial_0 B, \partial_1 B$ meet along their common boundary. Then we will construct an element of $\widetilde{\mathcal{S}}^s_{B,\partial_0 B}(M)$.

Definition 3.1.3. By a **stratified set** over B with coefficients in X we mean a pair (Σ, ψ) where Σ is a compact smooth oriented q manifold together with a smooth mapping $\pi : \Sigma \to B$ sending $\partial \Sigma$ to ∂B and $\psi : \Sigma \to X$ is a continuous mapping so that $\pi : \Sigma \to B$ has only fold singularities and the singular set Σ_0 is a $q - 1$ submanifold of Σ transverse to $\partial \Sigma$. Let Σ_+ and Σ_- be the closures of the subsets of $\Sigma - \Sigma_0$ on which the map $\pi : \Sigma \to B$ is orientation preserving and orientation reversing, respectively. Thus $\Sigma_- \cap \Sigma_+ = \Sigma_0$ and $\Sigma_- \cup \Sigma_+ = \Sigma$.

We say that (Σ, ψ) is a **stratified subset** of a smooth bundle M over B if Σ is a smooth submanifold of M and $\pi : \Sigma \to B$ is the restriction of $p : M \to B$.

Let $SD^X_{B,\partial_0}(M)$ be the set of stratified deformation classes of stratified subsets (Σ, ψ) of M over B with coefficients in X so that $\pi \Sigma$ is disjoint from $\partial_0 B$. By a **stratified deformation** of stratified subsets $(\Sigma, \psi) \simeq (\Sigma', \psi')$ of M we mean a stratified subset (L, φ) of $M \times I$ over $B \times I$ with coefficients in X so that the image of $\Sigma \times I$ in $B \times I$ is disjoint from $\partial_0 B \times I$ and so that $(\Sigma, \psi), (\Sigma', \psi')$ are (M, φ) restricted to $B \times 0, B \times 1$ respectively.

The coefficients that we are interested in are $X = BSO$, classifying oriented stable vector bundles over Σ and $X = G/O = SG/SO$ classifying vector bundles with homotopy trivializations of the corresponding spherical fibration. This latter is the input for Hatcher construction and the Arc de Triomph construction will be a mapping

$$AdT : SD^{G/O}_{B,\partial_0}(M) \to \widetilde{\mathcal{S}}^s_{B,\partial_0}(M)$$

The claim is that this map is rationally split surjective. In other words, rationally stably, all exotic tangential smoothings on M are given by the construction that we will now give.

The idea of the construction is to attach negative Hatcher handles along Σ_- and positive Hatcher handles along Σ_+ and have them cancel along Σ_0. The map $\psi : \Sigma \to G/O$ gives the bundle ξ in the Hatcher handle.

Suppose that $m > n > q$ and $M \to B$ is a smooth bundle with fiber dimension $m + n$ which we assume is odd ($2q + 3$ is the minimum). Suppose we have a stratified subset $\Sigma \subset M$ with coefficient map $\psi : \Sigma \to G/O$. This gives a stable vector bundle ξ over Σ. Let η be the unique m-plane bundle over Σ isomorphic to the pull-back of the vertical tangent bundle of M and let η_-, η_+, η_0 be the restrictions of η to $\Sigma_-, \Sigma_+, \eta_0$. Then we have an embedding

$$D(\tilde\pi_+) : D^n \times D^m(\eta_+) \hookrightarrow M$$

lying over the restriction $\pi_+ : \Sigma_+ \to B$ of π to Σ_+. This gives a tubular neighborhood of Σ_+. Replacing $+$ with $-$ we get $D(\tilde\pi_-)$ lying over π_- giving a thickening of Σ_-. The embeddings $D(\tilde\pi_+)$ and $D(\tilde\pi_-)$ are disjoint except near Σ_0. To correct this we move $D(\tilde\pi_-)$ slightly to in the fiber direction near Σ_0 so that the images of $D(\tilde\pi_+)$ and $D(\tilde\pi_-)$ are disjoint everywhere. We do this move systematically by moving in the direction of, say, the last coordinate vector e_n in D^n. The result will be that the image of $D(\tilde\pi_-)$ will no longer contain Σ_- close to Σ_0.

Do this in such a way that there is an embedding

$$D(\tilde\pi_0) : D^n \times D(\eta_0) \to M$$

So that $D(\tilde\pi_-)(x, y) = D(\tilde\pi_0)(\frac{1}{3}(x + e_n), y)$ and $D(\tilde\pi_+)(x, y) = D(\tilde\pi_0)(\frac{1}{3}(x - e_n), y)$. Or, start with embedding $D(\tilde\pi_0)$ and move the mappings $D(\tilde\pi_+), D(\tilde\pi_-)$ vertically (along the fibers) so that they land in the two halves of the image of $D(\tilde\pi_0)$ as indicated.

Take the bundle $M \times I$ over B and, using the map $D(\tilde\pi_+)$ we attach the positive Hatcher handle $B^{n,m}(\xi, \eta_+)$ along its base $\partial_0 B^{n,m}(\xi, \eta_+) = D^n \times D^m(\eta_+) \times 0$ to the top $M \times 1$ of $M \times I$. Then we

attach the negative Hatcher handle $A^{n,m}(\xi, \eta_-)$ to the top of $M \times I$ using the composite map

$$E^{n,m}(\xi, \eta_-) \xrightarrow{F(j)} D^n \times D^m(\eta_-) \xrightarrow{D(\tilde{\pi}_-)} M$$

Since the images of $D(\tilde{\pi}_+)$ and $D(\tilde{\pi}_-)$ are disjoint, these attachments are disjoint.

Over $\pi(\Sigma_0)$ we have a positive and negative Hatcher handle attached on the interior of the image of $D(\tilde{\pi}_0)$. Next, we slide the attachment map for the negative Hatcher handle until it "cancels" the positive Hatcher handle. It is very easy to see how this works. Over Σ_0 the negative Hatcher handle $A^{n,m}(\xi, \eta_0)$ is attached along its base $\partial_0 A^{n,m}(\xi, \eta_0) = E^{n,m}(\xi, \eta_0)$ and the positive Hatcher handle is

$$B^{n,m}(\xi, \eta_0) = D^n \times D^m(\eta_0) \cup_{h \times 1} E^{n,m}(\xi, \eta_0) \times [1, 2]$$

We can slide the base of $E^{n,m}(\xi, \eta_0)$ of $A^{n,m}(\xi, \eta_0)$ along the top of the $M \times 1 \cup B^{n,m}(\xi, \eta_+)$ until it is equal to $E^{n,m}(\xi, \eta_0) \times 2 \subseteq B^{n,m}(\xi, \eta_0)$. We can do this in a precise way since we are working inside of the model which is the image of $D(\tilde{\pi}_0)$ in $M \times 1$. We extend this deformation to $A^{n,m}(\xi, \eta_-)$. Then we will have the desired bundle over B whose fibers are h-bordisms with base equal to the original bundle M. We call this new bundle $W(\Sigma, \psi)$ (suppressing n, m):

$$W(\Sigma, \psi) = M \times I \cup B^{n,m}(\xi, \eta_+) \cup A^{n,m}(\xi, \eta_-)$$

To be sure, we need to round off the corners. And we also need to taper off the cancelling Hatcher handles along Σ_0. But, along Σ_0, the two Hatcher handles cancel and we have a local diffeomorphism of $W(\Sigma, \psi)$ with $M \times I$ near Σ_0. Using this diffeomorphism we can identify W with $M \times I$ along this set and we have a smooth bundle over B. The local diffeomorphism exists by Proposition 2.3.4. The reason that we have a bundle at the end is because, in a neighborhood of the AdT construction along Σ_0 we either have two Hatcher handles, which are a smooth continuation of what we have at Σ_0 or we have $M \times I$ locally (which means we are only looking at the portion in the image of $D(\tilde{\pi}_0)$) and there we are using the diffeomorphism given by Proposition 2.3.4 to identify $M \times I$ with the $M \times I$ with the pair of Hatcher handles attached. So, we have local triviality and thus a smooth bundle. $W \to B$. Let

$$AdT(\Sigma, \psi) = top(W(\Sigma, \psi))$$

with tangential homeomorphism given by W. If we have any deformation of (Σ, ψ) then we can apply the same construction to this stratified set over $B \times I$ and we get a isotopy between the two constructions showing that $AdT(\Sigma, \psi)$ changes by an isotopy. Thus we get the following lemma.

Proposition 3.1.4. *The AdT construction as described above gives a well defined mapping*

$$AdT : SD^{G/O}_{B,\partial_0}(M) \to \pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M)$$

from the set of stratified deformation classes of stratified subsets (Σ, ψ) of M with coefficients in G/O to the space of stable tangential smoothings of M. Furthermore, this mapping is a homomorphism of additive groups where addition in $SD^{G/O}_{B,\partial_0}(M)$ is given by disjoint union and addition in $\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M)$ is given by the little cubes operad on the stabilization.

Proof. It is clear that $SD^{G/O}_{B,\partial_0}(M)$ is a monoid with addition given by disjoint union using transitivity to make any two stratifies subsets of M disjoint by a small perturbation. We also have additive inverse given modulo 2 by changing the orientation of Σ which switches $\Sigma_- \leftrightarrow \Sigma_+$. By "modulo 2" we mean that the sum of (Σ, ψ) with (Σ, ψ) with the orientation of Σ reversed is an element of order 2 (twice of it is zero). The proof of this is as follows.

If we add Σ with another copy Σ' having Σ_- and Σ_+ reversed then we can cancel the interiors of Σ_- with its copy Σ'_+ which has orientation $+$ producing a fold line Σ_a and similarly for Σ_+ cancelling

its copy Σ'_- along Σ_b. This leaves a small tubular neighborhood of Σ_0 having a cross-section in the shape of a figure 8:
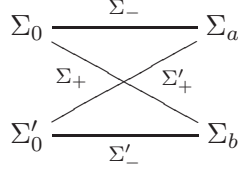
Since the width is arbitrarily small the bundle data ξ, η only changes along the length of Σ_0 not along this cross section. This cannot be cancelled since it has an odd number of crossing in the picture. (There is no actual intersection.) However, two copies of this stratified set will cancel since it is equivalent to its mirror image. The two copies can simply be merged together.

To show that the mapping AdT is additive, we take two smooth structures θ_1, θ_2 on the stabilized $M \times D^{2k-1} \times I$ which by the stabilization construction are equal to the original smooth structure on $\partial^v(M \times D^{2k-1}) \times I \cup M \times D^{2k-1} \times 0$ and on the complements of $E_1 \times D^{2k}$ and $E_2 \times D^{2k}$ respectively. By transversality, these two subsets, the supports of the two exotic smooth structures are disjoint. Therefore, by Proposition A.5.10, $\theta_1 + \theta_2$ is given by changing the smooth structure of both E_1 and E_2. This show that SdT is additive. $\qquad\square$

Remark 3.1.5. The proof above shows that the inverse of $(\Sigma, \psi) \in SD^{G/O}_{B,\partial_0}(M)$ is given by (Σ', ψ) which is (Σ, ψ) with the orientation of Σ reversed plus an element of the form $(\Sigma_0, \psi_0) \times F$ where F is the "figure 8" one-dimensional stratified set in the diagram above (in the proof of Lemma 3.1.4).

Proposition 3.1.6. *If $\psi : \Sigma \to G/O$ is trivial then so is $AdT(\Sigma, \psi)$. Therefore, AdT induces a homomorphism*

$$\overline{AdT} : \overline{SD}^{G/O}_{B,\partial_0}(M) \to \pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(M)$$

Where $\overline{SD}^{G/O}_{B,\partial_0}(M)$ is the quotient of $SD^{G/O}_{B,\partial_0}(M)$ by all (Σ, ψ) where ψ is null homotopic.

Proof. If ψ is constant then the positive and negative Hatcher handles in the Arc de Triomph construction are standard disk bundles and attaching these to the top of $M \times I$ will not change its fiber diffeomorphism type. $\qquad\square$

3.1.2. *Homotopy calculation.* To prove Theorem 3.1.1 we need calculations in the form of more commuting diagrams.

Let

$$\widetilde{ch} : SD^{G/O}_{B,\partial_0}(M) \to \bigoplus_{k>0} H_{q-4k}(M, \partial_1 M)$$

be the mapping given by sending (Σ, ψ) to the image of the normalized chern character of the bundle ξ under the mapping

$$\widetilde{ch}(\xi) \in \bigoplus H^{4k}(\Sigma) \cong \bigoplus H_{q-4k}(\Sigma, \partial\Sigma) \xrightarrow{j_*} \bigoplus H_{q-4k}(M, \partial_1 M)$$

induced by the inclusion $j : (\Sigma, \partial\Sigma) \to (M, \partial_1 M)$. Since ξ is an oriented bundle, the Framing Principle applies to prove the following.

Lemma 3.1.7. *The following diagram commutes where $\tau^{IK} = \sum \tau^{IK}_{2k}$ is the total IK-higher torsion.*

This assumes that $n + m$ is odd. Homology and cohomology have coefficients in $\mathbb{R}$.

Although we claim that the Framing Principle implies this lemma, we don't need to verify it since this lemma follows from the next lemma.

Lemma 3.1.8. *Every element of $\overline{SD}_{B,\partial_0}^{G/O}(M)$ is in the image of a homomorphism*

$$\Sigma_{\tilde{\lambda}} : G(L,\partial_0 L) \to \overline{SD}_{B,\partial_0}^{G/O}(M)$$

where $\lambda : (L,\partial_1 L) \to (B,\partial_1 B)$ is a codimension 0 immersion covered by an embedding $\tilde{\lambda} : L \to M$ which makes the following diagram commute.

$$
\begin{array}{ccc}
G(L,\partial_0 L) & \xrightarrow{\quad (-1)^n 2\widetilde{ch} \quad} & \bigoplus_{k>0} H_{q-4k}(L,\partial_1 L) \\
\Big\downarrow{\scriptstyle \Sigma_{\tilde{\lambda}}} \quad {\scriptstyle top\, E_+^n(M,\tilde{\lambda},-)} & & \quad {\scriptstyle \Sigma_{\tilde{\lambda}}} \\
\overline{SD}_{B,\partial_0}^{G/O}(M) \xrightarrow[\overline{AdT}]{} \pi_0\widetilde{\mathcal{S}}_{B,\partial_0}^s(M) & \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}) &
\end{array}
$$

$$\underset{(-1)^n 2\widetilde{ch}}{}$$

Proof of Lemma 3.1.7. First we note that both maps coming out of $SD_{B,\partial_0}^{G/O}(M)$ factor through $\overline{SD}_{B,\partial_0}^{G/O}(M)$. Each element then lifts to $G(L,\partial_0 L)$. Next we chase the diagram at the beginning of the proof of Theorem 3.0.5 to show that the two images of this element in $\bigoplus H_{q-4k}(B,\partial_1)$ are equal. The diagram in Lemma 3.1.8 above shows that the two images obtained are the same as the two images in the diagram of Lemma 3.1.7 which we are proving. $\square$

Proof of Lemma 3.1.8. The mapping $\Sigma_{\tilde{\lambda}}$ takes a map $\xi : L \to G/O$ which is trivial over $\partial_0 L$ and produces a stratified subset

$$\Sigma_{\tilde{\lambda}}(\xi) = (\Sigma, \psi)$$

where Σ is two copies of L, thus $\Sigma_- \cong \Sigma_+ \cong L$, glued together along $\partial_0 L$ and embedded in M using two small perturbations of the embedding $\tilde{\lambda} : L \to M$. The mapping psi is equal to ξ on Σ_+ and is trivial on Σ_-. Since ψ is trivial on Σ_-, the negative Hatcher handles in $W(\Sigma,\psi)$ are standard disk bundles. So, the bundle $AdT(\Sigma,\psi)$ will not change if we remove these "trivial" Hatcher handles. The result is then equivalent to the immersed Hatcher handle. This shows that the triangle in the diagram commutes. Commutativity of the (curved) square follows from the definition of $\widetilde{ch}(\xi)$ on $SD_{B,\partial_0 B}^{G/O}$, namely that it is the push-forward along the embedding $D(\tilde{\lambda}) : E \to M$ of the normalized chern character of ξ as a bundle over L.

It remains to prove the elementwise surjectivity statement. This follows from the stratified deformation lemma 3.2.1 whose proof we leave until the end. This lemma shows that any stratified subset (Σ,ψ) of M can be deformed so that every component of Σ_- is contained in a disjoint contractible subset of Σ. Then we can deform ψ so that it is constant on each component of Σ_- and therefore also on Σ_0. Then let $(L,\partial_0 L) = (\Sigma_+, \Sigma_0)$ and let $\lambda : L \to B$ be the map $\pi_+ : \Sigma_+ \to B$. Let $\tilde{\lambda} : L \to M$ be the inclusion map of Σ_+. Then we claim that the image of (Σ,ψ) in $\overline{SD}_{B,\partial_0}^{G/O}(M)$ is equal to the image $\Sigma_{\tilde{\lambda}}(\xi_+)$ of $\xi_+ = \xi|\Sigma_+ \in G(L,\partial_0 L)$. Since we started with an arbitrary element of $SD_{B,\partial_0}^{G/O}(M)$ this will prove the lemma.

To see that (Σ,ψ) and $\Sigma_{\tilde{\lambda}}(\xi_+)$ are equal in $\overline{SD}_{B,\partial_0}^{G/O}(M)$, we just take the difference $\Sigma_{\tilde{\lambda}}(\xi_+) - (\Sigma,\psi)$. The negative of (Σ,ψ) given in Remark 3.1.5 is $(\Sigma',\psi) + (\Sigma_0,\psi_0) \times F$ where Σ' is Σ with orientation reversed. But ψ is trivial on Σ_0. Therefore, the second summand is trivial. So $\Sigma_{\tilde{\lambda}}(\xi_+) - (\Sigma,\psi) = \Sigma_{\tilde{\lambda}}(\xi_+) + (\Sigma',\psi)$. But then, $\Sigma'_- = \Sigma_+$ has the same G/O coefficient map as $\Sigma_{\tilde{\lambda}}(\xi_+)$ has on its positive part. Therefore, the interior of the negative part of Σ' cancels the interior of the positive part of $\Sigma_{\tilde{\lambda}}(\xi_+)$ by a stratified deformation. The result has trivial coefficient map to G/O and therefore is trivial in $\overline{SD}_{B,\partial_0}^{G/O}(M)$ as claimed. $\square$

3.1.3. *Proof of the AdT Theorem.* The Arc de Triomph Theorem 3.1.1 will follow from the following first version of the theorem.

Lemma 3.1.9. *The mapping*

$$SD_{B,\partial_0}^{G/O}(M) \xrightarrow{\widetilde{2ch}} \bigoplus_{k>0} H_{\dim B - 4k}(M, \partial_1; \mathbb{R})$$

is rationally surjective in the sense that its image generates $\bigoplus_{k>0} H_{\dim B - 4k}(M, \partial_1; \mathbb{R})$ as a vector space over $\mathbb{R}$.

Proof. Consider the bundle $M \times I \to B$ and consider an arbitrary fiberwise generalized Morse function $f : M \times I \to I$ which agrees with the projection map over $\partial_0 B$ and in a neighborhood of the vertical boundary. Thus $f = pr_I$ on the set

$$A = \partial_0 M \times I \cup M \times \{0, 1\}$$

Let $\Sigma(f)$ be the fiberwise singular set. This is a stratified subset of $M \times I$ disjoint from the set A since the projection to B has only fold singularities and the Morse point set which are the regular points of the projection $\Sigma(f) \to B$ are stratified by index i. We will use just the sign $(-1)^i$ making Σ_+ into the set of Morse points of even index and Σ_- the set of odd index Morse points of f. It is important to note that $\Sigma(f)$ is a manifold with boundary and $\partial \Sigma(f) = \Sigma(f) \cap M_{\partial_1 B} \times I$.

The singular set is the inverse image of zero under the vertical derivative $D^v(f)$ of f and therefore a framed manifold with boundary. (Add the vertical normal bundle to see the framing.) Since the space of all smooth functions on $M \times I$ equal pr_I on A is contractible and contains a function without critical points, this framed manifold is framed null cobordant and represents the trivial element of the fiberwise framed cobordism group of M relative to $M_{\partial_1 B}$ which is $\pi_0 \Gamma_{B,\partial_0} Q_B(M)$ where $Q_B(M)$ is the bundle over B with fiber $Q(X_+) = \Omega^\infty \Sigma^\infty(X_+)$ over $b \in B$ if X is the fiber of $M \times I$ over b.

The negative eigenspace of $D^2(f)$ gives a stable vector bundle ξ over $\Sigma(f)$. So $\Sigma(f)$, together with ξ gives a stratified subset of $M \times I$ with coefficients in $BO = \operatorname{colim} BO(k)$. Since $\Sigma(f)$ is a framed manifold with boundary which is framed null cobordant when we ignore this vector bundle, we get an element of the kernel of the map from the fiberwise framed cobordism group of $BO \times M$ to that of M. This kernel is π_0 of the fiber of the map:

$$\gamma : \Gamma_{B,\partial_0} Q_B(BO \times M) \to \Gamma_{B,\partial_0} Q_B(M)$$

In [18], it is shown that the space of generalized Morse functions on a manifold X is $\dim X$-equivalent to $Q(BO \wedge X_+)$. If we apply that theorem fiberwise, we get that the space of fiberwise generalized Morse functions on $M \times I$ has the $n + m - q$ homotopy type of the fiber of the map γ above.

However, it is a standard homotopy argument to show that there is a split surjection

$$Q(BO \wedge X_+) \to \Omega^\infty(BO \wedge X_+)$$

which is rationally equivalent to the homology of X in every 4th degree since BO is rationally equivalent to $\prod_{k>0} K(\mathbb{Z}, 4k)$. Therefore, $\pi_0(fiber(\gamma))$ has a split summand which is rationally isomorphic to the group:

$$H := \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}; \mathbb{Q})$$

by the basic homotopy calculation (Corollary B.2.2) of Appendix B.

This implies that a set of generators for the vector space $H \otimes \mathbb{R}$ is given by taking the normalized chern character of all possible stratified sets $(\Sigma, \xi) \in SD_{B,\partial_0}^{BO}(M \times I)$ given by all fiberwise generalized Morse functions on $M \times I$ fixing the subspace A. Using the fact that the group $J(\Sigma)$ is finite with order, say k, we know that $J(\xi^k) = 0$ in $J(\Sigma)$ and therefore lifts to a map $\Sigma \to G/O$. So, these

various stratified sets $(\Sigma, \xi^k) \in SD_{B,\partial_0}^{G/O}(M \times I)$ will have normalized chern characters generating the vector space $H \otimes \mathbb{R}$ as claimed. $\square$

Lemma 3.1.10. *The following diagram commutes*

$$SD_{B,\partial_0}^{G/O}(M) \xrightarrow{\quad AdT \quad} \pi_0 \widetilde{\mathcal{S}}_{B,\partial_0}^s(M) \xrightarrow{\quad \Theta \quad} \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B})$$

$$(-1)^n 2\widetilde{ch}$$

where $\Theta : M' \mapsto \Theta(M', M)$ gives the rational exotic structure class of M'.

This lemma proves the Arc de Triomph Theorem 3.1.1 since we just proved in Lemma 3.1.9 that the normalized chern character is rationally surjective and we know by the smoothing theorem that Θ is a rational isomorphism.

Proof. Take the diagram from Lemma 3.1.8 and add the arrow Θ:

$$
\begin{array}{ccc}
G(L, \partial_0 L) & \xrightarrow{\quad (-1)^n 2\widetilde{ch} \quad} & \bigoplus_{k>0} H_{q-4k}(L, \partial_1 L) \\
\Sigma_{\tilde\lambda} \downarrow & {}^{top\,E_+^n(M,\tilde\lambda,-)} & \Sigma_{\tilde\lambda} \\
\overline{SD}_{B,\partial_0}^{G/O}(M) \xrightarrow{\overline{AdT}} \pi_0 \widetilde{\mathcal{S}}_{B,\partial_0}^s(M) \xrightarrow{\Theta} \bigoplus_{k>0} H_{q-4k}(M, M_{\partial_1 B}) \\
(-1)^n 2\widetilde{ch}
\end{array}
$$

The outside curved square commutes by Theorem 3.0.5. The map $\Sigma_{\tilde\lambda}$ can be chosen to hit any element of $\overline{SD}_{B,\partial_0}^{G/O}(M)$ by the previous lemma. Therefore, the curved triangle at the bottom commutes. This implies the lemma since the maps factor uniquely through $\overline{SD}_{B,\partial_0}^{G/O}(M)$. $\square$

3.2. **Stratified deformation lemma.** It remains to prove the following lemma which was used to show that each Arc de Triomph construction can be deformed into an immersed Hatcher construction.

Lemma 3.2.1 (Stratified Deformation Lemma)**.** *If the fiber dimension of M is $\geq q + 2$, then any element of $SD_{B,\partial_0}^{G/O}(M)$ is represented by a stratified subset (Σ, ψ) of M with the property that the components of Σ_- are contained in disjoint contractible subsets of Σ.*

Proof. This is the same proof which appears in [19] on page 446-447 with five figures and in [23] on page 73 with one figure. We repeat is here since the statements are not the same, only analogous.

The dimension hypothesis implies that all deformations of Σ in M can be made into isotopies of smooth embeddings over B by transversality. So, we will not concern ourselves with that point.

Suppose that $\partial_1 B$ is empty. Then we will deform any (Σ, ψ) into the desire shape. When $\partial_1 B$ is nonempty, we double B along $\partial_1 B$ and double M along $M_{\partial_1 B}$ and similarly for (Σ, ψ). Then do the deformation $\mathbb{Z}/2$ equivariantly. The fixed point sets of the $\mathbb{Z}/2$ action on the new B and new M are the original $\partial_1 B$ and $M_{\partial_1 B}$.

First choose an equivariant triangulation of Σ_- so that the fixed point set is a subcomplex and so that each simplex maps monomorphically into B. Then we will cut apart the set Σ_- by deleting a tubular neighborhood of each simplex starting with the lowest dimension. If we let S denote what is left of the original set Σ_-, then, at the end, the set S will be a disjoint union of q-disks.

At each step in the deformation, a new component of Σ_- will be introduced which will be contained in a q-disk subset of Σ which is disjoint from S and from all the other q-disks. Each step of the deformation will be $\mathbb{Z}/2$ equivariant and will alter the set Σ only in an arbitrarily small

29

neighborhood of the set S. Consequently, the q-disks containing the new components of Σ_- will not be touched.

The deformation starts with the vertices of $S = \Sigma_-$. Let v be a vertex in the interior of S. Then we can introduce a "trivial lens" (L, ψ) in M near v. By definition, this is an embedded q sphere in M which is the union of two q-disks $L = L_- \cup L_+$ so that the projection map $L \to B$ has fold singularities along $L_1 = L_- \cap L_+$ and so that the interior of each q disk maps diffeomorphically onto an open neighborhood of $p(v)$ in B. Choose the map $\psi : L \to BO$ to be constant and equal to $\psi(v)$. Then we can cancel a small neighborhood of v in S with a small neighborhood of the center point of L_+. This changes S in the desired way and introduces a new component of Σ_- which is a q-disk.

Suppose by induction that the $m - 1$ skeleton of S has been removed where $m \geq 1$. Let D^m be an m-disk embedded in S with boundary $S^{m-1} \subseteq \partial S$. In order to eliminate D^m we need another m-disk Δ embedded in Σ_+ so that the projections of Δ and D^m to B are equal and so that $D^m \cup \Delta^m$ forms a lens. We need to construct this disk.

Let S'^{m-1} be a sphere in the interior of Σ_+ which is parallel to S^{m-1} and lies over the image of D^m in B. Over that sphere, we create a "tube" T, a product of an $m - 1$ sphere with a trivial lens of dimension $q - m + 1$. Then $T_- \cong T_+ \cong S^{m-1} \times D^{q-m+1}$. The mapping to BO should agree with $\psi | S'^{m-1}$. Then we can cancel a tubular neighborhood of S'^{m-1} with a tubular neighborhood of the core $S^{m-1} \times *$ of T_-. Then the new component C of Σ_- will be a collar neighborhood of the boundary of $S^{m-1} \times D^{q-m+1}$:

$$ C \cong S^{m-1} \times S^{q-m} \times I $$

This has two boundary components. The outer boundary component $S^{m-1} \times S^{q-m} \times 1$ will also be the boundary of $T_+ \cong S^{m-1} \times D^{q-m+1}$ and C is an external collar for T_+ making $T_+ \cup C \cong T_+ \cong S^{m-1} \times D^{q-m+1}$. We can make this set contractible by adding an m-disk Δ_-. To do this we use the disk $D^m \subseteq S$ as a path along with to embed a disk Δ with map to BO given by $\psi | C^m$ and use this to do surgery on $S^{m-1} \times * \times 0$ on the inside boundary component of C. This performs a deformation of (Σ, ψ) which makes the new component of Σ_- together with T_+ into a disk and the new Σ_+ contains an m-disk Δ_+ which, together with D^m forms a lens which can be cancelled.

When D^m lies in the fixed point set, this can be done equivariantly and when D^m is not contained in the fixed point set, it will be disjoint from it and this can all be done away from the fixed point set and the mirror image of D^m can be eliminated in the analogous way making the deformation $\mathbb{Z}/2$ equivariant in both cases. Thus the deformation works as promised, proving the lemma. $\qquad \square$

This completes the proof of all the theorems in this paper.

APPENDIX A. SMOOTHING THEORY

This paper has two appendices. The first is an exposition on Dwyer-Weiss-Williams smoothing theory as outlined in the short notes of Williams [28], the second explains the homotopy calculation needed to convert the DWW homotopically defined smoothing invariant into a cohomology class.

The important results explained in Appendix A: Smoothing Theory are the following

- (Theorem A.5.14) The space of stable tangential smoothings of a compact manifold bundle is homotopy equivalent to the space of sections of the corresponding $\mathcal{H}^\%$ bundle.

$$ \gamma_M : \widetilde{\mathcal{S}}^s_{B, \partial_0 B}(M) \xrightarrow{\cong} \Gamma_{B, \partial_0 B} \mathcal{H}^\%_B(M) $$

- (Corollary A.5.19) If $\lambda : (L, \partial_1 L) \to (B, \partial_1 B)$ is a codimension 0 immersion covered by a codimension 0 embedding of bundles $D(\tilde{\lambda}) : E \to W$ then we get a commuting diagram:

$$
\begin{array}{ccc}
\widetilde{\mathcal{S}}^s_{L,\partial_0 L}(E) & \xrightarrow[\simeq]{\gamma_E} & \Gamma_{L,\partial_0 L}\mathcal{H}^{\%}_L(E) \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{S}}^s_{B,\partial_0 B}(M) & \xrightarrow[\simeq]{\gamma_M} & \Gamma_{B,\partial_0 B}\mathcal{H}^{\%}_B(M)
\end{array}
$$

Appendix A is divided into five parts:

(1) *Spaces of manifolds.* We describe the smoothing problem as the homotopy fiber of the forgetful mapping from the space of smooth manifolds to the space of topological manifold.
(2) *The functor EM.* Since every microbundle contains a Euclidean bundle, every paracompact topological manifold M has a tangent Euclidean bundle EM.
(3) *Linearized Euclidean bundle VM.* This is a vector bundle which is topologically equivalent to the tangent Euclidean bundle EM. I.e., VM is EM with additional structure making it into a vector bundle. By classical smoothing theory, this additional structure exists if and only if M is smoothable. If M is smooth, then we can take VM to be the usual tangent bundle TM.
(4) *Tangential smoothing.* This refers to a homeomorphism $f : M_0 \to M_1$ between smooth manifolds together with a vector bundle morphism $Tf : TM_0 \to TM_1$ between their tangent bundles which is compatible with the topological derivative $Ef : EM_0 \to EM_1$. There is also an equivalent formulation given by a continuous family of topological manifolds $M_t, t \in [0,1]$ which have linear Euclidean bundles VM_t so that $VM_0 = TM_0$ and $VM_1 = TM_1$.
(5) *Smoothing theorems.* After stabilization, the core of a linearized topological bundle has a unique smoothing up to contractible choice compatible with the linearization. Furthermore, the linearization extends uniquely up to homotopy to the complement of the core which is an internal fiberwise collar for the bundle. This finally leads to the main smoothing theorems as explained above.

A.1. **Spaces of manifolds.** First we recall the basic construction which is that a compact topological/smooth manifold bundle $M \to E \to B$ is equivalent to a mapping from B to the space of topological/smooth manifolds which are homeomorphic/diffeomorphic to M. This leads us to consider two spaces of manifolds: topological manifolds and smooth manifolds and the homotopy fiber of the forgetful map

$$\varphi : \mathcal{S}^d_\bullet(n) \to \mathcal{S}^t_\bullet(n)$$

which is the space of all smoothings of a fixed topological n-manifold. We also need one other space of manifolds: topological manifolds with linear structures on their tangent Euclidean bundles. We will discuss this after we recall the theory of Euclidean bundles in the next section.

A.1.1. *Topological manifolds.* Let $S^t(n)$ be the set of all compact topological n-submanifolds of $\mathbb{R}^\infty$ so that

$$(M, \partial M) \subset ([0,\infty) \times \mathbb{R}^\infty, 0 \times \mathbb{R}^\infty)$$

This condition allows us to attach a canonical external open collar $C = \partial M \times [0,1) \subset [0,1) \times \mathbb{R}^\infty$. Instead of defining a topology on the space $S^t(n)$ we will take the easy approach of making it into a simplicial set. So, let $\mathcal{S}^t_\bullet(n)$ be the simplicial set whose k-simplices are continuous Δ^k families of compact topological n-manifolds $M_t, t \in \Delta^k$. By a *continuous family* we mean a mapping $f : \Delta^k \to S^t(n)$ with the property that its adjoint

$$W = \{(t,x) \in \Delta^t \times \mathbb{R}^\infty \,|\, x \in M_t\} \subseteq \Delta^k \times \mathbb{R}^\infty$$

is a fiber bundle over Δ^k with fiber $f(t) = M_t$.

There is a *tautological bundle* over the geometric realization $|\mathcal{S}^t_\bullet(n)|$ whose fibers are compact n-manifolds embedded in $\mathbb{R}^\infty$. The inverse image of a simplex is the corresponding manifold W given above. Since this bundle contains all possible W, it is universal, i.e., any compact n-manifold bundle $p : E \to B$ over a triangulated space B is classified by a mapping $B \to S^t(n)$ which is continuous on each simplex of B in the above sense. Therefore, we get a simplicial map

$$simp\ B \to \mathcal{S}^t_\bullet(n)$$

from the set of simplices of B to $\mathcal{S}^t_\bullet(n)$ which sends a k-simplex σ to $p^{-1}(\sigma) \in \mathcal{S}^t_k(n)$ (assuming that a fiberwise embedding $E \hookrightarrow B \times \mathbb{R}^\infty$ has been chosen). This proves the following well-known theorem where $Homeo(M)$ is the group of homeomorphisms of M with the compact-open topology.

Theorem A.1.1. *The geometric realization of $\mathcal{S}^t_\bullet(n)$:*

$$|\mathcal{S}^t_\bullet(n)| \simeq \coprod BHomeo(M)$$

is the disjoint union over all homeomorphism classes of compact n-manifolds M of the classifying space of $Homeo(M)$. □

A.1.2. *Smooth manifolds.* We define a smoothing of a topological manifold M without boundary to be continuous mapping $\alpha : M \to \mathbb{R}^\infty$ whose image is an immersed smooth submanifold. We call α an "immersion".

Definition A.1.2. If M is a topological manifold and N is a smooth manifold we define an *immersion* to be a continuous mapping $\alpha : M \to N$ so that for every $x \in M$ there is an open neighborhood U of x in M so that α is an embedding on U with image $\alpha(U)$ a smooth submanifold of N with smooth boundary. If M_t is a family of topological manifolds without boundary forming the fibers of a bundle $W \to B$ over a smooth manifold B then by a *family of immersions* we mean a continuous family of maps $\alpha_t : M_t \to N$ giving an immersion $\alpha : W \to N \times B$.

If U is an open subset of the interior of a topological manifold M we define a *smoothing* of U to be an immersion $\alpha : U \to \mathbb{R}^\infty$. We defined a *smoothing* of a closed subset A of M to be the germ of a smoothing of a neighborhood of A in the interior of M. Thus a smoothing of A is an equivalence class of smoothings in open neighborhoods of A where two such smoothings are equivalent if they agree on a third smaller neighborhood of A. When we pass to manifold bundles we always define these open neighborhoods to be *uniform open neighborhoods* meaning they are open subsets of the total space of the bundle.

If M is a manifold with boundary then we define a smoothing of M to be the germ of a smoothing of a neighborhood of M in $M' = M \cup C$ where $C = \partial M \times [0, 1)$ is the standard external open collar for M. Smoothings of closed subsets of M are defined similarly.

The key point about this version of the definition of smoothing is that it is clearly *excisive* in the following sense.

Proposition A.1.3. *If M is the union of two closed subsets $M = A \cup B$, then a smoothing α of M is the same as a pair of smoothings α_A, α_B for A and B which agree on $A \cap B$.*

Proof. The smoothings α_A, α_B are given by immersions on neighborhoods of A, B in M' which agree on the intersection. This gives an immersion defined on a neighborhood of M in M'. □

Let $S^d(n)$ be the set of all pairs (M, α) where M is an element of $S^t(n)$, i.e. a compact topological submanifold of $\mathbb{R}^\infty$ and α is a smoothing of M. Let $\mathcal{S}^d_\bullet(n)$ denote the simplicial set whose k-simplices are pairs (W, α) where $W \in \mathcal{S}^t_\bullet(n)$ and $\alpha : W \to \mathbb{R}^\infty \times B$ is an immersion over B.

Note that there is a simplicial forgetful map

$$\varphi : \mathcal{S}^d_\bullet(n) \to \mathcal{S}^t_\bullet(n)$$

which is given in every degree by projection to the first coordinate: $\varphi_k(W, \alpha) = W$.

Theorem A.1.4. *The geometric realization of $\mathcal{S}_\bullet^d(n)$:*

$$|\mathcal{S}_\bullet^d(n)| \simeq \coprod BDiff(M)$$

is the disjoint union over all diffeomorphism classes of compact n-manifolds M of the classifying space of the group $Diff(M)$ of diffeomorphisms of M with the C^∞ topology. □

Our definition of smoothing also gives us an exponential map

$$\mu_M : TM \to M'$$

defined in some neighborhood $U(M)$ of the zero section of the tangent bundle $TM \subset M \times \mathbb{R}^\infty$. This is given by the inverse function theorem as the second coordinate of the inverse of $\pi : M \times M' \to TM$ given by $\pi(x,y) = \pi_x(\alpha'(y) - \alpha(x))$ where π_x is orthogonal projection to the tangent plane to $\alpha(M)$ at $\alpha(x)$ translated to the origin.

A.1.3. *Homotopy fiber of φ.* Given a single topological manifold X, the space of smoothings α of X is homeomorphic to the space of all pairs (X, α). This is the same as the inverse image of X (and its degeneracies) under the simplicial forgetful map $\varphi : \mathcal{S}_\bullet^d \to \mathcal{S}_\bullet^t$. An elementary argument shows that this space is the homotopy fiber of the simplicial map φ. More generally, the space of smoothings of a continuous Δ^k family W_0 of topological manifolds is homotopy equivalent to the space of smoothings of W_0 which we define as follows.

Definition A.1.5. Suppose W_0 is a Δ^k family of topological n-manifolds, i.e. $(W_0 \to \Delta^k) \in \mathcal{S}_k^t(n)$. Then, a *smoothing* of W_0 is defined to be a pair (W, α) where

    (1) W is a continuous $\Delta^k \times I$ family of topological manifolds,
    (2) $W|\Delta^k \times 0 = W_0$,
    (3) α is a smoothing of $W_1 = W|\Delta^k \times 1$.

Since W is topologically isomorphic to $W_0 \times I$, the space of smoothings of W_0 is homotopy equivalent to the space of actual smoothings of W_0. Homotopy smoothings have some additional properties which we state without proof.

Let $\mathcal{S}_\bullet^{t/d}(n)$ denote the simplicial set whose k-simplices are smoothings (W, α) of k-simplices $W_0 \in \mathcal{S}_k^t(n)$ as defined above. We have the following observation.

Proposition A.1.6. *The simplicial forgetful map $\mathcal{S}_\bullet^{t/d}(n) \to \mathcal{S}_\bullet^d(n)$ sending (W, α) to (W_1, α) is a homotopy equivalence.* □

Proposition A.1.7. *If $X \in \mathcal{S}_0^t(n)$ let $\mathcal{S}_\bullet^{t/d}(X)$ denote the simplicial subset of $\mathcal{S}_\bullet^{t/d}(n)$ whose k-simplices are the smoothings of $X \times \Delta^k$. Then $|\mathcal{S}_\bullet^{t/d}(X)|$ is the homotopy fiber of the forgetful map*

$$|\varphi| : |\mathcal{S}_\bullet^d(n)| \to |\mathcal{S}_\bullet^t(n)|$$

over $X \in \mathcal{S}_0^t(n)$. □

More generally, given any topological manifold bundle $W_0 \to B$ where B is a smooth manifold, we define a *smoothing* of W_0 to be a topological embedding $\alpha : W_0 \to B \times \mathbb{R}^\infty$ over B whose image is a smooth submanifold of $B \times \mathbb{R}^\infty$. Smooth Δ^k families of such embeddings form a simplicial set which represents the space of all homotopy liftings of the classifying map $B \to |\mathcal{S}_\bullet^t(n)|$ to $|\mathcal{S}_\bullet^d(n)|$, in other words a point in the homotopy fiber of the map

$$|\mathcal{S}_\bullet^d(n)|^B \to |\mathcal{S}_\bullet^t(n)|^B.$$

We call this the *space of fiberwise smoothings* of W_0.

By a *fiberwise smoothing* of W_0 we mean a pair (W, α) where W is a topological manifold bundle over $B \times I$ which is equal to W_0 over $B \times 0$ and α is a smoothing of $W|B \times 1$. Taking Δ^k families we can form a simplicial set which we call the *space of fiberwise smoothings* of W_0. As in Proposition

A.1.6, it is clear that this space is homotopy equivalent to the space of actual smoothings of a topological bundle. However, smoothings makes it easier to compare two different smoothings of the same bundle.

Suppose that $f : W_0 \to W_1$ is a homeomorphism between two smooth bundles over B. We can construct the associated smoothing

$$W = W_0 \times I \cup_f W_1$$

where W_1 is identified with $W_0 \times I$ using the homeomorphism f. The problem is to determine whether there is a smooth structure on W which extends the given smooth structure on W_0 and W_1. If f is smooth then we can take the smooth structure on W induced from $W_0 \times I$. Using our strict definitions, this would be given by the family of smoothings $h_t = (1 - t)\alpha_0 + t\alpha_1 \circ f : W_0 \to \mathbb{R}^\infty$. This will be a family of embeddings if we assume that the smoothings α_i of W_i have image in linearly independent subspaces of $\mathbb{R}^\infty$, if not we can simply pass through a third smoothing with this property.

Proposition A.1.8. *Suppose that $f : W_0 \to W_1$ is a fiberwise homeomorphism between smooth bundles over B and suppose that f is homotopic through continuous fiberwise embeddings over B to a smooth embedding. Then there is a smooth structure on the fiberwise interior of W which agrees with the smooth structure on W_0 and W_1.*

Proof. The continuous image of the fiberwise interior of W_t under a topological embedding $f_t : W_t \to W_1$ is a smooth manifold. Therefore, the image of f_t gives the desired smoothing. □

In classical smoothing theory, a smoothing of a topological manifold is given by a lifting of the tangent microbundle to a linear bundle. In other words, a vector bundle structure on the topological tangent bundle gives a smoothing of a single manifold. This is not true for a topological manifold bundle $W \to B$. If we choose a vector bundle structure on the vertical topological tangent bundle of the topological bundle $W \to B$ we have a further obstruction to smoothing. To study this question we need to construct a third space of manifolds: compact topological manifolds with vector bundle structures on their topological tangent bundles. We call these "linearized manifolds."

A.2. **The functor** *EM*. A compact topological manifold M has a topological tangent bundle

$$\pi : EM \to M$$

which is called the *tangent Euclidean bundle* of M. The fiber $\pi^{-1}(x)$ is homeomorphic to an open ball neighborhood of x in M. This section gives the basic properties of Euclidean bundles in general and the functorial properties of the tangent Euclidean bundle of a manifold.

A.2.1. *Euclidean bundles.* A *Euclidean bundle* is a fiber bundle

$$\pi : E \to B$$

with fiber $\mathbb{R}^n$ and structure group $\mathcal{H}omeo(\mathbb{R}^n, 0)$, the group of homeomorphisms of $\mathbb{R}^n$ fixing 0 with the compact open topology. By a *Euclidean subbundle* of E we mean an open subset $E_0 \subseteq E$ which includes the zero section and which is also a Euclidean bundle. A *morphism* of Euclidean bundles $E_0 \to E_1$ over B is a fiberwise open embedding which preserves the zero section, i.e., an isomorphism of E_0 with a subbundle of E_1.

Lemma A.2.1 (Kistor). *Let $E \to B$ be a Euclidean bundle over a finite cell complex B and let E_0 be a Euclidean subbundle of E. Then E_0 is fiberwise isotopic to E fixing a neighborhood of the zero section. I.e., there is a continuous one-parameter family of Euclidean morphisms $f_t : E_0 \to E$ which are the identity in a neighborhood of the zero section so that f_0 is the inclusion map and f_1 is a homeomorphism.* □

Remark A.2.2. The conclusion of this lemma can be rephrased as saying that there is a Euclidean subbundle of $E \times I \to B \times I$ which is equal to E_0 over $B \times 0$ and equal to E over $B \times 1$.

Proposition A.2.3. *If E, E' are Euclidean bundles over B with isomorphic subbundles $\varphi_0 : E_0 \cong E'_0$ then there is an isomorphism $\varphi : E \to E'$ which agrees with φ_0 in some neighborhood of the zero section.*

Proof. $E \cong E_0 \cong E'_0 \cong E'$ by the lemma. $\qquad\square$

Lemma A.2.4. *Let $E \to B$ be a Euclidean bundle over a finite dimensional CW-complex B and let E_0 be a subbundle of the restriction $E|A$ of E to a subcomplex $A \subseteq B$. Then there is a Euclidean subbundle E'_0 of E so that $E_0 = E'_0|A$.*

Proof. Assuming that we have constructed E'_0 over $A \cup B^k$ we can extend E'_0 to $A \cup B^{k+1}$ one $(k+1)$-cell at a time using the lemma as rephrased in the remark. $\qquad\square$

A.2.2. *Microbundles.* A *(topological) microbundle* over B is defined to be a space E containing B with inclusion map $s : B \to E$ and retraction $\pi : E \to B$ so that B is covered by open subsets U of E for which $\pi(U) = U \cap B$ and so that $\pi|U : U \to U \cap B$ is a Euclidean bundle.

A *morphism* of topological microbundles $E_0 \to E_1$ over B is defined to be the germ along B of a fiberwise homeomorphism $f : E_0 \to E_1$ which is the identity on B. Thus, f is only defined in some neighborhood of B and any two such maps are equivalent if they agree on some neighborhood of B in E_0.

Theorem A.2.5 (Kistor,Masur)**.** *Suppose that B is (the realization of) a finite dimensional, locally finite simplicial complex and $\pi : E \to B$ is a microbundle. Then E contains an open neighborhood E_0 of $s(B)$ so that $(E_0, s(B))$ is a Euclidean bundle over B. Furthermore, E_0 is unique up to isomorphism.*

Remark A.2.6. This implies that any topological microbundle over B is microbundle isomorphic to a Euclidean bundle which is uniquely determined up to isomorphism.

Proof. We can refine the triangulation of B so that each simplex is contained in one of the open sets U. This gives a Euclidean bundle E_σ over each simplex σ. Since B is locally finite and finite dimensional we can choose the E_σ so that $E_\sigma \subseteq E_\tau|\sigma$ for all $\sigma \subseteq \tau$. Then a Euclidean bundle E_n can be constructed over the n-skeleton of B by induction on n: E_0 already exists. Given E_n, we can extend E_n to each $n+1$ simplex using Lemma A.2.4. This shows existence.

To prove the uniqueness, we take any two Euclidean bundles and use the above argument to construct a third Euclidean bundle which is a subbundle of both. By Kistor's Lemma A.2.1, all three Euclidean bundles are isomorphic. $\qquad\square$

Corollary A.2.7. *If B is dominated by a finite dimensional locally finite simplicial complex (for example a paracompact topological manifold) then any microbundle over B contains a Euclidean bundle neighborhood of its section.*

Proof. If B is a retract of a nice space X then any microbundle over B pulls back to a microbundle over X which contains a Euclidean bundle which restricts to a Euclidean bundle neighborhood of the section of the original microbundle.

Any paracompact n-manifold M satisfies this condition since each component of M is second countable and therefore can be properly embedded in $\mathbb{R}^{2n+1}$. This is an exercise in Munkres. The Tietze extension theorem can be used to show that M is a retract of a neighborhood which we can take to be simplicial. $\qquad\square$

A.2.3. *Tangent Euclidean bundle.* The discussion above implies the well-known fact that every paracompact topological manifold M has a tangent Euclidean bundle: we first attach the standard external open collar $C = \partial M \times [0, 1)$ and embed M in the open manifold $M' = M \cup C$. The *tangent microbundle* of M is the equivalence class of $p_1 : M \times M' \to M$ (projection to the first coordinate), together with the diagonal section $\Delta : M \to M \times M'$.

Definition A.2.8. If M is a topological manifold with external open collar $C = \partial M \times [0, 1)$ and $\pi : E \to M$ is a Euclidean bundle then a *topological exponential map* is defined to be a continuous map

$$\eta : E \to M' = M \cup C$$

so that

    (1) $\eta(s(x)) = x$ for all $x \in M$ where $s : M \to E$ is the zero section,
    (2) η maps each fiber $E_x = \pi^{-1}(x)$ homeomorphically onto an open neighborhood of x in M'.

We call $(E, \eta) = (E, s, \pi, \eta)$ a *tangent Euclidean bundle* for M.

If (E, η) is a tangent Euclidean bundle for M then the germ of E around $s(B)$ is the tangent microbundle of M. To see this note that the two mappings π, η form an open embedding $(\pi, \eta) : E \hookrightarrow M \times M'$ by invariance of domain. The image of (π, η) determines (E, η) up to isomorphism. We say that (E_0, η_0) is *contained in* (E_1, η_1) if the image of E_0 in $M \times M'$ is contained in the image of E_1. Kistor's theorem on the existence and uniqueness of Euclidean bundles can be stated as follows in the case of the tangent microbundle.

Theorem A.2.9. *Any paracompact topological manifold M has a tangent Euclidean bundle. Furthermore, for any two tangent Euclidean bundles, there exists a third Euclidean bundle which is contained in both.* $\square$

In the case where M is a smooth manifold, we note that EM is also a smooth manifold although the smooth structure of EM along ∂M depends on a choice of extensions α' of the smooth structure of M to M'. Also we have a canonical exponential map $\mu_M : TM \to M'$ defined in a neighborhood $U(M)$ of the zero section and this gives us a diffeomorphism of bundles between EM and $U(M)$.

A.2.4. *Vertical tangent Euclidean bundle.* We are interested in the case when $p : M \to B$ is a bundle over a finite complex B with fiber X a compact topological manifold. In this case we first add the external collar $C = \partial^{\mathsf{v}} M \times [0, 1)$ where $\partial^{\mathsf{v}} M$ is the *vertical boundary* of M, i.e. the bundle over B with fiber ∂X. Then $M' = M \cup C$ is an open manifold bundle over B with fiber $X' = X \cup \partial X \times [0, 1)$. The fiber product $M \oplus M'$ is a bundle over B with fiber $X \times X'$ and the vertical tangent microbundle is the neighborhood germ of the fiberwise diagonal ΔM in $M \oplus M'$.

Since M is a compact and finite dimensional it is dominated by a finite complex. Therefore, any microbundle over M contains a Euclidean bundle. In particular, there exists a Euclidean bundle $\pi : E^{\mathsf{v}} M \to M$ unique up to isomorphism and an open embedding

$$(\pi, \eta) : E^{\mathsf{v}} M \hookrightarrow M \oplus M'$$

over M sending the zero section to the fiberwise diagonal.

We call $(E^{\mathsf{v}} M, \eta)$ a *vertical tangent Euclidean bundle* for $M \xrightarrow{p} B$ and $\eta : E^{\mathsf{v}} M \to M'$ the *fiberwise topological exponential map*. As before, $(E^{\mathsf{v}} M, \eta)$ is given up to isomorphism by the image of the embedding $(\pi, \eta) : E^{\mathsf{v}} M \hookrightarrow M \oplus M'$.

A.2.5. *Topological derivative.* Any homeomorphism $f : M_0 \to M_1$ induces a homeomorphism $f' : M_0' \to M_1'$ by sending $(x, t) \in \partial M_0 \times [0, 1)$ to $(f(x), t) \in \partial M_1 \times [0, 1)$. This gives a map of tangent microbundles:

$$
\begin{array}{ccc}
M_0 \times M_0' & \xrightarrow{f \times f'} & M_1 \times M_1' \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\
M_0 & \xrightarrow{\quad f \quad} & M_1
\end{array}
$$

This morphism of microbundles is the *topological derivative* of f. (If $E \to B$, $E' \to B'$ are bundles or microbundles, then a morphism $E \to E'$ over a map $f : B \to B'$ is defined to be a morphism between E and the pull-back $f^* E'$ over B.)

Choosing Euclidean subbundles of the microbundles, we can represent the topological derivative by an isomorphism of Euclidean bundles Ef. If $\pi_0 : EM_0 \to M_0$ is a tangent Euclidean bundle for M_0 with embedding $(\pi_0, \eta_0) : EM_0 \hookrightarrow M_0 \times M_0'$ then $\pi_1 = f \circ \pi_0 : EM_0 \to M_1$ is evidently a tangent Euclidean bundle for M_1 with embedding $(\pi_1, \eta_1) = (f \times f')(\pi_0, \eta_0) : EM_0 \to M_1 \times M_1'$. In other words, there is an isomorphism of Euclidean bundles $Ef : EM_0 \to EM_1$ over f as indicated in the following commuting diagrams.

$$
\begin{array}{ccc}
EM_0 & \xrightarrow{\ Ef\ } & EM_1 \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
M_0 & \xrightarrow{\ f\ } & M_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
EM_0 & \xrightarrow{\ Ef\ } & EM_1 \\
{\scriptstyle \eta_0}\downarrow & & \downarrow{\scriptstyle \eta_1} \\
M_0' & \xrightarrow{\ f'\ } & M_1'
\end{array}
$$

We call Ef the *topological (Euclidean) derivative* of f.

Similarly, if $f : M_0 \to M_1$ is a fiberwise homeomorphism between two topological manifold bundles over the same space B, we get an isomorphism of Euclidean bundles $E^{\mathsf{v}}f : E^{\mathsf{v}}M_0 \to E^{\mathsf{v}}M_1$ over f compatible with the fiberwise exponential maps as indicated in the following diagrams in which all arrows commute with the projection to B.

$$
\begin{array}{ccc}
E^{\mathsf{v}}M_0 & \xrightarrow{\ E^{\mathsf{v}}f\ } & E^{\mathsf{v}}M_1 \\
\downarrow & & \downarrow \\
M_0 & \xrightarrow{\ f\ } & M_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
E^{\mathsf{v}}M_0 & \xrightarrow{\ E^{\mathsf{v}}f\ } & E^{\mathsf{v}}M_1 \\
{\scriptstyle \eta_0}\downarrow & & \downarrow{\scriptstyle \eta_1} \\
M_0' & \xrightarrow{\ f'\ } & M_1'
\end{array}
$$

We call $E^{\mathsf{v}}f$ the *fiberwise* or *vertical topological (Euclidean) derivative* of f.

A.3. **Linearized Euclidean bundle** VM. The first step to finding a smooth structure on a topological manifold is to impose a linear structure on the tangent Euclidean bundle.

A.3.1. *Linearization.* We define a *linearization* of a Euclidean bundle $\pi : E \to M$ to be the germ along the zero section of a vector bundle structure on E. Since this is a germ, it is a structure on the microbundle of E. A linearization makes the microbundle linear. A linearization of E is the same as a lifting of the structure map of E to $BGL(n, \mathbb{R})$:

$$
\begin{array}{ccc}
 & & BGL(n, \mathbb{R}) \\
 & \nearrow & \downarrow \\
M & \longrightarrow & BHomeo(\mathbb{R}^n, 0)
\end{array}
$$

A linearization of E can also be viewed as an equivalence class $[\lambda]$ of a topological microbundle morphism $\lambda : V \to E$ where V is a vector bundle over B, two such morphisms $\lambda : V \to E, \lambda' : V' \to E$ being equivalent if $\lambda = \lambda' \circ \psi$ for some linear isomorphism $\psi : V \to V'$. In particular, V is well-defined up to isomorphism. We call ψ the *comparison map* for λ, λ'. If λ, λ' are inequivalent linearizations of E, we also get a comparison map $\psi = (\lambda')^{-1}\lambda : V \to V'$ which is a nonlinear map germ between vector spaces.

A *linearization* of a topological manifold M is defined to be a linearization of its tangent microbundle. This is given by a microbundle morphism

$$
\lambda : VM \to EM
$$

for some vector bundle VM. We call the pair (M, VM) a *linearized manifold*. A *fiberwise linearization* of a topological manifold bundle $W \to B$ is defined to be a linearization $V^{\mathsf{v}}W$ of the vertical tangent microbundle of W over B. Thus, a *homotopy* of linearizations of M is given by a fiberwise linearization of $M \times I$ over I.

We note that a microbundle morphism $\lambda : VM \to EM$ carries the same information as an exponential map $\mu : U \to M'$ where U is a neighborhood of the zero section of VM. As we remarked already, such a structure exists if M is a smooth manifold. Then we have a smooth exponential map (inverse to orthogonal projection in $\mathbb{R}^\infty$)

$$\mu_M : U(M) \to M'$$

where $U(M)$ is a neighborhood of the zero section in TM. This gives a microbundle morphism $TM \to EM$ making (M, TM) into a linearized topological manifold. We call this the *canonical linearization* of M.

There is a problem that the topological derivative of a smooth map $M_0 \to M_1$ is not covered by a linear map of canonical linearizations. So, instead we use smooth linearizations. A linearization (or fiberwise linearization) $\lambda : VM \to EM$ of a smooth manifold M will be called *smooth* if VM has a smooth structure compatible with the linear structure so that λ is a diffeomorphism in some neighborhood of the zero section. We note that the smooth structure on VM is unique if it exists and the comparison map ψ between any two smooth linearizations is also smooth. Note that the derivative of ψ along the zero section gives an isomorphism of vector bundles $V \to V'$. Therefore, any smooth linearization of a smooth manifold is canonically isomorphic to its tangent bundle as a vector bundle.

Proposition A.3.1. *For any compact smooth manifold M, the space of smooth linearizations $\mu : TM \to EM$ with fixed derivative along the zero section is convex and thus contractible.* ☐

Suppose M_0, M_1 are smooth manifolds with canonical linearizations $\lambda_i : TM_i \to EM_i$ and $f : M_0 \to M_1$ is a diffeomorphism with tangent map $Tf : TM_0 \to TM_1$. Then the map germs

$$\lambda_0, \ Ef^{-1} \circ \lambda_1 \circ Tf : TM_0 \to EM_0$$

are smooth linearizations of M_0 with the same derivative, namely the identity, along the zero section. Therefore, there is a 1-parameter family of smooth linearizations $\mu_t : TM_0 \to EM_0$ all having the same derivative going from $\mu_0 = \lambda_0$ to $\mu_1 = Ef^{-1} \circ \lambda_1 \circ Tf$. This is an example of a (smooth) tangential homeomorphism of smooth manifolds.

A.3.2. *Stabilizing linearizations.* Suppose that (M, VM) is a linearized manifold with exponential map germ $\mu : VM \to M'$. Then any vector bundle $p : L \to M$ will be seen to have an induced linearization on the (noncompact) manifold L. We will usually restrict to a disk bundle $D(L)$ which is compact.

We choose an extension $L' \to M'$ of the vector bundle L to M' and assume we have a Gauss map $\gamma : L' \to \mathbb{R}^N$, i.e. a continuous map which is a linear monomorphism on each fiber. This gives a metric on L' and allows us to take the ϵ-disk bundle $D_\epsilon(L')$. Over any two points $x, y \in M'$ we also have a linear map between fibers of L':

$$\pi_y^x : L_x \to L_y$$

given by orthogonal projection in $\mathbb{R}^N$. When $x = y$ this is the identity map on L_x. Therefore, for some neighborhood U of x in M' we get an isomorphism of vector bundles $\pi_U^x : U \times L_x \cong L'|U$ given by

$$\pi_U^x(y, w) = \pi_y^x(w) \in L_y \subseteq L'|U.$$

The vector bundle of the linearization of L induced by VM will be the pull-back $p^*(VM \oplus L)$ of the direct sum $VM \oplus L$. The exponential map on the restriction of $p^*(VM \oplus L)$ to the zero section $M \subset L$ is the map

$$\overline{\mu} : VM \oplus L \to L$$

given on the fiber $V_x M \times L_x$ over $x \in M$ by $\overline{\mu} = \pi_U^x \circ (\mu_x \times id)$ or

$$\overline{\mu}(v, w) = \pi_{\mu_x(v)}^x(w)$$

for some neighborhood U of x in M'.

Since this construction is continuous on the input data, it also works for vector bundles L over fiberwise linearized manifold bundles $(M, V^{\vee}M)$ over a manifold B to produce a linearization of the Euclidean bundle $V^{\vee}M \oplus L$ over M.

Proposition A.3.2. *An extension of this exponential map to all of L exists and is well-defined up to homotopy. Furthermore, if $M, V^{\vee}M, L, p, \gamma$ are smooth then $\overline{\mu}$ and its extension to L will be smooth.* $\qquad\square$

Since $M \subset L$ is a deformation retract, this follows from the following important lemma.

Lemma A.3.3 (Linearization extension lemma). *Suppose that $W \to B$ is a topological manifold bundle and $K \subseteq W$ is a fiberwise deformation retract of W. Then any linearization V of $E^{\vee}W|K$ extends to all of W and any two such extensions are homotopic rel K. Furthermore, if W, V are smooth, then this extension will also be smooth.*

Proof. Choose a fiberwise deformation retraction $r_t : W \to W$ of W to K. Then r_t is covered by a deformation retraction $\tilde{r}_t : E^{\vee}W \to E^{\vee}W$ of the Euclidean bundle $E^{\vee}W$ to $E^{\vee}W|K$ which we can take to be fiberwise smooth in the smooth case and $\tilde{r}_1$ gives an isomorphism $(\tilde{r}_1)_*$ between $E^{\vee}W$ and the pull-back $r_1^*(E^{\vee}W|K)$ of $E^{\vee}W|K$ to W. If $\lambda : V \to E^{\vee}W|K$ is a linearization of $E^{\vee}W|K$ then

$$r_1^* V \xrightarrow{r_1^*(\lambda)} r_1^*(E^{\vee}W|K) \xrightarrow{(\tilde{r}_1)_*^{-1}} E^{\vee}W$$

is a linearization of $E^{\vee}W$ which will be fiberwise smooth in the smooth case and $(\tilde{r}_t)_* \circ r_t^*(\overline{\lambda})$ is a deformation of any linearization $\overline{\lambda} : \overline{V} \to E^{\vee}W$ extending λ to this one. $\qquad\square$

A.3.3. *Tangential homeomorphisms.* Two linearizations of a manifold M are equivalent if they lie in the same path component of the space of linearizations of M, in other words there is a fiberwise linearization of $M \times I$ which agrees with these linearizations at the endpoints. Two linearized manifolds are equivalent if there is a homeomorphism between them so that the linearization of one manifold is equivalent to the pull back of the linearization of the other manifold. We will make these definitions more precise and extend them to manifold bundles over B.

i. *tangential homeomorphism*

By a *tangential homeomorphism* between linearized manifolds we mean a triple:

$$(f, Vf, \mu_t) : (M_0, VM_0, \lambda_0) \to (M_1, VM_1, \lambda_1)$$

where

    (1) $f : M_0 \to M_1$ is a homeomorphism,
    (2) $Vf : VM_0 \to VM_1$ is a nonsingular linear map over f, and
    (3) $\mu_t : VM_0 \to EM_0$ is a one parameter family of linearizations of M_0 going from $\mu_0 = \lambda_0$ to $\mu_1 = Ef^{-1} \circ \lambda_1 \circ Vf$:

$$
\begin{array}{ccc}
VM_0 & \xrightarrow{\ Vf\ } & VM_1 \\
\downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \lambda_1} \\
EM_0 & \xrightarrow{\ Ef\ } & EM_1
\end{array}
$$

A fiberwise tangential homeomorphism between fiberwise linearized manifolds bundles is defined similarly.

ii. *deformation of linearized manifolds*

Suppose that $W \to I$ is a compact manifold bundle over the unit interval together with a fiberwise linearization $V^{\vee}W \to E^{\vee}W$. This is equivalent to the one parameter family of linearized manifolds (M_t, VM_t) where $M_t = W|t$ with linearization $VM_t = V^{\vee}W|t$.

Note that a tangential homeomorphism $(f, Vf, \mu_t) : (M_0, VM_0) \to (M_1, VM_1)$ also gives a one parameter family of linearized manifolds

$$(M_t, VM_t) = (M_0, \mu_t : VM_0 \to EM_0)$$

going from (M_0, VM_0) to (M_1, VM_1) if we identify $(M_0, VM_0, \mu_1) \cong (M_1, VM_1, \lambda_1)$ via the isomorphism (f, Vf).

Conversely, we have the following.

Proposition A.3.4. *A one parameter family of linearized manifolds* (M_t, VM_t) *gives a tangential homeomorphism* $(M_0, VM_0) \simeq (M_1, VM_1)$ *which is uniquely determined up to a contractible choice.*

Proof. Since bundles over I are trivial, there exist homeomorphisms $f_t : M_0 \to M_t$ equal to the identity for $t = 0$ covered by nonsingular vector bundle maps $Vf_t : VM_0 \to VM_t$ giving us a one parameter family of linearizations $\mu_t : VM_0 \to EM_0$ making the following diagram commute.

$$
\begin{array}{ccc}
VM_0 & \xrightarrow{Vf_t} & VM_t \\
\downarrow{\mu_t} & & \downarrow{\lambda_t} \\
EM_0 & \xrightarrow{Ef_t} & EM_t
\end{array}
$$

So (M_t, VM_t) is isomorphic to $(M_1, \eta_t : VM_1 \to EM_1)$ by (g_t, Vg_t) for every $t \in I$. This in turn gives an equivalence of linearized manifolds

$$(g_0, Vg_0, \eta_t \circ Vg_0) : (M_0, VM_0) \simeq (M_1, VM_1).$$

The choices that we made are the product structures for bundles over I. The space of such product structures is contractible. $\qquad\square$

iii. *example*

Suppose that M_0 is a smooth manifold bundle over B and W is a smooth manifold bundle over B which is a topological manifold bundle over $B \times I$ so that $\pi^{-1}(0) = M_0$ and $\pi^{-1}(1) = M_1$ are smooth manifold bundles over B where π is the composition $\pi : W \to B \times I \to I$. Then we will obtain two tangential homeomorphisms between the smooth bundles $M_0 \times I, M_1 \times I$ and W.

(a) The first is

$$(f, V^vf, \mu_t^v) : (M_0 \times I, T^vM_0 \times I) \to (M_1 \times I, T^vM_1 \times I)$$

Since W is a bundle over $B \times I$, W is homeomorphic to $M_0 \times I$ over $B \times I$. This gives a homeomorphism

$$f_t : M_0 \to M_t = \pi^{-1}(t)$$

of bundles over B and we let

$$f = f_1 \times id_I : M_0 \times I \to M_1 \times I.$$

The smooth structure on W gives a linearization of the stabilized Euclidean bundle of M_t:

$$\mu_t : T^vW|M_t \to E^vM_t \oplus \epsilon^1$$

This gives a fiberwise linearization of the bundle $M_t \times I$ over B which is smooth for $t = 0, 1$. So, by the proposition above, we get a tangential homeomorphism of the stabilization

$$M_0 \times I \to M_1 \times I.$$

(b) The second tangential homeomorphism is between the two smooth manifold bundles

$$M_0 \times I \to W$$

First, we add an external collar $M_0 \times [-1.0]$ to the bottom of W. This gives $W^+ = W \cup_{M_0} M_0 \times [-1, 0]$ a bundle over B which is fiberwise diffeomorphic to W. But W^+ has a new projection map $\pi : W^+ \to [-1, 1]$. Let $W_t, t \in I$ be the 1-parameter family of topological manifolds given by

$W_t = \pi^{-1}[-1, t]$. Since W_t is topologically embedded in the smooth manifold bundle W^+ of the same dimension, W_t obtains a linearization from the smooth linearization of W^+. The linearization is smooth for $t = 0, 1$ since W_0, W_1 are smooth submanifolds of W^+. This gives a 1-parameter family of linearized topological manifold bundles going from $W_0 \cong M_0 \times I$ to $W_1 = W^+ \cong W$ as claimed.

A.3.4. *Tangential homeomorphism of smooth manifolds.* By the discussion in the last subsection, a tangential homeomorphism between two smooth manifolds (with canonical linearizations) is given up to contractible choice by a fiberwise linearized topological bundle over I which is smooth over the end points. We call this a "tangential (topological) concordance" between the two smooth manifolds. To avoid repetition, we give the formal definition only for bundles.

Definition A.3.5. By a *fiberwise tangential concordance* between two smooth manifold bundles $M_0 \to B$, $M_1 \to B$ over the same base we mean a linearized topological manifold bundle $W \to B \times I$ so that $W_0 = W|B \times 0$ and $W_1 = W|B \times 1$ are smooth fiberwise linearizations of (the underlying topological manifold bundles of) M_0 and M_1.

When we represent a fiberwise tangential concordance by a fiberwise tangential homeomorphism (f, Vf, μ_t) we would like to say that we can choose $Vf : T^vM_0 \to T^vM_1$ to be smooth. However, this is not possible without changing the smooth structure of T^vM_0 since Vf is a map over the continuous map f which is not, in general, homotopic to a diffeomorphism.

Proposition A.3.6. *If the vertical tangent bundle of M_0 is trivial then any fiberwise tangential concordance between M_0 and M_1 is represented by a fiberwise tangential homeomorphism (f, Vf, μ_t) where $Vf : T^vM_0 \to T^vM_1$ is smooth, using the smooth structure $T^vM_0 \cong M_1 \times \mathbb{R}^n$ and the given smooth structure on T^vM_1. Furthermore the space of all such tangential homeomorphisms representing the same tangential concordance is contractible.*

Proof. The space of smooth nonsingular bundle maps $M_1 \times \mathbb{R}^n \to T^vM_1$ over M_1 is homotopy equivalent to $Map(M_1, O(n))$ which is homotopy equivalent to the space of nonsingular continuous bundle maps $M_0 \times \mathbb{R}^n \to T^vM_1$ over f. □

In fact, we can choose $f : M_0 \to M_1$ to be a smooth embedding on the core of M_0 and we can choose Vf to be smooth over that core.

A.4. **Tangential smoothing.** Given a linearized topological manifold (M_0, VM_0), a *tangential smoothing* of (M_0, VM_0) is a smooth manifold M_1 together with a tangential homeomorphism $(M_0, VM_0) \simeq (M_1, TM_1)$. In this section we express this as a point in the homotopy fiber of a map between moduli spaces of manifolds with smooth and linear structures.

A.4.1. *Space of linearized manifolds.* Let $\widetilde{S}^t(n)$ be the set of all linearized n-manifolds. For concreteness, we take these to be triples (M, V, μ) where $M \subseteq \mathbb{R}^\infty$ is a compact topological n-manifold embedded in $\mathbb{R}^\infty$, $\pi : V \to M$ is an n-plane bundle over M and $\mu : V \to M' = M \cup C$ is a topological exponential map. As before, $C = \partial M \times [0, 1)$ is the standard external collar for M.

Let $\widetilde{S}^t_\bullet(n)$ be the simplicial set whose k-simplices are continuous Δ^k families of linearized n-manifolds. This is a space which lies between the moduli spaces $S^d_\bullet(n)$ and $S^t_\bullet(n)$ in the sense that the simplicial forgetful map $\varphi : S^d_\bullet(n) \to S^t_\bullet(n)$ factors through $\widetilde{S}^t_\bullet(n)$:

$$
\begin{array}{ccc}
& \widetilde{S}^t_\bullet(n) & \\
{}^{\widetilde{\varphi}}\nearrow & & \searrow^{\psi^t} \\
S^d_\bullet(n) \xrightarrow{\quad \varphi \quad} & & S^t_\bullet(n)
\end{array}
$$

The second map $\psi^t : \widetilde{\mathcal{S}^t_\bullet}(n) \to \mathcal{S}^t_\bullet(n)$ is the simplicial forgetful map given by projection to the first coordinate: $\psi^t(M, V, \mu) = M$. But, we are mainly interested in the first map

$$\widetilde{\varphi} : \mathcal{S}^d_\bullet(n) \to \widetilde{\mathcal{S}^t_\bullet}(n).$$

This simplicial map is defined by taking a Δ^k family of smooth manifolds to the underlying family of topological manifolds with canonical linearizations.

A.4.2. *Homotopy fiber of $\widetilde{\varphi}$.* If we write down the definition it will be obvious that the homotopy fiber of $\widetilde{\varphi}$ is the space of tangential smoothings of a fixed linearized topological manifold.

Definition A.4.1. For any linearized topological manifold (M, VM) let $\widetilde{\mathcal{S}^{t/d}_\bullet}(M, VM)$ be the simplicial set whose k-simplices are tangential smoothings of the trivial bundle $M \times \Delta^k \to \Delta^k$ with fiberwise linearization $\lambda \times id : VM \times \Delta^k \to EM \times \Delta^k$.

Proposition A.4.2. $\widetilde{\mathcal{S}^{t/d}_\bullet}(M, VM)$ *is the homotopy fiber of the forgetful functor*

$$\widetilde{\varphi} : \mathcal{S}^d_\bullet(n) \to \widetilde{\mathcal{S}^t_\bullet}(n)$$

over $(M, VM) \in \widetilde{\mathcal{S}^t_0}(n)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

More generally, if $(M, V^\vee M)$ is a fiberwise linearized manifold bundle over B, we can define a space $\widetilde{\mathcal{S}^{t/d}_B}(M)$ of fiberwise tangential smoothings of $(M, V^\vee M)$. This is the homotopy fiber of the map

$$\widetilde{\varphi}_* : |\mathcal{S}^d_\bullet(n)|^B \to |\widetilde{\mathcal{S}^t_\bullet}(n)|^B$$

over the map $B \to |\widetilde{\mathcal{S}^t_\bullet}(n)|$ classifying the linearized bundle $(M, V^\vee M)$ and we are interested in π_0 of this space. However, this set may be empty. We need at least one smoothing $(M_0, V^\vee M_0)$ to make it nonempty. This smoothing can be used as a base point and the other smoothings will be call "exotic (fiberwise tangential) smoothings."

Definition A.4.3. If $M_0 \to B$ is a smooth manifold bundle, then an *exotic (fiberwise tangential) smoothing* of M_0 is defined to be another smooth manifold bundle $M_1 \to B$ which is fiberwise tangentially homeomorphic to M_0.

If A is a submanifold of ∂B and P is a smooth subbundle of M then we define $\widetilde{\mathcal{S}^{t/d}_{B,A}}(M, P)$ to be the (simplicial) subspace of $\widetilde{\mathcal{S}^{t/d}_B}(M)$ consisting of fiberwise tangential smoothings of $(M, V^\vee M)$ which are equal to the given smoothing on $P \cup M_A$ where $M_A = p^{-1}(A)$.

A.4.3. *Stabilization.* By stabilization we mean taking direct limit with respect to all linear disk bundles. In particular we replace the tangential smoothing space $\widetilde{\mathcal{S}^{t/d}_{B,A}}(M_0)$ with the *stable tangential smoothing space*

$$\widetilde{\mathcal{S}^s_{B,A}}(M_0) := \varinjlim \widetilde{\mathcal{S}^{t/d}_{B,A}}(D(\xi))$$

where the direct limit is with respect to all linear disk bundles ξ over M_0.

Stabilization is used to make the tangent bundle trivial. We take the disk bundle of the normal bundle $D(\nu)$. If we replace M_0 with $D(\nu) \times D^m$ with corners rounded then this new M_0 has a smooth spine $J_0 = D(\nu)^-$ which is a smooth manifold bundle. It has a core K_0 which is fiberwise diffeomorphic to M_0 and whose complement $M_0 - K_0$ is an internal collar neighborhood of the fiberwise boundary $\partial^\vee M_0$. Thus $M_0 - K_0$ is diffeomorphic to $\partial^\vee M_0 \times [0, 1)$.

i. *corners*

There is one problem: We need to know that corners can be rounded off in a canonical way. But, for our purposes, this is easy since any two ways of rounding off corners will clearly be tangentially homeomorphic and we have the following lemma.

Lemma A.4.4. *If M_0, M_1 are two smooth bundles over B which are fiberwise tangentially homeomorphic then their tangential smoothing spaces are simplicially homotopy equivalent:*

$$\widetilde{\mathcal{S}_B^{t/d}}(M_0) \simeq \widetilde{\mathcal{S}_B^{t/d}}(M_1).$$

Proof. The fiberwise tangential homeomorphism gives a one parameter family of linearized topological manifold bundle W_t □

One important example was given in subsubsubsection A.3.3(iii). Given a smooth bundle $W \to B$ which fibers topologically over $B \times I$ in such a way that the inverse image of $B \times 0$ and $B \times 1$ are smooth bundles M_0 and M_1, then W defines a tangential homeomorphism $M_0 \times I \to M_1 \times I$. In other words $M_1 \times I$ becomes a point in the stable tangential smoothing space of M_0. We denote this by

$$top(W) \in \widetilde{\mathcal{S}}_{B,A}^s(M_0)$$

This is represented by M_1 in the sense that M_1 and $top(W)$ are stably equivalent.

ii. *flat sides*

Stabilization can also be given by the simple process of taking products with disks:

$$\widetilde{\mathcal{S}}_{B,A}^s(M_0) = \lim_{\to} \widetilde{\mathcal{S}_{B,A}^{t/d}}(M_0 \times D^n)$$

The reason is that trivial disk bundles are cofinal in the directed system of all disk bundles over M_0. Since corners are not a problem, we can also use cubes I^n instead of disks D^n.

Another cofinal system is given by the space of tangential smoothings W of $M_0 \times I^n \times I$ which are "flat" on $M_0 \times I^n \times 0$ in the sense that the tangential homeomorphism

$$(f, V^{\vee}f) : M \times I^n \times I \to W$$

is induced by a tangential homeomorphism

$$(\partial_0 f, V^{\vee}\partial_0 f) : M \times I^n \times 0 \to \partial_0 W$$

in a neighborhood of $M \times I^n \times 0$. This is only a restriction on the tangential map $V^{\vee}f$ and homotopy $\mu^{\vee}$ since any homeomorphism f as above induces a homeomorphism $\partial_0 f$ on $M \times I^n \times 0$. The "flatness" condition is that the maps

$$\mu_t^{\vee} : V^{\vee}(M \times I^n \times I) \to V^{\vee}(W)$$

should send $V^{\vee}(M \times I^n \times 0$ to $V^{\vee}(\partial_0 W)$.

It is easy to see that "flat on one side" smoothings form a cofinal system. When we pass from a smoothing of W to a smoothing of $W \times I$ we always have a tangential smoothing which is flat on one side (in fact on both sides). Similarly, when we stabilize "flat on one side" smoothings we first forget the flatness on one side then take the product with an interval.

iii. *smoothness of the boundary*

Finally, we need the fact that, after stabilization, tangential smoothings of M which are fixed on the vertical boundary $\partial^{\vee}M$ give the same thing as those which don't fix the boundary. We will formulate this more precisely and prove it in the next section.

Suppose that $\partial B = \partial_0 B \cup \partial_1 B$ where $\partial_0 B$ and $\partial_1 B$ meet along their boundary $\partial\partial_0 B = \partial\partial_1 B$. The the boundary of the total space M is equal to $\partial M = \partial_0 M \cup \partial_1 M$ where $\partial_0 M = \partial^{\vee}M \cup M_{\partial_0 B}$ be the union of the vertical boundary $\partial^{\vee}M$ of M and the restriction $M_{\partial_0 B}$ of M to $\partial_0 B$ and $\partial_1 M = M_{\partial_1 B}$.

Let $\widetilde{\mathcal{S}^{t/d}_{B,\partial_0 B}}(M, \partial^{\vee}M)$ be the space of tangential smoothings of M which are fixed on $\partial_0 M$. This is a subspace of $\widetilde{\mathcal{S}^{t/d}_{B,\partial_0 B}}(M)$.

Proposition A.4.5. *After stabilization we get a homotopy equivalence:*

$$\lim_{\to} \widetilde{\mathcal{S}^{t/d}_{B,\partial_0 B}}(D(M), D(\partial^{\vee}M)) \simeq \lim_{\to} \widetilde{\mathcal{S}^{t/d}_{B,\partial_0 B}}(D(M))$$

where both limits are with respect to all linear disk bundles $D(M)$ over M and $D(\partial^{\vee}M)$ is the restriction of $D(M)$ to $\partial^{\vee}M$.

A.5. **Smoothing theorems.** Given a smooth bundle $p : M_0 \to B$ and a smooth submanifold A of ∂B, we would like to determine the set of all isotopy classes of exotic fiberwise smoothings of M_0 which are equal to the given smoothing over A. By definition this is π_0 of the space $\widetilde{\mathcal{S}^{t/d}_{B,A}}(M_0)$ of all tangential smoothings of M_0:

$$(f, V^{\vee}f, \mu^{\vee}_t) : (M_0, T^{\vee}M_0, \lambda_0) \to (M_1, T^{\vee}M_1, \lambda_1)$$

which are trivial over A. This lies in the null component of $\widetilde{\mathcal{S}^{t/d}_{B,A}}(M_0)$ if and only if $f : M_0 \to M_1$ is isotopic to a fiberwise diffeomorphism and $\mu^{\vee}_t$ is isotopic to a family of smooth fiberwise linearizations of M_0.

When we stabilize M_0 will contain a core K_0 which is M_0 minus an internal fiberwise collar and a spine J_0 which is a high codimensional submanifold of K_0. Thus K_0 will be a smooth manifold bundle diffeomorphic to M_0 and both K_0 and J_0 will be fiberwise deformation retracts of M_0.

It will follow from standard immersion theory that $(f, V^{\vee}f)$ is isotopic to a smooth embedding on the core K_0 in such a way that $\mu^{\vee}_t$ becomes isotopic to a smooth linearization over the core. What will remain is the question of smoothability of the internal collar. The extension of the smooth linearization is automatic by the linearization extension lemma A.3.3. So we have a classical smoothing problem whose obstruction space is well known to be a homology theory.

A.5.1. *Smoothing of disk bundles.* We will go over the simplest example: disk bundles.

i. *the problem* We are given two smooth bundles M_0, M_1 over B with fiber D^n, a smooth n-dimensional disk, which are fiberwise diffeomorphic over a submanifold A of B and a fiberwise tangential homeomorphism

$$(f, V^{\vee}f, \mu^{\vee}_t) : (M_0, T^{\vee}M_0, \lambda_0) \to (M_1, T^{\vee}M_1, \lambda_1)$$

which agree with the given diffeomorphism over A. We want to find an isotopy of f rel A to a diffeomorphism over B in a way compatible with the tangential data given by $V^{\vee}f, \mu^{\vee}_t$.

ii. *spines and cores*

The first step is to choose a spine. The spine of a disk is any interior point and the spine of a disk bundle is any section which lies in the interior. Let s_0, s_1 be smooth sections of M_0, M_1 with images J_0, J_1 in the fiberwise interiors so that $s_1 = f \circ s_0$ over A. We will choose a small standard disk bundle neighborhood K_0 of the image J_0 of s_0.

Next, we deform $f : M_0 \to M_1$ so that it takes J_0 to J_1. Since the fibers are contractible, the sections s_1 and $f \circ s_0$ of M_1 are homotopic rel A. We can use the topological ambient isotopy theorem to extend this to an isotopy of f rel A to a homeomorphism taking J_0 to J_1. (However, this example of the ambient isotopy theorem is an easy exercise.)

Lemma A.5.1. *Any topological isotopy of f rel A can be extended to $V^{\vee}f$ and $\mu^{\vee}_t$ to give an isotopy of the tangential homeomorphism $(f, V^{\vee}f, \mu^{\vee}_t)$.*

Proof. Consider the tangential homeomorphism as a 1-parameter family of linearized manifolds (M_t, VM_t, μ_t) together with a family of homeomorphisms $f_t : M_0 \to M_t$ which is the identity on $B \times 0 \cup A \times I$. To prove the lemma we take the same family of linearized manifold bundles with a new family of homeomorphisms. $\qquad\square$

iii. *extending the smooth structure to the core*

Let $(M_t, V^{\vee}M_t, \mu_t^{\vee}, J_t)$ be the 1-parameter family of linearized manifolds over B with a given choice of spine which is smooth over $B \times 0 \cup A \times I \cup B \times 1$.

Let $V_t = V^{\vee}M_t|J_t, E_t = E^{\vee}M_t|J_t$ considered as bundles over B and let $\mu_t : V_t \to E_t$ the microbundle isomorphism given by the restriction of $\mu_t : V^{\vee}M_t \to E^{\vee}M_t$ to J_t. Then V_t will be a vector bundle over B and E_t will be a Euclidean bundle over B which is fiberwise homeomorphic to a neighborhood of J_t. Since continuous isomorphisms of smooth vector bundles are isotopic to smooth vector bundle isomorphisms, we can choose a family of vector bundle isomorphisms $V_0 \cong V_t$ which is the identity for $t = 0$ and smooth for $t = 1$ and we can do this relative to A. This gives a smooth structure on V_t for all t which agrees with the smooth structure over A and over the endpoints.

Using the microbundle isomorphism $\mu_t : V_t \to E_t$ we get a family of smoothings for a neighborhood of J_t in M_t which is compatible with $V^{\vee}M_t$ over J_t. By definition, μ_t will be a smooth linearization of E_t. By the linearization extension lemma A.3.3, we can extend this to a new fiberwise linearization of M_t which is smooth in a neighborhood of J_t (and everywhere where it was already smooth). Furthermore this new linearization will be isotopic to the old one.

iv. *smoothing the collar*

The situation is the following. We have a 1-parameter family of linearized manifold bundles $(M_t, V^{\vee}M_t, \mu_t^{\vee})$ together with a smoothing over the core K_t which is a disk bundle in the interior of M_t. We also have a smoothing over $B \times 0 \cup A \times I \cup B \times 1$ which is compatible with the linearization.

The key point is that smoothing is *excisive* (Proposition A.1.3). Therefore, we may remove the interior of the core K_t. If we stabilize M_t one more, replacing it with $M_t \times [-1, 1]$, we will have the smooth core $K_t \times [-1, 0]$ which meets the boundary. Then, after excising the interior of this new core and rounding off the corners, we get $M_t \times [-1, 1] - \operatorname{int} K_t \times [-1, 0)$ which is a topological h-cobordism bundle over B whose fibers are h-cobordisms of D^n and therefore homeomorphic to $D^n \times I$ which have a smooth structure on the base $D^n \times 0$ and sides $\partial D^n \times I$ and over $B \times 0 \cup A \times I \cup B \times 1$.

This can be rephrased as follows. We have a continuous mapping of pairs

$$(B \times I, B \times 0 \cup A \times I \cup B \times 1) \to (\mathcal{C}ob^t(D^n), \mathcal{C}ob^d(D^n))$$

where $\mathcal{C}ob^t(M)$ is the space of topological h-cobordisms $W \subseteq \mathbb{R}^{\infty}$ of M which are fixed on the base $M \times 0$ and the boundary, $\partial M \times I$ and $\mathcal{C}ob^d(M)$ is the space of pairs (W, α) where $W \in \mathcal{C}ob^t(M)$ and α is a smoothing of W which agrees with a fixed standard smoothing on $M \times 0 \cup \partial M \times I$. These spaces are topologized as geometric realizations of simplicial subsets of $\mathcal{S}^t(n + 1)$ and $\mathcal{S}^d(n + 1)$.

We use the following facts:

(1) $\mathcal{C}ob^t(D^n)$ is contractible by the Alexander trick.
(2) $\mathcal{C}ob^d(D^n)$ is an n-fold loop space since it has an action of the little n-cubes operad.
(3) The smooth structure to be fixed over $B \times 0$.

Therefore, we can trivialize the smooth structure over $B \times 0$ by multiplying by its inverse. The smooth structure over $A \times I$ can also be made trivial in homotopy unique way. The map to $\mathcal{C}ob^h(D^n)$ contains no homotopy information. So, we are reduced to a map

$$B/A \to \mathcal{C}ob^d(D^n)$$

where B/A means smashing A to a point.

Theorem A.5.2. *If M_0 is a smooth disk bundle over B and A is a submanifold of B then the space of stable fiberwise tangential smoothings of B rel A is homotopy equivalent to the space of all pointed maps*

$$B/A \to \mathcal{H}(*)$$

where

$$\mathcal{H}(*) := \operatorname{colim} \mathcal{C}ob^d(D^n)$$

is the stable smooth h-cobordism space of a point. $\qquad\qquad\square$

v. *higher torsion calculation*

We use the well-known fact that $\mathcal{H}(*)$ is rationally homotopy equivalent to BO. This was first shown by Farrell and Hsiang and later Hatcher gave an explicit map $G/O \to \mathcal{H}(*)$ and conjectured that it was nontrivial. This was first proved by Bökstedt and later by Igusa using higher Reidemeister torsion. See [22] for an elementary explanation of this.

We note that $\widetilde{\mathcal{S}}^s_{B,A}(B,0)$ is the space which classifies stable exotic smooth structures on linear disk bundles over B which are trivial over A. ($B \to B$ is the unique D^0 bundle over B with trivial vertical tangent bundle 0.) The theorem shows that

$$\widetilde{\mathcal{S}}^s_{B,A}(B) \simeq Map(B/A, \mathcal{H}(*)).$$

Corollary A.5.3. *$\pi_0 \widetilde{\mathcal{S}}^s_{B,A}(B)$ is an abelian group and we have an isomorphism*

$$\tau^{IK} : \pi_0 \widetilde{\mathcal{S}}^s_{B,A}(B) \otimes \mathbb{R} \cong \bigoplus_{k>0} H^{4k}(B,A;\mathbb{R})$$

given by sending any smooth disk bundle $E \to B$ which is linear over A and any tangential homeomorphism of E to a linear disk bundle to

$$\tau^{IK}(E) = \sum \tau^{IK}_{2k}(E) \in \bigoplus_{k>0} H^{4k}(B,A;\mathbb{R})$$

We note that $\tau^{\mathrm{IK}}(E)$ ignores the tangential data. If we took a different axiomatic higher torsion theory (such as the nonequivariant higher analytic torsion) we would need to subtract the higher torsion of the linear bundle for which E is an exotic smooth structure.

A.5.2. *immersion theory.* We are now looking at a stabilized exotic tangential smoothing (W, V^vW) of the bundle $M_0 \to B$ which is given by a tangential homeomorphism (f, V^vf, μ) between the two smooth bundles M_0, M_1 over B. After stabilizing M_0 has a high codimensional *spine* J_0 which is a smooth submanifold bundle of M_0 with trivial vertical normal bundle in M_0. We also have a *core* K_0 which is a small tubular neighborhood of J_0. We also need to assume that J_0 contains a submanifold bundle P_0 so that f is already a smooth embedding in a neighborhood of P_0 and W is smooth in a neighborhood of $P_0 \times I \cup W_A$ where A is a submanifold of B.

By standard immersion theory ([16],[14]), there is a fiberwise immersion $g : K_0 \to M_1$ over B which is regularly fiberwise homotopic to (f, V^vf) restricted to K_0. Since the spine has a high codimension, we have by transversality that g is an embedding on J_0. By replacing the core K_0 with a smaller core we may also assume that g is an embedding on K_0. We may also assume that g is equal to f in a neighborhood of P_0 and over A.

Immersion theory tells us that that f, g are fiberwise homotopic (fixing a neighborhood of P_0) by a one parameter family of continuous maps $h_t : K_0 \to M_1$ over B rel A and the fiberwise derivative $T^v g : T^v K_0 \to T^v M_1$ is homotopic through nonsingular linear maps $V^v h_t$ to the vector bundle map $V^v f$. Given any $\epsilon > 0$ we can choose the immersion g and the homotopy h_t to be within ϵ of f and that $h_t = f = g$ over A and near P_0 for all t.

Proposition A.5.4. *After stabilization, we can choose $(h_t, V^v h_t)$ so that $h_t : K_0 \to M_1$ is a fiberwise topological embedding for all $t \in I$.*

46

Proof. First, we can reduce the structure group of the linear bundle $V^{\vee}W$ over W to $O(n)$. Then we get a linear ϵ-disk bundle $D^{\vee}W$ over W which, as a bundle over $B \times I$, is linearized with vertical Euclidean bundle isomorphic to $V^{\vee}W \oplus V^{\vee}W$. This is a stabilization. So, it suffices to prove the theorem for $D^{\vee}W$.

The idea of the proof is the following. The interior of the disk bundle $D^{\vee}W$ is homeomorphic to the total space of the vertical tangent Euclidean bundle $E^{\vee}W$. So, the corresponding tangential homeomorphism, when restricted to the core of $D^{\vee}M_0$, is given by the topological vertical derivative $E^{\vee}f$ of f. The topological derivative $E^{\vee}g$ of g is a smooth embedding. Therefore, it suffices to show that $E^{\vee}f$ and $E^{\vee}g$ are homotopic through fiberwise topological embeddings.

The total space of $E^{\vee}K_0$ is the set of all pairs (x, y) in the same fiber of M_0 over B so that $x \in K_0$ and $y \in B_{\epsilon}^{\vee}(x)$ where $B_{\epsilon}^{\vee}(x)$ is the open ϵ-ball neighborhood of x in the fiber of $M_0 \to B$. Inside of this space we have the following two subspaces where $\delta << \epsilon$ is the number so that $K_0 = L_{2\delta}$ where L_{ω} the open ω-neighborhood of the spine J_0:

$$U_{\delta} = \{(x, y) \in E^{\vee}K_0 \,|\, x \in L_{\delta}, d(x, y) < \delta\}$$

$$U_{\delta/2} = \{(x, y) \in E^{\vee}K_0 \,|\, y \in L_{\delta/2}, d(x, y) < \delta/2\}$$

Then $U_{\delta/2} \subseteq U_{\delta}$. We will show that the restrictions of $E^{\vee}f, E^{\vee}g$ to $U_{\delta/2}$ are isotopic and that the isotopy agrees with the given homotopy.

The embedding $E^{\vee}f$ maps $(x, y) \in U_{\delta}$ to $(f(x), f(y)) \in E^{\vee}M_1$ by definition. For every fixed $x \in L_{\delta}$, this mapping sends $x \times B_{\delta}(x)$ to $f(x) \times E^{\vee}f_x(B_{\delta})$ by the mapping $E^{\vee}f_x$ ($E^{\vee}f$ restricted to the fiber of $E^{\vee}M_0 \to M_0$ over x) on the second factor. But the homotopy $V^{\vee}h_t$ is an isotopy from $E^{\vee}f$ to $E^{\vee}g$. Therefore, the embedding $E^{\vee}f$ is isotopic to the embedding which sends $x \times B_{\delta}(x)$ to $f(x) \times E^{\vee}g_x(B_{\delta})$ by the mapping $E^{\vee}g_x$ on the second factor.

By definition of $E^{\vee}g$, this new embedding is

$$f \times g : (x, y) \to (f(x), g(y)).$$

So, $E^{\vee}f|U_{\delta}$ is isotopic to $(f \times g)|U_{\delta}$. By the same argument, $E^{\vee}g|U_{\delta/2}$ is isotopic to $(f \times g)|U_{\delta/2}$. Since $U_{\delta/2} \subseteq U_{\delta}$, we get an isotopy from $E^{\vee}f|U_{\delta/2}$ to $E^{\vee}g|U_{\delta/2}$. And this isotopy will be fixed near P_0 and over A. As was already shown, these are the stabilized versions of f and g on a small tubular neighborhood of the spine of M_0. So, we are done. $\qquad\square$

A.5.3. *smoothing of the core.*

Theorem A.5.5. *There is no stable obstruction to finding a smoothing of the core of W. I.e., after stabilizing, the one parameter family of linearized topological manifolds bundles $(M_t, V^{\vee}M_t, \mu_t^{\vee})$ has a smoothing compatible with the linearization in some tubular neighborhood of the spine J_t. Furthermore, this smoothing will be equal to the given smoothing of W on $P_0 \times I \cup W_A$ if W is already smooth on this set.*

Remark A.5.6. By replacing B with $B \times I$ and A with $A \times I \cup B \times \{0, 1\}$, we conclude that the smoothing of the core is unique up to homotopy.

Proof. By the proposition above, we may assume that this 1-parameter family of linearized manifolds is given by a tangential homotopy equivalence $(f, V^{\vee}f, \mu_t^{\vee})$ where M_0, M_1 are smooth manifold bundles over B which are diffeomorphic over A and $f : M_0 \to M_1$ is a smooth embedding on the core K_0 and $V^{\vee}f$ is the vertical derivative of f along K_0. Returning to the 1-parameter family of manifold bundles $(M_t, V^{\vee}M_t)$, this implies that we have a continuous family of submanifolds $J_t \subseteq M_t$ which are smooth submanifolds for $t = 0, 1$ and these submanifolds have tubular neighborhoods which have product structures: $K_t \cong J_t^+ \times D^n$. where J_t^+ is J_t with an external closed collar attached.

After stabilization, we may assume that the vector bundle $V^{\vee}M_t$ is trivial: $V^{\vee}M_t \cong M_t \times \mathbb{R}^{k+n}$ where k is the dimension of the fiber of $J_t \to B$. The vertical tangent Euclidean bundle will also

be trivial: $E^{\lor}M_t \cong M_t \times \mathbb{R}^{k+n}$ and the linearization is given by a family of microbundle morphisms $\mu_t : V^{\lor}M_t \to E^{\lor}M_t$ which is smooth for $t = 0, 1$ and on the restriction of $V^{\lor}M_t$ to $V^{\lor}K_t$. This is equivalent to a family of mappings

$$\widehat{\mu}_t : M_t \to Homeo(\mathbb{R}^{k+n})$$

By the first lemma below, we can assume, after stabilization, that this map has image in the subgroup of all homeomorphisms of $\mathbb{R}^{k+n}$ having the form $g(x, y) = (x, g_x(y))$, i.e., they are $\mathbb{R}^k$ families of homeomorphisms of $\mathbb{R}^n$.

Along J'_t the spine with an open external collar, we now have a linearization

$$\mu_t : V^{\lor}M_t|J'_t \cong J'_t \times \mathbb{R}^{k+n} \to E^{\lor}M_t|J'_t \cong J'_t \times \mathbb{R}^{k+n}$$

which commutes with the projection to $J'_t \times \mathbb{R}^k$. Now restrict this to the fiber over $J'_t \times 0$. This gives a family of linearizations

$$\eta : J'_t \times \mathbb{R}^n \to J'_t \times \mathbb{R}^n$$

This is a map from a smooth linear bundle to a Euclidean bundle which, by the topological exponential map is homeomorphic to a neighborhood of J_t in M_t. We can use this map to change the smooth structure in this neighborhood so that η is a smooth map. By the second lemma below, we can deform the original linearization to a linearization which is smooth in a neighborhood of the spine. This will contain a somewhat smaller core but it is enough to prove the theorem. □

It remains to prove the two lemmas used in the theorem.

Lemma A.5.7. *The image of the stabilization map* $\sigma : Homeo(\mathbb{R}^{k+n}) \to Homeo(\mathbb{R}^{k+n+k})$ *given by* $\sigma(f)(x, y, z) = (f(x, y), z)$ *can be deformed into the subgroup of all homeomorphisms of* $\mathbb{R}^{k+n+k}$ *having the form* $g(x, y, z) = (x, g_x(y, z)))$. *Furthermore this deformation will always send smooth maps to smooth maps.*

Proof. The deformation is given by rotation. Let ρ_θ be the linear and thus smooth automorphism of $\mathbb{R}^{k+n+k}$ given by the matrix

$$\rho_\theta = \begin{pmatrix} \cos\theta I_k & 0 & -\sin\theta I_k \\ 0 & I_n & 0 \\ \sin\theta I_k & 0 & \cos\theta I_k \end{pmatrix}$$

Then, $\rho_\theta \circ \sigma(f) \circ \rho_{-\theta}, 0 \le \theta \le \pi/2$ is the desired deformation. □

Lemma A.5.8. *Let G be the subgroup of $Homeo(\mathbb{R}^{k+n})$ consisting of homeomorphisms g having the form $g(x, y) = (x, g_x(y))$ and so that g_0 is a smooth diffeomorphism. Let G_0 be the subgroup consisting of diffeomorphisms of $\mathbb{R}^{k+n}$ which lie in G. Then G_0 is a deformation retract of G.*

Proof. This is given by the Alexander trick:

$$g^t(x, y) = (x, g_{tx}(y))$$

for $0 \le t \le 1$. If g is smooth then so is g^t. □

A.5.4. *ignoring the boundary.* We are now ready to prove Proposition A.4.5 which says that, after stabilization, smoothings which are fixed on the vertical boundary and those which are not form homotopy equivalent spaces. The reason is that, after rounding corners, the core is diffeomorphic to the union of the core with the stabilized vertical boundary. So, after smoothing the core we cannot distinguish between the two spaces.

First, we use the "flat on one side" observation (subsubsection A.4.3) to stabilize and have a flat side $\partial_0 W \subseteq W$ which is tangentially homeomorphic to $D(M) \times 0 \subseteq D(M) \times I$ where $D(M)$ is a linear disk bundle over M. On the flat side we can make f smooth on the core $K_0 \times 0$ using the theorem above. (Equivalently, we can smooth the core of $D(M)$ and then stabilize to get a

smoothing of $K_0 \times I$ and then forget the smoothing on all but $K_0 \times 0$. However, uniqueness up to homotopy of this second method is not as easy to see.)

By construction of the core K_0, the complement of K_0 in $D(M)$ is a product of $\partial^v D(M)$ with an interval. Therefore, the pair $(D(M) \times I, K_0)$ is, after rounding corners, diffeomorphic to $(D(M) \times I, D(M) \times 0)$ and (W, K_0) is a tangential smoothing of that pair. If we apply the same construction to a tangential smoothing of $D(M) \times I$ which is fixed on $D(\partial^v M) \times I$, we make the homeomorphism f smooth on $D(\partial^v M) \times I \cup K_0 \times 0$ where K_0 is a neighborhood of the spine $J_0 \cong M$ which is the zero section of the disk bundle $D(M)$. But $D(\partial^v M) \times I$ is a disk bundle over $\partial^v M \times I$ which is an external collar for M and K_0 is a disk bundle over M of the same dimension. So, together they form a disk bundle over M with an external collar (after rounding corners). Thus there is a diffeomorphism of $D(M) \times I$ with corners rounded which takes $D(\partial^v M) \times I \cup K_0 \times I$ to $D(M) \times 0$, making the two stabilized tangential homeomorphisms equivalent. This proves the following extension of Proposition A.4.5.

Theorem A.5.9. *After stabilization we get homotopy equivalences:*

$$\lim_{\to} \widetilde{\mathcal{S}}^{t/d}_{B, \partial_0 B}(D(M), D(\partial^v M)) \simeq \lim_{\to} \widetilde{\mathcal{S}}^{t/d}_{B, \partial_0 B}(D(M)) \simeq \lim_{\to} \mathcal{S}^{t/d}_{B, \partial_0 B}(D(M) \times I, D(M) \times 0)$$

where all three limits are with respect to all linear disk bundles $D(M)$ over M and $D(\partial^v M)$ is the restriction of $D(M)$ to $\partial^v M$.

We removed the tilde from the last version of stabilization since, by the linearization extension lemma A.3.3, the linearization of a smoothing of $(D(M) \times I, D(M) \times 0)$ is unique up to contractible choice.

A.5.5. *little cubes operad.* Using the third form of stabilization given in the theorem above, we can see the infinite loop space structure on the stabilized smoothing space $\widetilde{\mathcal{S}}^s_{B, \partial_0 B}(M)$. Recall that the space $C_k(n)$ of k little n cubes in I^n is given by k disjoint embeddings $y_i : I^n \to I^n$ which are given by affine linear maps $y_i(x) = a_i x + b_i$ where a_i are positive real numbers and $b_i \in I^n$.

$$\alpha : C_k(n) \times \mathcal{S}^{t/d}_{B, \partial_0 B}(D(M) \times I^{n+1}, D(M) \times I^n)^k \to \mathcal{S}^{t/d}_{B, \partial_0 B}(D(M) \times I^{n+1}, D(M) \times I^n)$$

$$\alpha(y; W_1, \cdots, W_k) = D(M) \times I^{n+1} \cup_y \coprod W_i$$

where the base of each W_i is attached to the top of $D(M) \times I^{n+1}$ using the map

$$1_{D(M)} \times y_i : \partial_0 W_i = D(M) \times I^n \times 0 \to D(M) \times I^n \times 1$$

and the resulting corners are rounded.

There is an easier way to describe the addition operation in the case when the supports of the exotic smooth structures are disjoint. An element of $\widetilde{\mathcal{S}}^{t/d}_{B, \partial_0 B}(M, C)$ is said to have **support** in the closure of the complement of C in M.

Proposition A.5.10. *The addition operation on the stable smoothing space $\widetilde{\mathcal{S}}^s_{B, \partial_0 B}(M)$ given by the little cubes operad action described above is given unstably on smooth structures on M with disjoint supports $S_i \subseteq M - \partial^v M$ by $W = \sum W_i$ which is equal to M in the complement of $\coprod S_i$ and equal to W_i on S_i.*

We will only apply this proposition to the group structure on $\pi_0 \widetilde{\mathcal{S}}^s_{B, \partial_0 B}(M)$.

Proof. We will prove that the following diagram commutes up to homotopy for any $y \in C_k(n)$ where C_i is the closure of the complement of S_i in M.

$$
\begin{array}{ccc}
\prod \widetilde{\mathcal{S}^{t/d}_{B,\partial_0 B}}(M, C_i) & \xrightarrow{\Sigma} & \widetilde{\mathcal{S}^{t/d}_{B,\partial_0 B}}(M, \bigcap C_i) \\
\downarrow & & \downarrow \\
\prod \widetilde{\mathcal{S}^{s}_{B,\partial_0 B}}(M) & \xrightarrow{\alpha(y;-)} & \widetilde{\mathcal{S}^{s}_{B,\partial_0 B}}(M)
\end{array}
$$

Note that, except for Σ, these maps are only defined up to homotopy.

Suppose that (M_i) is an element of the upper left corner. Thus M_i is a tangential smoothing of M over $(B, \partial_0 B)$ with support in S_i. If we stabilize by taking a product with a disk, we get $M_i \times D^N \times I$ which is a smooth structure on $M \times D^N \times I$ with support in $S_i \times D^N \times I$. Using the smoothing of the core on one flat side argument explained in great depth in this section, we make the tangential homeomorphism smooth on the core which equivalent to $S_i \times D^N \times 0$. By conjugating by a smooth isotopy (after rounding corners) we can make the tangential homeomorphism smooth on $S_i \times (D^N \times 0 \cup \partial D^N \times I)$. Call this new bundle W_i with base $\partial_0 W_i \cong M \times D^N \times 0$. At this point we can use the uniqueness of linearization lemma to ignore the linearization. This brings us to the lower left corner of the diagram.

The little cubes operation now produces the smooth bundle

$$
M \times D^N \times I \cup_y \coprod W_i
$$

Since W_i has bottom and sides equal to $M \times (D^N \times 0 \cup \partial D^N \times I)$, we can lower W_i into $M \times D^N \times I$ to make

$$
M \times D^N \times I - \coprod M \times y_i(D^N) \times I \cup \coprod W_i
$$

This is a family of smooth bundles giving a homotopy of the mapping from the upper left to the lower right of our diagram. Since this new smooth structure on $M \times D^N \times I$ has support in the union of $S_i \times y_i(D^N) \times I$ and the S_i are disjoint, we can expand the embeddings $y_i : D^N \to D^N$ until they are the identity and obtain an isotopy of the structure. The result is a smooth structure on $M \times D^N \times I$ given by W_i on $S_i \times D^N \times I$ which is a description of the stabilization of $\sum M_i$. So, we have shown that the diagram commutes up to homotopy. $\square$

A.5.6. *Morlet's Theorem.* Since $D(M)$ can be chosen to have a trivial tangent bundle and can be stabilized by taking a limit with respect to all trivial disk bundles, we can now use the following theorem of Burghelea and Lashof which follows from Morlet's comparison theorem. (See [7], [8, Thm H].)

Theorem A.5.11. *Let X be a compact smooth manifold with trivial tangent bundle. Then the space of stable smooth structures on $X \times I$ equal to the standard smooth structure on $X \times 0 \cup \partial X \times I$ is a homology theory in X.*

Let $\mathcal{H}^{\%}(X)$ denote this homology theory of X. This the homology theory associated to the spectrum of $\mathcal{H}(*)$:

$$
\mathcal{H}^{\%}(X) = \Omega^{\infty}(X_+ \wedge \mathcal{H}(*)).
$$

(See the section on homotopy theory below.) Then the theorem above together with the smoothing of the core theorem gives the following.

Corollary A.5.12.

$$
\widetilde{\mathcal{S}^{s}_{D^k, S^{k-1}}}(X \times D^k) \simeq \Omega^k \mathcal{H}^{\%}(X)
$$

We can extend this calculation to the general case using the following lemma.

Lemma A.5.13 (fibration lemma)**.** *We have a fibration sequence:*

$$\widetilde{\mathcal{S}}^s_{B,\partial_0}(M) \to \widetilde{\mathcal{S}}^s_B(M) \to \widetilde{\mathcal{S}}^s_{\partial_0 B}(M)$$

Proof. If we choose a collar neighborhood C of $\partial_0 B$ in B and a topological product structure $M_C \cong M_{\partial_0 B} \times I$ we can easily extend deformations of smooth structures of M over $\partial_0 B$ to deformations (with support in M_C) of the smooth structure of M. $\square$

From the basic case given in Corollary A.5.12 and the fibration lemma we can conclude the general case:

Theorem A.5.14 (main smoothing theorem)**.** *Let $W \to B$ be a compact smooth manifold bundle. Then*

$$\widetilde{\mathcal{S}}^s_{B,\partial_0}(W) \simeq \Gamma_{B,\partial_0} \mathcal{H}^{\%}_B(W)$$

where $\mathcal{H}^{\%}_B(W)$ is the fiberwise $\mathcal{H}^{\%}$ homology bundle of W over B, i.e. the bundle whose fiber over $b \in B$ is $\mathcal{H}^{\%}(W_b)$ where $W_b = p^{-1}(b)$.

Remark A.5.15. Since the fibers of $\mathcal{H}^{\%}_B(W) \to B$ are infinite loop spaces and $\mathcal{H}(*)$ has finite type [10], this implies that $\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(W)$ is a finitely generated abelian group.

Proof. Choose a smooth triangulation of B so that $\partial_0 B$ and $\partial_1 B$ are subcomplexes. Let A be a subcomplex of B containing the $k-1$ skeleton and let σ be a k simplex of B. Then we have a mapping between two fibration sequences:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{S}}^s_{\sigma,\partial\sigma}(W_\sigma) & \longrightarrow & \widetilde{\mathcal{S}}^s_{A\cup\sigma}(W_{A\cup\sigma}) & \longrightarrow & \widetilde{\mathcal{S}}^s_A(W_A) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
\Gamma_{\sigma,\partial\sigma}\mathcal{H}^{\%}_\sigma(W_\sigma) & \longrightarrow & \Gamma_{A\cup\sigma}\mathcal{H}^{\%}_B(W) & \longrightarrow & \Gamma_A\mathcal{H}^{\%}_B(W)
\end{array}
$$

We use the excisiveness of smoothing to identify $\widetilde{\mathcal{S}}^s_{A\cup\sigma,A}(W_{A\cup\sigma}) = \widetilde{\mathcal{S}}^s_{\sigma,\partial\sigma}(W_\sigma)$. Since σ is contractible, W_σ is a product bundle $W_\sigma \cong \sigma \times W_b$. So, α is a homotopy equivalence by Corollary A.5.12. If γ is a homotopy equivalence then β will be a homotopy equivalence. Therefore, by induction $\widetilde{\mathcal{S}}^s_B(W) \simeq \Gamma_B \mathcal{H}^{\%}_B(W)$. Another map of fibration sequences proves the relative version stated in the theorem. $\square$

A.5.7. *Stratified smoothing theorem.* We will use the following trivial observation to extend the main smoothing theorem to the "stratified" case.

Lemma A.5.16 (additivity of smoothing)**.** *Suppose that E_i are disjoint smooth bundles over B. Then*

$$\widetilde{\mathcal{S}^{t/d}_{B,\partial_0}}\left(\coprod E_i, \coprod \partial^v E_i\right) = \prod \widetilde{\mathcal{S}^{t/d}_{B,\partial_0}}(E_i, \partial^v E_i)$$

The basic case of the stratified smoothing theorem is the following. Suppose that M is a smooth bundle over B and E is a codimension 0 subbundle of M which is a disjoint union of bundles: $E = \coprod E_i$. Then clearly,

$$\widetilde{\mathcal{S}^{t/d}_{B,\partial_0}}(E, \partial^v E) \cong \widetilde{\mathcal{S}^{t/d}_{B,\partial_0}}(M, C) \subseteq \widetilde{\mathcal{S}^{t/d}_{B,\partial_0}}(M, \partial^v M)$$

where C is the closure of the complement of E in M.

Lemma A.5.17. *After stabilization we get homotopy equivalences* γ_E, γ_M *compatible with inclusion in the sense that the following diagram commutes.*

$$
\begin{array}{ccc}
\widetilde{\mathcal{S}}^s_{B,\partial_0}(E) & \xrightarrow[\simeq]{\gamma_E} & \Gamma_{B,\partial_0}\mathcal{H}^{\%}_B(E) \\
\Big\downarrow{\scriptstyle\subseteq} & & \Big\downarrow{\scriptstyle\subseteq} \\
\widetilde{\mathcal{S}}^s_{B,\partial_0}(M) & \xrightarrow[\simeq]{\gamma_M} & \Gamma_{B,\partial_0}\mathcal{H}^{\%}_B(M)
\end{array}
$$

Suppose that W is a smooth bundle over B and we have a smooth triangulation of B so that $\partial_0 B$ is a subcomplex. Over each simplex σ of B, suppose we have a smooth codimension-0 compact manifold subbundle $E_\sigma \subseteq W_\sigma$. Since σ is contractible, $E_\sigma \cong F \times \sigma$ for some compact manifold F. Suppose that $E_\tau \subseteq E_\sigma$ for all $\tau \subseteq \sigma$. Suppose also that E_σ is empty for all $\sigma \subseteq \partial_0 B$. For example, we could let $E_\tau = \emptyset$ if $\sigma \subseteq \partial_0 B$ and $E_\sigma = W_\sigma$ otherwise. We let $E \subseteq W$ be the union over all simplices σ of the restriction of E_σ to the interior of σ. We say that E is a *stratified bundle* over B since the restriction of E to each open simplex is a (trivial) bundle. More precisely, E is a *stratified subbundle* of W.

Let $\widetilde{\mathcal{S}}^s_B(E)$ be the subspace of $\widetilde{\mathcal{S}}^s_{B,\partial_0}(W)$ of all tangential stable smoothings of W which have support in the interior of E_σ over the interior of each σ for every simplex σ. Thus, we consider smoothings of E which are fixed on the vertical boundary $\partial^v E$ of E.

Let $\mathcal{H}^{\%}_B(E)$ denote the stratified subbundle of $\mathcal{H}^{\%}_B(W)$ which is equal to $\mathcal{H}^{\%}_\sigma(E_\sigma)$ over the interior of σ for every simplex σ.

Theorem A.5.18 (stratified smoothing theorem)**.**

$$
\widetilde{\mathcal{S}}^s_B(E) \simeq \Gamma_B \mathcal{H}^{\%}_B(E).
$$

Proof. The argument is the same as in the main smoothing theorem, but it is short and worth repeating. Let A be a subcomplex of B containing the $k-1$ skeleton and let σ be a k-simplex. Then we have a mapping of fiber sequences:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{S}}^s_{\sigma,\partial\sigma}(E_\sigma) & \longrightarrow & \widetilde{\mathcal{S}}^s_{A\cup\sigma}(E_{A\cup\sigma}) & \longrightarrow & \widetilde{\mathcal{S}}^s_A(E_A) \\
{\scriptstyle\alpha}\Big\downarrow & & {\scriptstyle\beta}\Big\downarrow & & {\scriptstyle\gamma}\Big\downarrow \\
\Gamma_{\sigma,\partial\sigma}\mathcal{H}^{\%}_\sigma(E_\sigma) & \longrightarrow & \Gamma_{A\cup\sigma}\mathcal{H}^{\%}_B(E) & \longrightarrow & \Gamma_A\mathcal{H}^{\%}_B(E)
\end{array}
$$

α is a homotopy equivalence by the theorem and γ is a homotopy equivalence by induction on the size of A. So β is a homotopy equivalence and the theorem follows. $\qquad\square$

We are interested in the following special case. Suppose that L is a compact smooth q-manifold ($q = \dim B$) with $\partial L = \partial_0 L \cup \partial_1 L$ where $\partial_0 L, \partial_1 L$ meet along a corner set $\partial\partial L$. Suppose that $\lambda : L \to B$ is a smooth immersion so that $\lambda^{-1}(\partial_1 B) = \partial_1 L$ and $\lambda(\partial_0 L)$ meets ∂B transversely along $\lambda(\partial\partial L)$. (See Figure 4.) Assume that the immersion λ is self-transverse, so that there exists a smooth triangulation of B for which the number of inverse image points in L is constant on each open simplex.

Let $\pi : E \to L$ be a compact manifold bundle with the same dimension as W and let $\tilde{\lambda} : E \to W$ be a smooth codimension 0 embedding over $\lambda : L \to B$. It follows then that $\tilde{\lambda}^{-1}(W_{\partial_1 B}) = E_{\partial_1 L}$. Also, the image in W of the complement of $E_{\partial_0 L}$ in E is a stratified subbundle of W over B. Call this image E^δ. A fiberwise smooth structure for E over L which is equal to the given smooth structure over $\partial_0 L$ is equivalent to a fiberwise smooth structure on W with support in E^δ:

$$
\widetilde{\mathcal{S}}^s_{L,\partial_0}(E) \cong \widetilde{\mathcal{S}}^s_B(E^\delta) \subseteq \widetilde{\mathcal{S}}^s_{B,\partial_0}(W)
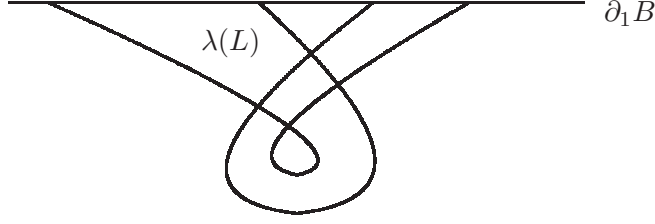$$

FIGURE 4. In this example, L is a square and $\lambda : L \to B$ maps $\partial_1 L =$ two opposite sides into $\partial_1 B$. The diamond shaped region is covered twice by $\lambda(L)$.

This implies that

$$\Gamma_{L,\partial_0}\mathcal{H}_L^{\%}(E) \simeq \Gamma_B \mathcal{H}_B^{\%}(E^\delta) \subseteq \Gamma_{B,\partial_0 B}\mathcal{H}_B^{\%}(W)$$

The statement that we need is the following.

Corollary A.5.19. *The following diagram commutes.*

$$
\begin{array}{ccccc}
\widetilde{\mathcal{S}}_{L,\partial_0 L}^s(E) & \xrightarrow[\simeq]{\mu} & \widetilde{\mathcal{S}}_B^s(E^\delta) & \xrightarrow{\subseteq} & \widetilde{\mathcal{S}}_{B,\partial_0 B}^s(W) \\
{\scriptstyle \gamma_E} \downarrow {\scriptstyle \simeq} & & {\scriptstyle \gamma_{E^\delta}} \downarrow {\scriptstyle \simeq} & & {\scriptstyle \gamma_W} \downarrow {\scriptstyle \simeq} \\
\Gamma_{L,\partial_0 L}\mathcal{H}_L^{\%}(E) & \xrightarrow[\simeq]{\mu} & \Gamma_B \mathcal{H}_B^{\%}(E^\delta) & \xrightarrow{\subseteq} & \Gamma_{B,\partial_0 B}\mathcal{H}_B^{\%}(W)
\end{array}
$$

APPENDIX B. HOMOTOPY THEORY

In Appendix B we calculate, rationally, π_0 of the spaces of sections obtained in Appendix A. The main results are the following.

- (Corollary B.2.2) Suppose that the base B and fiber X of $M \to B$ are oriented manifolds. Then

$$\pi_0 \Gamma_{B,\partial_0 B}\mathcal{H}_B^{\%}(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(M, \partial_1 M; \mathbb{R}) \cong \bigoplus_{k>0} H^{N+4k}(M, \partial_0 M; \mathbb{R})$$

  where $q = \dim B$, $N = \dim X$ and $\partial_0 M, \partial_1 M$ are the two parts of the boundary of M given by $\partial_1 M = M_{\partial_1 B}$ and $\partial_0 M = M_{\partial_0 B} \cup \partial^\nu M$.

- (Corollary B.4.2 and Corollary A.5.19) If $\lambda : (L, \partial_1 L) \to (B, \partial_1 B)$ and $D(\tilde\lambda) : E \to W$ are as above, we get the following commuting diagram where $\partial_1 E = E_{\partial_1 B}$ and all vertical maps are induced by $D(\tilde\lambda)$.

$$
\begin{array}{ccccc}
\pi_0 \widetilde{\mathcal{S}}_{L,\partial_0 L}^s(E) \otimes \mathbb{R} & \xrightarrow[\gamma_E]{\cong} & \pi_0 \Gamma_{L,\partial_0 L}\mathcal{H}_L^{\%}(E) \otimes \mathbb{R} & \xrightarrow[\theta_E]{\cong} & \bigoplus_{k>0} H_{q-4k}(E, \partial_1 E; \mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0 \widetilde{\mathcal{S}}_{B,\partial_0 B}^s(W) \otimes \mathbb{R} & \xrightarrow[\gamma_W]{\cong} & \pi_0 \Gamma_{B,\partial_0 B}\mathcal{H}_B^{\%}(W) \otimes \mathbb{R} & \xrightarrow[\theta_W]{\cong} & \bigoplus_{k>0} H_{q-4k}(W, \partial_1 W; \mathbb{R})
\end{array}
$$

The computation of $\pi_0 \Gamma_{B,\partial_0 B}\mathcal{H}_B^{\%}(M) \otimes \mathbb{R}$ is an exercise in elementary homotopy theory which we will now explain. First we need to recall the definition of generalized homology.

B.1. **Review of generalized homology.** We assume that all our spaces are Hausdorff and homotopy equivalent to CW-complexes. Suppose that G is a *prespectrum*, i.e., a collection of pointed spaces $G_0, G_1, \cdots$ and pointed maps $\Sigma G_n \to G_{n+1}$ (which is equivalent to a pointed map $G_n \to \Omega G_{n+1}$). Then, for any pointed space X, we get another prespectrum $X \wedge G$ with n-th space $X \wedge G_n$ since $\Sigma(X \wedge G_n) \cong X \wedge \Sigma G_n$. Two prespectra are considered to be the same if the spaces

G_n and structure maps $\Sigma G_n \to G_{n+1}$ agree for sufficiently large n. Therefore, G_n need only be defined for large n.

If ξ is an m-dimensional vector bundle and ϵ^k is the trivial k-plane bundle over the same base space then the Thom space $D(\xi \oplus \epsilon^k)/S(\xi \oplus \epsilon^k)$ of $\xi \oplus \epsilon^k$ is the k-fold suspension of the Thom space $D(\xi)/S(\xi)$ of ξ. Define a prespectrum $T(\xi)$ starting in degree m so that $T(\xi)_{m+k} = D(\xi \oplus \epsilon^k)/S(\xi \oplus \epsilon^k)$. This is the suspension spectrum of the formally desuspended usual Thom space:

$$T(\xi) = \Sigma^{-m} D(\xi)/S(\xi)$$

It is well-defined on the stable vector bundle associated to ξ. If ξ is oriented then the Thom Isomorphism Theorem tells us that the reduced homology of $T(\xi)$ is isomorphic to the homology of the base space of ξ.

Associated to any prespectrum G we have the space

$$\Omega^\infty G := \operatorname{colim} \Omega^n G_n$$

We will assume that the maps $G_n \to \Omega G_{n+1}$ are embeddings. Then $\Omega^\infty G$ is an infinite loop space since $\Omega^\infty G = \Omega(\Omega^\infty F)$ where $F_n = G_{n+1}$ is the *delooping* of G which we denote $F = \Omega^{-1} G$.

The homology/reduced homology groups of a space X with coefficients in the spectrum associated to G are defined to be the homotopy groups:

$$H_n(X; G) := \pi_n(\Omega^\infty(X_+ \wedge G)) = \operatorname{colim} \pi_{n+k}(X_+ \wedge G_k)$$

$$\overline{H}_n(X; G) := \pi_n(\Omega^\infty(X \wedge G)) = \operatorname{colim} \pi_{n+k}(X \wedge G_k)$$

where $X_+ = X \coprod *$ is X with an added disjoint basepoint. DWW, section 8, explains how any homotopy functor G gives an "excisive" functor $G^\%(X) \simeq X_+ \wedge G(*)$ and when $G(X)$ is a spectrum valued functor, meaning $G_n(X) \cong \Omega G_{n+1}(X)$, they used the notation,

$$G^\%(X) := \operatorname{colim} \Omega^n G_n^\%(X) \simeq \Omega^\infty(X_+ \wedge G(*))$$

We will also use the notation

$$\overline{G}^\%(X) := \Omega^\infty(X \wedge G(*))$$

so that $G^\%(X) = \overline{G}^\%(X_+)$. Then $\overline{G}^\%$ is a functor that takes cofibration sequences to fibration sequences and homotopy push-out squares to homotopy pull-back squares. In particular:

(B.1) $$\overline{G}^\%(X \vee Y) \simeq \overline{G}^\%(X) \times \overline{G}^\%(Y)$$

$$\Omega \overline{G}^\%(X) \simeq \overline{G}^\%(\Sigma X)$$

$$G^\%(D(\xi)/S(\xi)) \simeq \Omega^{\dim \xi} G^\%(T(\xi)).$$

B.2. **Fiberwise homology.** Suppose that $X \to E \xrightarrow{p} B$ is a fiber bundle where B is a compact oriented smooth q-manifold. Then let $G_B^\%(E)$ be the bundle over B with fiber $G^\%(X)$. Since the fibers are pointed, this bundle has a trivial section.

If $A \subseteq B$ is a cofibration let $\Gamma_{B,A} G_B^\%(E)$ be the space of sections of $G_B^\%(E)$ which are trivial on A. This is an infinite loop space since

$$\Gamma_{B,A} G_B^\%(E) \cong \Omega^k \Gamma_{B,A}(\Omega^{-k} G)_B^\%(E)$$

where $\Omega^{-k} G = \{G_{k+n}\}$. In particular, $\pi_0 \Gamma_{B,A} G_B^\%(E)$ is an abelian group.

B.2.1. *Theorem and corollary.* Suppose that ∂B is a union of two $q - 1$ dimensional submanifolds $\partial_0 B, \partial_1 B$ which meet along their common boundary $\partial \partial_0 B = \partial \partial_1 B$. For any $A \subseteq B$ we use the notation $E_A = p^{-1}(A)$. Then

Theorem B.2.1 (basic homotopy calculation)**.** *There is a natural homotopy equivalence*

$$\Gamma_{B,\partial_0 B} G_B^{\%}(E) \simeq \Omega^q \overline{G}^{\%}(T(\xi)/T(\xi_1))$$

where $T(\xi)$ is the Thom space of the pull-back ξ of the stable normal bundle of B to E and $T(\xi_1) \subseteq T(\xi)$ is the subspace given by restricting ξ to $E_{\partial_1 B}$.

   The proof of this fact is very similar to the proof of Poincaré duality and is explained below. Here is the example that we have in mind.

Corollary B.2.2. *Let $\mathcal{H}(X)$ be the space of stable h-cobordisms of X. Then*

$$\pi_0 \Gamma_{B,\partial_0 B} \mathcal{H}_B^{\%}(E) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1 B}; \mathbb{R})$$

where $q = \dim B$.

Proof. By the theorem we have

$$\pi_0 \Gamma_{B,\partial_0 B} \mathcal{H}_B^{\%}(E) \cong \overline{H}_q(T(\xi)/T(\xi_1); \mathcal{H}(*)).$$

But we have a rational equivalence of infinite loop spaces

$$\mathcal{H}(*) \simeq_{\mathbb{Q}} G/O \simeq_{\mathbb{Q}} \prod_{k>0} K(\mathbb{Z}, 4k).$$

So, rationally we have:

$$\overline{H}_q(T(\xi)/T(\xi_1); \mathcal{H}(*)) \cong_{\mathbb{Q}} \bigoplus_{k>0} \overline{H}_{q-4k}(T(\xi)/T(\xi_1)) \cong_{\mathbb{Q}} \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1 B})$$

using the Thom isomorphism theorem at the last step. Extend scalars to $\mathbb{R}$ to get the result. $\square$

B.2.2. *definition of θ, Θ.* To make a specific choice for the isomorphism in Corollary B.2.2 we need a specific rational homotopy equivalence

$$\mathcal{H}(*) \simeq_{\mathbb{Q}} \prod_{k>0} K(\mathbb{Z}, 4k)$$

This is equivalent to choosing a fixed rational cohomology class in $\prod_{k>0} H^{4k}(\mathcal{H}(*); \mathbb{Q})$. We take this to be the higher IK-torsion invariant which is a real cohomology class

$$\tau^{\mathrm{IK}} \in \prod_{k>0} H^{4k}(\mathcal{H}(*); \mathbb{R})$$

which comes from an element of $\prod_{k>0} H^{4k}(\mathcal{H}(*); \zeta(2k+1)\mathbb{Q})$ by [22].
   With this choice we get a natural isomorphism

$$\theta_E : \pi_0 \Gamma_{B,\partial_0 B} \mathcal{H}_B^{\%}(E) \otimes \mathbb{R} \xrightarrow{\cong} \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1 B}; \mathbb{R})$$

Combining this with the homotopy equivalence $\widetilde{\mathcal{S}}_{B,\partial_0}^s(E) \simeq \Gamma_{B,\partial_0 B} \mathcal{H}_B^{\%}(E)$ given by Theorem (A.5.14) we obtain a natural isomorphism

$$\Theta_E : \pi_0 \widetilde{\mathcal{S}}_{B,\partial_0}^s(E) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1 B}; \mathbb{R}).$$

Proposition B.2.3. *In the case when $E \to B$ is a linear disk bundle, the composition*

$$\pi_0 \widetilde{\mathcal{S}}^s_{B,\partial_0}(E) \xrightarrow{\Theta} \bigoplus_{k>0} H_{q-4k}(E, E_{\partial_1 B}; \mathbb{R}) \cong \bigoplus_{k>0} H_{q-4k}(B, \partial_1 B; \mathbb{R}) \cong \bigoplus_{k>0} H^{4k}(B, \partial_0 B; \mathbb{R})$$

is equal to the higher IK-torsion invariant:

$$\Theta_E(E') = \tau^{IK}(E', E) = \tau^{IK}(E') \in \bigoplus_{k>0} H^{4k}(B, \partial_0 B; \mathbb{R})$$

Proof. This follows from the definition of Θ_E and the fact that $\tau^{\mathrm{IK}}(E) = 0$ for linear disk bundles making $\tau^{\mathrm{IK}}(E', E) = \tau^{\mathrm{IK}}(E') - \tau^{\mathrm{IK}}(E) = \tau^{\mathrm{IK}}(E')$. $\qquad\square$

B.3. **Proof of the theorem.** We prove Theorem B.2.1 first in the special case when B is a compact q-manifold embedded in D^q and $\partial_0 B$ is empty (so $\partial_1 B = \partial B$). In that case the normal bundle of B is trivial, so $T(\xi) = E_+$ and $T(\xi)/T(\xi_1) = E/E_{\partial B}$. Let

$$\varphi_B : \Gamma_B G_B^{\%}(E) \to \Omega^q \overline{G}^{\%}(E/E_{\partial B})$$

be the map given as follows. Take the inclusion of E into the trivial bundle $B \times E$ via the map $(p, id_E) : E \to B \times E$. This induces a map

$$\psi_B : \Gamma_B G_B^{\%}(E) \to \Gamma_B G_B^{\%}(B \times E) = Map(B, G^{\%}(E)).$$

For any $\gamma \in \Gamma_B G_B^{\%}(E)$ the mapping $\psi_B(\gamma) : B \to G^{\%}(E)$ sends ∂B into $G^{\%}(E_{\partial B})$. So, it induces a mapping

$$\varphi_B(\gamma) : D^q/S^{q-1} \to D^q/(D^q - intB) = B/\partial B \xrightarrow{\psi_B(\gamma)} \overline{G}^{\%}(E/E_{\partial B})$$

representing an element of $\Omega^n \overline{G}^{\%}(E/E_{\partial B})$. In the relative case, $\psi_{(B,\partial_0 B)}(\gamma)$ sends $\partial_1 B$ into $G^{\%}(E_{\partial_1 B})$ and $\partial_0 B$ into $* = G^{\%}(\emptyset) \subset G^{\%}(E_{\partial_1 B})$. So $\psi_{(B,\partial_0 B)}(\gamma)$ induces a mapping

$$\varphi_{(B,\partial_0 B)}(\gamma) : D^q/S^{q-1} \to B/\partial B \to \overline{G}^{\%}(E/E_{\partial_1 B})$$

giving an element of $\Omega^q \overline{G}^{\%}(E/E_{\partial_1 B})$.

Lemma B.3.1. *Suppose that B is a compact q-manifold embedded in D^q. Then the mapping*

$$\varphi_B : \Gamma_B G_B^{\%}(E) \to \Omega^q \overline{G}^{\%}(E/E_{\partial B})$$

described above is a homotopy equivalence.

Suppose for a moment that this is true.

Proof of Theorem B.2.1. Consider the next case when B is a compact q-manifold embedded in the interior of D^q and $\partial B = \partial_0 B \cup \partial_1 B$. Let $C \cong \partial_0 B \times I$ be an external collar neighborhood for $\partial_0 B$ in D^q so that $J = B \cup C \simeq B$ and $B \cap C = \partial_0 B$. Then the bundle E over B extends a bundle $E_J \to J$ which is unique up to isomorphism and the mappings $\varphi_J, \varphi_C, \varphi_{(B,\partial_0 B)}$ are compatible making the following diagram commute.

$$
\begin{array}{ccccc}
\Gamma_{B,\partial_0 B} G_B^{\%}(E) & \longrightarrow & \Gamma_J G_J^{\%}(E_J) & \longrightarrow & \Gamma_C G_C^{\%}(E_C) \\
\downarrow{\scriptstyle \varphi_{(B,\partial B)}} & & \downarrow{\scriptstyle \varphi_J} & & \downarrow{\scriptstyle \varphi_C} \\
\Omega^q \overline{G}^{\%}(E/E_{\partial_1 B}) & \longrightarrow & \Omega^q \overline{G}^{\%}(E_J/E_{\partial J}) & \longrightarrow & \Omega^q \overline{G}^{\%}(E_C/E_{\partial C})
\end{array}
$$

The top row is a fibration sequence since $\Gamma_{B,\partial_0 B} G_B^{\%}(E) = \Gamma_{J,C} G_J^{\%}(E_J)$ and the bottom row is a fibration sequence since

$$E/E_{\partial_1 B} \to E_J/E_{\partial J} \to E_C/E_{\partial C}$$

is a cofibration sequence. Since φ_J, φ_C are homotopy equivalences by the lemma above, the induced map $\varphi_{(B,\partial_0 B)}$ is also a homotopy equivalence and $T(\xi)/T(\xi_1) = E/E_{\partial_1 B}$, so the theorem holds in this case.

In the general case we choose an embedding $B^q \hookrightarrow D^{q+n}$ and let ν be the n-dimensional normal bundle of B. Let ξ be the pull back of ν to E and let $D(\nu), S(\nu), D(\xi), S(\xi)$ be the corresponding disk and sphere bundles. Then $D(\xi), S(\xi)$ are the pull-backs of $D(\nu), S(\nu)$ to E and therefore, $D(\xi) \to D(\nu)$ is a fibration with fiber X and $S(\xi) = D(\xi)_{S(\nu)}$. Since $D(\nu)$ is an $q + n$ manifold in D^{q+n}, $(B, \partial_0 B) \simeq (D(\nu), D(\nu)_{\partial_0 B})$ and the closure of the complement of $\partial_0 D(\nu) = D(\nu)_{\partial_0 B}$ in $\partial D(\nu)$ is $\partial_1 D(\nu) = S(\nu) \cup D(\nu)_{\partial_1 B}$, we have:

$$
\begin{array}{ccc}
\Gamma_{B,\partial_0 B} G^{\%}_B(E) & \xrightarrow{\simeq} & \Omega^n \overline{G}^{\%}(T(\xi)/T(\xi_1)) \\
\simeq \downarrow & & \uparrow \simeq \\
\Gamma_{D(\nu),D(\nu)_{\partial_0 B}} G^{\%}_{D(\nu)}(D(\xi)) & \xrightarrow{\varphi} & \Omega^{q+n} \overline{G}^{\%}(D(\xi)/D(\xi)_{\partial_1 D(\nu)})
\end{array}
$$

where $\varphi = \varphi_{(D(\nu),D(\nu)_{\partial_0 B})}$ is a homotopy equivalence by the first part of the proof. $\qquad\square$

Proof of Lemma B.3.1. Suppose first that $B = D^q$. Then

$$\Gamma_B G^{\%}_B(E) \cong Map(D^q, G^{\%}(X)) \simeq G^{\%}(X) \simeq \Omega^q \overline{G}^{\%}(\Sigma^q(X_+)) \cong \Omega^q \overline{G}^{\%}(E/E_{\partial B})$$

and this homotopy equivalence is given by φ_B.

In general we can choose a finite covering of B by closed q-disks A_i which is a "good covering" in the sense that the intersection of any finite number of A_i is either empty or homeomorphic to an q-disk. Let $C = A_1 \cup \cdots \cup A_{k-1}$ and $B = C \cup A_k$. By induction on k we know that $\varphi_C, \varphi_{C\cap A_k}$ and φ_{A_k} are homotopy equivalences. Now look at the commuting cubical diagram given by mapping each object of the left hand square to the corresponding object of the right hand square in the following diagrams.

$$
\begin{array}{ccc}
\Gamma_B G^{\%}_B(E) & \longrightarrow & \Gamma_{A_k} G^{\%}_{A_k}(E_{A_k}) \\
\downarrow & & \downarrow \\
\Gamma_C G^{\%}_C(E_C) & \longrightarrow & \Gamma_{A_k\cap C} G^{\%}_{A_k\cap C}(E_{A_k\cap C})
\end{array}
\qquad
\begin{array}{ccc}
\Omega^q \overline{G}^{\%}(E|B) & \longrightarrow & \Omega^q \overline{G}^{\%}(E|A_k) \\
\downarrow & & \downarrow \\
\Omega^q \overline{G}^{\%}(E|C) & \longrightarrow & \Omega^q \overline{G}^{\%}(E|A_k \cap C)
\end{array}
$$

Here $E|C = E_C/E_{\partial C} \cong E/E_{B-intC}$ and similarly for C replaced with $B, A_k, A_k \cap C$. Since the functors $X \mapsto \Gamma_X G^{\%}_X(E_X)$ and $\Omega^n \overline{G}^{\%}$ send cofiber squares to fiber squares, both squares are fiber squares. This implies that $\varphi_B : \Gamma_B G^{\%}_B(E) \to \Omega^q \overline{G}^{\%}(E|B)$ is a homotopy equivalence as claimed. $\qquad\square$

B.4. **Stratified bundles.** We recall the setup used in the stratified smoothing subsection A.5.7. We have a codimension 0 immersion $\lambda : (L, \partial_1 L) \to (B, \partial_1 B)$ covered by an embedding of smooth bundles $\tilde{\lambda} : E \to W$ of the same dimension. By the Corollary A.5.19 of the stratified smoothing Theorem A.5.18 we have the following commuting diagram where the maps μ are homotopy equivalences.

$$
\begin{array}{ccccc}
\widetilde{\mathcal{S}}^s_{L,\partial_0 L}(E) & \xrightarrow[\simeq]{\mu} & \widetilde{\mathcal{S}}^s_B(E^\delta) & \xrightarrow{\subseteq} & \widetilde{\mathcal{S}}^s_{B,\partial_0 B}(W) \\
\gamma_E \downarrow & & \gamma_{E^\delta} \downarrow & & \gamma_W \downarrow \\
\Gamma_{L,\partial_0 L} \mathcal{H}^{\%}_L(E) & \xrightarrow[\simeq]{\mu} & \Gamma_B \mathcal{H}^{\%}_B(E^\delta) & \xrightarrow{\subseteq} & \Gamma_{B,\partial_0 B} \mathcal{H}^{\%}_B(W)
\end{array}
$$

We need to prove that our calculation of the bottom three terms is compatible with the two arrows.

Theorem B.4.1 (stratified homotopy calculation)**.** *The following diagram commutes for any homology theory $G^\%$*

$$\begin{array}{ccc}
\Gamma_{L,\partial_0 L} G_L^\%(E) & \xrightarrow[\simeq]{\mu} & \Gamma_B G_B^\%(E^\delta) \xrightarrow{\subseteq} \Gamma_{B,\partial_0 B} G_B^\%(W) \\
\varphi_L \downarrow & & \varphi_B \downarrow \\
\Omega^q \overline{G}^\%(T(\xi_E)/T(\xi_{\partial_1 E})) & \xrightarrow{\tilde{\lambda}_*} & \Omega^q \overline{G}^\%(T(\xi)/T(\xi_{\partial_1 W}))
\end{array}$$

Here $\xi = p^*\nu_B$ is the pull-back of the normal bundle ν_B of B to W and $\xi_E = \tilde{\lambda}^*\xi$. The bottom arrow is induced by the inclusion $T(\xi_E) \hookrightarrow T(\xi)$ given by $\tilde{\lambda} : E \to W$. The mapping μ is the natural homotopy equivalence described below.

Since $\lambda : L \to B$ is a codimension 0 immersion, the normal bundle of B pulls back to the normal bundle of L: $\nu_L = \lambda^*\nu_B$. Since $p \circ \tilde{\lambda} = \lambda \circ \pi : E \to B$, it follows that $\xi_E = \tilde{\lambda}^*\xi \cong \pi^*\nu_L$. So, both vertical arrows in the diagram are the homotopy equivalences of the previous theorem.

The mapping μ can be described as follows. For any $b \in B$ let $x_1, \cdots, x_k$ be the elements of $L - \partial_0 L$ which map to b. Then $E_b^\delta = \coprod \tilde{\lambda}(E_{x_i})$. So

$$G^\%(E_b^\delta) \simeq \prod G^\%(\tilde{\lambda} E_{x_i})$$

where the projection map $(E_b^\delta)_+ \to (\tilde{\lambda} E_{x_i})_+$ is the identity on $\tilde{\lambda} E_{x_i}$ and sends the other components to the disjoint base point. (Then apply $\overline{G}^\%(X_+) = G^\%(X)$.)

There is a sixth space which can be inserted in the middle of the bottom arrow of the above diagram: $\Omega^q \overline{G}^\%(T(\xi_{E^\delta})/T(\xi_{\partial_1 E^\delta}))$ where ξ_{E^δ} is the restriction of the bundle ξ to E^δ and $\xi_{\partial_1 E^\delta}$ is the restriction of ξ to $\partial_1 E^\delta = E^\delta \cap W_{\partial_1 B}$.

The case that interests us is $G = \mathcal{H}$ where, using the Thom Isomorphism Theorem we have the following.

Corollary B.4.2. *The following diagram commutes where both horizontal arrows are induced by the embedding $\tilde{\lambda} : E \to W$.*

$$\begin{array}{ccc}
\pi_0 \Gamma_{L,\partial_0 L} \mathcal{H}_L^\%(E) \otimes \mathbb{R} & \xrightarrow{\tilde{\lambda}_*} & \pi_0 \Gamma_{B,\partial_0 B} \mathcal{H}_B^\%(W) \otimes \mathbb{R} \\
\theta_E \downarrow & & \theta_W \downarrow \\
\bigoplus_{k>0} H_{q-4k}(E, \partial_1 E; \mathbb{R}) & \xrightarrow{\tilde{\lambda}_*} & \bigoplus_{k>0} H_{q-4k}(W, \partial_1 W; \mathbb{R})
\end{array}$$

Lemma B.4.3. *There is a homotopy equivalence $\mu : \Gamma_{L,\partial_0 L} G_L^\%(E) \to \Gamma_B G_B^\%(E^\delta)$ which sends γ to the section $\mu(\gamma)$ which sends $b \in B$ to $(\gamma(x_i))_i \in \prod G^\%(\tilde{\lambda} E_{x_i})$.*

As before the proof relies on the lemma which does the case when B^q is embedded in D^q.

Lemma B.4.4. *Suppose that B^q is embedded in the q-disk D^k. Then*

$$\Gamma_B G_B^\%(E^\delta) \simeq \Gamma_{L,\partial_0 L} G_L^\%(E) \simeq \Omega^q \overline{G}^\%(E^\delta/E_{\partial B}^\delta) \simeq \Omega^q \overline{G}^\%(E/E_{\partial_1 L})$$

Furthermore the mapping $\varphi_B : \Gamma_B G_B^\%(E^\delta) \to \Omega^q \overline{G}^\%(E^\delta/E_{\partial B}^\delta)$ giving this homotopy equivalence is natural with respect to restriction and inclusion as explained below.

Naturality with respect to inclusion means that the following diagram commutes assuming that $E^\delta_{\partial_0 B}$ is empty. The vertical maps in the diagram are induced by the inclusion $E^\delta \hookrightarrow W$.

$$
\begin{array}{ccc}
\Gamma_B G_B^{\%}(E^\delta) & \xrightarrow{\varphi_B} & \Omega^q \overline{G}^{\%}(E^\delta/E^\delta_{\partial B}) \\
\downarrow & & \downarrow \\
\Gamma_{B,\partial_0 B} G_B^{\%}(W) & \xrightarrow{\varphi_{B,\partial_0}} & \Omega^q \overline{G}^{\%}(W/W_{\partial_1 B})
\end{array}
$$

Naturality with respect to restriction means that the following diagram commutes assuming that $A \subseteq B$ is a q-submanifold transverse to the image of $\lambda : L \to B$.

$$
\begin{array}{ccc}
\Gamma_B G_B^{\%}(E^\delta) & \xrightarrow{\varphi_B} & \Omega^q \overline{G}^{\%}(E^\delta/E^\delta_{\partial B}) \\
\downarrow & & \downarrow \\
\Gamma_A G_A^{\%}(E^\delta_A) & \xrightarrow{\varphi_A} & \Omega^q \overline{G}^{\%}(E^\delta_A/E^\delta_{\partial A})
\end{array}
$$

The vertical arrow on the left is given by restriction of sections to A and the vertical arrow on the right is induced by the quotient map $E^\delta/E^\delta_{\partial B} \to E^\delta_A/E^\delta_{\partial A}$.

Proof. The proof is basically the same as the proof of Lemma B.3.1. First we consider the elemental case in which $B = D^q$ and L is a disjoint union of disks L_i with embeddings $\lambda_i : L_i \to B$ so that $\lambda_i^{-1}(\partial B) = \partial_1 L_i$. Let E_i be the image of E_{L_i}. Then each E_i falls into one of three elemental cases:

(0) $\partial_1 L_i$ is empty. Then

$$\Gamma_B G_B^{\%}(E_i) \cong \Gamma_{L_i, \partial L_i} G_{L_i}^{\%}(E) \cong Map(L_i/\partial L_i, G^{\%}(E)) \simeq \Omega^q G^{\%}(E_i) = \Omega^q \overline{G}^{\%}(E_{i+})$$

(1) $\partial_1 L_i$ and $\partial_0 L_i$ are $q-1$ disks. In this case,

$$\Gamma_B G_B^{\%}(E_i) \simeq * \simeq \Omega^q \overline{G}^{\%}(E_i/E_{i\partial_1 L_i})$$

(2) $\partial_1 L_i = \partial B = S^{q-1}$ and $\partial_0 L_i$ is empty. Then

$$\Gamma_B G_B^{\%}(E_i) \cong Map(B, G^{\%}(E_i)) \simeq G^{\%}(E_i) \simeq \Omega^q \overline{G}^{\%}(E_i/E_{i\,\partial B})$$

Therefore,

$$\Gamma_B G_B^{\%}(E_i) \simeq \Omega^q \overline{G}^{\%}(E_i/E_{i,\partial B})$$

for each i and we conclude that

$$\Gamma_B G_B^{\%}(E^\delta) \simeq \prod \Gamma_B G_B^{\%}(E_i) \simeq \prod \Omega^q \overline{G}^{\%}(E_i/E_{i,\partial B}) \simeq \Omega^q \overline{G}^{\%}(E^\delta/E^\delta_{\partial B}).$$

In general we can choose a finite covering of B by closed q-disks A_i which is a "good covering" in the sense that the intersection of any finite number of A_i is either empty or homeomorphic to an q-disk and the restriction of E to each of these disks is elemental as described above. It is easy to do this very explicitly. First subdivide once to make sure the triangulation is sufficiently fine. Choose any fixed positive $\epsilon < 1/q + 1$. For every simplex σ take the set $N(\sigma)$ of all points $b \in B$ so that $t_i \leq \epsilon$ for every barycentric coordinate t_i of b corresponding to a vertex v_i not in σ. Then $N(\sigma)$ is a polyhedron, being given by linear inequalities of barycentric coordinates and it is the closure of its interior which contains σ as a deformation retract and is thus contractible. Therefore $N(\sigma)$ is a q-disk. Also, it is obvious that $N(\sigma) \cap N(\tau) = N(\sigma \cap \tau)$. Also, similar arguments show that each component of L over $N(\sigma)$ is a q disk and falls into one of the three cases discussed above. So $A_i = N(\sigma_i)$ form a good covering.

The rest of the proof is almost word-for-word the same as the second half of the proof of Lemma B.3.1 except that we need E to be replaced with E^δ and we need two more commuting squares with B replaced by L and C and A_k replaced by their inverse images in L. Then we have four fiber

squares in which corresponding terms are homotopy equivalent by induction on k proving the first part of the lemma.

It remains to show that the mapping

$$\varphi_B : \Gamma_B G_B^{\%}(E^\delta) \to \Omega^q \overline{G}^{\%}(E^\delta/E_{\partial B}^\delta)$$

which gives the homotopy equivalence is natural with respect to inclusion and restriction. But this A section γ of $G_B^{\%}(E^\delta)$ sends a point $b \in B$ to $\gamma(b) \in G^{\%}(E_b) \subseteq G_B^{\%}(E^\delta)$. The corresponding map $\varphi_B(\gamma) : D^q/S^{q-1} \to \overline{G}^{\%}(E/E_{\partial B})$ sends $b \in B \subseteq D^q$ to $\gamma(b) \in G^{\%}(E_b) \to \overline{G}^{\%}(E/E_{\partial B})$. This is clearly compatible with inclusion: we simply map these images into larger sets. This is also compatible with restriction: the points $b \in A$ are sent to the same points as before and $b \notin A$ are sent to the basepoint by both γ and $\varphi_A(\gamma)$. $\qquad\square$

Proof of stratified homotopy calculation. In the general case we choose an embedding $B^q \hookrightarrow D^{q+n}$ and let ν be the n-dimensional normal bundle of B. Let ξ be the pull back of ν to E and let $D(\nu), S(\nu), D(\xi), S(\xi)$ be the corresponding disk and sphere bundles. Then $D(\xi), S(\xi)$ are the pull-backs of $D(\nu), S(\nu)$ to E and therefore, $D(\xi) \to D(\nu)$ is a fibration with fiber X and $S(\xi) = D(\xi)_{S(\nu)}$. Since $D(\nu)$ is an $q+n$ manifold in D^{q+n}, $(B, \partial_0 B) \simeq (D(\nu), D(\nu)_{\partial_0 B})$ and the closure of the complement of $\partial_0 D(\nu) = D(\nu)_{\partial_0 B}$ in $\partial D(\nu)$ is $\partial_1 D(\nu) = S(\nu) \cup D(\nu)_{\partial_1 B}$, we have:

$$
\begin{array}{ccc}
\Gamma_{B,\partial_0 B} G_B^{\%}(E) & \xrightarrow{\simeq} & \Omega^n \overline{G}^{\%}(T(\xi)/T(\xi_1)) \\
\simeq \downarrow & & \uparrow \simeq \\
\Gamma_{D(\nu),D(\nu)_{\partial_0 B}} G_{D(\nu)}^{\%}(D(\xi)) & \xrightarrow{\varphi} & \Omega^{q+n} \overline{G}^{\%}(D(\xi)/D(\xi)_{\partial_1 D(\nu)})
\end{array}
$$

where $\varphi = \varphi_{(D(\nu),D(\nu)_{\partial_0 B})}$ is a homotopy equivalence by the first part of the proof. $\qquad\square$

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