

Central matricvariate and matrix multivariate T distributions

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Abstract

Several distributions are studied, simultaneously in the real, complex, quaternion and octonion cases. Specifically, these are the central, nonsingular matricvariate and matrix multivariate T and beta type II distributions and the joint density of the singular values are obtained for real normed division algebras.

1 Introduction

The complex case has renewed interest in multivariate analysis in diverse areas of science and technology, see Mehta [29], Ratnarajah *et al.* [33] and Micheas *et al.* [30], among many others. Moreover, diverse works involving multivariate analysis have appeared in the context of the quaternion case, see Bhavsar [2], Forrester [17], Li and Xue [28], among others. However, with respect to the octonion case, only a few, theoretical results have been published, see Forrester [17] and Khatri [25]. This lack of widespread interest may be, because as stated by Baez [1], *...there is still no proof that the octonions are useful for understanding the real world.* Nevertheless, for the sake of completeness, we include results in the octonion case as conjectures.

Using definitions, properties and notation from abstract algebra, we propose a unified approach that enables the simultaneous study of the distribution of a random matrix in the real, complex, quaternion and octonion cases, which is generically termed the, distribution of a random matrix for real normed division algebras.

In particular, the matricvariate T distribution has been studied by many authors in the real case, see Dickey [12], Box and Tiao [3], Press [32], Kotz and Nadarajah [27] and Díaz-García and Gutiérrez-Jáimez [8], among many others. The matricvariate T distribution appears in the frequentist approach to normal regression as the distribution of the Studentised error, see Díaz-García and Gutiérrez-Jáimez [4] and Kotz and Nadarajah [27]. In

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Bayesian conjugate-prior and diffuse-prior analysis for the same sampling models, it appears as the marginal prior or posterior distribution of the unknown coefficients matrix, and also as the predictive distribution of a future data array, see Dickey [12], Box and Tiao [3], Press [32], Díaz-García and Ramos-Quiroga [11], Fang and Li [15] and Kotz and Nadarajah [27]. It has been applied in microeconomic modeling to describe the operation of a market for a particular economic commodity and in macroeconomic modeling to describe the interrelations between a large number of macroeconomic variables, as an application of the linear simultaneous equation model, see Kotz and Nadarajah [27]. In the complex case, the matrix-variate T distribution has been applied in Bayesian estimation of a multivariate regression model, see Kotz and Nadarajah [27]. No less important is the role of the T distribution, because if the matrix \mathbf{T} has a T distribution, then the matrix $\mathbf{T}\mathbf{T}^*$ (or $\mathbf{T}^*\mathbf{T}$) is distributed as beta type II; and the distribution of the latter, in particular, plays a fundamental role in the MANOVA model, see Srivastava & Khatri [34], Press [32], and Muirhead [31].

In this work, the nonsingular central matrix-variate T and the beta type II distributions and some of their basic properties are studied, see Section 2. The matrix multivariate T distribution and its corresponding beta type II distribution is studied in Section 3. Finally, the joint densities of the singular values are derived in Section 4. We emphasize that all these results are derived for real normed division algebras. Some concepts and the notation of abstract algebra and Jacobians are summarised as an Appendix.

2 Matrix-variate T distribution

We begin this section by distinguishing between *matrix-variate* and *matrix multivariate* (or *matrix variate*) distributions. We say that the random matrix \mathbf{X} has a *matrix-variate distribution* if the kernel $g(\cdot)$ of its density is written as a function solely in terms of the determinant operator $g(|\mathbf{X}|)$. In any other case it is said that \mathbf{X} has a *matrix multivariate distribution*. The term matrix-variate distribution was introduced by Dickey [12], but the expression matrix-variate distribution (or matrix variate distribution) was later used to describe any distribution of a random matrix, see Gupta, and Varga [21], Gupta and Nagar [20], and references therein. Alternatively, the term matrix multivariate (instead of matrix variate) has been used by Goodall and Mardia [18], and this is the approach adopted in our paper.

Theorem 2.1. *Let $\mathbf{T} \in \mathcal{L}_{m,n}^\beta$ defined as*

$$\mathbf{T} = \mathbf{L}^{-1}\mathbf{Y} + \boldsymbol{\mu}$$

where \mathbf{L} is any square root of \mathbf{V} such that $\mathbf{L}\mathbf{L}^* = \mathbf{V} \sim \mathcal{W}_m^\beta(\nu, \boldsymbol{\Xi})$, $\boldsymbol{\Xi} \in \mathfrak{P}_m^\beta$ and $\nu > \beta(m-1)$; independent of $\mathbf{Y} \sim \mathcal{N}_{m \times n}^\beta(\mathbf{0}, \mathbf{I}_m \otimes \boldsymbol{\Sigma})$, $\boldsymbol{\Sigma} \in \mathfrak{P}_n^\beta$. Then the density of \mathbf{T} is

$$\frac{\Gamma_m^\beta[\beta(n+\nu)/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[\beta\nu/2]|\boldsymbol{\Xi}|^{\beta\nu/2}|\boldsymbol{\Sigma}|^{\beta m/2}}|\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{T} - \boldsymbol{\mu})^*|^{-\beta(n+\nu)/2}, \quad (1)$$

which is termed the matrix-variate T distribution¹ and is denoted as

$$\mathbf{T} \sim \mathcal{T}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \boldsymbol{\Xi}, \boldsymbol{\Sigma}).$$

Proof. From Kabe [24] and Díaz-García and Gutiérrez-Jáimez [6, 10], the joint density of \mathbf{V} and \mathbf{Y} is

$$\propto |\mathbf{V}|^{\beta(\nu-m+1)/2-1} \text{etr}\{-\beta(\boldsymbol{\Xi}^{-1}\mathbf{V} + \boldsymbol{\Sigma}^{-1}\mathbf{Y}^*\mathbf{Y})/2\},$$

¹In the literature, it is customary to use the expressions real matrix-variate T distribution, complex matrix-variate T distribution, quaternion matrix-variate T distribution and octonion matrix-variate T distribution; here, however, we use simply matrix-variate T distribution as the generic term.

where the constant of proportionality is

$$c = \frac{1}{(2\beta^{-1})^{\beta m \nu/2} \Gamma_m^\beta[\beta \nu/2] |\Xi|^{\beta \nu/2}} \cdot \frac{1}{(2\pi\beta^{-1})^{\beta m n/2} |\Sigma|^{\beta m/2}}.$$

Making the change of variable $\mathbf{Y} = \mathbf{L}\mathbf{T}$, where $\mathbf{V} = \mathbf{L}\mathbf{L}^*$, then by (A-3)

$$(d\mathbf{V})(d\mathbf{Y}) = |\mathbf{L}\mathbf{L}^*|^{\beta n/2} (d\mathbf{V})(d\mathbf{T}) = |\mathbf{V}|^{\beta n/2} (d\mathbf{V})(d\mathbf{T}).$$

Thus, the joint density of \mathbf{V} and \mathbf{T} is

$$\propto |\mathbf{V}|^{\beta(\nu+n-m+1)/2-1} \text{etr}\{-\beta(\Xi^{-1} + (\mathbf{T} - \boldsymbol{\nu})\Sigma^{-1}(\mathbf{T} - \boldsymbol{\nu})^*)/2\}.$$

Finally, integrating over $\mathbf{V} \in \mathfrak{P}_m^\beta$, we have

$$(2\beta^{-1})^{\beta m(n+\nu)/2} \Gamma[\beta(n+\nu)/2] |\Xi^{-1} + (\mathbf{T} - \boldsymbol{\nu})\Sigma^{-1}(\mathbf{T} - \boldsymbol{\nu})^*|^{-\beta(n+\nu)/2},$$

from which the desired result is obtained. \square \square

Now, observe that by Dickey [12]

$$\frac{\Gamma_m^\beta[\beta(n+\nu)/2]}{\pi^{mn\beta/2} \Gamma_m^\beta[\beta\nu/2]} = \frac{\Gamma_n^\beta[\beta(n+\nu)/2]}{\pi^{mn\beta/2} \Gamma_n^\beta[\beta(n+\nu-m)/2]}$$

and

$$|\Xi^{-1} + (\mathbf{T} - \boldsymbol{\nu})\Sigma^{-1}(\mathbf{T} - \boldsymbol{\nu})^*| = |\Xi|^{-1} |\Sigma|^{-1} |\Sigma + (\mathbf{T} - \boldsymbol{\nu})^* \Xi (\mathbf{T} - \boldsymbol{\nu})|,$$

from which, alternatively, the density (1) can be expressed as

$$\frac{\Gamma_n^\beta[\beta(n+\nu)/2] |\Xi|^{\beta n/2} |\Sigma|^{\beta(n+\nu-m)/2}}{\pi^{mn\beta/2} \Gamma_n^\beta[\beta(n+\nu-m)/2]} |\Sigma + (\mathbf{T} - \boldsymbol{\nu})^* \Xi (\mathbf{T} - \boldsymbol{\nu})|^{-\beta(n+\nu)/2}, \quad (2)$$

Corollary 2.1. Let $\mathbf{T} \in \mathcal{L}_{m,n}^\beta$ defined as

$$\mathbf{T} = \mathbf{X}\mathbf{L}_1^{-1} + \boldsymbol{\mu}$$

where \mathbf{L}_1 is any square root of \mathbf{U} such that $\mathbf{L}_1\mathbf{L}_1^* = \mathbf{U} \sim \mathcal{W}_n^\beta(\nu+n-m, \Sigma^{-1})$, $\Sigma \in \mathfrak{P}_n^\beta$, independent of $\mathbf{X} \sim \mathcal{N}_{m \times n}^\beta(\mathbf{0}, \Xi^{-1} \otimes \mathbf{I}_n)$, with $\Xi \in \mathfrak{P}_m^\beta$. Then, $\mathbf{T} \sim \mathcal{T}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \Xi, \Sigma)$.

Proof. The proof is a verbatim copy of the proof of Theorem 2.1. \square \square

Now, assume that $\mathbf{T} \sim \mathcal{T}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ with $n \geq m$ and let $\mathbf{F} \in \mathfrak{P}_m^\beta$ defined as $\mathbf{F} = \mathbf{T}\mathbf{T}^*$ then, under the conditions of Theorem 2.1 and Corollary 2.1, we have

$$\begin{aligned} \mathbf{F} &= \mathbf{L}^{-1}\mathbf{Y}\mathbf{Y}^*(\mathbf{L}^{-1})^* = \mathbf{L}^{-1}\mathbf{S}(\mathbf{L}^{-1})^* \\ &= \mathbf{X}\mathbf{U}^{-1}\mathbf{X}^*, \end{aligned}$$

with $\mathbf{S} = \mathbf{Y}\mathbf{Y}^* \sim \mathcal{W}_m^\beta(n, \mathbf{I}_m)$, $n > \beta(m-1)$. Thus:

Theorem 2.2. The density of \mathbf{F} is

$$\frac{1}{\mathcal{B}_m^\beta[\beta\nu/2, \beta n/2]} |\mathbf{F}|^{\beta(n-m+1)/2-1} |\mathbf{I}_m + \mathbf{F}|^{-\beta(n+\nu)/2}, \quad (3)$$

where $\mathcal{B}_m^\beta[\cdot, \cdot]$ is given by (A-2) and \mathbf{F} is said to have a matricvariate beta type II distribution.

Proof. The proof follows from (1) by applying (A-1) and (A-6). \square \square

In addition, assume that $n < m$ and let $\tilde{\mathbf{F}} \in \mathfrak{P}_n^\beta$ defined as $\tilde{\mathbf{F}} = \mathbf{T}^* \mathbf{T}$ then, under the conditions of Theorem 2.1 and Corollary 2.1 we have

$$\begin{aligned}\tilde{\mathbf{F}} &= \mathbf{X}^* \mathbf{U}^{-1} \mathbf{X} \\ &= \mathbf{L}_1^{-1} \mathbf{X}^* \mathbf{X} (\mathbf{L}_1^{-1})^* = \mathbf{L}_1^{-1} \mathbf{S}_1 (\mathbf{L}_1^{-1})^*\end{aligned}$$

with $\mathbf{S}_1 = \mathbf{X}^* \mathbf{X} \sim \mathcal{W}_n^\beta(m, \mathbf{I}_n)$, $m > \beta(n-1)$, Thus:

Theorem 2.3. $\tilde{\mathbf{F}}$ has the density

$$\frac{1}{\mathcal{B}_n^\beta[\beta(\nu+n-m)/2, \beta m/2]} |\tilde{\mathbf{F}}|^{\beta(m-n+1)/2-1} |\mathbf{I}_n + \tilde{\mathbf{F}}|^{-\beta(n+\nu)/2-1}. \quad (4)$$

Furthermore, we say that $\tilde{\mathbf{F}}$ has a matricvariate beta type II distribution.

Proof. The proof is the same as that given in Theorem 2.2. Alternatively, observe that density (4) can be obtained from density (3) making the following substitutions, see Muirhead [31, Eq. (7), p. 455] and Srivastava & Khatri [34, p. 96],

$$m \rightarrow n, \quad n \rightarrow m, \quad \nu \rightarrow \nu + n - m. \quad \square \quad (5)$$

\square

Finally, assume that $\mathbf{M} \in \mathcal{L}_{m \times m}^\beta$ is any square root of constant matrix $\mathbf{\Delta} = \mathbf{M} \mathbf{M}^* \in \mathfrak{P}_m^\beta$. Also, define $\mathbf{Z} = \mathbf{M}^* \mathbf{F} \mathbf{M}$, therefore:

Corollary 2.2. The density of \mathbf{Z} is

$$\frac{|\mathbf{\Delta}|^{\beta\nu/2}}{\mathcal{B}_m^\beta[\beta\nu/2, \beta n/2]} |\mathbf{Z}|^{\beta(n-m+1)/2-1} |\mathbf{\Delta} + \mathbf{Z}|^{-\beta(n+\nu)/2}.$$

\mathbf{Z} is said to have a nonstandardised matricvariate beta type II distribution.

Proof. The proof follows from (3) by applying (A-4). \square \square

Densities (3) and (4) have been studied by several authors in the real case, see Khatri [26] and Srivastava & Khatri [34], Gupta and Nagar [20], among many others; and by James [23], Muirhead [31], Díaz-García and Gutiérrez-Jáimez [7] and Díaz-García and Gutiérrez-Jáimez [5] and Díaz-García and Gutiérrez-Jáimez [9], in the noncentral, doubly noncentral, singular and nonsingular and complex cases, among many other authors.

3 Matrix multivariate T distribution

Theorem 3.1. Let $\mathbf{T}_1 = S^{-1/2} \mathbf{Y} + \boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$ where $(S^{1/2})^2 = S \sim \Gamma^\beta(\nu, \rho)$ (that is, S has a gamma distribution with parameters ν and ρ), $\rho > 0$, independent of $\mathbf{Y} \sim \mathcal{N}_{m \times n}^\beta(\mathbf{0}, \mathbf{I}_m \otimes \boldsymbol{\Sigma})$, $\boldsymbol{\Sigma} \in \mathfrak{P}_n^\beta$. Then the density of \mathbf{T}_1 is

$$\frac{\Gamma_1^\beta[\beta(\nu+mn)/2] \rho^{\beta mn/2}}{\pi^{\beta mn/2} \Gamma_1^\beta[\beta\nu/2] |\boldsymbol{\Sigma}|^{\beta m/2}} [1 + \rho \operatorname{tr} \boldsymbol{\Sigma}^{-1} (\mathbf{T}_1 - \boldsymbol{\mu})^* (\mathbf{T}_1 - \boldsymbol{\mu})]^{-\beta(\nu+mn)/2}, \quad (6)$$

which is termed the matrix multivariate T distribution and is denoted as $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \mathbf{I}_m, \boldsymbol{\Sigma})$.

Proof. The joint density of S and \mathbf{Y} is

$$\propto s^{\beta\nu/2-1} \text{etr}\{-\beta(s/\rho + \text{tr } \mathbf{Y}\mathbf{Y}^*)/2\},$$

where the constant of proportionality is

$$c = \frac{1}{(2\beta-1)^{\beta\nu/2} \Gamma_1^\beta[\beta\nu/2] \rho^{\beta\nu/2}} \cdot \frac{1}{(2\pi\beta-1)^{\beta mn/2} |\boldsymbol{\Sigma}|^{\beta m/2}}.$$

Noting that by (A-3)

$$(ds)(d\mathbf{Y}) = s^{\beta mn/2} (ds)(d\mathbf{T}_1),$$

the desired result is obtained analogously to the proof of Theorem 2.1. \square \square

Corollary 3.1. *Assume that $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \mathbf{I}_m, \mathbf{I}_n)$, and let \mathbf{M} and \mathbf{N} be any square root of the constant matrices $\boldsymbol{\Delta} = \mathbf{M}\mathbf{M}^* \in \mathfrak{P}_m^\beta$ and $\boldsymbol{\Lambda} = \mathbf{N}\mathbf{N}^* \in \mathfrak{P}_n^\beta$, respectively. Also let $(\mathbf{M}^*)^{-1}\mathbf{T}_1\mathbf{N}^{-1} + \boldsymbol{\mu} = \mathbf{Q}_1 \in \mathcal{L}_{m,n}^\beta$, where $\boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$ is constant. Then,*

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]}{\pi^{\beta mn/2} \Gamma_1^\beta[\beta\nu/2]} |\boldsymbol{\Delta}|^{\beta n/2} |\boldsymbol{\Lambda}|^{\beta m/2} [1 - \text{tr } \boldsymbol{\Delta}(\mathbf{Q}_1 - \boldsymbol{\mu})\boldsymbol{\Lambda}(\mathbf{Q}_1 - \boldsymbol{\mu})^*]^{-\beta(\nu+mn)/2},$$

Hence, we write $\mathbf{Q}_1 \sim \mathcal{MT}_{m \times n}^\beta(\nu, \boldsymbol{\mu}, \boldsymbol{\Delta}, \boldsymbol{\Lambda})$.

Proof. The proof follows observing that, by (A-3)

$$(d\mathbf{T}_1) = |\mathbf{M}\mathbf{M}^*|^{\beta n/2} |\mathbf{N}\mathbf{N}^*|^{\beta m/2} (d\mathbf{Q}_1) = |\boldsymbol{\Delta}|^{\beta n/2} |\boldsymbol{\Lambda}|^{\beta m/2} (d\mathbf{Q}_1). \quad \square$$

\square

Now, assuming that $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$, with $n \geq m$ and defining $\mathbf{F}_1 = \mathbf{T}_1\mathbf{T}_1^* \in \mathfrak{P}_m^\beta$, then, under the conditions of Theorem 3.1 we have that

$$\mathbf{F}_1 = S^{-1}\mathbf{Y}\mathbf{Y}^* = S^{-1}\mathbf{W}$$

where $\mathbf{W} = \mathbf{Y}\mathbf{Y}^* \sim \mathcal{W}_m^\beta(n, \mathbf{I}_m)$, $n > \beta(m-1)$.

Theorem 3.2. *The density of \mathbf{F}_1 is*

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta\nu/2] \Gamma_m^\beta[\beta n/2]} |\mathbf{F}_1|^{\beta(n-m+1)/2-1} (1 + \text{tr } \mathbf{F}_1)^{-\beta(\nu+mn)/2}, \quad (7)$$

\mathbf{F}_1 is said to have a matrix multivariate beta type II distribution.

Proof. The proof follows from (6) by applying (A-1) and (A-6). \square \square

Similarly, if $n < m$ and $\tilde{\mathbf{F}}_1 = \mathbf{T}_1^*\mathbf{T}_1 \in \mathfrak{P}_n^\beta$.

Theorem 3.3. *$\tilde{\mathbf{F}}_1$ has the density*

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta\nu/2] \Gamma_n^\beta[\beta m/2]} |\tilde{\mathbf{F}}_1|^{\beta(m-n+1)/2-1} (1 + \text{tr } \tilde{\mathbf{F}}_1)^{-\beta(\nu+mn)/2}. \quad (8)$$

Thus, $\tilde{\mathbf{F}}_1$ is said to have a matrix multivariate distribution type II distribution.

Proof. The proof is the same as that given in Theorem 3.2. Alternatively, the density (8) can be obtained from density (7) by making the following substitutions,

$$m \rightarrow n, \quad n \rightarrow m. \quad \square \quad (9)$$

□

As in the matricvariate beta type II distributions, assume that $\mathbf{A} \in \mathcal{L}_{m \times m}^\beta$ is any square root of constant matrix $\mathbf{\Pi} = \mathbf{A}\mathbf{A}^* \in \mathfrak{P}_m^\beta$. Also, define $\mathbf{Z} = \mathbf{A}^*\mathbf{F}\mathbf{A}$, therefore:

Corollary 3.2. *The density of \mathbf{Z} is*

$$\frac{\Gamma_1^\beta[\beta(\nu + mn)/2]|\mathbf{\Pi}|^{\beta n/2}}{\Gamma_1^\beta[\beta\nu/2]\Gamma_n^\beta[\beta m/2]}|\mathbf{Z}|^{\beta(n-m+1)/2-1}(1 + \mathbf{\Pi}\mathbf{Z})^{-\beta(n+\nu)/2}.$$

\mathbf{Z} is said to have a nonstandardised matrix multivariate beta type II distribution.

Proof. The proof follows from (7) by applying (A-4). □ □

In the real and singular case, the matricvariate and matrix multivariate T distributions have been studied by Díaz-García and Gutiérrez-Jáimez [8].

4 Singular value densities

In this section, the joint densities of the singular values of matrices \mathbf{T} , $\tilde{\mathbf{T}}$, \mathbf{T}_1 and $\tilde{\mathbf{T}}_1$ are derived. In addition, and as a direct consequence, the joint densities of the eigenvalues of \mathbf{F} , $\tilde{\mathbf{F}}$, \mathbf{F}_1 and $\tilde{\mathbf{F}}_1$ are obtained for real normed division algebras.

Theorem 4.1. *Let $\delta_1, \dots, \delta_m$ be the singular values of $\mathbf{T} \sim \mathcal{T}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$, $\delta_1 > \dots > \delta_m > 0$. Then its joint density is*

$$\frac{2^m \pi^{\beta m^2 + \tau}}{\Gamma_m^\beta[\beta m/2]\mathcal{B}_m^\beta[\beta\nu/2, \beta n/2]} \prod_{i=1}^m \delta_i^{\beta(n-m+1)-1} (1 + \delta_i^2)^{-\beta(\nu+n)/2} \prod_{i < j}^m (\delta_i^2 - \delta_j^2)^\beta \quad (10)$$

where τ is defined in Lemma .3.

Proof. This follows immediately from (1), first using (A-5) and then applying (A-1). □ □

The joint density of the singular values of $\tilde{\mathbf{T}}$ is obtained from (10) after making the substitutions (5).

Theorem 4.2. *Assume that $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ and let $\alpha_1, \dots, \alpha_m$, $\alpha_1 > \dots > \alpha_m > 0$, be its singular values. Then its joint density is*

$$\frac{2^m \pi^{\beta m^2/2 + \tau} \Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta\nu/2]\Gamma_m^\beta[\beta m/2]\Gamma_m^\beta[\beta n/2]} \prod_{i=1}^m \alpha_i^{\beta(n-m+1)-1} \left(1 + \sum_{i=1}^m \alpha_i^2\right)^{-\beta(\nu+mn)/2} \prod_{i < j}^m (\alpha_i^2 - \alpha_j^2)^\beta \quad (11)$$

Proof. The proof is identical to that given for Theorem 4.1. □ □

Analogously, the joint density of the singular values of $\tilde{\mathbf{T}}_1$ is obtained from (11), making the substitutions (9).

Finally, observe that $\delta_i = \sqrt{\text{eig}_i(\mathbf{T}\mathbf{T}^*)}$ and $\alpha_i = \sqrt{\text{eig}_i(\mathbf{T}_1\mathbf{T}_1^*)}$, where $\text{eig}_i(\mathbf{A})$, $i = 1, \dots, m$, denotes the i -th eigenvalue of \mathbf{A} . Let $\lambda_i = \text{eig}_i(\mathbf{T}\mathbf{T}^*) = \text{eig}_i(\mathbf{F})$ and $\gamma_i = \text{eig}_i(\mathbf{T}_1\mathbf{T}_1^*) = \text{eig}_i(\mathbf{F}_1)$, observing that, for example, $\delta_i = \sqrt{\lambda_i}$. Then

$$\bigwedge_{i=1}^m d\delta_i = \bigwedge_{i=1}^m 2^{-m} \prod_{i=1}^m \lambda_i^{-1/2} d\lambda_i,$$

the corresponding joint density of $\lambda_1, \dots, \lambda_m$, $\lambda_1 > \dots > \lambda_m > 0$ is obtained from (10) as

$$\frac{\pi^{\beta m^2 + \tau}}{\Gamma_m^\beta[\beta m/2] \mathcal{B}_m^\beta[\beta \nu/2, \beta n/2]} \prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1} (1 + \lambda_i)^{-\beta(\nu+n)/2} \prod_{i < j}^m (\lambda_i - \lambda_j)^\beta.$$

Analogously, the joint density of $\gamma_1, \dots, \gamma_m$, $\gamma_1 > \dots > \gamma_m > 0$, is obtained from (11) as

$$\frac{\pi^{\beta m^2/2 + \tau} \Gamma_1^\beta[\beta(\nu + mn)/2]}{\Gamma_1^\beta[\beta \nu/2] \Gamma_m^\beta[\beta m/2] \Gamma_m^\beta[\beta n/2]} \prod_{i=1}^m \gamma_i^{\beta(n-m+1)/2-1} \left(1 + \sum_{i=1}^m \gamma_i\right)^{-\beta(\nu+mn)/2} \prod_{i < j}^m (\gamma_i - \gamma_j)^\beta.$$

Remark 4.1. Observe that $\mathbf{Y} \in \mathfrak{L}_{m,n}^\beta$ has a matrix multivariate elliptically contoured distribution for real normed division algebras if its density, with respect to the Lebesgue measure, is given by (see Díaz-García and Gutiérrez-Jáimez [6]):

$$\frac{C^\beta(m, n)}{|\Sigma|^{\beta n/2} |\Theta|^{\beta m/2}} h \left\{ \text{tr} \left[\Sigma^{-1} (\mathbf{Y} - \boldsymbol{\mu}) \Theta^{-1} (\mathbf{Y} - \boldsymbol{\mu})^* \right] \right\}, \quad (12)$$

where $\boldsymbol{\mu} \in \mathfrak{L}_{m,n}^\beta$, $\Sigma \in \mathfrak{P}_m^\beta$, $\Theta \in \mathfrak{P}_m^\beta$. The function $h : \mathfrak{F} \rightarrow [0, \infty)$ is termed the generator function, and it is such that $\int_{\mathfrak{F}_1^\beta} u^{\beta nm - 1} h(u^2) du < \infty$ and

$$C^\beta(m, n) = \frac{\Gamma[\beta mn/2]}{2\pi^{\beta mn/2}} \left\{ \int_{\mathfrak{F}_1^\beta} u^{\beta nm - 1} h(u^2) du \right\}$$

Such a distribution is denoted by $\mathbf{Y} \sim \mathcal{E}_{n \times m}^\beta(\boldsymbol{\mu}, \Sigma, \Theta, h)$, for the real case, see Fang and Zhang [16] and Gupta, and Varga [21]; and Micheas *et al.* [30] for the complex case. Observe that this class of matrix multivariate distributions includes normal, contaminated normal, Pearson type II and VII, Kotz, Jensen-Logistic, power exponential and Bessel distributions, among others; these distributions have tails that are more or less weighted, and/or present a greater or smaller degree of kurtosis than the normal distribution.

Assume that $\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim \mathcal{E}_{m \times n + \nu}^\beta(\mathbf{0}, \mathbf{I}_m, \mathbf{I}_{n+\nu}, h)$, $n, \nu \geq m$; and define, $\mathbf{T} = \mathbf{L}^{-1} \mathbf{Y}_1$, where \mathbf{L} is any square root of $\mathbf{V} = \mathbf{Y}_2 \mathbf{Y}_2^*$ such that $\mathbf{L} \mathbf{L}^* = \mathbf{V}$. Then $\mathbf{T} \sim \mathcal{T}_{m \times n}^\beta(\nu, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$. From (12) the density of \mathbf{Y} is

$$C^\beta(m, n + \nu) h \left\{ \text{tr} (\mathbf{Y}_1 \mathbf{Y}_1^* + \mathbf{Y}_2 \mathbf{Y}_2^*) \right\}.$$

Let $\mathbf{V} = \mathbf{Y}_2 \mathbf{Y}_2^*$ then by (.4), $(d\mathbf{Y}_2) = 2^{-m} |\mathbf{V}|^{\beta(\nu-m+1)/2-1} (d\mathbf{V})(\mathbf{H}_1 d\mathbf{H}_1^*)$. Thus, the marginal density of \mathbf{Y}_1 and \mathbf{V} is obtained by integrating over $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$ by using (A-1), hence

$$\frac{C^\beta(m, n + \nu) \pi^{\beta \nu m/2}}{\Gamma_m^\beta[\beta \nu/2]} |\mathbf{V}|^{\beta(\nu-m+1)/2-1} h \left\{ \text{tr} (\mathbf{Y}_1 \mathbf{Y}_1^* + \mathbf{V}) \right\}.$$

Now, let $\mathbf{T} = \mathbf{L}^{-1} \mathbf{Y}_1$, where $\mathbf{L} \mathbf{L}^* = \mathbf{V}$, then by (.1)

$$(d\mathbf{Y}_1)(d\mathbf{V}) = |\mathbf{V}|^{\beta n/2} (d\mathbf{T})(d\mathbf{V}).$$

Therefore, the joint density of \mathbf{T} and \mathbf{V} is

$$\frac{C^\beta(m, n + \nu)\pi^{\beta\nu m/2}}{\Gamma_m^\beta[\beta\nu/2]} |\mathbf{V}|^{\beta(n+\nu-m+1)/2-1} h\{\text{tr}(\mathbf{I}^m + \mathbf{T}\mathbf{T}^*)\mathbf{V}\}.$$

The desired result follows by applying the next equally,

$$\int_{\mathbf{V} \in \mathfrak{P}_m^\beta} |\mathbf{V}|^{\beta(n+\nu-m+1)/2-1} h\{\text{tr} \mathbf{A}\mathbf{V}\} (d\mathbf{V}) = \frac{\Gamma_m^\beta[\beta(n + \nu)2] |\mathbf{A}|^{-\beta(n+\nu)/2}}{\pi^{\beta m(n+\nu)/2} C^\beta(m, n + \nu)},$$

see Díaz-García and Gutiérrez-Jáimez [6] and Díaz-García and Gutiérrez-Jáimez [10]. \square

Observe that in this case, \mathbf{Y}_1 and \mathbf{Y}_2 (or $\mathbf{V} = \mathbf{Y}_2\mathbf{Y}_2^*$) are stochastically dependent. Furthermore, note that only when the particular matrix multivariate elliptical distribution is the matrix multivariate normal distribution, are \mathbf{Y}_1 and \mathbf{Y}_2 (or $\mathbf{V} = \mathbf{Y}_2\mathbf{Y}_2^*$) independent. Therefore, we can say that the matrix multivariate T distribution is invariant under the family of matrix multivariate elliptical distributions for real normed division algebras, and furthermore, its density is the same as when normality is assumed. Analogously, it can be proved that the matrix multivariate T , matrix multivariate and matrix multivariate beta type II distributions are invariant under the family of matrix multivariate elliptical distributions for real normed division algebras. Furthermore, this invariance prevails under other classes of elliptical models for real normed division algebras, see Fang and Zhang [16], Gupta, and Varga [21] and Díaz-García and Gutiérrez-Jáimez [6].

Conclusions

Any topic in statistical literature, is usually first studied in the real case, later in the complex case, later for the quaternion case and exceptionally for the octonion case. From the results presented in this paper, the real, complex, quaternion and octonion cases are obtained by simply replacing β with 1, 2, 4 or 8, respectively. Furthermore, as observed by Kabe [24], these results can be extended to hypercomplex cases, that is, for biquaternion and bioctonion algebras. Then, from the results presented here, the hypercomplex cases are obtained by replacing β with 2β .

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Appendix

A detailed discussion of real normed division algebras may be found in Baez [1] and Gross and Richards [19]. For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes, a **vector space** is always a finite-dimensional module over the field of real numbers. An **algebra** \mathfrak{F} is a vector space that is equipped with a bilinear map $m : \mathfrak{F} \times \mathfrak{F} \rightarrow \mathfrak{F}$ termed *multiplication* and a nonzero element $1 \in \mathfrak{F}$ termed the *unit* such that $m(1, a) = m(a, 1) = 1$. As usual, we abbreviate $m(a, b) = ab$ as ab . We do not assume \mathfrak{F} associative. Given an algebra, we freely think of real numbers as elements of this algebra via the map $\omega \mapsto \omega 1$.

An algebra \mathfrak{F} is a **division algebra** if given $a, b \in \mathfrak{F}$ with $ab = 0$, then either $a = 0$ or $b = 0$. Equivalently, \mathfrak{F} is a division algebra if the operation of left and right multiplications by any nonzero element is invertible. A **normed division algebra** is an algebra \mathfrak{F} that is also a normed vector space with $\|ab\| = \|a\| \|b\|$. This implies that \mathfrak{F} is a division algebra and that $\|1\| = 1$.

There are exactly four normed division algebras: real numbers (\mathfrak{R}), complex numbers (\mathfrak{C}), quaternions (\mathfrak{H}) and octonions (\mathfrak{O}), see Baez [1]. We take into account that \mathfrak{R} , \mathfrak{C} , \mathfrak{H} and \mathfrak{O} are the only normed division algebras; moreover, they are the only alternative division algebras, and all division algebras have a real dimension of 1, 2, 4 or 8, which is denoted by

β , see Baez [1, Theorems 1, 2 and 3]. In other branches of mathematics, the parameters $\alpha = 2/\beta$ and $t = \beta/4$ are used, see Edelman and Rao [14] and Kabe [24], respectively.

Let $\mathcal{L}_{m,n}^\beta$ be the linear space of all $m \times n$ matrices of rank $m \leq n$ over \mathfrak{F} with m distinct positive singular values, where \mathfrak{F} denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{F}^{m \times n}$ be the set of all $m \times n$ matrices over \mathfrak{F} . The dimension of $\mathfrak{F}^{m \times n}$ over \mathfrak{R} is βmn . Let $\mathbf{A} \in \mathfrak{F}^{m \times n}$, then $\mathbf{A}^* = \overline{\mathbf{A}}^T$ denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

Table 1: Notation

Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}_{m,n}^\beta$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^\beta(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	\mathfrak{S}_m^β

In addition, let \mathfrak{P}_m^β be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{F}^{m \times m}$; then \mathfrak{P}_m^β is an open subset of \mathfrak{S}_m^β .

Let \mathfrak{D}_m^β be the *diagonal subgroup* of $\mathcal{L}_{m,m}^\beta$ consisting of all $\mathbf{D} \in \mathfrak{F}^{m \times m}$, $\mathbf{D} = \text{diag}(d_1, \dots, d_m)$.

For any matrix $\mathbf{X} \in \mathfrak{F}^{n \times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}) . Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{F}^{m \times n}$, \mathfrak{S}_m^β , \mathfrak{D}_m^β or $\mathcal{V}_{m,n}^\beta$, see Dimitriu [13].

If $\mathbf{X} \in \mathfrak{F}^{m \times n}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{F}^{m \times n}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^m \bigwedge_{j=1}^n dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}_m^\beta$ (or $\mathbf{S} \in \mathfrak{T}_L^\beta(m)$ is a lower triangular matrix) then $(d\mathbf{S})$ (the Lebesgue measure in \mathfrak{S}_m^β or in $\mathfrak{T}_L^\beta(m)$, the set of lower triangular matrices) denotes the exterior product of the $m(m+1)\beta/2$ functionally independent variables (or denotes the exterior product of the $m(m-1)\beta/2 + n$ functionally independent variables, if $s_{ii} \in \mathfrak{R}$ for all $i = 1, \dots, m$)

$$(d\mathbf{S}) = \begin{cases} \bigwedge_{i \leq j} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, \\ \bigwedge_{i=1}^m ds_{ii} \bigwedge_{i < j} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, & \text{if } s_{ii} \in \mathfrak{R}. \end{cases}$$

The context generally establishes the conditions on the elements of \mathbf{S} , that is, if $s_{ij} \in \mathfrak{R}$, $\in \mathfrak{C}$, $\in \mathfrak{H}$ or $\in \mathfrak{O}$. It is considered that

$$(d\mathbf{S}) = \bigwedge_{i \leq j} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)} \equiv \bigwedge_{i=1}^m ds_{ii} \bigwedge_{i < j} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Observe, too, that for the Lebesgue measure $(d\mathbf{S})$ defined thus, it is required that $\mathbf{S} \in \mathfrak{P}_m^\beta$, that is, \mathbf{S} must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$ then $(d\mathbf{\Lambda})$ (the Lebesgue measure in \mathfrak{D}_m^β) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^n \bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^m \bigwedge_{j=i+1}^n \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1^* \mathbf{H}_2^*)^* = (\mathbf{h}_1, \dots, \mathbf{h}_m | \mathbf{h}_{m+1}, \dots, \mathbf{h}_n)^* \in \mathfrak{U}^\beta(n)$. It can be proved that this differential form does not depend on the choice of the \mathbf{H}_2 matrix. When $n = 1$; $\mathcal{V}_{m,1}^\beta$ defines the unit sphere in \mathfrak{F}^m . This is, of course, an $(m-1)\beta$ - dimensional surface in \mathfrak{F}^m . When $n = m$ and denoting \mathbf{H}_1 by \mathbf{H} , $(\mathbf{H} d\mathbf{H}^*)$ is termed the *Haar measure* on $\mathfrak{U}^\beta(m)$.

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^\beta$ is

$$\text{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathbf{H}_1 d\mathbf{H}_1^*) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \quad (\text{A-1})$$

where $\Gamma_m^\beta[a]$ denotes the multivariate *Gamma function* for the space \mathfrak{S}_m^β , and is defined by

$$\begin{aligned} \Gamma_m^\beta[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - (i-1)\beta/2], \end{aligned}$$

where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$, $|\cdot|$ denotes the determinant and $\text{Re}(a) > (m-1)\beta/2$, see Gross and Richards [19]. Similarly, from Herz [22] the *multivariate beta function* for the space \mathfrak{S}_m^β , can be defined as

$$\begin{aligned} \mathcal{B}_m^\beta[b, a] &= \int_{\mathbf{0} < \mathbf{B} < \mathbf{I}_m} |\mathbf{B}|^{a-(m-1)\beta/2-1} |\mathbf{I}_m - \mathbf{B}|^{b-(m+1)\beta/2-1} (d\mathbf{B}) \\ &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} |\mathbf{A}|^{a-(m-1)\beta/2-1} |\mathbf{I}_m + \mathbf{A}|^{-(a+b)} (d\mathbf{A}) \\ &= \frac{\Gamma_m^\beta[a] \Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b]}, \end{aligned} \quad (\text{A-2})$$

where $\mathbf{A} = (\mathbf{I} - \mathbf{B})^{-1} - \mathbf{I}$, $\text{Re}(a) > (m-1)\beta/2$ and $\text{Re}(b) > (m-1)\beta/2$.

Now, we show two Jacobians in terms of the β parameter, which are based on the work of Kabe [24] and Dimitriu [13]. These results are proposed as extensions of real, complex or quaternion cases, see James [23], Khatri [25], Mehta [29], Ratnarajah *et al.* [33] and Li and Xue [28].

Lemma .1. Let \mathbf{X} and $\mathbf{Y} \in \mathcal{L}_{m,n}^\beta$, and let $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{B} + \mathbf{C}$, where $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$, $\mathbf{B} \in \mathcal{L}_{n,n}^\beta$ and $\mathbf{C} \in \mathcal{L}_{m,n}^\beta$ are constant matrices. Then

$$(d\mathbf{Y}) = |\mathbf{A}^* \mathbf{A}|^{\beta n/2} |\mathbf{B}^* \mathbf{B}|^{\beta m/2} (d\mathbf{X}). \quad (\text{A-3})$$

Lemma .2. Let \mathbf{X} and $\mathbf{Y} \in \mathfrak{P}_m^\beta$, and let $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{C}$, where \mathbf{A} and $\mathbf{C} \in \mathcal{L}_{m,m}^\beta$ are constant matrices. Then

$$(d\mathbf{Y}) = |\mathbf{A}^* \mathbf{A}|^{\beta(m-1)/2+1} (d\mathbf{X}). \quad (\text{A-4})$$

Lemma .3 (Singular value decomposition, *SVD*). Let $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$, such that $\mathbf{X} = \mathbf{V}_1 \mathbf{D} \mathbf{W}^*$ with $\mathbf{V}_1 \in \mathcal{V}_{m,n}^\beta$, $\mathbf{W} \in \mathfrak{U}^\beta(m)$ and $\mathbf{D} = \text{diag}(d_1, \dots, d_m) \in \mathfrak{D}_m^1$, $d_1 > \dots > d_m > 0$. Then

$$(d\mathbf{X}) = 2^{-m} \pi^\tau \prod_{i=1}^m d_i^{\beta(n-m+1)-1} \prod_{i < j}^m (d_i^2 - d_j^2)^\beta (d\mathbf{D})(\mathbf{V}_1 d\mathbf{V}_1^*)(\mathbf{W} d\mathbf{W}^*), \quad (\text{A-5})$$

where

$$\tau = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

Lemma .4. Let $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$, and $\mathbf{S} = \mathbf{X}\mathbf{X}^* \in \mathfrak{P}_m^\beta$. Then

$$(d\mathbf{X}) = 2^{-m} |\mathbf{S}|^{\beta(n-m+1)/2-1} (d\mathbf{S})(\mathbf{V}_1 d\mathbf{V}_1^*). \quad (\text{A-6})$$