RADIAL EXTENSION OF A BI-LIPSCHITZ PARAMETRIZATION OF A STARLIKE JORDAN CURVE

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ABSTRACT. In this paper we discus the radial extension w of a bi-Lipschitz parameterization $F(e^{it}) = f(t)$ of a starlike Jordan curve γ w.r. to origin. We show that, if parameterization is bi-Lipschitz, then the extension is bi-Lipschitz and consequently quasiconformal. If γ is the unit circle, then $\mathrm{Lip}(f) = \mathrm{Lip}(F) = \mathrm{Lip}(w) = K_w$. If γ is not a circle centered at origin, and F is a polar parametrization of γ, then we show that $\mathrm{Lip}(f) = \mathrm{Lip}(F) < \mathrm{Lip}(w)$.

1. INTRODUCTION

By $\mathbf{U}$ we denote the unit disk in the complex plane $\mathbf{C}$. Its boundary is the unit circle $\mathbf{T}$. Let D and Ω be subdomains of the complex plane $\mathbf{C}$, and $w = u + iv : D \to \Omega$ be a function that has both partial derivatives at a point $z \in D$. By $\nabla w(z)$ we denote the matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. For the matrix $\nabla w(z)$ we define

$$|\nabla w(z)| := \max_{|h|=1} |\nabla w(z)h| = |w_z| + |w_{\bar z}| \tag{1.1}$$

and

$$l(\nabla w)(z) := \min_{|h|=1} |\nabla w(z)h| = ||w_z| - |w_{\bar z}||, \tag{1.2}$$

where

$$w_z := \frac{1}{2}\left(w_x + \frac{1}{i}w_y\right) \text{ and } w_{\bar z} := \frac{1}{2}\left(w_x - \frac{1}{i}w_y\right).$$

A mapping $f : X \to Y$, between metric spaces (X, d_X) and (Y, d_Y) is said to be $\mathcal{L}-$*Lipschitz* and $(\ell, \mathcal{L})-$*bi-Lipschitz*, for some constants $0 < \ell \le \mathcal{L}$, if

$$d_Y(f(x), f(y)) \le \mathcal{L} d_X(x, y), \quad x, y \in X,$$

and

$$\ell d_X(x, y) \le d_Y(f(x), f(y)) \le \mathcal{L}\, d_X(x, y), \quad x, y \in X$$

respectively. We define

$$\mathrm{Lip}(f) := \sup_{x \ne y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

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1.1. **Quasiconformal mappings.** We say that a function $u : D \to \mathbf{R}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has of course, partial derivatives u_x, u_y a.e. in D.

A sense-preserving homeomorphism $w\colon D \to \Omega$, where D and Ω are subdomains of the complex plane $\mathbf{C}$, is said to be K-quasiconformal (K-q.c), $K \geq 1$, if w is ACL in D in the sense that the real and imaginary part are ACL in D, and

$$D_w(z) := \frac{|w_z| + |w_{\bar z}|}{|w_z| - |w_{\bar z}|} \leq K \quad \text{a.e. on } D, \tag{1.3}$$

(cf. [1], pp. 23–24). Notice that the condition (1.3) can be written as

$$|\mu_w(z)| \leq k \quad \text{a.e. on } D \text{ where } k = \frac{K-1}{K+1} \text{ i.e. } K = \frac{1+k}{1-k}\ ,$$

and $\mu_w(z) := \frac{w_{\bar z}}{w_z}$ is the complex dilatation of w. Sometimes instead of K quasiconformal we write k quasiconformal.

1.2. **Quasiconformal extension.** A homeomorphism $\tilde f : \mathbf{R} \to \mathbf{R}$ is called $M-$quasisymmetric if for all x and $t > 0$

$$\frac{\tilde f(x+t) - \tilde f(x)}{\tilde f(x) - \tilde f(x-t)} \leq M,$$

and $f(\infty) = \infty$. We easy can modify the previous definition for self - homeomorphisms of the unit circle. It is well known that, every quasisymmetric function has quasiconformal extension to the half-plane. We want to point out two most important extensions: *Beurling- Ahlfors extension* [2], and the *barycentric extension* of Douady and Earle [5] (see also [10]).

Let Ω be a starlike Jordan domain with respect to the origin. Let $\gamma = \partial\Omega$ and let $F : \mathbf{T} \to \gamma$ be a homeomorphism. The *radial extension* of a homeomorphism is defined by $w(re^{it}) = rF(e^{it})$ and it defines a homeomorphism of the unit disk onto Ω. Radial extension maps piecewise-linearly, but not smoothly.

Note this important and simple fact, if Ω is not starlike w.r. to 0, then the radial extension is not a mapping between $\mathbf{U}$ and Ω. One of primary aims of this paper is to describe all homeomorphisms, whose radial extensions are quasiconformal. We will show that the extension is quasiconformal if and only if it is bi-Lipschitz. It is well known that every bi-Lipschitz is quasiconformal. The converse is not true. However, if the mapping is quasiconformal, then it is Hölder continuous under some conditions on the boundaries (see [11] and [12]). For connection between these two concepts (bi-Lipschitz mappings and quasiconformal mappings) we also refer to the paper [3].

We say that a mapping $F : \mathbf{T} \to \gamma$ is *polar parametrization*, if $\arg F(e^{it}) = t$. Thus $F(e^{it}) = r(t)e^{it}$, for some positive continuous function r, such that $r(0) = r(2\pi)$.

For a given homeomorphism $F : \mathbf{T} \to \gamma$ define

- $f : [0, 2\pi] \to \gamma$, $f(t) = F(e^{it})$

- $w : \mathbf{U} \to \Omega$, $w(z) = |z|F(z/|z|)$.

Take $z = e^{it}, w = e^{is} \in \mathbf{T}$. *The spherical and the chordal distance* between points z and w are defined by

$$d_1(z, w) = |\arg z - \arg w| = |t - s| \quad \text{and} \quad d_2(z, w) = |z - w| = 2\left|\sin\frac{t-s}{2}\right|.$$

Notice that $d_1 \geq d_2$. For a given function F define the following four constants:

- $l := \mathrm{Lip}(f)$,
- $L := \mathrm{Lip}(F)$,
- $\Lambda := \mathrm{Lip}(w)$ and
- $\mathcal{K} = \operatorname{ess\,sup}_z D_w(z)$.

In this paper we will compare these constants.

We will show that, if $\gamma = \mathbf{T}$, then $l = L = \Lambda = \mathcal{K}$ (Theorem 2.2). The condition $\gamma = \mathbf{T}$ is essential, see Example 2.6. However, if F is a polar parametrization of a starlike Jordan curve w.r. 0, then we will show the following interesting fact $l = L$ (Theorem 3.1); in addition we will show that, for polar parametrizations of a curve that is not a circle centered at origin $L < \Lambda$ (Theorem 3.5). In the last section, we will show that, the radial extension is quasiconformal if and only if it is bi-Lipschitz (Theorem 4.1). Finally we provide two explicit examples.

2. Preliminary results

In this section we will derive some auxiliary results. Further we will consider the case $\gamma = \mathbf{T}$. Since

$$|z - w| \leq |\arg z - \arg w|, \quad \text{for} \quad z, w \in \mathbf{T}$$

and

$$\mathbf{T} \subset \overline{\mathbf{U}}$$

it follows that

$$l \leq L \leq \Lambda. \tag{2.1}$$

Recall the following fundamental result of Rademacher: ([6, Theorem 6.15]). Every Lipschitz function in an open of $\mathbf{R}^n$ is differentiable almost everywhere.

Lemma 2.1. *If $\varphi : \mathbf{R} \to \mathbf{R}$ is a $\mathcal{L}$ Lipschitz mapping, such that $\varphi(x + a) = \varphi(x) + b$ for some a and b and every x, then there exist a sequence of C^∞ $\mathcal{L}$ Lipschitz functions $\varphi_n : \mathbf{R} \to \mathbf{R}$ such that φ_n converges uniformly to φ, and $\varphi_n(x + a) = \varphi_n(x) + b$.*

Proof. This result is well-known. We refer to [7]. $\square$

By Rademacher theorem, Lemma 2.1 and Mean value theorem, for a Lipschitz mappings f and w, we have the following simple facts

$$\mathrm{Lip}(f) = \operatorname*{ess\,sup}_t |f'(t)|$$

and

$$\mathrm{Lip}(w) = \operatorname*{ess\,sup}_{|z|<1} |\nabla w(z)|.$$

If $z = re^{it}$, then

$$e^{2it} = \frac{z}{\bar{z}}.$$

Thus

$$2ie^{2it}t_z = \frac{1}{\bar{z}} \text{ and } 2ie^{2it}t_{\bar{z}} = -\frac{z}{\bar{z}^2}.$$

Therefore

$$t_z = \frac{1}{2iz} \text{ and } t_{\bar{z}} = -\frac{1}{2i\bar{z}}.$$

Moreover

$$r^2 = z\bar{z}.$$

Thus

$$r_z = \frac{\bar{z}}{2r}, \quad \text{and} \quad r_{\bar{z}} = \frac{z}{2r}.$$

On the other hand

$$w(z) = rf(t).$$

Thus

$$w_z = r_z f(t) + rf'(t)t_z, \quad \text{and} \quad w_{\bar{z}} = r_{\bar{z}} f(t) + rf'(t)t_{\bar{z}}.$$

Hence

$$w_z = \frac{\bar{z}}{2r} f(t) + f'(t)\frac{r}{2iz} \quad \text{and} \quad w_{\bar{z}} = \frac{z}{2r} f(t) - f'(t)\frac{r}{2i\bar{z}}.$$

Thus

$$|w_z| = \frac{1}{2}|f(t) - if'(t)| \text{ and } |w_{\bar{z}}| = \frac{1}{2}|f(t) + if'(t)| \tag{2.2}$$

and

$$|\nabla w(z)| = \frac{1}{2}\left(|f(t) - if'(t)| + |f(t) + if'(t)|\right), \tag{2.3}$$

which implies that

$$\mathrm{Lip}(w) = \operatorname*{ess\,sup}_{t} \frac{1}{2}\left(|f(t) - if'(t)| + |f(t) + if'(t)|\right). \tag{2.4}$$

Now we have the following theorem:

Theorem 2.2. *If $F : \mathbf{T} \to \mathbf{T}$ is a Lipschitz surjective mapping, then*

$$l = L = \Lambda = \mathcal{K} = \mathrm{Lip}(\psi) = |\psi'|_\infty. \tag{2.5}$$

Here and in the sequel by $|g|_\infty$ we mean the L^∞ norm of g.

Proof. If f is a mapping of the unit circle onto itself, then

$$f(t) = e^{i\psi(t)}$$

for some increasing bijective function $\psi : [0, 2\pi] \to [0, 2\pi]$. Moreover,

$$f(t) - if'(t) = e^{i\psi(t)}(1 + \psi'(t)) \quad \text{and} \quad f(t) - if'(t) = e^{i\psi(t)}(1 - \psi'(t)).$$

Thus

$$|\nabla w(z)| = \max\{1, \psi'(t)\},$$

and

$$\mathrm{Lip}(w) = |\psi'|_\infty. \tag{2.6}$$

From (2.6) and (2.1) it follows the theorem. □

From the previous theorem we infer the following corollaries:

Corollary 2.3. [9] *Let $F : \mathbf{T} \to \mathbf{T}$. Then f is $L-$Lipschitz continuous with respect to spherical distance if and only if f is $L-$Lipschitz continuous with respect to chordal distance.*

Corollary 2.4. *Let $w(z) = |z|F(z/|z|) : B^2 \to B^2$, where $F : \mathbf{T} \to \mathbf{T}$. Then w is $L-$Lipschitz if and only if F is $L-$Lipschitz.*

Remark 2.5. The question arises, can we replace the unit circle by some other starlike Jordan curve γ in the previous statements. The following example shows that, in general we do not have that $l = L$.

Example 2.6. Let $F(e^{it}) = (-1+\min\{2\pi - t, t\}, 1/10\sin t)$. And $f(t) = F(e^{it})$. Then $\mathrm{Lip}(f) = \frac{\sqrt{101}}{10} < \mathrm{Lip}(F) = \frac{\pi}{2}$.

3. Polar parametrization

Let γ be a starlike Jordan curve w.r. to the origin. Let $f(t) = r(t)e^{it}$ be the polar parametrization of γ. In this section we will prove the following intrigue results.

- For all polar parametrization of starlike curves holds $l = L$ (Theorem 3.1).
- For a polar parametrization f we have $L = \Lambda$ if and only if $\gamma = s\mathbf{T}$ for some $s > 0$ (Theorem 3.5).

As we said before,

$$l = \sup_{t \neq s} \frac{|f(s) - f(t)|}{|s - t|} \le L = \sup_{e^{it} \neq e^{is}} \frac{|f(s) - f(t)|}{|e^{is} - e^{it}|}. \tag{3.1}$$

As $|s - t| > |e^{it} - e^{is}|$ for all $e^{it} \neq e^{is}$, one expect that, for some (or all) polar parametrizations f we should have $l < L$. However we have

Theorem 3.1 (The main result)**.** *Let $r : [0, 2\pi] \to \mathbf{R}_+$ be a continuous positive function with $r(0) = r(2\pi)$. Define $f(t) = r(t)e^{it}$. Then*

$$\sup_{t \neq s} \frac{|f(t) - f(s)|}{|e^{it} - e^{is}|} = \sup_s \limsup_{t \to s} \frac{|f(t) - f(s)|}{|e^{it} - e^{is}|}, \tag{3.2}$$

and consequently $l = L$.

Remark 3.2. The condition that r is a continuous positive function means that the Jordan curve $\gamma = \{f(t) : 0 \le t \le 2\pi\}$ is starlike. On the other hand only starlike curves w.r. to the origin have this representation.

We need the following simple lemma:

Lemma 3.3. *For $p, q > 0$ and $t \in [0, 2\pi]$ we have*

$$\lim_{\varepsilon\to 0+0} 2\frac{1 - \frac{(1-\varepsilon)p+\varepsilon q\cos t}{|(1-\varepsilon)p+\varepsilon q e^{it}|}}{\varepsilon^2} = \frac{q^2\sin^2 t}{p^2}.$$

Proof of Theorem 3.1. Denote by f the periodic extension of f in $\mathbf{R}$. We will show that, for $e^{it} \neq e^{is}$

$$\frac{|f(t) - f(s)|}{|e^{it} - e^{is}|} \le \sup_s \limsup_{t\to s} \frac{|f(t) - f(s)|}{|e^{it} - e^{is}|}.$$

Let $0 \le s < t < 2\pi$. Then either $t - s \le \pi$ or $2\pi - (s - t) \le \pi$. Put $\tilde{s} = 0$, $\tilde{t} = t - s$, in the first case, and $\tilde{s} = 0$, $\tilde{t} = 2\pi - (s - t)$ in the second case, and $\tilde{f}(x) = f(x + s)$. Then

$$\frac{|\tilde{f}(\tilde{t}) - \tilde{f}(\tilde{s})|}{|e^{i\tilde{t}} - e^{i\tilde{s}}|} = \frac{|f(t) - f(s)|}{|e^{it} - e^{is}|}.$$

For simplicity, denote $\tilde{s}$ by s, $\tilde{t}$ by t and $\tilde{f}$ by f.

Let Δ be a triangle with vertexes $O = 0$, $P = p = f(s) = r(s) > 0$ and $Q = qe^{it} = f(t)$ (see Figure 1). Assume without loos of generality that

$$p \ge q. \tag{3.3}$$

Let $0 < \varepsilon < 1$ and Λ be a segment with endpoints 0 and $L \in [P, Q]$, such that $|LP| = \varepsilon|PQ|$. Let λ be the length of Λ. Then

$$\lambda = |(1 - \varepsilon)p + \varepsilon e^{it} q|.$$

Moreover the angle t_ε between Λ and OP is given by

$$\cos t_\varepsilon = \frac{\mathrm{Re}\{p[(1 - \varepsilon)p + \varepsilon e^{it} q]\}}{p|(1 - \varepsilon)p + \varepsilon e^{it} q|}.$$

Let x and y be the lengths of chords of the unit circle that correspond to the angles t and t_ε, respectively. Then

$$x^2 = 2 - 2\cos t$$

and

$$y^2 = 2 - 2\cos t_\varepsilon.$$

Thus

$$\frac{y^2}{\varepsilon^2} = 2\frac{1 - \cos t_\varepsilon}{\varepsilon^2} = 2\frac{1 - \frac{(1-\varepsilon)p+\varepsilon q\cos t}{|(1-\varepsilon)p+\varepsilon q e^{it}|}}{\varepsilon^2}. \tag{3.4}$$

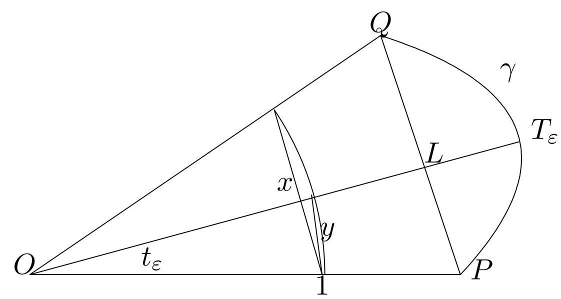

Figure 1

Let $\alpha = \angle P$ and $\beta = \angle Q$. As $p \geq q$, then $\beta \geq \alpha$. Thus $\alpha \leq (\pi - t)/2$. Let T_ε be the point of $\gamma = f[0, \pi]$ that belong to the half-line $[0, L)$. Since $\alpha \leq (\pi - t)/2$, we have the following simple geometric fact

$$\overline{\lim}_{\varepsilon\to 0}\frac{|T_\varepsilon P|}{|LP|} \geq \cos\frac{t}{2}. \tag{3.5}$$

On the other hand

$$|LP| = \varepsilon|f(t) - f(s)|. \tag{3.6}$$

From (3.2), (3.3), (3.4), (3.5) and (3.6) it follows that

$$\begin{aligned}
\overline{\lim}_{\varepsilon\to 0}\frac{|f(t_\varepsilon) - f(s)|^2}{|e^{it_\varepsilon} - e^{is}|^2} &= \overline{\lim}_{\varepsilon\to 0}\frac{|T_\varepsilon P|^2}{y^2} \\
&\geq \lim_{\varepsilon\to 0}\frac{|LP|^2\cos^2\frac{t}{2}}{y^2} \\
&= \lim_{\varepsilon\to 0}\frac{\varepsilon^2|f(t) - f(s)|^2\cos^2\frac{t}{2}}{y^2} \\
&= \frac{|f(t) - f(s)|^2\cos^2\frac{t}{2}}{\frac{q^2\sin^2 t}{p^2}} \\
&= \frac{|f(t) - f(s)|^2}{|e^{it} - e^{is}|^2}\frac{p^2}{q^2} \\
&\geq \frac{|f(t) - f(s)|^2}{|e^{it} - e^{is}|^2}.
\end{aligned}$$

Having in mind the fact that

$$\lim_{\varepsilon\to 0} t_\varepsilon = 0 = s,$$

it follows the desired conclusion. To show that $l = L$ we only need to point out that

$$\lim_{s\to t}\frac{|e^{it} - e^{is}|}{|t - s|} = \lim_{x\to 0}\frac{\sin x}{x} = 1.$$

□

Corollary 3.4. *Together with the assumptions of Theorem 3.1 assume that r is a smooth function. Then*

$$\sup_{e^{it}\neq e^{is}}\frac{|f(s) - f(t)|}{|e^{is} - e^{it}|} = \max_t\sqrt{r^2(t) + r'^2(t)}.$$

By Theorem 3.1 we obtain the following theorem:

Theorem 3.5. *Let γ be a Lipschitz starlike curve w.r. the the origin, parameterized by polar coordinates $F(e^{it}) = r(t)e^{it} : \mathbf{T} \to \gamma$. Let*

$$w(z) = |z|F\left(\frac{z}{|z|}\right) : \mathbf{U} \to \Omega$$

be its extension between the unit disk $\mathbf{U}$ *and the Jordan domain* $\Omega = \mathrm{int}(\gamma)$. *If* $\mathrm{Lip}(F) = \mathrm{Lip}(w)$, *then* γ *is a circle with the center at origin.*

Proof. Assume first that r is a smooth Lipschitz function. Let $f(t) = F(e^{it})$. Proceeding as in (2.3) we have

$$|\nabla w| = \frac{1}{2}\left(|r'(t)| + |2r(t) - ir'(t)|\right)$$

and

$$f'(t) = ie^{it}r(t) + e^{it}r'(t).$$

So

$$\left|f'(t)\right| \le |\nabla w|.$$

Whence

$$\left|f'(t)\right| = |\nabla w|$$

if and only if

$$|r(t) - ir'(t)| = \frac{1}{2}\left(|r'(t)| + |2r(t) - ir'(t)|\right). \tag{3.7}$$

Let

$$\chi(t) = \frac{r'(t)}{r(t)}.$$

Then (3.7) is equivalent to

$$|2 - 2i\chi(t)| = |\chi(t)| + |2 - i\chi(t)|. \tag{3.8}$$

By Theorem 3.1, $\mathrm{Lip}(f) = \mathrm{Lip}(F)$. It follows from (3.8) that, if $\mathrm{Lip}(f) = \mathrm{Lip}(w)$, then

$$\max_{0\le t\le 2\pi} |2 - 2i\chi(t)| = \max_{0\le t\le 2\pi} (|\chi(t)| + |2 - i\chi(t)|).$$

Therefore there exists $a \in [0, 2\pi]$ such that

$$|2-2i\chi(a)| = \max_{0\le t\le 2\pi} |2-2i\chi(t)| = \max_{0\le t\le 2\pi} (|\chi(t)|+|2-i\chi(t)|) = |\chi(a)|+|2-i\chi(a)|.$$

From

$$|2 - 2i\chi(a)| = |\chi(a)| + |2 - i\chi(a)|$$

we obtain that $\chi(a) = 0$. Thus

$$\max_{0\le t\le 2\pi} (|\chi(t)| + |2 - i\chi(t)|) = 2.$$

This implies that $\chi(t) \equiv 0$ and therefore

$$w(z) = az$$

for some complex constant a. The general case follows from Lemma 2.1. $\square$

4. A GENERAL PARAMETRIZATION

Let $\gamma = \partial\Omega$ be a smooth starlike Jordan curve w.r. to the origin in $\mathbf{C}$. We will recall some properties of γ. Let $s \to r(s)e^{is}$ be the polar parametrization of γ. The tangent t_s of γ at $\zeta = r(s)e^{is}$ is defined by

$$y = r(s)e^{is} + (r'(s) + ir(s))e^{is}(x - r(s)e^{is}).$$

Following the notations in [8], the angle α_s between ζ and the positive oriented tangent at ζ is defined by

$$\cos\alpha_s = \frac{\mathrm{Re}(r(s)e^{-is} \cdot (r'(s) + ir(s))e^{is})}{r(s)\sqrt{r^2(s) + r'^2(s)}} = \frac{r'(s)}{\sqrt{r^2(s) + r'^2(s)}}. \tag{4.1}$$

Hence

$$\sin\alpha_s = \frac{r(s)}{\sqrt{r^2(s) + r'^2(s)}}.$$

Consequently

$$\cot\alpha_s = \frac{r'(s)}{r(s)}. \tag{4.2}$$

Observe that for smooth starlike Jordan curve γ, we have

$$0 < \alpha_1 = \min_t \alpha_t \le \max_t \alpha_t = \alpha_2 < \pi.$$

Put

$$\alpha_\gamma = \min\{\alpha_1, \pi - \alpha_2\}. \tag{4.3}$$

Let $G : \mathbf{T} \to \gamma$ be a continuous locally injective function from the unit circle $\mathbf{T}$ onto the star-like Jordan curve γ. Then

$$g(t) = \rho(t)e^{i\psi(t)} = G(e^{it}),\ t \in [0, 2\pi)$$

is a parametrization of γ which represents g. If g is a orientation preserving then ψ obviously is monotone increasing. Suppose that g is differentiable. Since $r(\psi(t)) = \rho(t)$, we deduce that $\rho'(t) = r'(\psi(t)) \cdot \psi'(t)$. Hence

$$r'(\psi(t)) = \frac{\rho'(t)}{\psi'(t)}. \tag{4.4}$$

From (4.4) and (4.2) we obtain

$$\rho'(t) = \rho(t)\psi'(t)\cot\alpha_t. \tag{4.5}$$

Theorem 4.1. *Let γ be a smooth starlike Jordan curve with respect to origin parameterized by a homeomorphism $G(e^{it}) = g(t) = \rho(t)e^{i\psi(t)} : \mathbf{T} \to \gamma$. Let*

$$w(z) = |z|G\left(\frac{z}{|z|}\right) : \mathbf{U} \to \Omega.$$

Then the following conditions are equivalent

a) *G is bi-Lipschitz,*
b) *ψ is bi-Lipschitz,*

c) *w is bi-Lipschitz,*
d) *w is quasiconformal.*

Moreover if $L = \max\{|\psi'|_\infty, |1/\psi'|_\infty\}$. Then:
(i) *w is $(\ell, \mathcal{L})$ bi-Lipschitz continuous, where*

$$\mathcal{L} = \frac{|\rho|_\infty}{2\sin\alpha_\gamma}\left(\sqrt{L^2+\sin^2\alpha_\gamma(1-2L)}+\sqrt{L^2+\sin^2\alpha_\gamma(1+2L)}\right)$$

and

$$\ell = \frac{\mathrm{dist}^2(\gamma, 0)}{L\mathcal{L}}.$$

(ii) *w is $k-$quasiconformal, where*

$$k = \sqrt{\frac{L^2+\sin^2\alpha_\gamma(1-2L)}{L^2+\sin^2\alpha_\gamma(1+2L)}}.$$

On the other hand if w is $k-$quasiconformal, then
(i') *ψ is $(\frac{1-k}{1+k}, \frac{1+k}{1-k})$ bi-Lipschitz*
and
(ii') $\sin\alpha_\gamma \geq \frac{1-k}{1+k}$*, where α_γ is defined in* (4.3).

Proof. From (4.5) we obtain

$$|g'+ig| \pm |g'-ig| = \rho\left(\sqrt{\frac{\psi'^2}{\sin^2\alpha_t}+1+2\psi'} \pm \sqrt{\frac{\psi'^2}{\sin^2\alpha_t}+1-2\psi'}\right),$$

and consequently

$$(|g'+ig|+|g'-ig|)(|g'+ig|-|g'-ig|) = 4\rho^2\psi'.$$

By making use of (2.2), we obtain (i). Further by (2.2) we have

$$\kappa(t) := |\mu_w(re^{it})|^2 = \frac{(1-\psi')^2\rho^2+\rho'^2(t)}{(1+\psi')^2\rho^2+\rho'^2(t)}.$$

Since

$$\rho'(t) = \rho(t)\psi'(t)\cot\alpha_t$$

is follows that

$$\kappa(t) = \frac{\psi'^2+(1-2\psi')\sin^2\alpha_t}{\psi'^2+(1+2\psi')\sin^2\alpha_t}. \tag{4.6}$$

Furthermore, since $\psi[0,2\pi] = [0,2\pi]$, it follows that $L \geq 1$.

Let $0 \leq t_1, t_2 \leq 2\pi$. For $\frac{1}{L} \leq \psi'(t_1) \leq 1$ and $1 \leq \psi'(t_2) \leq L$, we have

$$\kappa(t_1) \leq \frac{1+(L^2-2L)\sin^2\alpha_t}{1+(L^2+2L)\sin^2\alpha_t} := \kappa_1$$

and

$$\kappa(t_2) \leq \frac{L^2+(1-2L)\sin^2\alpha_t}{L^2+(1+2L)\sin^2\alpha_t} := \kappa_2,$$

respectively. On the other hand

$$\kappa_2 - \kappa_1 = \frac{4(1-\sin^2\alpha_t)L(L^2-1)}{\left(\frac{L^2}{\sin^2\alpha_t}+1+2L\right)\left(\frac{1}{\sin^2\alpha_t}+L^2+2L\right)}$$

i.e. $\kappa_2 \geq \kappa_1$. Thus f is $k-$quasiconformal where

$$k = \sqrt{\frac{L^2+\sin^2\alpha_\gamma(1-2L)}{L^2+\sin^2\alpha_\gamma(1+2L)}}.$$

This concludes (ii).

Moreover the previous proof shows that $a) \Leftrightarrow b) \Leftrightarrow c) \Rightarrow d)$. To show $d) \Rightarrow c)$, and the last assertion of the theorem we do as follows. If f is $k-$quasiconformal, then f is differentiable almost everywhere. Assume that $\psi'(t) \geq 1$. Then from (4.6), we obtain that

$$\frac{\psi'(t)^2+(1-2\psi'(t))}{\psi'(t)^2+(1+2\psi'(t))} \leq \kappa(t) \leq k^2.$$

Thus

$$\psi'(t) \leq \frac{1+k}{1-k}.$$

By using (4.6) again, we obtain that

$$\sin\alpha_t \geq \frac{1-k}{1+k}. \tag{4.7}$$

Then we infer that

$$\psi' \geq \frac{\sin\alpha_t((1+k)\sin\alpha_t - \sqrt{-(1-k)^2+(1+k)^2\sin^2\alpha_t})}{1-k}.$$

Since the right hand side of the last inequality is increasing in $\alpha_t \in [\arcsin\frac{1-k}{1+k}, \frac{\pi}{2}]$, by making use of (4.7) it follows that

$$\psi' \geq \frac{1-k}{1+k}.$$

This finishes the proof of (i') and (ii'). $\square$

A question. The question arises, which homeomorphism of the unit circle onto a smooth starlike Jordan curve γ induces a radial quasiconformal mapping with the smallest constant of quasiconformality. It follows from Theorem 4.1 that, if there exists a K-quasiconformal radial mapping between the unit disk and a smooth starlike domain, then $K \geq \csc\alpha_\gamma$. To motivate the previous question, recall the Teichmüller problem. For a given M-quasisymmetric selfmapping of the unit circle or (equivalently) of the real line, find an extension with minimal constant of quasiconformality. This problem is related to unique extremality. For this topic we refer to the paper [4].

Remark 4.2. If γ is the unit circle, then $\alpha_\gamma = \frac{\pi}{2}$, and consequently the Theorem 4.1 contains Theorem 2.2.

Example 4.3. Let $0 < b \le a$ and let

$$w = r\phi(t) = r\varphi(e^{it}) = \left(\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2}\right)^{-1/2} re^{it}$$

be a radial mapping of the unit disk onto the interior of the ellipse

$$E(a,b) = \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}.$$

Then for $a^2 \ge 2b^2$ we have

$$\mathrm{Lip}(\phi) = \mathrm{Lip}(\varphi) = \frac{4(a^2+b^2)^3}{27a^2b^2}.$$

If $a^2 \le 2b^2$ we have

$$\mathrm{Lip}(\phi) = \mathrm{Lip}(\varphi) = a.$$

On the other hand by Theorem 3.5

$$\mathrm{Lip}(\varphi) < \mathrm{Lip}(w).$$

Moreover w is a K quasiconformal mapping where

$$K = \frac{a^4 + 6a^2b^2 + b^4 + \sqrt{14a^2 + a^4 + b^4}|a^2 - b^2|}{8a^2b^2}. \tag{4.8}$$

To show (4.8), we begin by

$$\frac{\|\nabla w\|^2}{J_w} = \frac{7 + 2a^2 + 7a^4 - 8(-1+a^4)\cos(2t) + (-1+a^2)^2\cos(4t)}{8(\cos^2 t + a^2 \sin^2 t)^2}.$$

The minimum is 2 and is achieved for $t = 0$, $t = \pi$, $t = \pi/2$ or $t = -\pi/2$ and the maximum for

$$t = \pm\text{arccos}(\pm\frac{c}{\sqrt{1+c^2}}),$$

where $c = \frac{a}{b}$. The maximum is equal to

$$\frac{1+6c^2+c^4}{4c^2}.$$

The larger solution of the equation

$$\left(K + \frac{1}{K}\right) = \frac{1+6c^2+c^4}{4c^2}$$

is given by (4.8).

Example 4.4. Let

$$f(s) = F(e^{it}) = \min\left\{\frac{1}{|\sin s|}, \frac{1}{|\cos s|}\right\} e^{is}.$$

Then f is a polar parametrization of the unit square $Q = \{(x,y) : \max\{|x|,|y|\} = 1\}$. Moreover, since $\alpha_\gamma = \frac{\pi}{4}$, and $L = 1$, by Theorem 4.1, we have

$$\mathrm{Lip}(f) = \mathrm{Lip}(F) = 2 < \mathrm{Lip}(w) = \frac{1}{2}(\sqrt{2} + \sqrt{10}) \approx 2.28825.$$

The constant of quasiconformality of w is

$$\mathcal{K} = \frac{1}{2}(3+\sqrt{5}) \approx 2.61803.$$

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