

**THE HOMOTOPY LIMIT PROBLEM AND (ETALE)  
HERMITIAN  $K$ -THEORY**

MARCO SCHLICHTING

ABSTRACT. Let  $X$  be a noetherian separated scheme with 2 invertible in the ring of regular functions. Assume further that  $X$  has finite Krull dimension and there is a global bound on the virtual 2 cohomological dimension of all its residue fields. We show that the map from hermitian  $K$ -theory of  $X$  to homotopy fixed points of  $K$ -theory under the natural  $\mathbb{Z}/2$ -action is a 2-adic equivalence. If  $-1$  is a sum of squares in the ring of regular functions, then the map is an integral equivalence. Furthermore, we show that the map from hermitian  $K$ -theory of  $X$  to its etale version is an isomorphism in the same range as it is for  $K$ -theory.

1. INTRODUCTION

Let  $X$  be a noetherian separated scheme of finite Krull dimension such that  $\frac{1}{2} \in \Gamma(X, O_X)$ . Write  $vcd_2(X)$  for  $vcd_2(X) = \max\{vcd_2(k(x)) | x \in X\}$  where for a field  $k$ , the number  $vcd_2(k)$  is the etale 2-cohomological dimension of the field  $k[\sqrt{-1}]$ . For instance, every separated scheme of finite type over  $\mathbb{Z}$  or over a field  $k$  with  $vcd_2(k) < \infty$  has  $vcd_2(X) < \infty$ .

Let  $GW^n(X, L)$  be the higher Grothendieck-Witt spectrum of  $X$  with coefficients in the  $n$ -th shifted line-bundle  $L[n]$ . It is the Grothendieck-Witt spectrum of the category  $\text{Ch}^b \text{Vect}(X)$  of bounded chain complexes of vector bundles over  $X$  equipped with the duality  $E \mapsto E^{\#L} = \text{Hom}(E, L[n])$  and quasi-isomorphisms as weak equivalences. The zero-th space of the associated  $\Omega$ -spectrum is the one defined in [Sch10, §8 Definition 7] and for an affine  $X = \text{Spec}(R)$ ,  $n = 0$ ,  $L = O_X$ , this space agrees with Karoubi's hermitian  $K$ -theory space of  $R$ . For the spectrum  $GW$ , we use the delooping constructed in [Sch] whose negative homotopy groups  $GW_i^n(X, L)$  are naturally isomorphic to Balmer's triangular Witt groups  $W^{n-i}(X, L)$  for  $i < 0$ ; see [Sch]. We could equally well work with the delooping constructed in [Sch10, §10] but the negative homotopy groups of this spectrum are less well-understood.

Write  $K^{[n]}(X, L)$  for the  $K$ -theory spectrum  $K(X)$  of  $X$  equipped with the  $C_2 = \mathbb{Z}/2$ -action induced by the duality  $\#_n^L$ . There is a natural map of spectra

$$GW^n(X, L) \rightarrow (K^{[n]}(X, L))^{hC_2}$$

from hermitian  $K$ -theory to homotopy fixed points of  $K$ -theory [Sch], [Kob99]. In this note we shall prove the following.

---

*Date:* November 24, 2010.

**1.1. Theorem.** *Let  $X$  be a noetherian separated scheme of finite Krull dimension with  $\frac{1}{2} \in \Gamma(X, O_X)$ . Assume that  $X$  has an ample family of line-bundles and  $\mathrm{vcd}_2(X) < \infty$ . Then for all  $\nu \geq 1$ , the natural map*

$$GW^n(X, L; \mathbb{Z}/2^\nu) \rightarrow (K^{[n]}(X, L; \mathbb{Z}/2^\nu))^{hC_2}$$

*is an equivalence.*

This theorem was proved for fields in characteristic 0 by Hu-Kriz-Ormsby in [HKO]. The theorem was conjectured by Williams in [Wil05, 3.4.2 Examples 3] for affine  $X$  and without the restriction on cohomological dimensions. However, without this restriction, there are counter examples; see [HKO].

If  $-1$  is a sum of squares we also prove the following integral version.

**1.2. Theorem.** *Let  $X$  be a noetherian separated scheme of finite Krull dimension with  $\frac{1}{2} \in \Gamma(X, O_X)$ . Assume that  $X$  has an ample family of line-bundles and  $\mathrm{vcd}_2(X) < \infty$ . Assume furthermore that  $-1 \in \Gamma(X, O_X)$  is a sum of squares. Then the natural map*

$$GW^n(X, L) \rightarrow (K^{[n]}(X, L))^{hC_2}$$

*is an equivalence.*

This holds, for instance when  $X$  is of finite type over a field that is not formally real, e.g., a finite field or an algebraically closed field.

Write  $GW^n(X_{et}, L)$  for the value at  $X$  of a globally fibrant replacement of  $GW^n(-, L)$  on a sufficiently large étale site. We prove the following.

**1.3. Theorem.** *Let  $X$  be a noetherian separated scheme of finite Krull dimension with  $\frac{1}{2} \in \Gamma(X, O_X)$ . Assume that  $X$  has an ample family of line-bundles and  $\mathrm{vcd}_2(X) < \infty$ . Then for all  $\nu \geq 1$ , the natural map*

$$GW^n(X, L; \mathbb{Z}/2^\nu) \rightarrow GW^n(X_{et}, L; \mathbb{Z}/2^\nu)$$

*is an equivalence on  $\mathrm{vcd}_2(X) - 2$ -connected covers.*

This is the evident analogue for hermitian  $K$ -theory of the  $K$ -theoretical Quillen-Lichtenbaum conjecture. The statement of the theorem was conjectured in [BKØ] where they show that the map is split surjective in high degrees. The  $K$ -theoretical analogue was proved in this generality in [RØ05].

**1.4. Remark.** All the theorems above remain valid if one replaces  $GW$  and  $K$  with their “non-connective” versions  $\mathbb{G}W$  and  $\mathbb{K}$  as defined in [Sch10], [Sch] for  $GW$  and [TT90], [Sch06] for  $K$ ; see 2.5.

**1.5. Remark.** The theorems above remain valid if one drops the assumption “ample family of line-bundles” and replaces vector-bundle  $GW$ - and  $K$ -theories by their versions that use perfect complexes instead of complexes of vector bundles.

**1.6. Remark.** The odd-primary analogues of Theorems 1.1 and 1.3 are essentially of no interest because

$$GW_i^n(X, L) \otimes \mathbb{Z}[1/2] = K_i^{[n]}(X, L)^{C_2} \otimes \mathbb{Z}[1/2] \oplus W^{n-i}(X) \otimes \mathbb{Z}[1/2]$$

where  $W^n$  denotes Balmer’s Witt groups.

**Acknowledgements.** I would like to thank Thomas Unger for useful conversations related to this work.

## 2. PROOFS

Fix a line-bundle  $L$  on  $X$ , and a two power integer  $l = 2^\nu$ . To simplify notation, we will write  $GW(X)$ ,  $K(X)$ , etc for  $GW^n(X, L)$ ,  $K^{[n]}(X, L)$ ,  $GW/l(X)$  for  $GW(X; \mathbb{Z}/l)$ ,  $GW^{et}/l(X)$  for  $GW(X_{et}; \mathbb{Z}/l)$  etc.

**2.1. Preliminaries.** We recall a few facts from [Sch]. The affine non-connective versions can also be found in [Kob99], [Wil05]. They generalize to schemes via the Mayer-Vietoris principle [Sch10], though we take a different approach in [Sch].

- (1) Let  $\eta \in GW_{-1}^{-1}(\mathbb{Z}[\frac{1}{2}]) \cong W^0(\mathbb{Z}[\frac{1}{2}])$  be the element corresponding to  $1 \in W^0(\mathbb{Z}[\frac{1}{2}])$ . Then there is a homotopy fibration of spectra

$$K(X)_{hC_2} \rightarrow GW(X) \rightarrow GW(X)[\eta^{-1}]$$

where  $GW(X)[\eta^{-1}] = \mathbb{L}$  is a spectrum whose homotopy groups  $\pi_i \mathbb{L}^{[n]}(X, L)$  are naturally isomorphic to Balmer's Witt groups  $W^{n-i}(X, L)$  for all  $n, i \in \mathbb{Z}$ .

- (2) There is a homotopy cartesian square of spectra

$$\begin{array}{ccc} GW(X) & \longrightarrow & GW(X)[\eta^{-1}] \\ \downarrow & & \downarrow \\ K(X)^{hC_2} & \longrightarrow & K(X)^{hC_2}[\eta^{-1}] \end{array}$$

where  $K(X)^{hC_2}[\eta^{-1}] = \hat{H}(C_2, K)$  is the Tate spectrum of  $C_2$  acting on  $K^{[n]}(X, L)$ . Both spectra  $\mathbb{L}$  and  $\hat{H}(C_2, K(X))$  are 4-periodic, and the map  $\mathbb{L} \rightarrow \hat{H}(C_2, K(X))$  commutes with the periodicity maps [Sch], [WW00]. In particular, the homotopy fibre  $\mathcal{F}(X)$  of  $GW(X) \rightarrow (K(X))^{hC_2}$  satisfies

$$\pi_i \mathcal{F}(X) \cong \pi_{i+4} \mathcal{F}(X)$$

for all  $i \in \mathbb{Z}$ .

**2.2. Lemma.** *Theorem 1.1 holds for fields.*

*Proof.* Denote by  $\mathcal{F}$  the homotopy fibre of  $GW/l \rightarrow (K/l)^{hC_2}$ . We have to show that  $\mathcal{F}(k) \sim 0$  for all fields  $k$  (with  $vcd_2(k) < \infty$ ). By a results of [HKO], we have  $\mathcal{F}(k) \sim 0$  for fields  $k$  of characteristic 0 (with  $vcd_2(k) < \infty$ ).

For a field  $k$  of characteristic  $p > 0$  (with  $vcd_2(k) < \infty$ ), the reduction to characteristic 0 is standard. Here are the details. We can assume that  $k$  is perfect since a purely transcendental extension of fields doesn't change  $K/l$ , Witt groups nor  $GW/l$ . Let  $V$  be the ring of Witt-vectors over  $k$ . It is a complete (hence henselian) DVR with residue field  $k$  and fraction field  $F$  of characteristic 0, and  $vcd_2(F) < \infty$ .

Let  $\pi \in V$  be a uniformizing element. I claim that the map  $V[T, T^{-1}] \rightarrow F : T \mapsto \pi$  induces an equivalence  $\mathcal{F}(V[T, T^{-1}]) \simeq \mathcal{F}(F)$ . It is known that the map induces an equivalence for  $K/l$  (same argument as for Witt groups below), hence it induces an equivalence for  $(K/l)^{hC_2}$  (and  $(K/l)_{hC_2}$ ). Thus, we need to see that the map induces an equivalence for  $GW/l$ . From the homotopy fibration of spectra

$K_{hC_2} \rightarrow GW \rightarrow \mathbf{L}$  (2.1 (1)) it suffices to show that the map induces an equivalence for  $\mathbf{L}$ . Balmer Witt groups (that is, the homotopy groups of  $\mathbf{L}$ ) are periodic of period 4, they are trivial for local rings in degrees  $\neq 0 \pmod{4}$  and are homotopy invariant for regular rings. From the localization sequence for  $V[T] \rightarrow V[T, T^{-1}]$ , we see that  $W^i(V[T, T^{-1}]) = 0$  for  $i \neq 0 \pmod{4}$ . Thus, we only need to check that the map  $W^0(V[T, T^{-1}]) \rightarrow W^0(F)$  is an isomorphism. But this follows from comparing the localization sequences for  $V[T] \rightarrow V[T, T^{-1}]$  and  $V \rightarrow F$  which reduce to a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & W^0(V[T]) & \longrightarrow & W^0(V[T, T^{-1}]) & \longrightarrow & W^0(V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & W^0(V) & \longrightarrow & W^0(F) & \longrightarrow & W^0(k) \longrightarrow 0 \end{array}$$

where the left two vertical maps are induced by  $T \mapsto \pi$  and the right vertical map is reduction mod  $\pi$ . By homotopy invariance, the left vertical map is an isomorphism. By Rigidity, the right vertical map is an isomorphism. It follows that the middle vertical map is an isomorphism. Thus, in the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(V) & \xrightarrow{\sim} & \mathcal{F}(V[T]) & \longrightarrow & \mathcal{F}(V[T, T^{-1}]) \\ & & \downarrow & & \downarrow \\ & & \mathcal{F}(V) & \longrightarrow & \mathcal{F}(F) \end{array}$$

the vertical maps are equivalences, and the upper horizontal map has a retraction. Therefore,  $\mathcal{F}(V)$  is a retract of  $\mathcal{F}(F)$ . Since,  $F$  has characteristic 0, we have  $\mathcal{F}(F) \simeq 0$ , hence,  $\mathcal{F}(V) \simeq 0$ . By Rigidity, the map  $V \rightarrow k$  induces an equivalence for  $\mathcal{F}$ . Hence,  $\mathcal{F}(k) \simeq \mathcal{F}(V) \simeq 0$ .  $\square$

**2.3. Lemma.** *The map*

$$GW^{et}/l(X) \rightarrow (K^{et}/l(X))^{hC_2}$$

*is an equivalence on  $(-1)$ -connected covers.*

*Proof.* There is a Bott element  $\beta \in (GW/l)_p(\mathbb{Z}[\frac{1}{2}])$  which maps to (a power of the) usual  $K$ -theoretical Bott element under the forgetful map  $GW/l \rightarrow K/l$ . The element is constructed in [BKØ] where it is shown that the map  $GW/l[\beta^{-1}] \rightarrow GW^{et}/l[\beta^{-1}]$  is an equivalence, and  $GW^{et}/l \rightarrow GW^{et}/l[\beta^{-1}]$  is an equivalence on  $(-1)$ -connected covers. The same is true for  $K$ -theory in place of higher Grothendieck-Witt groups. From the homotopy cartesian square 2.1 (2), we obtain the homotopy cartesian square

$$\begin{array}{ccc} GW/l(X)[\beta^{-1}] & \longrightarrow & GW/l(X)[\eta^{-1}][\beta^{-1}] \\ \downarrow & & \downarrow \\ K/l(X)^{hC_2}[\beta^{-1}] & \longrightarrow & K/l(X)^{hC_2}[\eta^{-1}][\beta^{-1}] \end{array}$$

In [BKØ] it is also shown that  $\beta\eta^2 = 0$ . In particular, both right hand terms are zero, and thus, the left vertical arrow is an equivalence. Finally, the natural map

$$(K/l)^{hC_2}[\beta^{-1}] \rightarrow (K/l[\beta^{-1}])^{hC_2}$$

is an equivalence because cup-product with  $\beta$  is an isomorphism in high degrees, by the solution of the  $K$ -theoretical Quillen-Lichtenbaum conjecture [RØ05].  $\square$

**2.4. Lemma.** *If Theorem 1.3 holds for the residue fields of  $X$  then it holds for  $X$ .*

*Proof.* This is standard from  $K$ -theory. Here are the details. Both sides satisfy Nisnevich descent [TT90], [Sch]. The map on the  $E^2$ -terms of the corresponding (strongly convergent) Nisnevich descent spectral sequences has the form

$$H_{Nis}^p(X, a_{Nis}(GW/l)_{-q}) \rightarrow H_{Nis}^p(X, a_{Nis}(GW^{et}/l)_{-q}).$$

The map of Nisnevich sheaves

$$a_{Nis}(GW/l)_q \rightarrow a_{Nis}(GW^{et}/l)_q$$

on  $X$  is an isomorphism for  $q \geq vcd_2 X - 1$ , by Rigidity and the assumption of the lemma. The result now follows from the Nisnevich descent spectral sequences.  $\square$

*Proofs of Theorems 1.1 and 1.3.* Consider the commutative diagram

$$(1) \quad \begin{array}{ccc} GW/l(X) & \longrightarrow & GW^{et}/l(X) \\ \downarrow & & \downarrow \\ K/l(X)^{hC_2} & \longrightarrow & (K^{et}/l(X))^{hC_2} \end{array}$$

From the solution of the  $K$ -theoretical Quillen-Lichtenbaum conjecture [RØ05], we know that the lower horizontal map is an equivalence on  $vcd_2(X) - 2$ -connected covers. From Lemma 2.3, we know that the right vertical map is an equivalence on  $(-1)$ -connected covers. From Lemma 2.2, we know that the left vertical map is an equivalence for fields. This implies that Theorem 1.3 holds for fields. From Lemma 2.4, we see that Theorem 1.3 holds for schemes. This implies that the homotopy fibre of the left vertical map in diagram (1) is trivial in high degrees. Since the homotopy fibre is periodic (2.1 (2)) we see that Theorem 1.1 holds.  $\square$

*Proof of theorem 1.2.* If  $-1 \in \Gamma(X, O_X)$  is a sum of squares, then  $2^n W^0(X) = 0$  for some  $n$ . Now,  $W^0$  acts on all homotopy groups of the spectrum  $\mathbb{L}$  representing Balmer Witt groups, and  $W^0$  acts on the homotopy groups of the Tate spectrum  $\hat{H}(C_2, K)$ ; see [WW00], [Sch]. It follows that the homotopy groups of the fibre  $\mathcal{F}(X)$  of  $\mathbb{L} \rightarrow \hat{H}(C_2, K)$  are annihilated by  $l = 2^{2n}$ . However, by Theorem 1.1, the homotopy cofibre of the multiplication map  $l : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$  is trivial. That is, multiplication by  $l$  is an isomorphism on the homotopy groups of  $\mathcal{F}(X)$ . But we just noticed that this map is the zero map. Hence, the homotopy groups of  $\mathcal{F}(X)$  are zero.  $\square$

**2.5. Proof of Remark 1.4.** The square

$$\begin{array}{ccc} GW(X) & \longrightarrow & \mathbb{G}W(X) \\ \downarrow & & \downarrow \\ K(X)^{hC_2} & \longrightarrow & \mathbb{K}(X)^{hC_2} \end{array}$$

is homotopy cartesian; see [Sch].  $\square$

## REFERENCES

- [BKØ] J. Berrick, M. Karoubi, and P. A. Østvær. Periodicity of hermitian  $k$ -groups. to appear.
- [HKO] Po Hu, Igor Kriz, and Kyle Ormsby. Equivariant and Real Motivic Stable Homotopy Theory. K-theory preprint archive.
- [Kob99] Damjan Kopal.  $K$ -theory, Hermitian  $K$ -theory and the Karoubi tower. *K-Theory*, 17(2):113–140, 1999.
- [RØ05] Andreas Rosenschon and Paul Arne Østvær. The homotopy limit problem for two-primary algebraic  $K$ -theory. *Topology*, 44(6):1159–1179, 2005.
- [Sch] Marco Schlichting. Hermitian  $K$ -theory, derived equivalences and Karoubi’s fundamental theorem. in preparation.
- [Sch06] Marco Schlichting. Negative  $K$ -theory of derived categories. *Math. Z.*, 253(1):97–134, 2006.
- [Sch10] Marco Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *Invent. Math.*, 179(2):349–433, 2010.
- [TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic  $K$ -theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.
- [Wil05] Bruce Williams. Quadratic  $K$ -theory and geometric topology. In *Handbook of  $K$ -theory. Vol. 1, 2*, pages 611–651. Springer, Berlin, 2005.
- [WW00] Michael S. Weiss and Bruce Williams. Products and duality in Waldhausen categories. *Trans. Amer. Math. Soc.*, 352(2):689–709, 2000.

MARCO SCHLICHTING, MATHEMATICS INSTITUTE, ZEEMAN BUILDING, UNIVERSITY OF WARWICK,  
COVENTRY, CV4 7AL, UK

*E-mail address:* m.schlichting@warwick.ac.uk