# Approximation of functions and their derivatives by analytic maps on certain Banach spaces

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Abstract. Let $X$ be a separable Banach space which admits a separating polynomial; in particular $X$ a Hilbert space. Let $f : X \to \mathbb{R}$ be bounded, Lipschitz, and $C^1$ with uniformly continuous derivative. Then for each $\varepsilon > 0$, there exists an analytic function $g : X \to \mathbb{R}$ with $|g - f| < \varepsilon$ and $\|g' - f'\| < \varepsilon$.

## 1. Introduction

The problem of approximating a smooth function and its derivatives by a function of higher order smoothness on a Banach space $X$ has been investigated by several authors, although the number of such results is limited. When $X$ is finite dimensional excellent results are known, and in fact Whitney in his classical paper [**W**] provides essentially a complete answer by showing: for every $C^k$ function $f : \mathbb{R}^n \to \mathbb{R}^m$ and every continuous $\varepsilon : \mathbb{R}^n \to (0, +\infty)$ there exists a real analytic function $g$ such that $\|D^j g(x) - D^j f(x)\| \leq \varepsilon(x)$ for all $x \in \mathbb{R}^n$ and $j = 1, ..., k$. This is the so-called $C^k$ fine approximation of $f$.

The first results for $X$ infinite dimensional concern the smooth, non-analytic case, and are due to Moulis [**M**]. She proves, in particular, a $C^1$ fine approximation theorem; namely, that for $X = c_0$ or $l_p$ with $1 < p < \infty$, and $Y$ an arbitrary Banach space, given a $C^1$ map $f : X \to Y$, and a continuous function $\varepsilon : X \to (0, \infty)$ , there exists a $C^k$ smooth map $g : X \to Y$ (where the optimal value of $k \geq 1$ depends on the choice of $X$) such that $|g(x) - f(x)| < \varepsilon(x)$ and $\|g'(x) - f'(x)\| < \varepsilon(x)$ . This result was later extended in [**AFGJL**] to the case where $X$ has an unconditional basis and admits a Lipschitz, $C^k$ smooth bump function. Further work along this line can be found in [**HJ**], where using ideas from [**F1**] it is shown that for

2000 *Mathematics Subject Classification.* Primary 46B20.

*Key words and phrases.* fine analytic approximation, separating polynomial, Banach space.

The second named author partly supported by NSERC (Canada).

certain range spaces $Y$, one can relax the conditions on $X$ in [**AFGJL**] and, for example, take $X$ to be merely separable.

It is important to note that all the results mentioned above require, in a very essential way, a theorem concerning the approximation of Lipschitz functions $f$ by more regular, Lipschitz functions $g$, where the Lipschitz constant of $g$ is fixedly proportional to the Lipschitz constant of $f$, regardless of the precision in the approximation. In [**M**] and [**AFGJL**] this is achieved by reducing the problem to the finite dimensional case using the unconditional basis, but otherwise without this reduction traditional methods of smooth approximation, such as smooth partitions of unity, do not work in addressing this problem. A new approach was found in [**F1**], and further developed in [**AFM**], [**F2**], [**AFK2**], and [**HJ**]. This technique has been called the method of *sup-partitions of unity* in [**HJ**]. It seems that $C^k$ fine approximation must rely on such results.

Concerning $C^k$ fine approximation by analytic functions for $X$ infinite dimensional, nothing is known. In view of the remarks in the preceding paragraph, it would appear that first one needs the ability to approximate Lipschitz functions by Lipschitz, analytic functions with good control over the Lipschitz constant. That is, one requires a kind of analytic sup-partition of unity. Only very recently has this been possible with the work of [**AFK1**], where it is proven that if $X$ is separable and admits a separating polynomial, then for every Lipschitz function $f : X \to \mathbb{R}$ and $\varepsilon > 0$ there exists a Lipschitz, analytic function $g : X \to \mathbb{R}$ with $|f - g| < \varepsilon$ and $\mathrm{Lip}(g) \leq C\mathrm{Lip}(f)$, where the constant $C > 1$ depends only on $X$ (for a precursor to this work see [**FK**]). Using this, we are able in this note to give the first results on the $C^1$ fine analytic approximation problem in infinite dimensions. We remark that this work is new even for $X$ a separable Hilbert space. We establish,

**Theorem 1.** *Let $X$ be a separable Banach space which admits a separating polynomial. Let $f : X \to \mathbb{R}$ be bounded and Lipschitz, with uniformly continuous derivative, and $\varepsilon > 0$. Then there exists an analytic function $g : X \to \mathbb{R}$ such that $|f - g| < \varepsilon$ and $\|f' - g'\| < \varepsilon$.*

Our notation is standard, with $X$ denoting a Banach space, and an open ball with centre $x$ and radius $r$ denoted $B_r(x)$. If $\{f_j\}_j$ is a sequence of Lipschitz functions defined on $X$, then we will at times say this family is *equi-Lipschitz* if there is a common Lipschitz constant for all $j$. A *homogeneous polynomial of degree* $n$ is a map, $P : X \to \mathbb{R}$, of the form $P(x) = A(x, x, ..., x)$, where $A : X^n \to \mathbb{R}$ is $n-$multilinear and continuous. For $n = 0$ we take $P$ to be constant. A *polynomial of degree* $n$ is a sum $\sum_{i=0}^{n} P_i(x)$, where $i \geq 1$ the $P_i$ are $i$-homogeneous polynomials.

Let $X$ be a Banach space, and $G \subset X$ an open subset. A function $f : G \to \mathbb{R}$ is called *analytic* if for every $x \in G$, there are a neighbourhood $N_x$, and homogeneous polynomials $P_n^x : X \to \mathbb{R}$ of degree $n$, such that

$$f(x+h) = \sum_{n\geq 0} P_n^x(h) \text{ provided } x+h \in N_x.$$

Further information on polynomials may be found, for example, in [**SS**].

For a Banach space $X$, we define its (Taylor) complexification $\widetilde{X} = X \bigoplus iX$ with norm

$$\|x+iy\|_{\widetilde{X}} = \sup_{0\leq\theta\leq 2\pi} \|\cos\theta\, x - \sin\theta\, y\|_X = \sup_{T\in X^*, \|T\|\leq 1} \sqrt{T(x)^2 + T(y)^2}.$$

If $L : E \to F$ is a continuous linear mapping between two real Banach spaces then there is a unique continuous linear extension $\widetilde{L} : \widetilde{E} \to \widetilde{F}$ of $L$ (defined by $\widetilde{L}(x+iy) = L(x) + iL(y)$) such that $\|\widetilde{L}\| = \|L\|$. For a continuous $k$-homogeneous polynomial $P : E \to \mathbb{R}$ there is also a unique continuous $k$-homogeneous polynomial $\widetilde{P} : \widetilde{E} \to \mathbb{C}$ such that $\widetilde{P} = P$ on $E \subset \widetilde{E}$ (but the norm of $P$ is not generally preserved: one has that $\|\widetilde{P}\| \leq 2^{k-1}\|P\|$). It follows that if $q(x)$ is a continuous polynomial on $X$, there is a unique continuous polynomial $\widetilde{q}(z) = \widetilde{q}(x+iy)$ on $\widetilde{X}$ where for $y = 0$ we have $\widetilde{q} = q$. For more information on complexifications (and polynomials) we recommend [**MST**]. In the sequel, all extensions of functions from $X$ to $\widetilde{X}$, as well as subsets of $\widetilde{X}$, will be embellished with a tilde.

## 2. Main Results

**2.1. The functions $\varphi_n$.** To prove Theorem 1, we start with a lemma which is a variation of [**AFK1**, Lemma 3], where here we have made three changes: added part $(4')$; included constants $M_n$ for the estimate in $(5)$; and relaxed the condition that $r \geq 1$ to $r > 0$. To obtain $(4')$, we replace the function $b_n$ in the proof of [**AFK1**, Lemma 3] with a $C^1$ version; the change in (5) is easily handled; and requiring merely $r > 0$ means that certain constants will depend on $r$, but as we shall apply the lemma with $r$ fixed throughout, this causes no problem.

First we need some definitions and notation. If $X$ posseses an $n^{\text{th}}$ order separating polynomial, then it admits a $2n$-homogeneous polynomial $q$ such that

$$\|x\|^{2n} \leq q(x) \leq A\|x\|^{2n}, \tag{2.1}$$

for some $A > 1$ (see e.g., [**AFK1**]). In [**AFK1**, Lemma 2] it is proved that the function $Q(x) = (q(x)+1)^{1/2n} - 1$ satisfies:

1. $Q$ is (real) analytic on $X$,
2. $Q$ is Lipschitz on $X$, where we can take $\mathrm{Lip}(Q) > 1$,
3. $\inf\{Q(x) : \|x\| \geq 1\} > 0 = Q(0)$,

(4) $Q(x) < 4\rho \Rightarrow \|x\| < 8\rho$ when $\rho \geq 1$; otherwise $Q(x) < 4\rho \Rightarrow \|x\| < \delta(\rho) \equiv \left((1+4\rho)^{2n} - 1\right)^{1/2n}$, this latter implication simply using (2.1) and the definition of $Q$.

(5) there exists $\delta > 0$ such that $Q$ extends to a Lipschitz, holomorphic map $\widetilde{Q}$ on the uniform strip $X \subset W_\delta \subset \widetilde{X}$ given by,

$$W_\delta = \left\{x + z : x \in X,\ z \in \widetilde{X},\ \|z\|_{\widetilde{X}} < \delta\right\}.$$

We use the notion of a *$Q$-body*, which for $\rho > 0$ is defined by

$$D_Q(x, \rho) = \{y \in X : Q(y - x) < \rho\}.$$

Let $\|\cdot\|_{c_0}$ be an equivalent analytic norm on $c_0$, with $\|x\|_\infty \leq \|x\|_{c_0} \leq A_1 \|x\|_\infty$ for all $x \in c_0$, and some $A_1 > 1$ (see e.g., [**FPWZ**], and also [**AFK1**], [**FK**] where it is referred to as the Preiss norm). It may be worth pointing out that the Preiss norm $\|\cdot\|_{c_0}$ is obtained as the restriction of a holomorphic map $\widetilde{\lambda}$ defined on a neighbourhood of $c_0 \setminus \{0\}$ in $\widetilde{c}_0$.

For the remainder of the proof, we fix a dense sequence $\{x_n\}_{n=1}^\infty$ in $X$.

**Lemma 1.** *Let $\widetilde{V} = W_\delta$ be an open strip around $X$ in $\widetilde{X}$ in which the function $\widetilde{Q}$ given above is defined. Given $\varepsilon \in (0,1)$, $r > 0$, and a covering $\{D_Q(x_n, r)\}_{n=1}^\infty$ of $X$, there exists a sequence of holomorphic functions $\widetilde{\varphi}_n = \widetilde{\varphi}_{n,r,\varepsilon} : \widetilde{V} \to \mathbb{C}$, whose restrictions to $X$ we denote by $\varphi_n = \varphi_{n,r,\varepsilon}$, with the following properties:*

- **1:** *The collection $\{\varphi_{n,r,\varepsilon} : X \to [0,2] \mid n \in \mathbb{N}\}$ is equi-Lipschitz on $X$, with Lipschitz constant of the form $L_\varphi = L_1 Lip(Q)/r > 1$ (where $L_1 \geq 1$ is independent of $\varepsilon$ and $n$),*
- **2:** *$0 \leq \varphi_{n,r,\varepsilon}(x) \leq 1 + \varepsilon$ for all $x \in X$.*
- **3:** *For each $x \in X$ there exists $m = m_{x,r} \in \mathbb{N}$ (independent of $\varepsilon$) with $\varphi_{m,r,\varepsilon}(x) > 1/2$.*
- **4:** *$0 \leq \varphi_{n,r,\varepsilon}(x) \leq \varepsilon$ for all $x \in X \setminus D_Q(x_n, 5r)$.*
- **4′:** *$\left\|\varphi'_{n,r,\varepsilon}(x)\right\| \leq \varepsilon$ for all $x \in X \setminus D_Q(x_n, 5r)$.*
- **5:** *For each $x \in X$ there exist $\delta_{x,r} > 0$, $a_{x,r} > 0$, and $n_{x,r} \in \mathbb{N}$ (all independent of $\varepsilon$) such that*

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| < \frac{1}{M_n n! a_{x,r}^n} \quad \text{for } n > n_{x,r},\ \ z \in \widetilde{X} \text{ with } \|z\|_{\widetilde{X}} < \delta_{x,r},$$

*where $M_n = e^{2C^2\kappa}(1 + \|x_n\|)$, and the $\kappa = \kappa(r) > 1$ and $C > 1$ are constants that will be specified in the proof of Theorem 1.*

- **6:** *For each $x \in X$, there exists $\delta_{x,r} > 0$ (independent of $\varepsilon$) and $n_{x,\varepsilon,r} \in \mathbb{N}$ such that for $\|z\|_{\widetilde{X}} < \delta_{x,r}$ and $n > n_{x,\varepsilon,r}$ we have $|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| < \varepsilon$.*

**7:** *For each $x \in X$, there exists $\delta_{x,\varepsilon,r}$ such that*

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| \leq 1+2\varepsilon \quad \text{for } n \in \mathbb{N}, \text{ and } z \in \widetilde{X} \text{ with } \|z\|_{\widetilde{X}} \leq \delta_{x,\varepsilon,r}.$$

**Proof.** We largely follow the proof of [**AFK1**, Lemma 3], with the few noted changes. As the proof in [**AFK1**] is rather long and technical, we here indicate only the key constructions, referring the reader to the above cited paper for full details. Note that because $r$ is fixed throughout, for ease of notation, we shall often suppress dependences on $r$. Define subsets $A_{1,r} = \{y_1 \in \mathbb{R} : -1 \leq y_1 \leq 4r\}$, and, for $n \geq 2$,

$$A_{n,r} = \{y = \{y_j\}_{j=1}^n \in \ell_\infty^n : -1-r \leq y_n \leq 4r,\ 2r \leq y_j \leq M_{n,r} + 2r \text{ for } 1 \leq j \leq n-1\},$$

$$A'_{n,r} = \{y = \{y_j\}_{j=1}^n \in \ell_\infty^n : -1 \leq y_n \leq 3r,\ 3r \leq y_j \leq M_{n,r} + r \text{ for } 1 \leq j \leq n-1\},$$

$$\text{where } M_{n,r} = \sup\{Q(x-x_j) : x \in D_Q(x_n, 4r),\ 1 \leq j \leq n\}.$$

Let $\mu \in C^\infty(\mathbb{R},[0,1+\varepsilon])$ be Lipschitz such that $\mu(t) = 0$ iff $t \geq 1$, and $\mu(t) = 1+\varepsilon$ iff $t \leq 1/2$. Let $b^n \in C^\infty(\mathbb{R},[0,1])$ be Lipschitz such that $b^n(t) = 1$ iff $t \notin (2r, M_{n,r}+2r)$, and $b^n(t) = 0$ iff $t \in [3r, M_{n,r}+r]$. Let $\widehat{b} \in C^\infty(\mathbb{R},[0,1])$ be Lipschitz such that $\widehat{b}(t) = 1$ iff $t \notin (-1-r, 4r)$, and $\widehat{b}(t) = 0$ iff $t \in [-1, 3r]$. Now define a Lipschitz, $C^\infty$ smooth map $b_n : c_{00} \subset c_0 \to [0,1]$ by $b_n(y_1,...,y_n) = \mu\left(\left\|\left(b^n(y_1),...,b^n(y_{n-1}),\widehat{b}(y_n)\right)\right\|_{c_0}\right)$. Then support$(b_n) = \overline{A}_n$, and $b_n = 1+\varepsilon$ on $A'_n$. Moreover, $b_n$ is Lipschitz with constant of the form $L_1/r$, where $L_1 \geq 1$ is independent of $n$.

Now one defines, on $\mathbb{R}^n$, the map

$$h_n(x) = \frac{1}{T_n}\int_{\mathbb{R}^n} b_n(y) e^{-k\sum_{j=1}^n 2^{-j}(x_j-y_j)^2}\,dy,$$

$$T_n = \int_{\mathbb{R}^n} e^{-k\sum_{j=1}^n 2^{-j}{y_j}^2}\,dy.$$

Because $b_n = b_{n,\varepsilon}$ has compact support, is bounded, Lipschitz, and $C^1$, one can choose $k = k_{n,\varepsilon} > 0$ sufficiently large that

$$|b_n(x) - h_n(x)| \leq \varepsilon/2 \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{2.2}$$

and

$$|b'_n(x) - h'_n(x)| \leq \varepsilon/2 \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{2.3}$$

Next one defines (real) analytic maps $\varphi_n : X \to \mathbb{R}$ by,

$$\varphi_n(x) = h_n(Q(x-x_1),...,Q(x-x_n)) = \frac{1}{T_n}\int_{\mathbb{R}^n} b_n(y)\, e^{-k_n\sum_{j=1}^n 2^{-j}(Q(x-x_j)-y_j)^2}.$$

It is more or less standard to show that $\mathrm{Lip}(\varphi_n) \leq \frac{L_1}{r}\,\mathrm{Lip}(Q)$. We can extend the maps $\varphi_{n,r,\varepsilon}$ to complex valued maps defined on $W_Q$ (see above) calling them $\widetilde{\varphi}_n$. Namely (where $x \in X$, $z \in \widetilde{X}$),

$$\widetilde{\varphi}_n(x+z) = \frac{1}{T_n}\int_{\mathbb{R}^n} b_n(y)\, e^{-k_n \sum_{j=1}^{n} 2^{-j}\left(\widetilde{Q}(x-x_j+z)-y_j\right)^2} dy$$

Note that the $\widetilde{\varphi}_n$ are well defined (as the $b_n$ have compact supports) and are holomorphic where $\widetilde{Q}$ is (namely on $\widetilde{W}_\delta$).

To see (4) and (4′), note that if $Q(x-x_n) \geq 5r$, then there is a neighbourhood $N$ of $x$ for which $y \in N$ implies that the point,

$$\widehat{x} = (Q(y-x_1), ..., Q(y-x_n)) \in \mathbb{R}^n \backslash A_n,$$

from which we have $b_n(\widehat{x}) = 0$ and $b_n'(\widehat{x}) = 0$. Hence, by (2.2) and (2.3), we have, $|\varphi_n(x)| < \varepsilon/2$ and $\|\varphi_n'(x)\| < \varepsilon/2$.

The remaining parts are handled as in [**AFK1**], noting that for (5) we choose $\kappa_n$ larger if need be to ensure the stated estimate involving the $M_n$. □

We return now to the proof of the theorem. Let $\varepsilon > 0$ be given and choose $\varepsilon'$ satisfying

$$0 < \varepsilon' < \min\{\frac{1}{8}, 1/(132C_0A_1^2L_1Lip(Q)), 1/(10A_1r)\},$$

where $L_1$ is as in part (1) of the preceding lemma, where  is defined immediately below, and where $C_0$ is a constant, only depending on $X$, which will be fixed later on (see page 9 below). Because $f$ is bounded, we may suppose that $1 \leq f \leq 2$. As $f'$ is uniformly continuous on $X$, we can find a fixed $\rho > 0$ so that for all $n$, $x \in B_\rho(x_n)$ implies $\|f'(x_n) - f'(x)\| < \varepsilon'$. Now, considering property (4) of $Q$, and noting that $\delta(r) \to 0^+$ as $r \to 0^+$, we can choose $r \in (0,1)$ (independent of $n$) so that $D_Q(x_n, 5r) \subset B_\rho(x_n)$ for all $n$. It will be convenient to write $D_n \equiv D_Q(x_n, 5r)$. This $r$ shall be fixed for the remainder of the proof.

**2.2. The functions $\nu_n$.** Next let $\overline{\nu} \in C^\infty(\mathbb{R}, [0,1])$ be Lipschitz such that $\overline{\nu}(t) = 1$ iff $|t| \leq 5r$, and $\overline{\nu}(t) = 0$ iff $|t| \geq \frac{11}{2}r$. Put $L = \mathrm{Lip}(f)$. Fix a sequence of functions $\{\varphi_{n,r,\varepsilon_1}\}_{n=1}^\infty$ with respect to the covering $\{D_Q(x_n, r)\}_{n=1}^\infty$ of $X$ as given by Lemma 1, where $r$ is fixed as above and the $\varepsilon$ of the Lemma is chosen to be

$$\varepsilon_1 := \min\left\{\varepsilon' r/3C_0 L\mathrm{Lip}(\overline{\nu}), \varepsilon' r/25L\mathrm{Lip}(\overline{\nu})\right\},$$

We write $\varphi_{n,r,\varepsilon_1}$ as $\varphi_n$ for ease of notation, and, as in Lemma 1 (1), $L_\varphi = \mathrm{Lip}(\varphi_n) = L_1\mathrm{Lip}(Q)/r \geq 1$, which we recall is independent of $n$. Often we will subsume dependence on $\varepsilon_1$ as dependence on $\varepsilon'$ and $L$.

Put $\Delta(t) = \left((|t|+1)^{2n} - 1\right)^{1/2n} \geq 0$. Now via convolution integrals between $\overline{\nu}$ and Gaussian kernels, we can find Lipschitz, analytic functions $\nu$, with $\mathrm{Lip}(\nu) = \mathrm{Lip}(\overline{\nu})$, and which $C^1$-fine approximate $\overline{\nu}$ in the following sense,

$$\begin{aligned} &\left|\nu(t) - \overline{\nu}(t)\right| < \frac{\varepsilon' r/2LL_\varphi}{1+\Delta(t)}, \\ &\left|\nu'(t) - \overline{\nu}'(t)\right| < \frac{\varepsilon' r/2LL_\varphi}{1+\Delta(t)}. \end{aligned} \tag{2.4}$$

Indeed, we can take $\nu$ to be of the form,

$$\nu(t) = \frac{1}{a}\int_{\mathbb{R}} \overline{\nu}(s)\, e^{-\kappa(t-s)^2} ds,$$

$$a = \int_{\mathbb{R}} e^{-\kappa s^2} ds,$$

where $\kappa > 1$ is chosen sufficiently large and is independent of $t$ (although it does depend on $\max\{\Delta(t) : t \in \operatorname{supp}(\overline{\nu})\} < \infty$). This is possible because $\overline{\nu}$ is $C^\infty$ with compact support, and the function $t \to \frac{\varepsilon'/2L}{1+\Delta(t)}$ is strictly positive, continuous and decreases slowly enough with respect to $e^{-\kappa t^2}$ (namely, $\lim_{t\to\infty} \Delta(t)/e^{\kappa t^2} = 0$). Moreover, since $\overline{\nu}$ has compact support, $\nu$ has a holomorphic extension,

$$\widetilde{\nu}(z) = \frac{1}{a}\int_{\mathbb{R}} \overline{\nu}(s)\, e^{-\kappa(z-s)^2} ds,$$

to $\mathbb{C}$. Next observe that for $t, s \in \mathbb{R}$ and $z \in \mathbb{C}$ with $|z| \le \eta$, we have,

$$\begin{aligned} \operatorname{Re}(t+z-s)^2 &= (t-s)^2 + 2(t-s)\operatorname{Re} z + \operatorname{Re}\left(z^2\right) \\ &= (t-s+\operatorname{Re} z)^2 - (\operatorname{Re} z)^2 + \operatorname{Re}\left(z^2\right) \\ &\ge (t-s+\operatorname{Re} z)^2 - 2\eta^2. \end{aligned}$$

Therefore when $|z| < \eta$ we get,

$$
\begin{aligned}
|\widetilde{\nu}(t+z)| &= \frac{1}{a}\left|\int_{\mathbb{R}} \overline{\nu}(s)\, e^{-\kappa(t+z-s)^2} ds\right| \\
&\leq \frac{1}{a}\int_{\mathbb{R}} e^{-\kappa \operatorname{Re}(t+z-s)^2} ds \\
&\leq \frac{1}{a}\int_{\mathbb{R}} e^{-\kappa(t-s+\operatorname{Re} z)^2-2\eta^2} ds \qquad (2.5)\\
&= \frac{e^{2\kappa\eta^2}}{a}\int_{\mathbb{R}} e^{-\kappa(t+\operatorname{Re} z-s)^2} ds \\
&= e^{2\kappa\eta^2},
\end{aligned}
$$

where we have used a variable change to obtain the last line. Now define Lipschitz, analytic functions $\nu_n : X \to \mathbb{R}$ by,

$$\nu_n(x) = \nu(Q(x-x_n)).$$

Clearly $\nu_n$ has the holomorphic extension $\widetilde{\nu}_n(z) = \widetilde{\nu}\left(\widetilde{Q}(z-x_n)\right)$. It will be convenient to put $\overline{\nu}_n(x) = \overline{\nu}(Q(x-x_n))$. Observe that, writing $\widehat{D}_n = D_Q(x_n, 6r)$,

$$|\nu_n(x)| < \frac{\varepsilon' r/2LL_\varphi}{1+\Delta(Q(x-x_n))}, \quad \text{for } x \notin \widehat{D}_n, \tag{2.6}$$

and

$$\left|\nu_n'(x)\right| < \frac{\operatorname{Lip}(Q)\,\varepsilon' r/2LL_\varphi}{1+\Delta(Q(x-x_n))}, \quad \text{for } x \notin \widehat{D}_n. \tag{2.7}$$

Note that

$$
\begin{aligned}
\frac{\varepsilon' r/2LL_\varphi}{1+\Delta(Q(x-x_n))} &= \frac{\varepsilon' r/2LL_\varphi}{1+q(x-x_n)^{1/2n}} \qquad (2.8)\\
&\leq \frac{\varepsilon' r/2LL_\varphi}{1+\|x-x_n\|}.
\end{aligned}
$$

Now we estimate $|\widetilde{\nu}_n(x+z)| = \left|\widetilde{\nu}\left(\widetilde{Q}(x-x_n+z)\right)\right|$, for $\|z\|_{\widetilde{X}} < \eta$. From [**AFK1**, Lemma 2], we can write

$$\widetilde{Q}(x-x_n+z) = Q(x-x_n) + Z_n,$$

where $Z_n \in \mathbb{C}$ with $|Z_n| \leq C \|z\|_{\widetilde{X}}$, for some constant $C > 1$. Then from the calculation (2.5) we get, for $\|z\|_{\widetilde{X}} < \eta$,

$$
\begin{aligned}
|\widetilde{\nu}_n (x+z)| &= \left|\widetilde{\nu}\left(\widetilde{Q}(x - x_n + z)\right)\right| \\
&= |\widetilde{\nu}(Q(x - x_n) + Z_n)| \qquad (2.9)\\
&\leq e^{2C^2 \kappa \eta^2}.
\end{aligned}
$$

It is also worthwhile to note that $\nu(t) < 1 + \varepsilon'$ for all $t$.

Let $T_n(x) = f'(x_n)(x - x_n) + f(x_n)$ be the first order Taylor polynomial of $f$ at $x_n$. Note that $\|T_n'(x)\| = \|f'(x_n)\| \leq L$. Observe that $T_n - f$ is Lipschitz on $B_\rho(x_n)$, with $\mathrm{Lip}(T_n - f) \leq \left\|(T_n - f)'\right\| = \|f'(x_n) - f'(x)\| \leq \varepsilon'$ on $B_\rho(x_n)$. It follows that $T_n - f$ is Lipschitz on $D_n \subset B_\rho(x_n)$ with constant not greater than $\varepsilon'$. Denote by $\overline{T_n - f}$ a bounded and Lipschitz extension of $(T_n - f)|_{D_n}$ to all of $X$, having the same bound and Lipschitz constant. For example, one can take, temporarily writing $h = (T_n - f)|_{D_n}$,

$$
\left(\overline{T_n - f}\right)(x) = \max\{-\|h\|_\infty, \min\{\|h\|_\infty, \inf_{y \in D_n}\{h(y) + \mathrm{Lip}(h)\|x - y\|\}\}\}.
$$

Write $\epsilon_n(x) = \left(\overline{T_n - f}\right)(x)$. We now apply [**AFK1**, Proposition 3] to $\epsilon_n(x)$, along with the standard 'scaling argument' that appears at the very end of the proof of [**AFK1**, Theorem 1], to obtain the following: there exists a constant $C_0 > 1$, depending only on $X$, a neighbourhood $X \subset \widetilde{W} \subset \widetilde{X}$, where $\widetilde{W} = \widetilde{W}_{\varepsilon', r}$ depends only on $\varepsilon'$ and $r$ (the dependence on $L_\varphi$ written as a dependence on $r$), and an analytic map $\delta_n : X \to \mathbb{R}$ such that

(1) $|\epsilon_n(x) - \delta_n(x)| < \varepsilon' r / L_\varphi$ for all $x \in X$,
(2) $\mathrm{Lip}(\delta_n) \leq C_0 \,\mathrm{Lip}(\epsilon_n) \leq C_0 \varepsilon'$,
(3) the map $\delta_n$ extends to a holomorphic map $\widetilde{\delta}_n$ on $\widetilde{W}$ (where in particular, $\widetilde{W}$ is independent of $n$),
(4) $|\widetilde{\delta}_n(x + iy) - \delta_n(x)| \leq M_\Delta$ for all $x + iy \in \widetilde{W}$, where $M_\Delta$ depends on $\varepsilon'$ and is independent of $n$.

Now we define analytic functions on $X$ by,

$$
\psi_n(x) = (T_n(x)\,\nu_n(x) - \delta_n(x))\,\varphi_n(x).
$$

Observe that from property (3) of $\delta_n$ and Lemma 1, $\psi_n$ extends to a holomorphic map $\widetilde{\psi}_n(z) = \left(\widetilde{T}_n(z)\,\widetilde{\nu}_n(z) - \widetilde{\delta}_n(z)\right)\widetilde{\varphi}_n(z)$, where

$$
\widetilde{T}_n(z) = \widetilde{T}_n(x + iy) = \widetilde{f'(x_n)}(x + iy - x_n) + f(x_n)
$$

($\widetilde{f'(x_n)}$ being the canonical extension of $f'(x_n)$ to all of $\widetilde{X}$), on a neighbourhood $X \subset \widetilde{W} \subset \widetilde{X}$, where $\widetilde{W}$ is independent of $n$.

**2.3. The approximating function $g$.** Let us define the function $g : X \to \mathbb{R}$ by,

$$g(x) = \frac{\|\{\psi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}}$$

We next show that $g$ is analytic. Since the norm $\|\cdot\|_{c_0}$ is real analytic on $c_0 \setminus \{0\}$, it is sufficient to check that the mappings $x \mapsto \{\varphi_n(x)\}_{n=1}^{\infty}$ and $x \mapsto \{\psi_n(x)\}_{n=1}^{\infty}$ are real analytic from $X$ into $c_0$ and do not take the value $0 \in c_0$. Using Lemma 1 it is easy to show that the function $x \mapsto \{\varphi_n(x)\}_{n=1}^{\infty}$ has such properties (see [**AFK1**, Lemma 4]).

As for the function $x \mapsto \{\psi_n(x)\}_{n=1}^{\infty}$, let us first show that it does not take the value 0. In fact we show that for each $x \in X$ there exists an $n$ so that the number $(T_n(x)\nu_n(x) - \delta_n(x))\varphi_n(x)$ is bounded above $1/4$. Indeed, for each $x \in X$, there is a minimal $n = n_x$ with $x \in D_Q(x_{n_x}, 3r)$, and via the proof of [**AFK1**, Lemma 3 (3)], for such $n_x$ we have $\varphi_{n_x}(x) \geq 1/2$. Note also that $D_Q(x_{n_x}, 3r) \subset D_Q(x_{n_x}, 5r) = D_{n_x}$, and $\epsilon_{n_x}(x) = T_{n_x}(x) - f(x)$ on $D_{n_x}$. So, from this and property (1) of $\delta_n$, we have $|T_{n_x}(x)\overline{\nu}_{n_x}(x) - f(x) - \delta_{n_x}(x)| = |\epsilon_{n_x}(x) - \delta_{n_x}(x)| \leq \varepsilon'$. Now to replace $\overline{\nu}_{n_x}$ with $\nu_{n_x}$, we observe by (2.4) and (2.8),

$$\begin{aligned}
|T_n(x)\nu_n(x) - T_n(x)\overline{\nu}_n(x)| &= |T_n(x)|\,|\nu_n(x) - \overline{\nu}_n(x)| \\
&\leq (L\|x - x_n\| + |f(x_n)|)\ \frac{\varepsilon' r/2LL_\varphi}{1 + \|x - x_n\|} \\
&\leq (L\|x - x_n\| + 2)\ \frac{\varepsilon' r/2LL_\varphi}{1 + \|x - x_n\|} \\
&\leq \varepsilon' r/2L_\varphi + \varepsilon' r/LL_\varphi \\
&\leq 3\varepsilon' r/L_\varphi \leq 3\varepsilon'.
\end{aligned} \tag{2.10}$$

Therefore, these estimates give, $|T_{n_x}(x)\nu_{n_x}(x) - f(x) - \delta_{n_x}(x)| \leq 4\varepsilon'$, and because $f \geq 1$, we have our desired bound

$$\begin{aligned}
|T_{n_x}(x)\nu_{n_x}(x) - \delta_{n_x}(x)|\,\varphi_{n_x}(x) &\geq |T_{n_x}(x)\nu_{n_x}(x) - \delta_{n_x}(x)|\,(1/2) \\
&\geq \left(f(x) - 4\varepsilon'\right)(1/2) > 1/4.
\end{aligned}$$

We remark that it follows from this that for any $x$,

$$\left\|\{\psi_n(x)\}_n\right\|_{c_0} \geq \left\|\{\psi_n(x)\}_n\right\|_\infty = \left\|\{(T_n(x)-\delta_n(x))\,\varphi_n(x)\}_n\right\|_\infty \geq 1/4. \tag{2.11}$$

Next, to show that the function $x \mapsto \{\psi_n(x)\}_{n=1}^\infty$ is real analytic from $X$ into $c_0$, we shall require that for each $x$, there exists $n_x$ and $\delta_x \in (0,1)$ so that for $n \geq n_x$ and $\|z\|_{\widetilde{X}} < \delta_x$, we have

$$\frac{\left|\widetilde{T}_n(x+z)\,\widetilde{\nu}_n(x+z) - \widetilde{\delta}_n(x+z)\right|}{M_n} \leq M_x, \tag{2.12}$$

where $M_x$ depends on $x$, but is independent of $n$.

Recalling that $\widetilde{T}_n(w) = \widetilde{f'(x_n)}\,(x - x_n + w) + f(x_n)$, and using (2.9) and property (1) and (4) of $\delta_n$, where we may suppose that $x + z \in \widetilde{W}$ when $\|z\|_{\widetilde{X}} < \delta_x < 1$, we obtain,

$$\begin{aligned}
&\left|\widetilde{T}_n(x+z)\,\widetilde{\nu}_n(x+z) - \widetilde{\delta}_n(x+z)\right| \\
&\leq \left|\widetilde{T}_n(x+z)\right|\left|\widetilde{\nu}_n(x+z)\right| + \left|\widetilde{\delta}_n(x+z) - \delta_n(x)\right| + \delta_n(x) \\
&\leq \left|\widetilde{f'(x_n)}\,(x - x_n + z) + f(x_n)\right| e^{2C^2\kappa\delta_x^2} + M_\Delta + \varepsilon' \\
&< \left(L\left(\|x - x_n\| + \|z\|_{\widetilde{X}}\right) + |f(x_n)|\right) e^{2C^2\kappa\delta_x^2} + 2M_\Delta \\
&< \left(L\left(\|x - x_n\| + 1\right) + 2\right) e^{2C^2\kappa} + 2M_\Delta \\
&\leq \left(3L\left(\|x - x_n\| + 1\right)\right) e^{2C^2\kappa} + 2M_\Delta
\end{aligned}$$

Now recalling that $M_n = e^{2C^2\kappa}\left(1 + \|x_n\|\right)$, we see that

$$\begin{aligned}
&\frac{3L\left(\|x - x_n\| + 1\right) e^{2C^2\kappa}}{M_n} \\
&\leq \frac{3L\left(\|x\| + \|x_n\| + 1\right)}{1 + \|x_n\|} \\
&\leq 3L\left(1 + \|x\|\right).
\end{aligned}$$

Putting $M_x = 2M_\Delta + 3L\left(1 + \|x\|\right)$, we have established $(2.12)$.

Now, to show the analyticity of $\{\psi_n(x)\}_{n=1}^{\infty}$, we first note that property (5) of Lemma 1 together with (2.12) yield

$$|\widetilde{\psi}_n(x+z)| = |\widetilde{T}_n(x+z)\widetilde{\nu}_n(x+z) - \widetilde{\delta}_n(x+z)|\,|\widetilde{\varphi}_n(x+z)| \leq \frac{M_x}{n!a_{x,r}^n}$$

whenever $n \geq n_x$ and $\|z\|_{\widetilde{X}} < \delta_x$.

Because the numerical series $\sum_{n=1}^{\infty} M_x/n!a_{x,r}^n$ is convergent, we then have that the series of functions $\sum_{n=1}^{\infty} |\widetilde{\psi}_n(x+z)|$ is uniformly convergent on the ball $B_{\widetilde{X}}(0,\delta_x)$, which clearly implies that the series

$$\sum_{n=1}^{\infty} \widetilde{\psi}_n(z)e_n = \{\widetilde{\psi}_n(z)\}_{n=1}^{\infty}$$

is uniformly convergent for $z \in B_{\widetilde{X}}(x,\delta_x)$. Then it is clear that $\{\widetilde{\psi}_n(z)\}_{n=1}^{\infty}$, being a series of holomorphic mappings which converges uniformly on the ball $B_{\widetilde{X}}(x,\delta_x)$, is a holomorphic mapping on this ball. Since $x \in X$ is arbitrary, this shows that $x \mapsto \{\psi_n(x)\}_{n=1}^{\infty}$ is real analytic.

**2.4. Showing that $g$ does the job.** Now we move on to our final estimates; $|g-f|$ and $\|g'-f'\|$. Fix $x \in X$, and put $\mathcal{N} = \mathcal{N}_x = \{n : x \in D_n\}$. Now we have (using $f \geq 1 > 0$), that

$$\begin{aligned}
|g(x) - f(x)| &= \left| \frac{\|\{\psi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} - f(x) \right| \\
&= \left| \frac{\|\{\psi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} - \frac{\|\{f(x)\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} \right| \\
&= \frac{1}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} \|\{(T_n(x)\nu_n(x) - f(x) - \delta_n(x))\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0} \\
&\leq 2\,\|\{(T_n(x)\nu_n(x) - f(x) - \delta_n(x))\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0},
\end{aligned}$$

the last line by Lemma 1 $(3)$. We proceed in cases.

**Case 1:** For $n \in \mathcal{N}$, we have $\epsilon_n(x) = T_n(x) - f(x) = T_n(x)\overline{\nu}_n(x) - f(x)$, and so by property (1) of $\delta_n$ and Lemma 1 $(7)$, we obtain the estimate, $|T_n(x)\overline{\nu}_n(x) - f(x) - \delta_n(x)|\,\varphi_n(x) \leq (\varepsilon' r/L_\varphi)\,3$. Then using $(2.10)$, we have

$$|T_n(x)\nu_n(x) - f(x) - \delta_n(x)|\,\varphi_n(x) \leq 6r\varepsilon'/L_\varphi. \tag{2.13}$$

**Case 2:** For $n \notin \mathcal{N}$, recall $\varphi_n(x) \leq \varepsilon_1 \leq \varepsilon' r/25L$.

Now, for $n$ such that $x \in \widehat{D}_n$, we have $Q(x-x_n) < 6r$, and so $\|x - x_n\| \le (q(x-x_n))^{1/2n} < \left((6r+1)^{2n} - 1\right)^{1/2n} < 7$, as $r < 1$. Hence, for such $n$ we have,

(2.14)
$$|T_n(x)\,\nu_n(x)| \le (L\,\|x - x_n\| + |f(x_n)|)\,\nu_n(x) \le (7L+2)\left(1+\varepsilon'\right) \le 18L.$$

On the other hand, for $n$ such that $x \notin \widehat{D}_n$, by (2.6) we obtain,

$$|T_n(x)\,\nu_n(x)| \le (L\,\|x-x_n\| + |f(x_n)|)\,\frac{\varepsilon' r/2LL_\varphi}{1+\|x-x_n\|}$$

$$\le (L\,\|x-x_n\| + 2)\,\frac{\varepsilon'/2L}{1+\|x-x_n\|}$$

$$\le \varepsilon'/2 + \varepsilon'/L \le 2\varepsilon'.$$

In any event, for all $n$ we have,

$$|T_n(x)\,\nu_n(x)| \le 18L. \tag{2.15}$$

Therefore, for $n \notin \mathcal{N}$, using again property (1) of $\delta_n$, we have,

$$|T_n(x)\,\nu_n(x) - f(x) - \delta_n(x)|\,\varphi_n(x) \le (|T_n(x)\,\nu_n(x)| + |f(x)| + \delta_n(x))\,\varphi_n(x)$$

(2.16)
$$\le \left(18L + 2 + 2\varepsilon'\right)\left(\varepsilon' r/25L\right) \le \varepsilon' r.$$

It follows that, $|g(x) - f(x)| \le 10A_1\varepsilon' r < \varepsilon$.

We now establish some derivative estimates. Fix $x$ and consider the expression

$$(T_n(x)\,\nu_n(x))' = T_n'(x)\,\nu_n(x) + T_n(x)\,\nu_n'(x).$$

From an estimate analogous to (2.15), using (2.4) and (2.7), we have that for all $n$, $\|T_n(x)\,\nu_n'(x)\| \le 9L\mathrm{Lip}(Q)\mathrm{Lip}(\nu)$. Also, $\|T_n'(x)\,\nu_n(x)\| \le L(1+\varepsilon') \le 2L$. Hence,

$$\left\|(T_n(x)\,\nu_n(x))'\right\| \le 2L + 9L\mathrm{Lip}(Q)\,\mathrm{Lip}(\nu) \le 11L\mathrm{Lip}(Q)\,\mathrm{Lip}(\nu).$$

Using this, and property (2) of $\delta_n$, we have,

(2.17)
$$\mathrm{Lip}(T_n\nu_n - f - \delta_n) \le 11L\mathrm{Lip}(Q)\,\mathrm{Lip}(\nu) + L + C_0\varepsilon' \le 13C_0L\mathrm{Lip}(Q)\,\mathrm{Lip}(\nu).$$

Next, for $x \in D_n$, $\overline{\nu}_n(x) = 1$, and again by property (2) of $\delta_n$,

(2.18)
$$\mathrm{Lip}\left((T_n\overline{\nu}_n - f - \delta_n)\,|_{D_n}\right) = \mathrm{Lip}\left((T_n - f - \delta_n)\,|_{D_n}\right) \le \varepsilon' + C_0\varepsilon' \le 2C_0\varepsilon'.$$

Next we compute, using (2.4) ,

$$\begin{aligned}\left\|(T_n(x)(\nu_n-\overline{\nu}_n))'\right\| &= \left\|T_n'(x)\right\| |\nu_n(x)-\overline{\nu}_n(x)| + |T_n(x)| \left\|\nu_n'(x)-\overline{\nu}_n'(x)\right\| \\ &\leq L\frac{\varepsilon' r/2LL_\varphi}{1+\|x-x_n\|} + (L\|x-x_n\|+2)\frac{\mathrm{Lip}(Q)\,\varepsilon' r/2LL_\varphi}{1+\|x-x_n\|} \\ &\leq \varepsilon' r/2 + \mathrm{Lip}(Q)\ \ \varepsilon' r/2 + \mathrm{Lip}(Q)\ \ \varepsilon' r/L \\ &\leq 2\mathrm{Lip}(Q)\,\varepsilon'.\end{aligned}$$

It follows from this and (2.18) that,

$$\mathrm{Lip}\left((T_n\nu_n - f - \delta_n)\,|_{D_n}\right) \leq 2C_0\varepsilon' + 2\mathrm{Lip}(Q)\,\varepsilon' r \leq 4C_0\mathrm{Lip}(Q)\,\varepsilon'. \tag{2.19}$$

Finally we turn to $\|g'(x)-f'(x)\|$ with the help of the above estimates. Again fix $x \in X$, and we obtain,

$$\begin{aligned}&\left\|g'(x)-f'(x)\right\| \\ &= \left(\frac{\|\{(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))\,\varphi_n(x)\}_{n=1}^\infty\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^\infty\|_{c_0}}\right)' \\ &= \frac{1}{\|\{\varphi_n(x)\}_{n=1}^\infty\|_{c_0}^2} \times \\ &\Big(\|\{\varphi_n(x)\}_{n=1}^\infty\|_{c_0}\,\|\{(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))\,\varphi_n(x)\}_{n=1}^\infty\|_{c_0}' \\ &-\|\{\varphi_n(x)\}_{n=1}^\infty\|_{c_0}'\,\|\{(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))\,\varphi_n(x)\}_{n=1}^\infty\|_{c_0}\Big)\end{aligned}$$

Let us first consider

$$\begin{aligned}&((T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))\,\varphi_n(x))' \\ &= (T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))'\,\varphi_n(x) + (T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))\,\varphi_n'(x).\end{aligned}$$

For the first term, and $n \in \mathcal{N}$, we have, using property (2) of Lemma 1 and (2.19) ,

$$\begin{aligned}\left\|\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)'\varphi_n(x)\right\| &\leq \left\|\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)'\right\|\varphi_n(x)\\ &\leq 4C_0\mathrm{Lip}\,(Q)\,\varepsilon'\,(1+\varepsilon_1)\\ &\leq 8C_0\mathrm{Lip}\,(Q)\,\varepsilon'.\end{aligned}$$

For $n\notin\mathcal{N}$, using Lemma 1 $(4')$ and $(2.17)$, we obtain,

$$\begin{aligned}\left\|\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)'\varphi_n(x)\right\| &\leq \left\|\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)'\right\|\varphi_n(x)\\ &\leq 13C_0L\mathrm{Lip}\,(Q)\,\mathrm{Lip}\,(\nu)\left(\varepsilon'/6C_0L\mathrm{Lip}\,(\nu)\right)\\ &\leq 3\mathrm{Lip}\,(Q)\,\varepsilon'.\end{aligned}$$

In any event, $\left\|\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)'\varphi_n(x)\right\|_{c_0}\leq 8C_0A_1\mathrm{Lip}(Q)\,\varepsilon'$.

Next we consider the second term, $\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n'(x)$. For $n\in\mathcal{N}$, from the estimate giving $(2.13)$, we have,

$$\begin{aligned}\left\|\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n'(x)\right\| &\leq \left|T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right|\left\|\varphi_n'(x)\right\|\\ &\leq 6\varepsilon' r/L_\varphi\,(L_\varphi)\leq 6\varepsilon'.\end{aligned}$$

For $n\notin\mathcal{N}$, just as in $(2.16)$ we obtain,

$$\begin{aligned}\left\|\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n'(x)\right\| &\leq \left|T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right|\left\|\varphi_n'(x)\right\|\\ &\leq \varepsilon'.\end{aligned}$$

Hence, altogether we see that,

$$\begin{aligned}\left\|\left(\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n(x)\right)'\right\|_{c_0} &\leq 8C_0A_1\mathrm{Lip}\,(Q)\,\varepsilon'+6A_1\varepsilon'\\ &\leq 14C_0A_1\mathrm{Lip}\,(Q)\,\varepsilon'.\end{aligned}$$

Lastly, we examine $\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}'\,\|\{(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))\,\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}$. Recall our estimate of $|f-g|$ found $\|\{(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x))\,\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}\leq 3A_1\varepsilon' r$. Therefore we have,

$$\left\|\{\varphi_n(x)\}_{n=1}^{\infty}\right\|_{c_0}' \left\|\{(T_n(x)\,\nu_n(x) - f(x) - \delta_n(x))\,\varphi_n(x)\}_{n=1}^{\infty}\right\|_{c_0}$$

$$\leq A_1 L_{\varphi} \left\|\{(T_n(x)\,\nu_n(x) - f(x) - \delta_n(x))\,\varphi_n(x)\}_{n=1}^{\infty}\right\|_{\infty}$$

$$\leq A_1 \frac{L_1 \mathrm{Lip}\,(Q)}{r}\left(5A_1\varepsilon' r\right) = 5A_1^2 L_1 \mathrm{Lip}\,(Q)\,\varepsilon'$$

Finally, because $\left\|\{\varphi_n(x)\}_{n=1}^{\infty}\right\|_{c_0}^2 \geq \left\|\{\varphi_n(x)\}_{n=1}^{\infty}\right\|_{\infty}^2 \geq 1/4$ as noted above, putting all the above estimates together yields,

$$\left\|g'(x) - f'(x)\right\| \leq \frac{(2A_1)\,14C_0A_1\mathrm{Lip}\,(Q)\,\varepsilon' + 5A_1^2L_1\mathrm{Lip}\,(Q)\,\varepsilon'}{1/4}$$

$$\leq \left(132C_0A_1^2L_1\mathrm{Lip}\,(Q)\right)\varepsilon' < \varepsilon. \quad \square$$

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