HYPERPLANE ARRANGEMENTS OF TORELLI TYPE

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Abstract. We give a necessary and sufficient condition in order for a hyperplane arrangement to be of Torelli type, namely that it is recovered as the set of unstable hyperplanes of its Dolgachev sheaf of logarithmic differentials. Decompositions and semistability of non-Torelli arrangements are investigated.

Introduction

An arrangement of hyperplanes in $\mathbb{P}^n$ is the union D of ℓ distinct hyperplanes $H_1, \dots, H_\ell$ of $\mathbb{P}^n$, so $H_i = \{f_i = 0\}$, where f_i is a linear form. The topology, the geometry, and the combinatorial properties of the pair $(\mathbb{P}^n, D)$ are interesting from many points of view, we refer to [OT92] for a comprehensive treatment. Let us only mention that Arnold, in his foundational paper [Arn69], first used the algebra of differential forms df_i/f_i, to give an explicit description of the cohomology ring of $\mathbb{P}^n \setminus D$, an approach generalized by Brieskorn, see [Bri73].

More generally, Deligne defined and extensively used in [Del70] the sheaf $\Omega_X(\log D)$ of forms with logarithmic poles along D, when D is a normal crossing divisor of a smooth variety X, while Saito in [Sai80] gave a definition of $\Omega_X(\log D)$ for more general divisors. Anyway $\Omega_X(\log D)$ is the dual of the sheafified derivation module, and as such it is a reflexive sheaf, in fact locally free if D is normal crossing.

Let again D be a hyperplane arrangement with normal crossings (also called a *generic* arrangement, namely D is such that any k hyperplanes meet along a $\mathbb{P}^{n-k}$). The sheaf $\Omega_{\mathbb{P}^n}(\log(D))$ is then associated to D. The main question asked (and solved) by Dolgachev and Kapranov in [DK93], is whether one can reconstruct D from $\Omega_{\mathbb{P}^n}(\log(D))$. We say that D is a *Torelli arrangement* in this case (or simply D is Torelli). They proved that if $\deg(D) \geq 2n+3$, then D is Torelli if and only if D do not osculate a rational normal curve. The result was extended to the range $\deg(D) \geq n+2$ in [Val00].

However this result only covers generic arrangement, while the most interesting arrangements are far from being so. On the other hand, Catanese-Hosten-Khetan-Sturmfels in [CHKS06] and Dolgachev in [Dol07] defined a subsheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$ of $\Omega_{\mathbb{P}^n}(\log(D))$, fitting in the residue exact sequence:

$$0 \to \Omega_{\mathbb{P}^n} \to \tilde{\Omega}_{\mathbb{P}^n}(\log(D)) \to \bigoplus_{i=1,\dots,\ell} \mathscr{O}_{H_i} \to 0.$$

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Dolgachev in [Dol07] formulated the Torelli problem for the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$, and proposed the following conjecture:

Conjecture (Dolgachev)**.** Assume $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$ is a semi-stable sheaf in the sense of Gieseker. Then D is Torelli if and only if the points given by the H_i's in the dual $\mathbb{P}^n$ do not belong to a stable rational curve of degree n.

A *stable rational curve* here means a connected curve of arithmetic genus 0 which is the union of s smooth rational curves $C_1, \dots, C_s$, with $\deg(C_i) = d_i$ and $d_1 + \dots + d_s = n$, each C_i spanning a $\mathbb{P}^{d_i}$, and the union of the $\mathbb{P}^{d_i}$'s spanning the dual $\mathbb{P}^n$. He also showed that the conjecture holds in the plane for up to 6 points.

In this paper we study in detail the Torelli problem for the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log(D))$. We denote by Z a finite set of points, say ℓ points $z_1, \dots, z_\ell$, lying in the dual space $\mathbb{P}_n$ of $\mathbb{P}^n$, and by D_Z the union of the corresponding hyperplanes $H_{z_1}, \dots, H_{z_\ell}$. In order to state our result, we need to introduce what we call *Kronecker-Weierstrass varieties* (a reason for this name will be apparent later on). If $(d, n_1, \dots, n_s)$ is a string of $s+1$ integers such that $n = d + n_1 \cdots + n_s$, we say that $Y \subset \mathbb{P}_n$ is a *Kronecker-Weierstrass (KW) variety of type* $(d; s)$ if $Y = C \cup L_1 \cup \dots \cup L_s \subset \mathbb{P}_n$, where the L_i's are linear subspaces of dimension $1 \le n_i \le n-1$ and C is a smooth rational curve of degree d, with $0 \le d \le n$ spanning a linear space L of dimension d such that:

i) for all i, $L \cap L_i$ is a single point which lies in C;
ii) the spaces L_i's are mutually disjoint.

In the case $d = 0$ (so C is reduced to a single point y), we replace the conditions by the fact that all the linear spaces L_i meet only at y. The point y in this case is called the distinguished point of Y.

We formulate now our main result. We give it here also for subschemes with multiple structure, we will see how to make sense of this further on.

Theorem 1. *Let $Z \subset \mathbb{P}_n$ be a finite-length, set-theoretically non-degenerate subscheme.*

Then Z fails to be Torelli if and only if Z is contained in a KW variety $Y \subset \mathbb{P}_n$ of type $(d; s)$ whose distinguished point (for $d = 0$) does not lie in Z.

The main ingredient that we bring in the proof is a functorial definition of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ as the dualized direct image of the sheaf of linear forms vanishing at Z in $\mathbb{P}_n$, under the natural point-hyperplane incidence variety. The key point is that this has to be taken with a grain of salt, namely all functors have to be derived in order to make the correspondence work smoothly.

As a corollary of the theorem above, we get that if Z is contained in a stable rational curve in $\mathbb{P}_n$, then Z is not Torelli, as conjectured by Dolgachev.

As another corollary, we will see that the converse implication holds on $\mathbb{P}^2$, even without the assumption that $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ is semistable. In higher dimension, this implication no longer holds, regardless of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ being semistable or not. To understand why, one first remarks that in many examples Z is contained in a KW variety Y without lying on a stable rational curve. Yet one has to prove semistability of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ for some of these examples. One way to do this is to provide a filtration of $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ associated to the decomposition of Y into irreducible components. This is the content of Theorem 3. Some exceptions to the "if" direction of Dolgachev's conjecture are Example 3.5 and 3.6.

0.1. **Structure of the paper.** In the next section we set up our framework for dealing with logarithmic sheaves, based on direct images of ideal sheaves. In section 2 we prove our main theorem, already stated above. This section also contains a result on the maximal number of unstable hyperplanes of a Steiner sheaf, see Theorem 2. Section 3 is devoted to build a decomposition tool for non-Torelli arrangements. In this last section we will outline some examples with interesting non-Torelli phenomena.

0.2. **Notations.** We refer to [OT92] for basic notions on hyperplane arrangements. As a matter of notation, we let $\mathbb{P}^n$ be the space of 1-dimensional quotients of a $\mathbf{k}$-vector space V of dimension $n+1$ over a field $\mathbf{k}$, and we write $\mathbb{P}^n = \mathbb{P}(V)$. We let $\mathbb{P}_n = \mathbb{P}(V^*)$ be the dual of $\mathbb{P}^n$, namely the space of hyperplanes of $\mathbb{P}^n$. Given a point $y \in \mathbb{P}_n$, we let H_y be the hyperplane of $\mathbb{P}^n$ given by y. We use the variables $x_0, \ldots, x_n$ for the polynomial ring of $\mathbb{P}^n$, and the variables $z_0, \ldots, z_n$ for the polynomial ring of $\mathbb{P}_n$.

Let Z be a finite length subscheme of the dual space $\mathbb{P}_n$ of $\mathbb{P}^n$. The scheme Z consists of finitely many points $y_1, \ldots, y_s$, each y_i supporting a subscheme of length m_i. Then Z defines the divisor D_Z in $\mathbb{P}^n$, namely the set $H_{y_1}, \ldots, H_{y_s}$ of hyperplanes of $\mathbb{P}^n$, each H_{y_i} counted with multiplicity m_i. Namely:

$$D_Z = m_1 H_{y_1} + \cdots + m_s H_{y_s}.$$

We will have to deal with complexes of coherent sheaves on $\mathbb{P}^n$. A natural framework for them is the derived category $\mathbf{D}^b(\mathbb{P}^n)$ of complexes of sheaves with bounded coherent cohomology. We refer to [GM96] for a comprehensive treatment.We will denote by $[i]$ the i-th shift to the right of a complex in the derived category. To shorten notations, we will denote by $(a \to b \to c \xrightarrow{[1]})$ the exact triangle $(a \to b \to c \to a[1])$. We will write $\mathbf{R}F$ for the right derived functor of a functor F, with image in the derived category.

1. The Steiner sheaf associated to a hyperplane arrangements

We consider the incidence variety $\mathbb{F}^n_n$ of pairs $(x, y) \in \mathbb{P}^n \times \mathbb{P}_n$ where x lies in H_y. We let p and q be the projections from $\mathbb{F}^n_n$ respectively to $\mathbb{P}^n$ and to $\mathbb{P}_n$. These projections are $\mathbb{P}^{n-1}$-bundles. We have the natural exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^n \times \mathbb{P}_n}(-1,-1) \to \mathscr{O}_{\mathbb{P}^n \times \mathbb{P}_n} \to \mathscr{O}_{\mathbb{F}^n_n} \to 0. \tag{1.1}$$

We consider the complex $\mathbf{R}p_*(q^*(\mathcal{I}_Z(1)))$ as an element of the derived category of complexes of coherent sheaves on $\mathbb{P}^n$. We set here the definition of a sheaf $\mathscr{F}_Z$ on $\mathbb{P}^n$ attached to Z, although it will turn out (Proposition 1.3) that $\mathscr{F}_Z$ is in fact isomorphic to the sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ introduced by Dolgachev. However we will stick to the shorter notation $\mathscr{F}_Z$ all over the paper.

Definition 1.1. Given a finite length subscheme Z of $\mathbb{P}_n$ we define

$$\mathscr{F}_Z = \mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1)).$$

Whenever the vector space V underlying $\mathbb{P}^n$ is unclear, we will rather write $\mathscr{F}^V_Z$.

Proposition 1.2. *Let $Z \subset \mathbb{P}_n$ be (schematically) non-degenerate subscheme of length ℓ. Then $\mathscr{F}_Z$ is a sheaf having the following resolution:*

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-(n+1)} \to \mathscr{O}^{\ell-1}_{\mathbb{P}^n} \to \mathscr{F}_Z \to 0.$$

Moreover, $\mathscr{F}_Z$ is torsion-free if, locally around any point $z \in Z$, we have $\mathcal{I}^2_z \subset \mathcal{I}_Z$.

Proof. Working on the product $\mathbb{P}^n \times \mathbb{P}_n$, we tensor (1.1) with $q^*(\mathcal{I}_Z(1))$, obtaining thus the exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{I}_Z \to \mathscr{O}_{\mathbb{P}^n} \boxtimes \mathcal{I}_Z(1) \to q^*(\mathcal{I}_Z(1)) \to 0 \tag{1.2}$$

Since Z has finite length, we have $\mathrm{H}^k(\mathbb{P}_n, \mathcal{I}_Z(t)) = 0$ for all $k > 1$ and for all $t \in \mathbb{Z}$. Further, we have $\mathrm{H}^0(\mathbb{P}_n, \mathcal{I}_Z) = 0$ for Z is not empty and $\mathrm{H}^0(\mathbb{P}_n, \mathcal{I}_Z(1)) = 0$ since Z is non-degenerate. Therefore, taking direct image onto $\mathbb{P}^n$, we get the following distinguished triangle:

$$\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))) \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-1} \xrightarrow{M_Z} \mathscr{O}_{\mathbb{P}^n}^{\ell-(n+1)} \to \mathbf{R}p_*(q^*(\mathcal{I}_Z(1)))[1]$$

where M_Z is obtained applying $\mathbf{R}p_*(-)$ to the inclusion appearing in (1.2). Therefore $\mathbf{R}p_*(q^*(\mathcal{I}_Z(1)))$ has cohomology only in degree 0 and 1, and is isomorphic to the cone of:

$$\mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-1} \xrightarrow{M_Z} \mathscr{O}_{\mathbb{P}^n}^{\ell-(n+1)}.$$

Taking $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$, we get that $\mathscr{F}_Z$ is isomorphic to the cone of:

$$\mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-(n+1)} \xrightarrow{M_Z^t} \mathscr{O}_{\mathbb{P}^n}^{\ell-1}.$$

Further, the sheaf $\mathbf{R}^1p_*(q^*(\mathcal{I}_Z(1)))$ is supported at the points x of $\mathbb{P}^n$ such that $\mathrm{H}^1(H_x, \mathcal{I}_{Z\cap H_x}(1)) \neq 0$. In particular, it is a torsion sheaf. Therefore, the map M_Z^t is injective, hence $\mathscr{F}_Z$ is concentrated in degree zero, and we have the exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-(n+1)} \xrightarrow{M_Z^t} \mathscr{O}_{\mathbb{P}^n}^{\ell-1} \to \mathscr{F}_Z \to 0. \tag{1.3}$$

It remains to prove that $\mathscr{F}_Z$ is torsion-free under our assumptions. Unwinding the double complex $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))), \mathscr{O}_{\mathbb{P}^n}(-1))$, we get two short exact sequences:

$$0 \to \mathcal{E}xt^1_{\mathbb{P}^n}(\mathbf{R}^1p_*q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}^n}(-1)) \to \mathscr{F}_Z \to \mathscr{K} \to 0, \tag{1.4}$$

$$\mathscr{K} \hookrightarrow \mathcal{H}om_{\mathbb{P}^n}(p_*q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}^n}(-1)) \to \mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1p_*q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}^n}(-1)) \to 0. \tag{1.5}$$

The coherent sheaf $\mathscr{K}$ is always torsion-free, and it differs from $\mathscr{F}_Z$ if and only if $\mathbf{R}^1p_*q^*(\mathcal{I}_Z(1))$ is supported in codimension 1. A necessary and sufficient condition for $\mathbf{R}^1p_*q^*(\mathcal{I}_Z(1))$ to be supported in codimension 1, is that there is $z \in Z$ such that, for all $x \in H_z$, we have $\mathrm{H}^1(H_x, \mathcal{I}_{Z\cap H_x}(1)) \neq 0$. This is equivalent to say that, given any linear form f vanishing at z, the ideal of Z modulo f contains all the quadrics of R/f.

In order to check the above condition, we can assume that the reduced support of Z is a single point, for H_x generically avoids all other points. Working locally around this point $z \in Z$, our hypothesis is thus that all quadrics of vanishing at z are in the ideal of Z. Therefore, the same thing takes place modulo f, and we are done. □

Let us describe briefly the relationship between our sheaf $\mathscr{F}_Z$ and the sheaves $\Omega_{\mathbb{P}^n}(\log D_Z)$ and $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$. First, let us recall a definition of $\Omega_{\mathbb{P}^n}(\log D_Z)$ (we refer for instance to [Sch03]). Let f be a polynomial defining D_Z, where Z consists of ℓ points of $\mathbb{P}_n$. We consider the sheafified derivation module $\mathscr{D}_0(Z)$, defined by the exact sequence:

$$0 \to \mathscr{D}_0(Z) \to \mathscr{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{(\partial_0 f, \dots, \partial_n f)} \mathscr{O}_{\mathbb{P}^n}(\ell - 1). \tag{1.6}$$

Then the sheaf $\Omega_{\mathbb{P}^n}(\log D_Z)$ is defined as:

$$\Omega_{\mathbb{P}^n}(\log D_Z) = \mathcal{H}om_{\mathbb{P}^n}(\mathscr{D}_0(Z), \mathscr{O}_{\mathbb{P}^n}(-1)).$$

Proposition 1.3. *Assume that Z is reduced and non-degenerate. Then $\mathscr{F}_Z$ is isomorphic to Dolgachev's sheaf $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$. Moreover, we have:*

$$\Omega_{\mathbb{P}^n}(\log D_Z) \cong \mathcal{H}om_{\mathbb{P}^n}(p_* q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}^n}(-1)) \cong \mathscr{F}_Z^{**}. \tag{1.7}$$

Proof. Let us first prove the claim regarding $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$. We apply the functor $\mathbf{R}p_* q^*$ to the exact sequence:

$$0 \to \mathcal{I}_Z(1) \to \mathscr{O}_{\mathbb{P}_n}(1) \to \mathscr{O}_Z \to 0.$$

Using (1.2), we obtain the distinguished triangle:

$$\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))) \to \mathcal{T}_{\mathbb{P}^n}(-1) \to \mathbf{R}p_*(q^*(\mathscr{O}_Z)) \xrightarrow{[1]} \tag{1.8}$$

Now if Z is reduced we have $Z = \{z_1, \dots, z_\ell\}$. Note that:

$$q^*(\mathscr{O}_Z) \cong \mathscr{O}_{q^{-1}(Z)} \cong \mathscr{O}_{\cup_{j=1,\dots,\ell} H_{z_j}}.$$

This sheaf lies above the divisor D_Z, and $p : q^{-1}(Z) \to D_Z$ is a resolution of singularities of D_Z. By Grothendieck duality, we have that $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathscr{O}_Z)), \mathscr{O}_{\mathbb{P}^n}(-1))[1]$ is isomorphic to:

$$\mathbf{R}p_*(\mathbf{R}\mathcal{H}om_{\mathbb{F}_n^n}(q^*(\mathscr{O}_Z), \mathscr{O}_{\mathbb{F}_n^n}(0,-n)))[n].$$

For each z_j in Z we have:

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\mathbb{F}_n^n}(q^*(\mathscr{O}_{z_j}), \mathscr{O}_{\mathbb{F}_n^n}(0,-n))[n] &\cong \mathcal{E}xt^n_{\mathbb{F}_n^n}(\mathscr{O}_{H_{z_j}}, \mathscr{O}_{\mathbb{F}_n^n}(0,-n))) \cong \\ &\cong \mathscr{O}_{H_{z_j}} \otimes \omega^*_{\mathbb{F}_n^n} \otimes \mathscr{O}_{\mathbb{F}_n^n}(0,-n) \cong \mathscr{O}_{H_{z_j}}. \end{aligned}$$

Therefore, taking $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$ of the triangle (1.8), we have the exact sequence:

$$0 \to \Omega_{\mathbb{P}^n} \to \mathscr{F}_Z \to p_*(\mathscr{O}_{q^{-1}(Z)}) \to 0. \tag{1.9}$$

We will be done if we can prove that this is the residue exact sequence defining $\tilde{\Omega}_{\mathbb{P}^n}(\log D_Z)$ according to [Dol07]. This will be accomplished by proving that there is in fact a unique functorial extension of $\Omega_{\mathbb{P}^n}(1)$ by $p_*(q^*(\mathscr{O}_Z))$, and observing that both the residue exact sequence and (1.9) are clearly functorial.

Claim 1.4. *We have a natural isomorphism:*

$$\mathrm{Ext}^1_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \Omega_{\mathbb{P}^n}) \cong \mathrm{Hom}_{\mathbb{P}_n}(\mathscr{O}_{\mathbb{P}_n}, \mathscr{O}_Z)^*.$$

Since $\mathscr{O}_Z$ is naturally a quotient of $\mathscr{O}_{\mathbb{P}_n}$, this claim will complete our argument. To prove the claim, we write the isomorphisms:

$$\begin{aligned} \mathrm{Ext}^1_{\mathbb{P}^n}(p_*(q^*(\mathscr{O}_Z)), \Omega_{\mathbb{P}^n}) &\cong \mathrm{Ext}^{n-1}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}(n+1), p_*(q^*(\mathscr{O}_Z)))^* \cong \\ &\cong \mathrm{Ext}^{n-1}_{\mathbb{F}_n^n}(p^*(\Omega_{\mathbb{P}^n}(n+1)), q^*(\mathscr{O}_Z))^*, \end{aligned}$$

where the first one is Serre duality and the second one is adjunction. Now we use the left adjoint functor to q^*, namely the functor $\mathbf{R}q_*(- \otimes \mathscr{O}_{\mathbb{F}_n^n}(-n,1))[n-1]$. Thus the latter group above is

$$\begin{aligned} &\cong \mathrm{Hom}_{\mathbb{P}_n}(\mathbf{R}q_*(p^*(\Omega_{\mathbb{P}^n}(1))) \otimes \mathscr{O}_{\mathbb{P}^n}(1), \mathscr{O}_Z)^* \cong \\ &\cong \mathrm{Hom}_{\mathbb{P}_n}(\mathscr{O}_{\mathbb{P}_n}, \mathscr{O}_Z)^*. \end{aligned}$$

Let us now turn to $\Omega_{\mathbb{P}^n}(\log D_Z)$. Let again $f = \prod_{i=1}^{\ell} f_i$ be an equation defining D_Z. Recall that the image of the rightmost map in (1.6) (the gradient map) is the Jacobian ideal $\mathscr{J}$ of D_Z. Denote by $\mathscr{J}_{D_Z}$ the image of $\mathscr{J}$ in $\mathscr{O}_{D_Z}$ (so $\mathscr{J}_{D_Z} = \mathscr{J} \cdot \mathscr{O}_{D_Z}$). Recall the natural exact sequence relating $\mathscr{J}_{D_Z}$ and $\mathscr{D}_0(Z)$ (see e.g. [Dol07, Section 2]):

$$0 \longrightarrow \mathscr{D}_0(Z) \longrightarrow \mathcal{T}_{\mathbb{P}^n}(-1) \longrightarrow \mathscr{J}_{D_Z}(\ell - 1) \longrightarrow 0. \tag{1.10}$$

Note also that we have:

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{P}^n}(p_*q^*(\mathscr{O}_Z), \Omega_{\mathbb{P}^n}) &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathcal{T}_{\mathbb{P}^n}, \mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(p_*q^*(\mathscr{O}_Z), \mathscr{O}_{\mathbb{P}^n})[1]) \cong \\ &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathcal{T}_{\mathbb{P}^n}, p_*q^*(\mathscr{O}_Z)(1)),\end{aligned}$$

so the last homomorphism group contains a unique functorial element. Further, from [Dol07, Proposition 2.4] we get an inclusion of $\mathscr{J}_{D_Z}(\ell)$ into $p_*(\omega_{q^{-1}Z} \otimes \omega^*_{\mathbb{P}^n}) \cong p_*(q^*\mathscr{O}_Z)(1)$.

Therefore, both $\mathscr{D}_0(Z)$ (by (1.10)) and $p_*(q^*(\mathcal{I}_Z(1)))$ (by the cohomology sequence of (1.8)) are the kernel of the unique functorial map $\mathcal{T}_{\mathbb{P}^n}(-1) \to p_*q^*(\mathscr{O}_Z)$. This gives an isomorphism:

$$p_*(q^*(\mathcal{I}_Z(1))) \cong \mathscr{D}_0(Z). \tag{1.11}$$

Note also that we have the exact sequence:

$$0 \to \mathscr{F}_Z \to \Omega_{\mathbb{P}^n}(\log D_Z) \to \mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1 p_*q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}^n}(-1)) \to 0.$$

The desired isomorphisms (1.7) easily follow from the above sequence and (1.11). □

Remark 1.5. The support of the cokernel sheaf $\mathcal{E}xt^2_{\mathbb{P}^n}(\mathbf{R}^1 p_*q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}^n}(-1))$ sits in codimension $k > 1$ if and only Z contains a subscheme of length $(n+1)$, contained in a linear subspace $\mathbb{P}_{k-1}$. Further, this shows again that $\mathscr{F}_Z$ and $\Omega_{\mathbb{P}^n}(\log D_Z)$ agree if D_Z is normal crossing in codimension 2, see [Dol07, Corollary 2.8].

Example 1.6. Consider the ideal $(z_0 z_2^2, (z_1 + z_1)z_1 z_2, z_0 z_1 z_2, z_0 z_1^2)$. This defines a subscheme $Z \subset \mathbb{P}_2$, which is the union of the first infinitesimal neighbourhood of $(1:0:0)$ and the three collinear points $(0:1:0)$, $(0:0:1)$, $(0:1:-1)$. Then we have:

$$M_Z = \begin{pmatrix} -x_0 & 0 & x_1 & 0 & 0 \\ x_0 & 0 & 0 & x_1 - x_2 & -x_2 \\ 0 & x_0 & 0 & 0 & x_2 \end{pmatrix}.$$

In this case $\mathscr{F}_Z$ is still torsion-free and we have:

$$0 \to \mathscr{F}_Z \to \Omega_{\mathbb{P}^n}(\log D_Z) \to \mathscr{O}_{x_1,\ldots,x_4} \to 0, \qquad \Omega_{\mathbb{P}^n}(\log D_Z) \cong \mathscr{O}_{\mathbb{P}^2}(2) \oplus \mathscr{O}_{\mathbb{P}^2}(1),$$

where $x_1, \ldots, x_4$ are $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, $(0:1:1)$, the 4 points corresponding to the 4 lines in $\mathbb{P}_2$ which are 3-secant to Z. The arrangement given by Z is thus free (i.e. $\Omega_{\mathbb{P}^n}(\log D_Z)$ splits as a direct sum of line bundles).

Example 1.7. Consider the scheme Z defined as the union of the second infinitesimal neighbourhood of $(0:1:0)$ and the two points $(1:0:0)$, $(0:0:1)$. Namely, the ideal of Z is ideal$(x_0 x_2^2, x_0^2 x_2, x_1 x_2^3, x_0^3 x_1)$. In this case, we obtain the matrix:

$$M_Z = \begin{pmatrix} 0 & 0 & -x_1 & 0 & 0 & 0 & x_2 \\ x_0 & 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & x_1 & 0 & 0 \\ -x_1 & x_0 & 0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & 0 & 0 & x_1 & 0 \end{pmatrix}.$$

Here we get the line L defined as $\{x_1 = 0\}$ as support of the torsion part of $\mathscr{F}_Z$. We have $\Omega_{\mathbb{P}^n}(\log D_Z) \cong \mathscr{O}_{\mathbb{P}^2}(2) \oplus \mathscr{O}_{\mathbb{P}^2}(2)$, i.e. Z is a free arrangement. The exact sequences (1.4) and (1.5) become:

$$0 \to \mathscr{O}_L(-2) \to \mathscr{F}_Z \to \mathscr{O}_{\mathbb{P}^2}(2)^2 \to \mathscr{O}_{Z_1 \cup Z_2} \to 0,$$

where Z_1, Z_2 are two length-2 subschemes, supported at the points $(1:0:0)$ and $(0:0:1)$, accounting for the two 4-secant lines to Z in $\mathbb{P}_2$, namely $\{z_0 = 0\}$ and $\{z_2 = 0\}$.

2. Unstable hyperplanes of logarithmic sheaves

The goal of this section is to prove our main result, stated in the introduction. We will first need some definitions.

Definition 2.1. Let $\mathscr{E}$ be a Steiner sheaf on $\mathbb{P}^n$, namely a sheaf $\mathscr{E}$ fitting into an exact sequence of the form:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^a \to \mathscr{O}_{\mathbb{P}^n}^b \to \mathscr{E} \to 0,$$

for some integers a, b. Then a hyperplane H is *unstable* for $\mathscr{E}$ if:

$$\mathrm{H}^{n-1}(H, \mathscr{E}_{|H}(-n)) \neq 0.$$

A point y of $\mathbb{P}_n$ is unstable for $\mathscr{E}$ if the hyperplane H_y is unstable for $\mathscr{E}$.

We can give a scheme structure to the set $\mathrm{W}(\mathscr{E})$ of unstable hyperplanes of $\mathscr{E}$, considering them as the scheme-theoretic support of the sheaf $\mathbf{R}^{n-1}q_*(p^*(\mathscr{E}(-n)))$.

Definition 2.2. A finite length subscheme Z of $\mathbb{P}_n$ is said to be *Torelli* if Z gives rise to a Torelli arrangement, namely if the *set* of unstable hyperplanes of $\mathscr{F}_Z$ is the support of Z, i.e. if we have a set-theoretic equality:

$$\mathrm{W}(\mathscr{F}_Z) = Z.$$

Lemma 2.3. *Let Z be a finite length subscheme of $\mathbb{P}_n$. Then we have a scheme-theoretic inclusion:*

$$Z \subset \mathrm{W}(\mathscr{F}_Z).$$

Proof. By Grothendieck duality, we have:

$$\mathscr{F}_Z(-n) \cong \mathbf{R}p_*(\mathbf{R}\mathcal{H}om_{\mathbb{F}_n^n}(q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{F}_n^n}(-n,-n)))[n-1],$$

from which we get an epimorphism:

$$\mathbf{R}q_*(p^*(\mathscr{F}_Z(-n)))[n-1] \to \mathbf{R}q_*(\mathbf{R}\mathcal{H}om_{\mathbb{F}_n^n}(q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{F}_n^n}(-n,-n)))[n-1].$$

Applying again Grothendieck duality, we get an isomorphism of the right hand side above and:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathbf{R}q_*q^*(\mathcal{I}_Z(1)), \mathscr{O}_{\mathbb{P}_n}(-n)),$$

which projects onto:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathcal{I}_Z(1), \mathscr{O}_{\mathbb{P}_n}(-n)).$$

Summing up, we have an epimorphism:

$$\mathbf{R}q_*(p^*(\mathscr{F}_Z(-n)))[n-1] \twoheadrightarrow \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathcal{I}_Z(1), \mathscr{O}_{\mathbb{P}_n}(-n)),$$

and taking cohomology in degree $n-1$ we get:

$$\mathbf{R}^{n-1}q_*(p^*(\mathscr{F}_Z(-n))) \twoheadrightarrow \mathcal{E}xt^{n-1}_{\mathbb{P}_n}(\mathcal{I}_Z, \mathscr{O}_{\mathbb{P}_n}(-n-1)) \cong \mathscr{O}_Z,$$

which proves our claim. $\square$

Remark 2.4. It was already proved in [Dol07] that any $z \in Z$ is unstable for $\mathscr{F}_Z$, hence Z is not Torelli if and only if the *set* of unstable hyperplanes of $\mathscr{F}_Z$ strictly contains Z.

One could say that Z is *scheme-theoretically Torelli* if the subscheme of unstable hyperplanes is Z itself. A criterion analogous to Theorem 1 for Z to be scheme-theoretically Torelli is lacking at the time being.

Remark 2.5. We point out that $\mathrm{W}(\mathscr{F}_Z) = \mathrm{W}(\Omega_{\mathbb{P}^n}(\log D_Z))$ if and only if Z does not possess a subscheme of length $(n+1)$ contained in a line, as explained in Remark 1.5. This remark makes more precise Proposition 3.2 of [Dol07].

2.1. **Kronecker-Weierstrass varieties and unstable hyperplanes.** In order to prove Theorem 1, we introduce some geometric objects that we call Kronecker-Weierstrass varieties. The name is inspired on the tool that classifies them. Indeed, the isomorphism classes of these varieties are given by the standard Kronecker-Weierstrass forms of a matrix of homogeneous linear forms in two variables. We recall the definition given in the introduction.

Definition 2.6. Let $(d, n_1, \dots, n_s)$ be a string of $s+1$ integers such that $n = d + n_1 \cdots + n_s$, and $1 \le d \le n$. Then $Y \subset \mathbb{P}_n$ is a *Kronecker-Weierstrass (KW) variety of type* $(d; s)$ if $Y = C \cup L_1 \cup \cdots \cup L_s \subset \mathbb{P}_n$, where the L_i's are linear subspaces of dimension $1 \le n_i \le n-1$ and C is a smooth rational curve of degree d (called the *curve part* of Y) spanning a linear space L of dimension d such that:

i) for all i, $L \cap L_i$ is a single point which lies in C;
ii) the spaces L_i's are mutually disjoint.

If $d = 0$, a KW variety of type $(0; s)$ is defined as $Y = L_1 \cup \cdots \cup L_s \subset \mathbb{P}_n$, where the L_i's are linear subspaces of dimension $1 \le n_i \le n-1$ and all the linear spaces L_i meet only at a point y, which is called *the distinguished point of* Y.

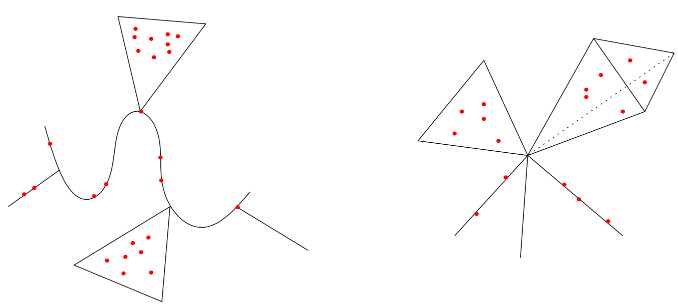

FIGURE 1. Points contained in a Kronecker Weierstrass variety.

Example 2.7. We outline some examples of KW variety.

1) A rational normal curve is a KW variety of type $(n; 0)$.
2) A union of two lines in $\mathbb{P}^2$ is a KW variety in three ways, two of them of type $(1; 1)$, and one of type $(0; 2)$ (the intersection point is the distinguished point).

Having this setup, we can move towards the proof of our Theorem 1. We need a series of lemmas and the following construction.

Given a point y of $\mathbb{P}_n$, we consider the Koszul complex resolving the ideal sheaf $\mathcal{I}_y$, namely a long exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}_n}(-n) \xrightarrow{d_n} \mathscr{O}_{\mathbb{P}_n}^n(-n+1) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_3} \mathscr{O}_{\mathbb{P}_n}^n(-2) \xrightarrow{d_2} \mathscr{O}_{\mathbb{P}_n}(-1) \xrightarrow{d_1} \mathcal{I}_y \to 0.$$

We let $\mathcal{S}_y$ be the sheaf $\mathrm{Im}(d_{n-1})$, twisted by $\mathscr{O}_{\mathbb{P}_n}(n)$. We have:

$$0 \to \mathscr{O}_{\mathbb{P}_n} \xrightarrow{(h_1,\ldots,h_n)} \mathscr{O}_{\mathbb{P}_n}^n(1) \to \mathcal{S}_y \to 0, \tag{2.1}$$

where the h_i's are linear forms on $\mathbb{P}_n$ and y is defined by $\{h_1 = \cdots = h_n = 0\}$.

The following lemma is the key to our argument. It is inspired on a generalization of [Val10, Proposition 6.1]

Lemma 2.8. *Let y be a point of $\mathbb{P}_n$, and let Z be a finite length subscheme of $\mathbb{P}_n$ not containing y. Then y is unstable for $\mathscr{F}_Z$ if and only if $\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z) \neq 0$.*

Proof. By definition y is unstable for $\mathscr{F}_Z$ if and only if:

$$\mathrm{H}^{n-1}(H_y, \mathscr{F}_Z(-n)) \neq 0.$$

In view of the exact sequence (1.3), this is equivalent to say that, restricting the matrix M_Z^t to H_y and taking cohomology, we get a non-zero cokernel of:

$$\mathrm{H}^{n-1}(H_y, \mathscr{O}_{H_y}^{\ell-(n+1)}(-n-1)) \xrightarrow{(M_Z^t)_{|H_y}} \mathrm{H}^{n-1}(H_y, \mathscr{O}_{H_y}^{\ell-1}(-n)).$$

By Serre duality, this means that

$$\mathrm{H}^0(H_y, \mathscr{O}_{H_y}^{\ell-1}) \xrightarrow{(M_Z)_{|H_y}} \mathrm{H}^0(H_y, \mathscr{O}_{H_y}^{\ell-(n+1)}(1))$$

has non-trivial kernel. Recalling by the proof of Proposition 1.2 that $\mathbf{R}p_*(q^*(\mathcal{I}_Z(1))$ is the cone of the map M_Z, we see that this is equivalent to say that:

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathscr{O}_{H_y}, \mathbf{R}p_*(q^*(\mathcal{I}_Z(1))) \neq 0.$$

Since $(p^*, \mathbf{R}p_*)$ is an adjoint pair, the above extension group is isomorphic to:

$$\mathrm{Ext}^1_{\mathbb{F}_n^n}(p^*(\mathscr{O}_{H_y}), q^*(\mathcal{I}_Z(1))).$$

We use again the left adjoint functor to q^* (recall that it is $\mathbf{R}q_*(- \otimes \mathscr{O}_{\mathbb{F}_n^n}(-n,1))[n-1]$). The above group is thus isomorphic to:

$$\mathrm{Ext}^{2-n}_{\mathbb{P}_n}(\mathbf{R}q_* p^*(\mathscr{O}_{H_y}(-n)), \mathcal{I}_Z). \tag{2.2}$$

Note also that we can compute (2.2) as:

$$\mathrm{H}^{\cdot}(\mathbb{P}_n, \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}\left(\mathbf{R}q_* p^*(\mathscr{O}_{H_y}(-n)[2-n]), \mathscr{O}_{\mathbb{P}_n}\right) \otimes \mathcal{I}_Z). \tag{2.3}$$

Let us now compute $\mathbf{R}q_* p^*(\mathscr{O}_{H_y}(-n))$. Making use of (1.1), we get a distinguished triangle:

$$\mathbf{R}q_* p^*(\mathscr{O}_{H_y}(-n)) \to \mathscr{O}_{\mathbb{P}_n}(-1)^n[-n+2] \xrightarrow{P_y} \mathscr{O}_{\mathbb{P}_n}[-n+2] \xrightarrow{[1]}$$

Here, it is easy to see that P_y is a matrix of linear forms defining y in $\mathbb{P}_n$. Dualizing the above diagram, we get an exact sequence (of sheaves):

$$0 \to \mathscr{O}_{\mathbb{P}_n} \xrightarrow{P_y^t} \mathscr{O}_{\mathbb{P}_n}(1)^n \to \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}\left(\mathbf{R}q_* p^*(\mathscr{O}_{H_y}(-n)), \mathscr{O}_{\mathbb{P}_n}\right)[-n+2] \to 0.$$

By the definition of the sheaf $\mathcal{S}_y$, we have thus an isomorphism:

$$\mathcal{S}_y \cong \mathbf{R}\mathcal{H}om_{\mathbb{P}_n}\left(\mathbf{R}q_* p^*(\mathscr{O}_{H_y}(-n)), \mathscr{O}_{\mathbb{P}_n}\right)[-n+2].$$

Then the space appearing in (2.3) is non-zero if and only if

$$\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \overset{\mathbf{L}}{\otimes} \mathcal{I}_Z) \neq 0,$$

where the notation above stands for left-derived tensor product. But one easily proves that $\mathcal{T}or_j(\mathcal{S}_y, \mathcal{I}_Z) = 0$ for $j > 0$, so (2.3) is non-zero if and only if

$$\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z) \neq 0.$$

So y is unstable if and only if the above vector space is not zero, and the lemma is proved. □

Lemma 2.9. *Let y be a point and Z be a finite-length, non-degenerate subscheme of $\mathbb{P}_n$, not containing y. Then $\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z) \neq 0$ if and only if Z is contained in the rank-1 locus of a $2 \times n$ matrix M of linear forms having non-proportional rows, with one row defining y.*

Proof. Recalling the exact sequence (2.1) defining $\mathcal{S}_y$, we let $h_1, \dots, h_n$ be a regular sequence defining $y \in \mathbb{P}_n$, and we note that a section in $\mathrm{H}^0(\mathbb{P}_n, \mathcal{S}_y \otimes \mathcal{I}_Z)$ is given by a global section s of $\mathcal{S}_y$ such that s vanishes along Z. In turn, s lifts to $\tilde{s}$ as in the diagram:

$$\begin{array}{ccccccccc} & & & & & & \mathscr{O}_{\mathbb{P}_n} & & \\ & & & & & \overset{\tilde{s}}{\swarrow} & \downarrow s & & \\ 0 & \longrightarrow & \mathscr{O}_{\mathbb{P}_n} & \xrightarrow{(h_1,\dots,h_n)} & \mathscr{O}_{\mathbb{P}_n}^n(1) & \longrightarrow & \mathcal{S}_y & \longrightarrow & 0. \end{array}$$

Now $\tilde{s}$ is given by $(g_1, \dots, g_n)$, where the g_i's are linear forms and the row $(g_1, \dots, g_n)$ is not proportional to $(h_1, \dots, h_n)$. Then in order for s to vanish on Z, we must have that Z is contained in the locus Y cut by the 2×2 minors of the matrix:

$$M = \begin{pmatrix} h_1 & \cdots & h_n \\ g_1 & \cdots & g_n \end{pmatrix},$$

Note that Y is not all of $\mathbb{P}_n$, because the two rows of M are not proportional. Since all the construction is reversible, the lemma is proved. □

Lemma 2.10. *Let Z be a finite-length, set-theoretically non-degenerate subscheme of $\mathbb{P}^n$ and $y \in \mathbb{P}_n$. Then the equivalent conditions of the previous lemma are satisfied if and only Z is contained in a KW variety Y of type $(d; s)$ with either $d > 0$ and y is in the curve part of Y, or $d = 0$, and y is the distinguished point of Y.*

Proof. Let us assume that the conditions of the previous lemma are satisfied, and look for the KW variety Y. So let us consider the matrix M given by the above lemma as a morphism of sheaves:

$$\mathscr{O}_{\mathbb{P}_n}(-1)^n \to \mathscr{O}_{\mathbb{P}_n}^2.$$

We have that Z is contained in the rank-1 locus of M, hence in the support of the cokernel sheaf $\mathscr{T}$ of the above map, hence in the image in $\mathbb{P}_n$ of the natural map $\mathbb{P}(\mathscr{T}) \to \mathbb{P}_n$.

The matrix M can be written in coordinates as $M_{i,j} = \sum_{k=0}^{n} a_{i,j,k} z_k$ for some scalars $a_{i,j,k}$, with $i = 0, 1$ and $j = 0, \dots, n-1$. This gives a matrix N of size $n \times (n+1)$, this time over $\mathbf{k}[\xi_0, \xi_1]$, by:

$$N_{j,k} = \sum_{i=0,1} a_{i,j,k} \xi_i. \tag{2.4}$$

Therefore, we think of the above matrix N as a map:

$$N : \mathscr{O}_{\mathbb{P}^1}(-1)^n \to \mathscr{O}_{\mathbb{P}^1}^{n+1}, \tag{2.5}$$

where the target space is identified with $V \otimes \mathscr{O}_{\mathbb{P}^1}$, with $V = \mathrm{H}^0(\mathbb{P}_n, \mathscr{O}_{\mathbb{P}_n}(1))$.

Note that this map is injective. Indeed, if y is defined by the forms $h_1, \dots, h_n$, up to a change of basis we may assume $h_i = z_i$, so that the identity matrix of size n is a submatrix N evaluated at $(1:0)$. The sheaf $\mathscr{L} = \mathrm{Cok}(N)$ decomposes as:

$$\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^1}(d) \oplus \mathscr{O}_{\mathbb{P}^1, p_1}^{n_1} \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1, p_s}^{n_s}, \tag{2.6}$$

for some distinct points $p_i \in \mathbb{P}^1$, and some integers $d, n_1, \dots, n_s \in [0, n]$. Since the sheaf $\mathscr{L}$ has degree n, we must have $d + n_1 + \cdots + n_s = n$.

The matrix N is classified by its standard Kronecker-Weierstrass (KW) form (hence the name of Y); we refer for this standard form for instance to [BCS97, Chapter 19]. This means that N can be written, in an appropriate basis, in block form like:

$$N = \left(\begin{array}{c|c|c|c} N_0 & 0 & \cdots & 0 \\ \hline 0 & N_1 & & 0 \\ \hline \vdots & & \ddots & \\ \hline 0 & 0 & & N_s \end{array}\right). \tag{2.7}$$

Here, N_0 is of size $d \times (d+1)$, with $\mathrm{Cok}(N_0) \cong \mathscr{O}_{\mathbb{P}^1}(d)$ and N_i is a square matrix of size n_i that degenerates on p_i only. For $i > 0$, each N_i can be further decomposed into its normal Jordan blocks, which are all of size one if and only if N_i is diagonal. Note also that N_0 can be written as:

$$N_0 = \begin{pmatrix} \xi_0 & 0 & \cdots & 0 \\ \xi_1 & \xi_0 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \xi_1 & \xi_0 \\ 0 & \cdots & 0 & \xi_1 \end{pmatrix}. \tag{2.8}$$

Let us show that, with these elements, one can define Y.

Case $d > 0$: In this case, since $d + n_1 + \cdots + n_s = n$, we have $1 \le n_j \le n-1$ for all j. We define then the curve C as the image of $\mathbb{P}(\mathscr{L})$ in $\mathbb{P}_n$ obtained by taking global sections of the quotient $\mathscr{O}_{\mathbb{P}^1}(d)$ of $\mathscr{L}$. Namely, C is just $\mathbb{P}^1$ mapped to $\mathbb{P}_n$ by $\mathscr{O}_{\mathbb{P}^1}(d)$, and spans the d-dimensional linear subspace $L = \mathbb{P}(\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}))$ corresponding to the projection $\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}) \to \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$. In an appropriate basis, the curve C is cut in the space $L = \{z_{d+1} = \cdots = z_n = 0\}$ as the rank-1 locus of:

$$\begin{pmatrix} z_1 & \cdots & z_d \\ z_0 & \cdots & z_{d-1} \end{pmatrix}.$$

We define then L_j as the cone over the image in $\mathbb{P}_n$ of p_j and the space given by the projection $\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}) \to \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}^{n_j}_{\mathbb{P}^1,p_j})$. Each L_j meets L only at p_j, and the p_j's are all distinct if $d > 0$. Since L_i meets L_j only along C, all linear spaces L_j's are mutually disjoint for $d > 0$. This defines the KW variety $Y = C \cup L_1 \cup \cdots \cup C_s$.

Note that y belongs to C. Indeed, in the basis under consideration, we have that $y = (1 : 0 : \ldots : 0)$, and C goes through this point. Note also that Y clearly contains the image of $\mathbb{P}(\mathscr{L}) \cong \mathbb{P}(\mathscr{T})$ in $\mathbb{P}_n$ under the natural map $\mathbb{P}(\mathscr{L}) \to \mathbb{P}(\mathrm{H}^0(\mathbb{P}^1, \mathscr{L}))$. But this image is the rank-1 locus of M, which contains Z. So Y contains Z.

Case $d = 0$: In this case, under the decomposition (2.7), we have $N_0 = 0$. The sheaf $\mathscr{L}$ defines a projection of $\mathbb{P}^1$ to a point of $\mathbb{P}_n$, which in the basis under consideration has coordinates $(1 : 0 : \ldots : 0)$, i.e. this point is y. In this case, each linear space L_j is a cone over y and $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}^{n_j}_{\mathbb{P}^1,p_j}))$, hence all the L_j's meet only at y. Once we prove that $1 \leq n_j \leq n - 1$ for all j, we can define $Y = L_1 \cup \cdots \cup L_s$, and clearly Z is contained in Y.

So let us show $1 \leq n_j \leq n-1$ for all j, in other words let us prove $s \geq 2$. Assume thus $s = 1$, and note that $\mathscr{L} \cong \mathscr{O}^{n}_{\mathbb{P}^1,p}$, with $p_1 = p = (a : b)$, so that N_1 degenerates on $(a : b)$ only. Note that the standard KW form of N_1 cannot be a multiple of the $n \times n$ identity matrix, times $b\xi_0 - a\xi_1$, for the two rows of the corresponding matrix M would be proportional. Hence the KW form of N_1 has at least one non-trivial Jordan block (i.e. of size at least 2). Then, the corresponding rank-1 locus of M is a multiple structure over a linear space of dimension at most $n - 1$. But then Z is contained in a multiple structure over a hyperplane, a contradiction, since Z is set-theoretically non-degenerate.

To prove the converse implication, let us be given a KW variety Y of type $(d; s)$ containing Z, with $d > 0$, let L_0 be the span of the curve part C of Y and let $L_1, \ldots, L_s$ the linear spaces of Y. For each L_i, we choose a basis of an $(n_i - 1)$-dimensional linear subspace disjoint from L_0, and we complete this to a basis of V by stacking a basis of L_0. We take N_0 as in (2.8), and, for $i = 1, \ldots, s$, we let (a_i, b_i) be the points on $\mathbb{P}^1$ corresponding to the intersection $C \cap L_i$ under the parametrization $\mathbb{P}^1 \to C$. We define N_i as a square matrix of size n_i having $b_i\xi_0 - a_i\xi_1$ on the diagonal and zero anywhere else. We have thus presented the matrix as is (2.4), hence we have a $2 \times n$ of the form $M_{i,j} = \sum_{k=0}^{n} a_{i,j,k} z_k$ in the coordinates given by the chosen basis. The first row of M thus defines y, and the rank-1 locus of M is Y.

If $d = 0$ we choose a projection $\mathbb{P}^1 \to \{y\}$, and we choose s distinct points $(a_i : b_i)$ in $\mathbb{P}^1$. We still have the matrices N_i, and the matrix N_0 is the zero matrix with one row. Constructing N as in (2.7), the same choice of basis for V allows to write the matrix M, whose first row defines y and whose rank-1 locus is Y. □

We can now prove our main result, Theorem 1. Namely, let $Z \subset \mathbb{P}_n$ be a finite-length, set-theoretically non-degenerate subscheme. Then we have to prove that the set of unstable hyperplanes $\mathrm{W}(\mathscr{F}_Z)$ contains at least another point $y \notin Z$ if and only if Z is contained in a KW variety Y of type $(d; s)$ whose distinguished point (if $d = 0$) does not lie in Z.

Proof of Theorem 1. Let us assume that Z is not Torelli, and prove that Z is contained in a KW variety. Since Z is not Torelli, there is a point $y \in \mathbb{P}_n$, not belonging to Z, unstable for $\mathscr{F}_Z$. We can apply Lemmas 2.8, 2.9, 2.10 since Z is set-theoretically non-degenerate. Then, there is a KW variety Y containing Z, and we are done.

Conversely given a KW variety Y of type $(d; s)$ containing Z, we look at two cases. If $d = 0$, then by assumption Z does not contain the distinguished point y of Y. But by Lemmas 2.8, 2.9, 2.10, the point y is unstable for $\mathscr{F}_Z$, so Z is not Torelli. If $d > 0$, we let y be any point of the curve part C of Y. By Lemmas 2.8, 2.9, 2.10, y is unstable for Z. But Z is of finite length, so there is $y \in C \setminus Z$ and Z is not Torelli. □

Recall Dolgachev's conjecture from the introduction (see [Dol07, Conjecture 5.8]). It states that a semi-stable arrangement of hyperplanes Z (i.e. such that $\mathscr{F}_Z$ is a semi-stable sheaf) fails to be Torelli if and only if Z belongs to a stable rational curve of degree n.

Corollary 2.11. *The "only if" implication of Dolgachev's conjecture is true.*

Proof. If Z belongs to a curve $C = C_0 \cup \cdots \cup C_s$ as above, then we fix one component $C = C_0$ and we define L_i as the span of C_i, for $i > 0$. The variety $Y = C \cup L_1 \cup \cdots \cup L_s$ is a KW variety containing Z, so Z is not Torelli. □

Corollary 2.12. *A finite length subscheme Z of $\mathbb{P}^2$, whose set-theoretic support is not contained in a line, is Torelli if and only if it is not contained in a conic.*

Hence Dolgachev's conjecture (see [Dol07, Chapter 5]) holds on $\mathbb{P}^2$. In fact something quite stronger is true, for no stability condition is required in our result; in fact $\mathscr{F}_Z$ needs not even be torsion-free.

We note in the next corollary that, for generic arrangements, our approach gives a quick proof of some of the main results of [DK93] and [Val00]. Also, we note some simple examples of non-generic Torelli arrangements.

Corollary 2.13. *Let Z be a subscheme of length $\ell < \infty$ of $\mathbb{P}_n$.*

i) If the subscheme Z is contained in no quadric, then Z is Torelli;
ii) assume that Z is in general linear position and $\ell \geq n + 3$. Then Z is contained in a smooth rational normal curve of degree n if and only if Z is not Torelli.

Proof. The statement (i) is clear, since all 2×2 minors of the matrix M of the previous lemma are quadrics.

For (ii), we want to show that, if $\ell \geq n + 3$ and Z is in general linear position, then Z is contained in a KW variety Y if and only if it is contained in a rational normal curve of degree n. One direction is clear, so we assume that there are $C, L_1, \dots, L_s$ as in Theorem 1, such that $Y = C \cup L_1 \cup \cdots \cup L_s$ contains Z, with $s \geq 1$. Note that the span L' of $C \cup L_1 \cup \cdots \cup L_{s-1}$ has dimension $d + a_1 + \cdots + a_{s-1}$, hence there are at most $d + a_1 + \cdots + a_{s-1} + 1$ points of Z in L'. Also, L_s contains at most $a_s + 1$ points of Z. Hence Y contains at most $d + a_1 + \cdots + a_s + 2 = n + 2$ points of Z, so $\ell \geq n + 3$ contradicts that Z be contained in Y. □

2.2. **Maximal number of unstable hyperplanes.** One can ask, given a Steiner sheaf $\mathscr{E}$, how to recognize if $\mathscr{E}$ is isomorphic to $\mathscr{F}_Z$, for some Z in $\mathbb{P}_n$. The next theorem gives an answer to this question.

Theorem 2. *Let $\mathscr{E}$ be a sheaf having resolution:*

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-n-1} \to \mathscr{O}_{\mathbb{P}^n}^{\ell-1} \to \mathscr{E} \to 0.$$

Assume that $\mathrm{W}(\mathscr{E})$ *contains* ℓ *distinct points* $\{z_1,\ldots,z_\ell\} = Z$*, and that* $\mathscr{O}_{H_{z_i}}$ *is not a direct summand of* $\mathscr{E}$*, for any* j*. Then* $\mathscr{E}$ *is isomorphic to* $\mathscr{F}_Z$*.*

Proof. Let H be an unstable hyperplane of $\mathscr{E}$, hence assume $\mathrm{H}^{n-1}(H,\mathscr{E}_{|H}(-n)) \neq 0$, i.e. $\mathrm{H}^{n-1}(\mathbb{P}^n,\mathscr{E}\otimes\mathscr{O}_H(-n)) \neq 0$. We have:

$$\begin{aligned}\mathrm{H}^{n-1}(\mathbb{P}^n,\mathscr{E}\otimes\mathscr{O}_H(-n)) &\cong \mathrm{Ext}^{n-1}_{\mathbb{P}^n}(\mathscr{O}_{\mathbb{P}^n},\mathscr{E}\otimes\mathbf{R}\mathcal{H}om(\mathscr{O}_H(n+1)[-1],\mathscr{O}_{\mathbb{P}^n}))\cong\\ &\cong \mathrm{Ext}^{n-1}_{\mathbb{P}^n}(\mathscr{O}_H(n+1)[-1],\mathscr{E})\cong\\ &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathscr{E},\mathscr{O}_H)^*.\end{aligned}$$

Looking at the resolutions of $\mathscr{E}$ and $\mathscr{O}_H$, one sees that any non-zero map $\mathscr{E}\to\mathscr{O}_H$ is surjective, and that the kernel $\mathscr{E}'$ of such a map is again a Steiner sheaf.

Let now $H' \neq H$ be another unstable hyperplane of $\mathscr{E}$. By the induced map $\mathrm{H}^{n-1}(H',\mathscr{E}'_{|H'}(-n)) \twoheadrightarrow \mathrm{H}^{n-1}(H',\mathscr{E}_{|H'}(-n))$ we see that H' is unstable for $\mathscr{E}'$ as well. Let $\mathscr{K}$ by the kernel of the (surjective) map $\mathscr{E}'\to\mathscr{O}_{H'}$. Then $\mathscr{K}$ injects in $\mathscr{E}$, and we let $\mathscr{C}$ be $\mathscr{E}/\mathscr{K}$. We claim that $\mathscr{C}$ is isomorphic to $\mathscr{O}_H\oplus\mathscr{O}_{H'}$. Indeed, we have $\mathscr{E}'/\mathscr{K}\cong\mathscr{O}_{H'}$, hence we get an exact sequence:

$$0\to\mathscr{O}_{H'}\to\mathscr{C}\to\mathscr{O}_H\to 0.$$

Switching the roles of H and H' provides a splitting of the above sequence, so that $\mathscr{C}\cong\mathscr{O}_H\oplus\mathscr{O}_{H'}$.

Iterating this procedure, we find a surjective map:

$$\mathscr{E}\twoheadrightarrow\bigoplus_{i=1,\ldots,\ell}\mathscr{O}_{H_{z_i}}.$$

Note that the kernel of this map is $\Omega_{\mathbb{P}^n}$. Indeed, by diagram chasing, it is the kernel of a surjective map $\mathscr{O}_{\mathbb{P}^n}(-1)^{n+1}\twoheadrightarrow\mathscr{O}_{\mathbb{P}^n}$. Therefore we have and exact sequence:

$$0\to\Omega_{\mathbb{P}^n}\to\mathscr{E}\to\bigoplus_{i=1,\ldots,\ell}\mathscr{O}_{H_{z_i}}\to 0.$$

To conclude we can use Claim 1.4. Indeed, $\mathscr{F}_Z$ is given, up to isomorphism, as the only extension of $\bigoplus_{i=1,\ldots,\ell}\mathscr{O}_{H_{z_i}}$ by $\Omega_{\mathbb{P}^n}$ associated by Claim 1.4 to the canonical surjection $\mathscr{O}_{\mathbb{P}_n}\to\mathscr{O}_Z$. An extension of $\bigoplus_{i=1,\ldots,\ell}\mathscr{O}_{H_{z_i}}$ by $\Omega_{\mathbb{P}^n}$ not isomorphic to $\mathscr{F}_Z$ corresponds then to a map $\mathscr{O}_{\mathbb{P}_n}\to\mathscr{O}_Z$ which is not surjective, say $\mathscr{O}_{z_j}$ is not in the image. Such extension contains $\mathscr{O}_{H_{z_j}}$ as a direct summand, which contradicts our hypothesis on $\mathscr{E}$. □

We get the following bound on the number of unstable hyperplanes of a Steiner sheaf.

Corollary 2.14. *Let* $\mathscr{E}$ *be a torsion-free Steiner sheaf with resolution:*

$$0\to\mathscr{O}_{\mathbb{P}^n}(-1)^{\ell-n-1}\to\mathscr{O}_{\mathbb{P}^n}^{\ell-1}\to\mathscr{E}\to 0.$$

Assume that $\mathrm{W}(\mathscr{E})$ *contains* ℓ *distinct points* $\{z_1,\ldots,z_\ell\} = Z$ *not contained in a KW variety in* $\mathbb{P}_n$*. Then* $\mathrm{W}(\mathscr{E}) = Z$*.*

The following proposition gives an elementary way to write down the matrix M_Z.

Proposition 2.15. *Let* $Z = \{z_1,\ldots,z_\ell\}$ *be a non-degenerate Torelli arrangement, and consider the equations* $f_1,\ldots,f_\ell$ *of the* ℓ *hyperplanes of* $\mathbb{P}^n$*. Then, up to permutation of*

$1, \dots, \ell$, *there are constants* $\alpha_{i,j}$ *such that:*

$$f_\ell = \sum_{i=1,\dots,\ell-1} \alpha_{i,j} f_i, \tag{2.9}$$

for all $j = 1, \dots, \ell - n - 1$, *and the matrix* M_Z *can be written as:*

$$M = \begin{pmatrix} \alpha_{1,1} f_1 & \cdots & \alpha_{\ell,1} f_{\ell-1} \\ \vdots & & \vdots \\ \alpha_{1,\ell-n-1} f_1 & \cdots & \alpha_{\ell,\ell-n-1} f_{\ell-1} \end{pmatrix}.$$

Proof. The ℓ forms $f_1, \dots, f_\ell$ span the space V that has dimension $n+1$, hence up to reordering there are $\ell - n - 1$ linearly independent ways of writing f_ℓ as combination of $f_1, \dots, f_{\ell-1}$, and we have the constants $\alpha_{i,j}$.

Now, the i-th column of the matrix M above vanishes identically on the hyperplane H_i, which implies that H_i is unstable for the cokernel $\mathscr{E}$ of M^{t} for $i = 1, \dots, \ell - 1$. Further, in view of (2.9), we have that H_ℓ is also unstable for $\mathscr{E}$. Therefore, since Z is Torelli we conclude that $\mathrm{W}(\mathscr{E}) = Z$, hence, by the previous theorem, M_Z can be taken to be precisely M. □

3. Decomposition of logarithmic sheaves

Here we develop a tool for studying semistability of non-Torelli arrangements. This tool will take the form of a filtration associated to any non-Torelli arrangement. We will use this to provide some exceptions to Dolgachev's conjecture.

3.1. **Blowing up a linear subspace.** Let U be a $k+1$-dimensional subspace of V, with $1 \le k \le n-1$, and consider the subspace $\mathbb{P}_k = \mathbb{P}(U^*)$ of $\mathbb{P}_n = \mathbb{P}(V^*)$, embedded by $i : \mathbb{P}(U^*) \hookrightarrow \mathbb{P}_n$. Define $U^\perp$ as the kernel of the projection $V^* \to U^*$, and note that $U^\perp \cong (V/U)^*$. Denote by $\tilde{\mathbb{P}}^n_U$ the blowing up of $\mathbb{P}^n$ along $\mathbb{P}^{n-k-1} = \mathbb{P}(V/U) \subset \mathbb{P}^n$, and write $\pi_U : \tilde{\mathbb{P}}^n \to \mathbb{P}^k$ and $\sigma_U : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ for the two natural projections (we will drop this index U whenever possible). In our convention, points of $\mathbb{P}(V)$ and $\mathbb{P}(U)$ are quotients of V and U, so one can write:

$$\tilde{\mathbb{P}}^n = \{(x,u) \in \mathbb{P}^n \times \mathbb{P}^k \mid x_{|U} = u\}.$$

We consider $\mathbb{F}^k_k = \{(u,v) \in \mathbb{P}^k \times \mathbb{P}_k \mid u \in H_v\}$ and p_U and q_U are the natural projections to $\mathbb{P}^k$ and $\mathbb{P}_k$. In order to compare the incidence varieties $\mathbb{F}^n_n$ over $\mathbb{P}^n$ and $\mathbb{F}^k_k$ over $\mathbb{P}^k$, we consider the blown-up flag:

$$\tilde{\mathbb{F}}^n_n = \{(x,u,y) \in \mathbb{P}^n \times \mathbb{P}^k \times \mathbb{P}_n \mid x_{|U} = u, x \in H_y\}.$$

This blown-up flag contains the relative blown-up flag:

$$\tilde{\mathbb{F}}^n_k = \{(x,u,v) \in \mathbb{P}^n \times \mathbb{P}^k \times \mathbb{P}_k \mid x_{|U} = u, x \in H_v\}.$$

Projecting onto the different coordinates we get the commutative diagrams:

(3.1)
$$
\begin{array}{ccccc}
 & & \tilde{\mathbb{F}}^n_n & \longrightarrow & \mathbb{F}^n_n \\
 & \nearrow & \downarrow & & \downarrow p \\
\mathbb{F}^k_k & \longrightarrow & \tilde{\mathbb{P}}^n & \xrightarrow{\sigma} & \mathbb{P}^n \\
 & {\scriptstyle q_U}\searrow & \downarrow \pi & & \\
 & & \mathbb{P}^k & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & \mathbb{F}^k_k & & \\
 & \nearrow & \downarrow & \searrow {\scriptstyle p_U} & \\
\tilde{\mathbb{F}}^n_k & \longrightarrow & \tilde{\mathbb{F}}^n_k & \longrightarrow & \mathbb{P}_k \\
\downarrow & & \downarrow & & \downarrow i \\
\tilde{\mathbb{F}}^n_n & \longrightarrow & \mathbb{F}^n_n & \xrightarrow[q]{} & \mathbb{P}_n
\end{array}
$$

Let us analyze the sheaf $\mathscr{F}_Z$ when Z is degenerate, namely Z spans a proper subspace $\mathbb{P}(U^*) = \mathbb{P}_k \subset \mathbb{P}_n$. We may think that the last $n-k$ coordinates in $\mathbb{P}_n$ vanish on $\mathbb{P}_k$. This amounts to ask that the equations of the hyperplanes of Z only depend on the variables $x_0, \ldots, x_k$. The same happens to the matrix M_Z, that now naturally defines the Steiner sheaf $\mathscr{F}_Z^U$ over $\mathbb{P}^k$ associated to $Z \subset \mathbb{P}_k$. Note that we have the rational map:

$$\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$$

It is tempting to look at $\rho^*(\mathscr{F}_Z^U)$ as a component of $\mathscr{F}_Z$, defined by the same matrix M_Z, pulled back on $\mathbb{P}^n$ by ρ. The following lemma proves that this can be done (up to resolving the indeterminacy of ρ), and that the remaining component is $(n-k)$ copies of $\mathscr{O}_{\mathbb{P}^n}(-1)$.

Lemma 3.1. *Let Z be a finite length subscheme of $\mathbb{P}_n$, assume that Z spans a $\mathbb{P}_k = \mathbb{P}(U^*)$ with $1 \leq k \leq n-1$, and let $\sigma = \sigma_U, \pi = \pi_U$. Then we have:*

$$\mathscr{F}_Z \cong V/U \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \oplus \sigma_* \pi^*(\mathscr{F}_Z^U).$$

Proof. Assume that Z is contained in $\mathbb{P}_k = \mathbb{P}(U^*)$ and consider the exact sequence:

$$0 \to \mathcal{I}_{\mathbb{P}_k, \mathbb{P}_n}(1) \to \mathcal{I}_{Z, \mathbb{P}_n}(1) \to i_*(\mathcal{I}_{Z, \mathbb{P}_k}(1)) \to 0,$$

and the Koszul complex resolving $\mathcal{I}_{\mathbb{P}_k, \mathbb{P}_n}(1)$, namely:

$$0 \to \mathscr{O}_{\mathbb{P}_n}(k-n+1) \to \cdots \to \wedge^2 U^\perp \otimes \mathscr{O}_{\mathbb{P}_n}(-1) \to U^\perp \otimes \mathscr{O}_{\mathbb{P}_n} \to \mathcal{I}_{\mathbb{P}_k, \mathbb{P}_n}(1) \to 0.$$

Applying $\mathbf{R}p_*(q^*(-))$ to these exact sequences, in view of the vanishing $\mathbf{R}p_*(q^*(\mathscr{O}_{\mathbb{P}_n}(t)))$ for $2-n \leq t \leq -1$, we get a distinguished triangle:

$$U^\perp \otimes \mathscr{O}_{\mathbb{P}^n} \to \mathbf{R}p_* q^*(\mathcal{I}_Z(1)) \to \mathbf{R}p_* q^*(i_*(\mathcal{I}_{Z, \mathbb{P}_k}(1))) \xrightarrow{[1]}$$

Taking $\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(-, \mathscr{O}_{\mathbb{P}_n}(-1))$, we obtain the distinguished triangle:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}_n}(\mathbf{R}p_* q^*(i_*(\mathcal{I}_{Z, \mathbb{P}_k}(1))), \mathscr{O}_{\mathbb{P}_n}(-1)) \to \mathscr{F}_Z \to V/U \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \xrightarrow{[1]}$$

Our task is thus to prove that the leftmost complex in the triangle above is a sheaf isomorphic to $\sigma_* \pi^*(\mathscr{F}_Z^U)$. Let $\mathscr{E}_Z$ be this complex, for the remaining part of the proof.

Using repeatedly commutativity of the diagrams (3.1) together with projection formula, it is easy to get a natural transformation:

$$\mathbf{R}\sigma_* (\mathbf{R}\tilde{p}_U)_* \alpha^* q_U^* \cong \mathbf{R}p_* q^* i_*,$$

where α is the projection $\tilde{\mathbb{F}}^n_k \to \mathbb{F}^k_k$. By smooth base change, we also have:

$$(\mathbf{R}\tilde{p}_U)_* \alpha^* \cong \pi^* (\mathbf{R}p_U)_*,$$

where $\tilde{p}_U$ is the projection $\tilde{\mathbb{F}}^n_n \to \tilde{\mathbb{P}}^n$. This gives at once the natural isomorphism:

$$\mathbf{R}\sigma_* \pi^*(\mathbf{R}p_U)_* q_U^*(\mathcal{I}_{Z,\mathbb{P}_k}(1)) \cong \mathbf{R}p_* q^* i_*(\mathcal{I}_{Z,\mathbb{P}_k}(1)). \tag{3.2}$$

Therefore, in order to compute $\mathscr{E}_Z$, we have to apply $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$ to the left hand side. But we have seen that this simply amounts to transpose a matrix of linear forms of size $(\ell - 1) \times (\ell - k - 1)$, just as well as transposition is needed to define $\mathscr{F}_Z^U$ from $\mathbf{R}(p_U)_* q_U^*(\mathcal{I}_{Z,\mathbb{P}_k}(1))$ on $\mathbb{P}^k$, so that dualization of these complexes commutes with taking $\mathbf{R}\sigma_*\pi^*$. Hence we have shown that $\mathscr{E}_Z$ is isomorphic to $\mathbf{R}\sigma_*\pi^*(\mathscr{F}_Z^U)$, and therefore to $\sigma_*\pi^*(\mathscr{F}_Z^U)$.

This provides a short exact sequence:

$$0 \to \sigma_*\pi^*(\mathscr{F}_Z^U) \to \mathscr{F}_Z \to V/U \otimes \mathscr{O}_{\mathbb{P}^n}(-1) \to 0.$$

We will be done once this sequence splits, which in turn would be ensured by the vanishing:

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathscr{O}_{\mathbb{P}^n}(-1), \sigma_*\pi^*(\mathscr{F}_Z^U)) = 0.$$

But this vanishing is clear since $\sigma_*\pi^*(\mathscr{F}_Z^U)$ is a Steiner sheaf. □

In the above situation, we set:

$$\mathscr{E}_Z^U = \sigma_*\pi^*(\mathscr{F}_Z^U).$$

3.2. **Decomposing non-Torelli arrangements.** Let us borrow the notations from the previous paragraph. In particular, recall that, given a $(k+1)$-dimensional subspace U of V, and Z in $\mathbb{P}(U^*)$, we have a sheaf $\mathscr{F}_Z^U$ over $\mathbb{P}(U)$, and hence a sheaf $\sigma_*\pi^*(\mathscr{F}_Z^U)$ over $\mathbb{P}^n = \mathbb{P}(V)$, where $\sigma = \sigma^U$ and $\pi = \pi_U$ are the natural projections to $\mathbb{P}^n$ and $\mathbb{P}(U)$ from the blow-up $\tilde{\mathbb{P}}^n$ of $\mathbb{P}^n$ along $\mathbb{P}(V/U)$.

Lemma 3.2. *Assume that Z is contained in a rational normal curve C spanning $\mathbb{P}(U^*) \subset \mathbb{P}_n$. Then $\mathscr{F}_Z^U$ is isomorphic to $\mathscr{F}_{Z'}^U$, for any other subscheme Z' contained in C having the same length as Z.*

Proof. Let ℓ be the length of Z. We consider the exact sequence:

$$0 \to \mathcal{I}_{C,\mathbb{P}(U^*)}(1) \to \mathcal{I}_{Z,\mathbb{P}(U^*)}(1) \to \mathscr{O}_C((d-\ell)p) \to 0,$$

where, given an integer a, we write $\mathscr{O}_C(ap)$ for a divisor of degree a in C, namely a times a point $p \in C \cong \mathbb{P}^1$. Looking at the sheafified minimal graded free resolution of $\mathcal{I}_{C,\mathbb{P}(U^*)}(1)$ over $\mathbb{P}(U^*)$, we see immediately that:

$$\mathbf{R}(p_U)_* q_U^*(\mathcal{I}_{C,\mathbb{P}(U^*)}(1)) = 0.$$

Therefore the complex $\mathbf{R}(p_U)_* q_U^*(\mathcal{I}_{Z,\mathbb{P}(U^*)}(1))$ only depends on the value ℓ, hence so does $\mathscr{F}_Z^U$. □

By the previous lemma, if C_d is a rational normal curve of degree d spanning a $\mathbb{P}_d = \mathbb{P}(U^*)$, we can set:

$$\mathscr{E}_\ell^{C_d} = \sigma_*(\pi^*(\mathscr{F}_Z^U)),$$

for any subscheme Z of length ℓ of C_d.

The next result gives a decomposition tool for an arrangement Z which is contained in a KW-variety Y. So, let $Y = C \cup L_1 \cup \cdots \cup L_s$, where $L_i = \mathbb{P}(U_i) = \mathbb{P}_{n_i}$ and C is a smooth rational curve of degree $d > 0$, and the conditions (i) and (ii) of the introduction are satisfied. Let $y_i = C \cap L_i$.

Theorem 3. *Let $Z = Z_0 \cup \cdots \cup Z_s \subset \mathbb{P}_n$ be a subscheme of length ℓ, smooth at y_i for all i. Assume that L_i is the span of Z_i, and that $Z_0 \subset C \setminus \{y_1, \ldots, y_s\}$. Set ℓ_i for the length of Z_i. Then:*

i) we have a natural exact sequence:

$$0 \to \bigoplus_{i=1,\ldots,s} \mathscr{E}_{Z_i}^{U_i} \to \mathscr{F}_Z \to \mathscr{E}_{\ell_0+s}^{C_d} \to 0; \tag{3.3}$$

ii) we have the resolutions:

$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell_i - n_i - 1} \to \mathscr{O}_{\mathbb{P}^n}^{\ell_i - 1} \to \mathscr{E}_{Z_i}^{U_i} \to 0,$$
$$0 \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\ell_0 + s - d - 1} \to \mathscr{O}_{\mathbb{P}^n}^{\ell_0 + s - 1} \to \mathscr{E}_{\ell_0+s}^{C_d} \to 0.$$

Proof. Since Z lies in $Y = C \cup L_1 \cup \cdots \cup L_s$, we have the sequences:

$$0 \to \mathcal{I}_{Y,\mathbb{P}_n}(1) \to \mathcal{I}_{Z,\mathbb{P}_n}(1) \to \mathcal{I}_{Z,Y}(1) \to 0. \tag{3.4}$$

The following claim ensures that $\mathcal{I}_{Y,\mathbb{P}_n}(1)$ does not contribute to $\mathscr{F}_Z$.

Claim 3.3. *Given Y as above, we have $\mathbf{R}p_* q^*(\mathcal{I}_{Y,\mathbb{P}_n}(1)) = 0$.*

Let us postpone the proof of the claim above, and assume it for the time being. Set $\mathbb{L} = L_1 \cup \cdots \cup L_s$, $Z' = Z_1 \cup \cdots \cup Z_s$ and $Z'_0 = Z_0 \cup y_1 \cup \cdots \cup y_s$.

By the definition of Y and the hypothesis on Z we deduce the following exact commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z'_0,C}(1) & \longrightarrow & \mathscr{O}_C((d-s)p) & \longrightarrow & \mathscr{O}_{Z_0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z,Y}(1) & \longrightarrow & \mathscr{O}_Y(1) & \longrightarrow & \mathscr{O}_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z',\mathbb{L}}(1) & \longrightarrow & \mathscr{O}_{\mathbb{L}}(1) & \longrightarrow & \mathscr{O}_{Z'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.5}$$

Here, p is a point in $C \cong \mathbb{P}^1$. Moreover, clearly we have:

$$\mathcal{I}_{Z',\mathbb{L}}(1) \cong \bigoplus_{i=1,\ldots,s} \mathcal{I}_{Z_i,L_i}(1). \tag{3.6}$$

Hence, we may rewrite the leftmost column of the above diagram as:

$$0 \to \mathscr{O}_C((-s - \ell_0 + d)p) \to \mathcal{I}_{Z,Y}(1) \to \bigoplus_{i=1,\ldots,s} \mathcal{I}_{Z_i,L_i}(1) \to 0. \tag{3.7}$$

Notice also that we can switch the roles of C and $\mathbb{L}$, to obtain:

$$0 \to \bigoplus_{i=1,\ldots,s} \mathcal{I}_{y_i,L_i}(1) \to \mathscr{O}_Y(1) \to \mathscr{O}_C(1) \to 0. \tag{3.8}$$

Applying the functor $\mathbf{R}p_*(q^*(-))$ to the exact sequence (3.4) and dualizing, we have, in view of Claim 3.3:

$$\mathscr{F}_Z \cong \mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_{Z,Y}(1))), \mathscr{O}_{\mathbb{P}^n}(-1)).$$

Applying $\mathbf{R}p_*(q^*(-))$ and $\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(-, \mathscr{O}_{\mathbb{P}^n}(-1))$ to (3.7) gives the desired exact sequence (3.3). Indeed, For each of the terms $\mathcal{I}_{y_i,L_i}(1)$ appearing in the isomorphisms (3.6), we can use the argument used in Lemma 3.1, that gives:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathcal{I}_{y_i,L_i}(1))), \mathscr{O}_{\mathbb{P}^n}(-1)) \cong \sigma_*^{U_i}\pi_{U_i}^*(\mathscr{F}_{Z_i}^{U_i}) = \mathscr{E}_{Z_i}^{U_i}.$$

For $\mathscr{O}_C(d-\ell_0-s)$ we use the same argument and Lemma 3.2 to obtain:

$$\mathbf{R}\mathcal{H}om_{\mathbb{P}^n}(\mathbf{R}p_*(q^*(\mathscr{O}_C(d-\ell_0-s))), \mathscr{O}_{\mathbb{P}^n}(-1)) \cong \sigma_*^{U_0}\pi_{U_0}^*(\mathscr{F}_{Z_0'}^{U_0}) = \mathscr{E}_{\ell_0+s}^{C_d}.$$

We thus proved (i). The resolutions required for (ii) are provided by Lemma 3.1. It remains to prove Claim 3.3. □

Proof of Claim 3.3. Looking at (1.1), we see that the claim follows if we prove that $\mathcal{I}_Y(1)$ is the cohomology of a complex where only the sheaves $\mathscr{O}_{\mathbb{P}_n}(1-n), \dots, \mathscr{O}_{\mathbb{P}_n}(-1)$ appear. We can use Beilinson's theorem to prove that this is the case. In fact we merely have to prove the following vanishing results:

$$\mathrm{H}^k(\mathbb{P}_n, \mathcal{I}_Y(t)) = 0, \qquad \text{for all } k, \text{ and for } t = 0, 1. \tag{3.9}$$

To show this, we look at (3.8). Since $d + n_1 + \cdots + n_s = n$, taking cohomology of this sequence, we get :

$$\mathrm{H}^k(\mathbb{P}_n, \mathscr{O}_Y(1)) = 0, \qquad \text{for all } k > 0, \qquad \dim_{\mathbf{k}} \mathrm{H}^0(\mathbb{P}_n, \mathscr{O}_Y(1)) = n+1.$$

Hence we have (3.9) for $t = 1$, for Y is non-degenerate.

Taking cohomology of (3.8), twisted by $\mathscr{O}_{\mathbb{P}_n}(-1)$ immediately gives (3.9) for $t = 0$, and we are done. □

Corollary 3.4. *With the notations of the previous theorem, $\mathscr{E}_{Z_i}^{U_i}$ is a direct summand of $\mathscr{F}_Z$ if y_i belongs to Z.*

Proof. Order $1, \dots, s$ so that $y_1, \dots, y_r$ belong to Z and $y_{r+1}, \dots, y_s$ do not. Using (3.8) and a diagram similar to (3.5), we get an exact sequence:

$$0 \to \bigoplus_{i=1,\dots,r} \mathcal{I}_{Z_i,L_i}(1) \oplus \bigoplus_{i=r+1,\dots,s} \mathcal{I}_{Z_i \cup y_i, L_i}(1) \to \mathcal{I}_{Z,Y}(1) \to \mathscr{O}_C((d-r-\ell_0)p) \to 0.$$

Comparing with (3.7), we see that, for $i = 1, \dots, r$, $\mathcal{I}_{Z_i,L_i}(1)$ is a direct summand of $\mathcal{I}_{Z,Y}(1)$, so that $\mathscr{E}_{Z_i}^{U_i}$ is a direct summand of $\mathscr{F}_Z$. □

3.3. **Exceptions to Dolgachev's conjecture.** We conclude the paper with some examples of hyperplane arrangements having interesting unstable loci, giving some counterexamples to the "only if" implication of Dolgachev's conjecture. Namely, we describe finite sets Z in $\mathbb{P}_n$ such that $\mathrm{W}(\mathscr{F}_Z)$ is the union of Z and a line in $\mathbb{P}_3$, or Z and a plane in $\mathbb{P}_4$, or Z and a point in $\mathbb{P}_4$. The results of this section are used to prove semistability in some cases.

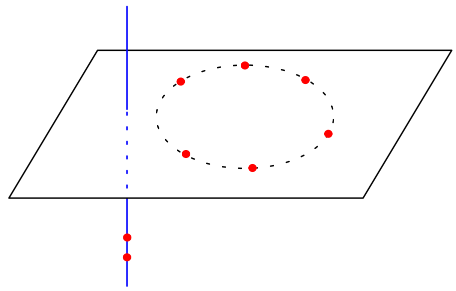

FIGURE 2. Seven points in $\mathbb{P}_3$ with an unstable line.

Example 3.5. We consider the union Z_1 of 5 points on a unique conic, spanning a plane L_1 in $\mathbb{P}_3$, and the union Z_0 of 2 more points on a line L_0. We assume that L_0 does not meet the conic $D \subset L_1$ passing through Z_1, and that $Z_0 \cap L_1 = \emptyset$. We let $Z = Z_0 \cup Z_1$.

Consider a point y of L_0. Then there are a rational normal curve through y (take L_0) and a plane (take L_1) such that $L_0 \cup L_1$ contains Z, and satisfying (i) and (ii). Thus all points of L_0 are unstable, and Z is not Torelli.

On the other hand, if $y \notin Z$ does not lie in L_0, then y is not unstable for $\mathscr{F}_Z$. Indeed, any subvariety $Y \subset \mathbb{P}_n$ through y and Z as in Theorem 1 would have to contain Z_1 and L, hence be $L_0 \cup L_1$. So y has to lie in L_1. But even the points of $L_1 \setminus Z$ are not unstable, for we should have a conic in L_1 through y and Z_1 (hence the conic is D) and a line through Z_1 (hence the line is L_0) meeting at a single point; but D does not pass through $L_0 \cap L_1$.

Finally, note that $\mathscr{F}_Z$ is a stable sheaf, at least for most choices of the 5 points of Z_1. In fact, let us prove it under the assumption that $Z_1 = \{\zeta_1, \dots, \zeta_5\}$ is such that ζ_3 lies in intersection of the lines N_1 and N_2 through ζ_1, ζ_2 and ζ_4, ζ_5 (still $D = N_1 \cup N_2$ disjoint from L_0). In this case, Theorem 3 applies to give a short exact sequence:

$$0 \to \mathscr{F}_1 \to \mathscr{F}_Z \to \mathscr{F}_0 \to 0,$$

where $\mathscr{F}_1$ is $\mathscr{E}_{Z_1}^{U_1}$ (we set $L_i = \mathbb{P}(U_i)$) and $\mathscr{F}_0$ is $\mathscr{E}_{-3}^{L_0}$, which in this case is isomorphic to $\mathcal{I}_{M_0}(1)$, where M_0 is the line dual to L_0. Here $\mathscr{F}_1$ splits, in view of Corollary 3.4, as $\mathcal{I}_{M_1}(1) \oplus \mathcal{I}_{M_2}(1)$, where the M_i's are the lines dual to the N_i's. Then, it is straightforward to check that $\mathscr{F}_Z$ is strictly semistable, for the graded object associated to the above filtration of $\mathscr{F}_Z$ is $\mathcal{I}_{M_0}(1) \oplus \mathcal{I}_{M_1}(1) \oplus \mathcal{I}_{M_2}(1)$.

In coordinates, we could take L_0 as $\{z_2 = z_3 = 0\}$ and L_1 as $\{z_1 = 0\}$. Further, N_1 and N_2 can be taken as $\{z_0 - z_2 = z_1 = 0\}$ and $\{z_0 - z_3 = z_1 = 0\}$, so that $\zeta_3 = (1 : 0 : 1 : 1)$. The matrix M_Z in this case is:

$$M_Z = \begin{pmatrix} x_0 + x_1 & -x_1 & 0 & x_3 & 0 & x_2 \\ 0 & 0 & x_0 + x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 + x_3 & x_2 \end{pmatrix},$$

Example 3.6. With a little more work one can modify the above example so that $\mathscr{F}_Z$ is even stable. This can be achieved adding a point on L_0 and a further point on L_1, outside $N_1 \cup N_2$.

In coordinates, we can add $(1:2:0:0)$ and $(0:0:1:1)$. This gives rise (up to permutation) to the matrix M_Z:

$$\begin{pmatrix} x_0+x_1 & 0 & -x_1 & 0 & x_3 & 0 & x_2 & 0 \\ 0 & x_0+2x_1 & -2x_1 & 0 & x_3 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_0+x_2 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0+x_3 & x_2 & 0 \\ x_0+x_1 & 0 & -x_1 & 0 & 0 & 0 & 0 & x_2+x_3 \end{pmatrix}$$

Stability of $\mathscr{F}_Z$ can be deduced by the following resolutions:

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-3) \oplus \mathscr{O}_{\mathbb{P}^3}(-2) \to \mathscr{O}_{\mathbb{P}^3}(-2) \oplus \mathscr{O}_{\mathbb{P}^3}(-1)^4 \to \mathscr{F}_Z^{**}(-2) \to 0,$$
$$0 \to \mathscr{O}_{\mathbb{P}^3}(-4) \to \mathscr{O}_{\mathbb{P}^3}(-3) \oplus \mathscr{O}_{\mathbb{P}^3}(-1)^3 \to \mathscr{F}_Z^{*}(1) \to 0.$$

Example 3.7. Let L_1 and L_2 be two planes in $\mathbb{P}_4$, meeting at a single point y. Then y is the distinguished point of the KW variety $L_1 \cup L_2$. Let $Z_1 \subset L_1$ and $Z_2 \subset L_2$ be subschemes of length $\ell_1, \ell_2 < \infty$, both disjoint from y. Then $Z = Z_1 \cup Z_2$ cannot be Torelli, for y is always an unstable hyperplane of $\mathscr{F}_Z$.

If there is no conic through Z_1 and y nor through Z_2 and y, then y is the *only point of* $\mathbb{P}_4$ *outside* Z *giving an unstable hyperplane for* $\mathscr{F}_Z$. If Z_1 consists of 3 points such that $Z_1 \cup y$ is in general linear position, then for a general point z of L_1, there is a conic C through $z \cup y \cup Z_1$, and Z is contained in the KW variety $C \cup L_2$. Hence any point of C is unstable. So *all the points of* L_1 *give unstable hyperplanes* in this case.

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