ON GENERALIZATION OF KIBLE-SLEPIAN KERNEL FORMULA

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Abstract. We study generalization of Kible-Slepian (K-S) expansion formula in 3 dimensions. The generalization is obtained by replacing in K-S formula Hermite polynomials by $q-$Hermite ones. If such replacement would lead to non-negativity of values for all allowed values of parameters and for values of variables from certain Cartesian product of compact intervals then we would deal with generalization of 3 dimensional normal distribution. We show that this is not the case. We indicate some values of parameters and some compact set in $\mathbb{R}^3$ of positive measure, such that values of the extension K-S formula are on this set negative. Nevertheless we indicate other applications of so generalized K-S formula, namely we use this formula to sum certain kernels built of Al-Salam-Chihara polynomials for cased that were not considered by other authors. One of such kernels sum up to Askey-Wilson density. Hence we are able to obtain generalization of 2 dimensional Poisson-Mehler formula. We also pose several open questions.

1. Introduction

In 1945 W.F. Kibble [2] and later independently Slepian [1] have extended Poisson-Mehler formula to higher dimensions, expanding ratio of the standardized multidimensional Gaussian density divided by the product of one dimensional marginal densities in multiple sum involving only constants (correlation coefficients) and Hermite polynomials. The symmetry of this beautiful formula encourages further generalizations in the sense that Hermite polynomials in the Kible-Slepian formula are substituted by their generalization. In recent years such nice generalization of Hermite polynomials was intensively studied. These are so called $q-$Hermite polynomials a one parameter family of orthogonal polynomials that for $q=1$ are exactly equal to classical Hermite polynomials. Thus the question is if similar sum is nonnegative with ordinary 'probabilistic' Hermite polynomials substituted by $q-$Hermite ones for some values of parameter q (say $-1<q\leq 1$) and all values of correlation coefficients that make variance-covariance matrix positive definite. It will turn out that not. We will indicate particular values of q and correlation coefficients such that in 3 dimensions such a sum is negative. Nevertheless we also indicate that it is worth to study described above sums since we obtain a nice simple tool to study properties of different kernels involving so called Al-Salam-Chihara (ASC) polynomials. Kernels built of ASC polynomials were studied by Askey, Rahman and Suslov in [4]. These kernels have many applications in quantum physics in particular in studying different so called $q-$oscillators. See e.g. [5], [6].

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Kernels obtained by studying described above sums of $q-$Hermite polynomials are different then those studied in [4]. Hence we obtain new results related to important problem of summing kernels. One of this new results is generalization of Poisson-Mehler expansion formula in the sense that $q-$Hermite polynomials are replaced by ASC polynomials. Of course sum is different. Instead of the density of the measure than makes ASC polynomials orthogonal (classical case) we get density of measure that makes Askey-Wilson polynomials orthogonal. We also analyze other non-symmetric kernels built of $q-$Hermite polynomials and sum them.

The paper is organized as follows. In the next section we introduce all necessary auxiliary information concerning so called $q-$series theory, $q-$Hermite and ASC polynomials. In the following section we present our main results while less interesting or longer proofs are shifted to the last section. We also include special section with open problems since not all questions that appeared when studying this beautiful object we were able to answer.

2. Notation and auxiliary results

Let us introduce notation traditionally used in so called $q-$series theory. q is a parameter in most cases $-1 < q \leq 1$, however in some applications one considers also $q > 1$ or q complex and $|q| < 1$. Having q we define $[0]_q = 0$; $[n]_q = 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$, $[n]_q! = \prod_{i=1}^{n}[i]_q$, with $[0]_q! = 1$, $\genfrac{[}{]}{0pt}{}{n}{k}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q![k]_q!} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases}$. It will be useful to use so called $q-$Pochhammer symbol for $n \geq 1: (a|q)_n = \prod_{i=0}^{n-1}\left(1 - aq^i\right)$, with $(a|q)_0 = 1$, $(a_1, a_2, \ldots, a_k|q)_n = \prod_{i=1}^{k}(a_i|q)_n$. Often $(a|q)_n$ as well as $(a_1, a_2, \ldots, a_k|q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if it will not cause misunderstanding.

It is easy to notice that $(q)_n = (1-q)^n [n]_q!$ and that
$\genfrac{[}{]}{0pt}{}{n}{k}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases}$.
Notice that for $n \geq k \geq 0$ we have: $[n]_1 = n, [n]_1! = n!$, $\genfrac{[}{]}{0pt}{}{n}{k}_1 = \binom{n}{k}$ (Newton symbol $\binom{n}{k} = \frac{n!}{(n-k)!k!}$), $(a|1)_n = (1-a)^n$ and $[n]_0 = \begin{cases} 1 & if \quad n \geq 1 \\ 0 & if \quad n = 0 \end{cases}$, $[n]_0! = 1$, $\genfrac{[}{]}{0pt}{}{n}{k}_0 = 1$, $(a|0)_n = \begin{cases} 1 & if \quad n = 0 \\ 1-a & if \quad n \geq 1 \end{cases}$.

To define briefly and swiftly these one-dimensional distributions that later will be used to construct possible multidimensional generalizations of normal distributions, let us define the following sets for $|q| < 1:$

$$S(q) = [-2/\sqrt{1-q}, 2/\sqrt{1-q}], S(1) = \mathbb{R}.$$

Let us also define the following sets of polynomials:

-the $q-$Hermite polynomials defined by

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q), \tag{2.1}$$

for $n \geq 1$ with $H_{-1}(x|q) = 0$, $H_0(x|q) = 1$, and

-the so called Al-Salam-Chihara polynomials defined by the relationship for $n \geq 0:$

$$P_{n+1}(x|y,\rho,q) = (x - \rho y q^n)P_n(x|y,\rho,q) - (1-\rho^2 q^{n-1})[n]_q P_{n-1}(x|y,\rho,q), \tag{2.2}$$

with $P_{-1}\left(x|y,\rho,q\right)=0,\ P_0\left(x|y,\rho,q\right)=1.$

-the Chebyshev polynomials of the second kind defined by the relationship for $n\geq 1$

$$2xU_n\left(x\right)=U_{n+1}\left(x\right)+U_{n-1}\left(x\right),$$

with $U_{-1}\left(x\right)=0$, $U_0\left(x\right)=1$.

Notice that $H_n\left(x|1\right)=H_n\left(x\right), H_n\left(x|0\right)=U_n\left(x/2\right)$ and $P_n\left(x|y,\rho,1\right)=H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\left(1-\rho^2\right)^{n/2}$, where $H_n\left(x\right)$ denotes so called 'probabilistic' Hermite polynomials i.e. monic polynomials that are orthogonal with respect to measure with density $\exp(-x^2/2)/\sqrt{2\pi}$, below we will show that

$$P_n\left(x|y,\rho,0\right)=U_n\left(x/2\right)-\rho yU_{n-1}\left(x/2\right)+\rho^2U_{n-2}\left(x/2\right).$$

Polynomials (2.1) satisfy the following very useful identity originally formulated for so called continuous $q-$Hermite polynomials h_n (can be found in e.g. [12] Thm. 13.1.5) and here below presented for polynomials H_n using the relationship

$$h_n\left(x|q\right)=\left(1-q\right)^{n/2}H_n\left(\frac{2x}{\sqrt{1-q}}|q\right),\ n\geq 1, \tag{2.3}$$

$$H_n\left(x|q\right)H_m\left(x|q\right)=\sum_{j=0}^{\min(n,m)}\begin{bmatrix}m\\j\end{bmatrix}_q\begin{bmatrix}n\\j\end{bmatrix}_q\left[j\right]_q!H_{n+m-2k}\left(x|q\right). \tag{2.4}$$

One can find also in the literature (e.g. [11], [9]) the following useful formula:

$$H_{n+m}\left(x\right)\sum_{j=0}^{\min(n,m)}(-1)^jq^{\binom{j}{2}}\begin{bmatrix}m\\j\end{bmatrix}_q\begin{bmatrix}n\\j\end{bmatrix}_q\left[j\right]_q!H_{n-j}\left(x\right)H_{m-j}\left(x\right) \tag{2.5}$$

that was originally formulated for so called Rogers-Szegö polynomials defined by:

$$W_n\left(x|q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q x^j, \tag{2.6}$$

that are related to continuous $q-$Hermite polynomials by

$$h_n\left(x|q\right)=e^{in\theta}W_n\left(e^{-i2\theta}|q\right),$$

with $x=\cos\theta$. To simplify notation let us introduce the following auxiliary polynomials of order at most 2.

$$r_k\left(y|q\right)=(1+q^k)^2-(1-q)yq^k, \tag{2.7}$$

$$v_k\left(x|y,\rho,q\right)=(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})xy+(1-q)\rho^2(x^2+y^2)q^{2k} \tag{2.8}$$

It is known (see e.g. [8]) that $q-$Hermite polynomials are monic and orthogonal with respect to the measure that has density given by:

$$f_N\left(x|q\right)=\frac{(q)_\infty\sqrt{1-q}}{2\pi\sqrt{r_0\left(x^2|q\right)}}\prod_{k=0}^{\infty}r_k\left(x^2|q\right)I_{S(q)}\left(x\right), \tag{2.9}$$

defined for $|q|<1$, $x\in\mathbb{R}$, We will set also

$$f_N\left(x|1\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right). \tag{2.10}$$

Similarly it is known (e.g. from [10]) that $\{P_n(x|y,\rho,q)\}_{n\geq -1}$ are monic and orthogonal with respect to measures that for $q\in(-1,1]$ have density:

$$f_{CN}(x|y,\rho,q)=\frac{\sqrt{1-q}\,(q)_\infty\left(\rho^2\right)_\infty}{2\pi\sqrt{r_0\left(x^2|q\right)}}\prod_{k=0}^{\infty}\frac{r_k\left(x^2|q\right)}{v_k(x|y,\rho,q)}I_{S(q)}(x) \tag{2.11a}$$

defined for $|q|<1$, $|\rho|<1$, $x\in\mathbb{R}$, $y\in S(q)$. For $q=1$ we set

$$f_{CN}(x|y,\rho,1)=\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}}\exp\left(-\frac{(x-\rho y)^2}{2\left(1-\rho^2\right)}\right).$$

It is known (see e.g. [12] formula 13.1.10) that for $|q|<1:$

$$\sup_{x\in S(q)}|H_n(x|q)|\leq W_n(1|q)(1-q)^{-n/2}. \tag{2.12}$$

where W_n is given by (2.6).

We will need also the polynomials $\{B_n(x|q)\}_{n\geq -1}$ defined by the following 3-term recurrence:

$$B_{n+1}(y|q)=-q^n yB_n(y|q)+q^{n-1}[n]_q B_{n-1}(y|q)\,;n\geq 0, \tag{2.13}$$

with $B_{-1}(y|q)=0$, $B_0(y|q)=1$. One can show (see e.g. [10]) the $B_n(x|1)=i^nH_n(ix)$ and $B_n(x|0)=\left\{\begin{array}{ccc}1 & if & n=0\vee 2\\ -x & if & n=1\\ 0 & if & n>2\end{array}\right.$.

Further facts concerning q-Hermite and Al-Salam-Chihara polynomials are collected in the following Lemma

Lemma 1. *Assume that* $0<q\leq 1$, $|\rho|,|\rho_1|,|\rho_2|<1$, $n,m\geq 0$, $x,y,z\in S(q)$, *then*

i) $H_n(x|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\rho^{n-k}H_{n-k}(y|q)P_k(x|y,\rho,q)$

ii) $\forall n\geq 1:P_n(x|y,\rho,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\rho^{n-k}B_{n-k}(y|q)H_n(x|q)$, $\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}B_{n-j}(x|q)H_j(x|q)=0$,

iii) $\int_{S(q)}H_n(x|q)H_m(x|q)f_N(x|q)dx=\left\{\begin{array}{ccc}0 & when & n\neq m\\ [n]_q! & when & n=m\end{array}\right.$.

iv) $\int_{S(q)}P_n(x|y,\rho,q)P_m(x|y,\rho,q)f_{CN}(x|y,\rho,q)dx=\left\{\begin{array}{ccc}0 & when & n\neq m\\ \left(\rho^2\right)_n[n]_q! & when & n=m\end{array}\right.$.

v) $\int_{S(q)}H_n(x|q)f_{CN}(x|y,\rho,q)dx=\rho^nH_n(y|q)$,

vi) $\int_{S(q)}f_{CN}(x|y,\rho_1,q)f_{CN}(y|z,\rho_2,q)dy=f_{CN}(x|z,\rho_1\rho_2,q)$

vii) $\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n(x|q)H_n(y|q)=f_{CN}(x|y,\rho,q)/f_N(x|q)$.

viii) For $|t|,|q|<1:$

$$\sum_{i=0}^{\infty}\frac{W_i(1|q)t^i}{(q)_i}=\frac{1}{(t)_\infty^2},\sum_{i=0}^{\infty}\frac{W_i^2(1|q)t^i}{(q)_i}=\frac{\left(t^2\right)_\infty}{(t)_\infty^4},$$

convergence is absolute, where $W_i(x|q)$ *is defined by (2.6).*

ix) $\forall x,y\in S(q):0<C(y,\rho,q)\leq\frac{f_{CN}(x|y,\rho,q)}{f_N(x|q)}\leq\frac{\left(\rho^2\right)_\infty}{(\rho)_\infty^4}$

Proof. ii) was proved in [10]. i) is proved in [13] (formula 4.7) for polynomials h_n and $Q_n(x|a,b,q)$ related to polynomials P_n by the relationship $Q_n(x|a,b,q)=$

$(1-q)^{n/2}P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}},\sqrt{b},q\right)$. However one can also derive it easily it from ii). iii) and iv) are known (see e.g. [12]) for polynomials h_n and Q_n. v) and vi) can be found in [8], but its particular cases in different form were also shown in [7]. vii) is in fact famous Poisson-Mehler formula which has many proofs. One of them is in [12] the other e.g. in [3]. viii) see Exercise 12.2(b) and 12.2(c) of [12]. ix) was proved in [16] (assertion vii) of Proposition 1). □

Let us remark, following [16], that

$$f_{AW}(x|y,\rho_1,z,\rho_2,q)=\frac{f_{CN}\left(y|x,\rho_1,q\right)f_{CN}\left(x|z,\rho_2,q\right)}{f_{CN}\left(y|z,\rho_1\rho_2,q\right)} \tag{2.14}$$

is the density of measure that makes re-scaled Askey-Wilson (AW) polynomials for some complex values of parameter (related to y,ρ_1,z,ρ_2, for details see formula (2.5) of [16]). We will call it AW density.

As mentioned in the Introduction the main subject of this paper is the generalization of Kibble-Slepian formula. It rather difficult to express in its generality formula expanding ratio of the non-degenerated multidimensional Gaussian density divided by the product of its one dimensional marginals in the multiple series (in fact involving $n(n+1)/2$ (n being the dimension) fold sum of Hermite polynomials of one variable with coefficients that are powers of off diagonal elements of variance-covariance matrix.

We will analyze only generalization of its three dimensional version. It is simple to express and general enough to expose interesting properties and applications. Namely we will analyze the following function

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)=\prod_{i=1}^{3}f_N\left(x_i|q\right) \tag{2.15}$$

$$\times\sum_{i,j,k\geq 0}\frac{\rho_{12}^{j}\rho_{23}^{k}\rho_{13}^{i}}{[i]_q!\,[j]_q!\,[k]_q!}H_{i+j}\left(x_1|q\right)H_{j+k}\left(x_2|q\right)H_{i+k}\left(x_3|q\right), \tag{2.16}$$

where $|\rho_{12}|,|\rho_{13}|,|\rho_{23}|<1$ and $\Delta\overset{df}{=}\det\left[\begin{array}{ccc}1&\rho_{12}&\rho_{13}\\\rho_{12}&1&\rho_{23}\\\rho_{13}&\rho_{23}&1\end{array}\right]=1+2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2$ $-\rho_{13}^2-\rho_{23}^2=>0$. Notice that

$$\Delta=(1-\rho_{12}^2)(1-\rho_{23}^2)-(\rho_{13}-\rho_{12}\rho_{23})^2, \tag{2.17}$$

and similarly for other pairs of indices $(1,2)$ and $(2,3)$.

The above mentioned assumptions concerning parameters $\rho_{12},\rho_{13,}\ \rho_{23}$ will be assumed throughout the remaining part of the paper. Similarly we will assume that $x_i\in S\left(q\right),\ i=1,2,3$ unless otherwise stated. Besides from assertions viii) ix) of Lemma 1 it follows that all considered in this paper series for $|q|<1$ are absolutely convergent, hence we will not repeat this statement unless it will be necessary.

Let us immediately observe that when $q=1$, that is when $H_n\left(x|q\right)$ is substituted be $H_n\left(x\right)$ and $f_N\left(x_i\right)=\exp\left(-x_i^2/2\right)/\sqrt{2\pi}$, then by Kibble-Slepian formula (see e.g. [1] Example 2)) for $n=3$ and matrix $\boldsymbol{\varrho}=\left[\begin{array}{ccc}1&\rho_{12}&\rho_{13}\\\rho_{12}&1&\rho_{23}\\\rho_{13}&\rho_{23}&1\end{array}\right]$,

$f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},1\right)$ is the density of normal distribution $N\left(\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1&\rho_{12}&\rho_{13}\\\rho_{12}&1&\rho_{23}\\\rho_{13}&\rho_{23}&1\end{bmatrix}\right)$.

Let us present some immediate remarks concerning properties of f_{3D}.

Remark 1. i) $\int_{S(q)}\int_{S(q)}\int_{S(q)}f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)dx_1dx_2dx_3=1$,

ii) $\int_{S(q)\times S(q)}f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)dx_1dx_2=f_N\left(x_3|q\right)$,

$\int_{S(q)\times S(q)\times S(q)}H_n\left(x_1|q\right)H_m\left(x_2|q\right)f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)dx_3dx_1dx_2$
$=\begin{cases}0 & if\quad n\neq m\\ \rho_{12}^n\left[n\right]_q! & if\quad m=n\end{cases}$ and similarly for other pairs $(1,3)$ and $(2,3)$

iii) $\int_{S(q)}f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)dx_3=f_{CN}\left(x_1|x_2,\rho_{12},q\right)f_N\left(x_2|q\right)$

$\int_{S(q)\times S(q)\times S(q)}x_1f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)dx_1dx_2dx_3=0$,
$\int_{S(q)\times S(q)\times S(q)}x_1^2f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)dx_1dx_2dx_3=1$ and again similarly for the remaining indices 2 and 3.

iv) If additionally $\rho_{13}=\rho_{23}=0$, then $f_{3D}\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)=$
$f_{CN}\left(x_1|x_2,\rho_{12},q\right)f_N\left(x_2|q\right)f_N\left(x_3|q\right)$ and similarly for other pairs $(1,3)$ and $(2,3)$

Proof. In all assertions we apply assertion iii) of Lemma 1 , the fact that $H_1\left(x|q\right)$ $=x$ and $H_2\left(x\right)=x^2-1$ and also formulae (2.4) and assertion v) of Lemma 1. iv) follows directly from assertion vii) of Lemma 1. □

As it follows from the above mentioned Remark f_{3D} is a serious candidate for a 3 dimensional density. To analyze its properties deeper we will need the following Lemma.

Lemma 2. *Let us denote* $\gamma_{m,k}\left(x,y|\rho,q\right)=\sum_{i=0}^{\infty}\frac{\rho^i}{[i]_q!}H_{i+m}\left(x|q\right)H_{i+k}\left(y|q\right)$. *Then*

$$\gamma_{m,k}\left(x,y|\rho,q\right)=\gamma_{0,0}\left(x,y|\rho,q\right)Q_{m,k}\left(x,y|\rho,q\right), \tag{2.18}$$

where $Q_{m,n}$ *is a polynomial in* x *and* y *of order at most* $m+k$.
Further denote $C_n\left(x,y|\rho_1,\rho_2,\rho_3,q\right)=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\rho_1^{n-i}\rho_2^iQ_{n-i,i}\left(x,y|\rho_{3,}q\right)$.
Then we have in particular

i) $Q_{m,k}\left(x,y|\rho,q\right)=\sum_{s=0}^{k}(-1)^sq^{\binom{s}{2}}\begin{bmatrix}k\\s\end{bmatrix}_q\rho^sH_{k-s}\left(y|q\right)P_{m+s}(x|y,\rho,q)/(\rho^2)_{m+s}$,

ii) $C_n\left(x,y|\rho_1,\rho_2,\rho_3,q\right)=\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_qH_{n-s}\left(y|q\right)P_s\left(x|y,\rho_3,q\right)\rho_1^{n-s}\rho_2^s\left(\rho_1\rho_3/\rho_2\right)_s/\left(\rho_3^2\right)_s$.

iii) In particular $\gamma_{m,0}\left(x,y,|\rho,q\right)=\gamma_{0,0}\left(x,y,|\rho,q\right)P_m\left(x|y,\rho,q\right)/\left(\rho^2\right)_m$ *for all* $x,y\in S\left(q\right)$ *and* $q\in(-1,1)$.

Proof. Assertions i) and iii) are proved in [16]. Thus we will present the proof of ii). We have $C_n\left(x,y|\rho_1,\rho_2,\rho_3,q\right)=$
$\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\rho_1^{n-i}\rho_2^i\sum_{j=0}^{n-i}(-1)^j\begin{bmatrix}n-i\\j\end{bmatrix}_qq^{\binom{j}{2}}\rho_3^jH_{n-i-j}\left(y|q\right)P_{i+j}\left(x|y,\rho_3,q\right)/(\rho_3^2)_{i+j}=$
$\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_qH_{n-s}\left(y|q\right)P_s\left(x|y,\rho_3,q\right)/\left(\rho_3^2\right)_s\sum_{j=0}^{s}\begin{bmatrix}s\\j\end{bmatrix}_q(-1)^jq^{\binom{j}{2}}\rho_1^{n-s+j}\rho_2^{s-j}\rho_3^j=$
$\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_qH_{n-s}\left(y|q\right)P_s\left(x|y,\rho_3,q\right)\rho_1^{n-s}\rho_2^s/\left(\rho_3^2\right)_s\sum_{j=0}^{s}\begin{bmatrix}s\\j\end{bmatrix}_q(-1)^jq^{\binom{j}{2}}(\rho_1\rho_3/\rho_2)^j=$
$\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_qH_{n-s}\left(y|q\right)P_s\left(x|y,\rho_3,q\right)\rho_1^{n-s}\rho_2^s\left(\rho_1\rho_3/\rho_2\right)_s/\left(\rho_3^2\right)_s$. On the way we have used the following identity $\left(a\right)_n=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\left(-1\right)^iq^{\binom{i}{2}}a^j$. □

We get immediate observations:

Corollary 1. *For all $n \geq 1$ we have:*

i) $P_n(x|y,\rho,q) = (\rho^2)_n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2}} H_{n-i}(x|q) P_i(y|x,\rho,q) / (\rho^2)_i$

ii) $\int_{S(q)} P_n(x|y,\rho,q) f_{CN}(y|x,\rho,q)\, dy = (\rho^2)_n H_n(x|q)$.

iii) $C_n(x,y|\rho_2\rho_3,\rho_2,\rho_3,q) = \rho_2^n H_n(x|q)$

iv) $C_n(x,y|\rho_1,\rho_1\rho_3,\rho_3,q) = \rho_1^n H_n(y|q)$

v) $C_n(x,y|\rho_1,\rho_2,0,q) = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q \rho_1^{n-s}\rho_2^s H_{n-s}(y|q) H_s(x|q)$, $C_n(x,y|0,\rho_2,\rho_3,q)$ $= \rho_2^n P_n(x|y,\rho_3,q) / (\rho_3^2)_n$, $C_n(x,y|\rho_1,0,\rho_3,q) = \rho_2^n P_n(y|x,\rho_3,q) / (\rho_3^2)_n$,

Proof. i) see Corollary 2 of [16] . ii) We use previous assertion and assertion iv) of Lemma 1. iii) We have $(\rho_2\rho_3^2/\rho_2)_s = (\rho_3^2)_s$, hence $C_n(x,y|\rho_2\rho_3,\rho_2,\rho_3,q) = \rho_2^n \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q H_{n-s}(y|q) P_s(x|y,\rho_3,q) \rho_3^{n-s} = \rho_2^n H_n(x|q)$ by assertion i) of Lemma 1. iv) $(\rho_1\rho_3/\rho_1\rho_3)_s = \begin{cases} 1 & if \quad s=0 \\ 0 & if \quad s>0 \end{cases}$ v) First two statements are direct consequence of the assumptions that $\rho_3 = 0$ or $\rho_1 = 0$. The third one follows the fact that $\rho_2^s(\rho_1\rho_3/\rho_2)_s = \prod_{i=1}^{s}(\rho_2 - q^{i-1}\rho_1\rho_3)$, which for $\rho_2 = 0$ is equal to $(-1)^s q^{\binom{s}{2}} \rho_1^s\rho_3^s$. Then we apply assertion i) of this Corollary. □

3. Main results

Applying Lemma 2 to function f_{3D} we have the following Proposition

Proposition 1. *i)*

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q) = f_{CN}(x_3|x_1,\rho_{13},q) f_N(x_1|q) f_N(x_2|q) \times \sum_{s\geq 0} \frac{1}{[s]_q!} H_s(x_2|q) C_s(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q), \tag{3.1}$$

similarly for other pairs $(1,3)$ *and* $(2,3)$,

ii)

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q) = f_{CN}(x_1|x_3,\rho_{13},q) f_{CN}(x_3|x_2,\rho_{23},q) f_N(x_2|q) \times \sum_{s=0}^{\infty} \frac{\rho_{12}^s(\rho_{13}\rho_{23}/\rho_{12})_s}{[s]_q!(\rho_{13}^2)_s(\rho_{23}^2)_s} P_s(x_1|x_3,\rho_{13},q) P_s(x_2|x_3,\rho_{23},q), \tag{3.2}$$

similarly for other pairs $(1,3)$ *and* $(2,3)$.

Proof. Lengthy proof is shifted to section 5. □

As an immediate corollary we have the following formula:

Corollary 2.

$$\sum_{s=0}^{\infty} \frac{\rho_{12}^s(\rho_{13}\rho_{23}/\rho_{12})_s}{[s]_q!(\rho_{13}^2)_s(\rho_{23}^2)_s} P_s(x_1|x_3,\rho_{13},q) P_s(x_2|x_3,\rho_{23},q)$$
$$= \frac{f_{CN}(x_1|x_2,\rho_{12},q)}{f_{CN}(x_1|x_3,\rho_{13},q)} \sum_{k\geq 0} \frac{\rho_{13}^k(\rho_{12}\rho_{23}/\rho_{13})_k}{[k]_q!(\rho_{12}^2)_k(\rho_{23}^2)_k} P_k(x_1|x_2,\rho_{12},q) P_k(x_3|x_2,\rho_{23},q).$$

Proof. From assertion ii) of the Proposition 1 we get

$$\sum_{s=0}^{\infty}\frac{\rho_{12}^{s}\left(\rho_{13}\rho_{23}/\rho_{12}\right)_{s}}{[s]_{q}!\left(\rho_{13}^{2}\right)_{s}\left(\rho_{23}^{2}\right)_{s}}P_{s}\left(x_{1}|x_{3},\rho_{13},q\right)P_{s}\left(x_{2}|x_{3},\rho_{23},q\right)=$$

$$\frac{f_{CN}\left(x_{1}|x_{2},\rho_{12},q\right)f_{CN}\left(x_{2}|x_{3},\rho_{23},q\right)f_{N}\left(x_{3}|q\right)}{f_{CN}\left(x_{1}|x_{3},\rho_{13},q\right)f_{CN}\left(x_{3}|x_{2},\rho_{23},q\right)f_{N}\left(x_{2}|q\right)}\sum_{k\geq0}\frac{\rho_{13}^{k}\left(\rho_{12}\rho_{23}/\rho_{13}\right)_{k}}{[k]_{q}!\left(\rho_{12}^{2}\right)_{k}\left(\rho_{23}^{2}\right)_{k}}P_{k}\left(x_{1}|x_{2},\rho_{12},q\right)P_{k}\left(x_{3}|x_{2},\rho_{23},q\right).$$

Now we use the fact that: $f_{CN}\left(x_{2}|x_{3},\rho_{23},q\right)f_{N}\left(x_{3}|q\right)=f_{CN}\left(x_{3}|x_{2},\rho_{23},q\right)f_{N}\left(x_{2}|q\right).$ □

Corollary 3. *i) If* $\rho_{12}=\rho_{13}\rho_{23}$ *then*

$$f_{3D}\left(x_{1},x_{2},x_{3}|\rho_{13}\rho_{23},\rho_{13},\rho_{23},q\right)=f_{CN}\left(x_{1}|x_{3},\rho_{13},q\right)f_{CN}\left(x_{3}|x_{2},\rho_{23},q\right)f_{N}\left(x_{2}|q\right).$$

ii)If $\rho_{12}=\rho_{13}\rho_{23}$ *then*

(3.3)
$$\sum_{s\geq0}\frac{\rho_{13}^{3}}{[s]_{q}!\left(\rho_{13}^{2}\rho_{23}^{2}\right)_{s}}P_{s}\left(x_{1}|x_{2},\rho_{13}\rho_{23},q\right)P_{s}\left(x_{3}|x_{2},\rho_{23},q\right)=\frac{f_{CN}\left(x_{1}|x_{3},\rho_{13},q\right)}{f_{CN}\left(x_{1}|x_{2},\rho_{13}\rho_{23},q\right)}$$

iii) If $\rho_{12}=0$ *then*

(3.4)
$$\sum_{s=0}^{\infty}\frac{\left(-1\right)^{s}q^{\binom{s}{2}}\rho_{13}^{s}\rho_{23}^{s}}{[s]_{q}!\left(\rho_{13}^{2}\right)_{s}\left(\rho_{23}^{2}\right)_{s}}P_{s}\left(x_{1}|x_{3},\rho_{13},q\right)P_{s}\left(x_{2}|x_{3},\rho_{23},q\right)$$
$$=\frac{f_{CN}\left(x_{1}|x_{2},\rho_{13},q\right)}{f_{CN}\left(x_{1}|x_{3},\rho_{13},q\right)}\sum_{k\geq0}\frac{\rho_{13}^{k}}{[k]_{q}!\left(\rho_{23}^{2}\right)_{k}}H_{k}\left(x_{1}|q\right)P_{k}\left(x_{3}|x_{2},\rho_{23},q\right)$$

Proof. i) We use Corollary 1 and deduce that $\sum_{s\geq0}\frac{1}{[s]_{q}!}H_{s}\left(x_{2}|q\right)C_{s}\left(x_{1},x_{3}|\rho_{12},\rho_{23},\rho_{13},q\right)$ $=\sum_{s\geq0}\frac{\rho_{23}^{s}}{[s]_{q}!}H_{s}\left(x_{2}|q\right)H_{s}\left(x_{3}|q\right)=f_{CN}\left(x_{3}|x_{2},\rho_{23},q\right)/f_{N}\left(x_{3}|q\right)$ or we notice that $\left(\rho_{13}\rho_{23}/\rho_{12}\right)_{s}$ when $\rho_{12}=\rho_{13}\rho_{23}$ is equal to 0 for $s\geq1$.

ii) We use equivalent form of (3.2) that is

$$\begin{aligned}f_{3D}(x_{1},x_{2},x_{3}|\rho_{12},\rho_{13},\rho_{23},q) &= f_{CN}\left(x_{1}|x_{2},\rho_{12},q\right)f_{CN}\left(x_{2}|x_{3},\rho_{23},q\right)f_{N}\left(x_{3}|q\right)\\ &\quad\times\sum_{s=0}^{\infty}\frac{\rho_{13}^{s}\left(\rho_{12}\rho_{23}/\rho_{13}\right)_{s}}{[s]_{q}!\left(\rho_{12}^{2}\right)_{s}\left(\rho_{23}^{2}\right)_{s}}P_{s}\left(x_{1}|x_{2},\rho_{13},q\right)P_{s}\left(x_{3}|x_{2},\rho_{23},q\right)\end{aligned}$$

apply assumption, observing that then $\left(\rho_{12}\rho_{23}/\rho_{13}\right)_{s}=\left(\rho_{23}^{2}\right)_{s}$ and using the assertion i) of this corollary.

iii) We use the fact that $\rho_{12}^{s}\left(\rho_{13}\rho_{23}/\rho_{12}\right)_{s}=\prod_{i=1}^{s}(\rho_{12}-q^{i-1}\rho_{13}\rho_{23})$ which for ρ_{12} $=0$ equals $\left(-1\right)^{s}q^{1+\ldots+s-1}\rho_{13}^{s}\rho_{23}^{s}$ and the fact that $P_{s}\left(x|y,0,q\right)=H_{s}\left(x|q\right).$ □

We have also the following remark concerning ordinary probabilistic Hermite polynomials

Remark 2. We have

$$\sqrt{\frac{(1-\rho_{13}^2)(1-\rho_{23}^2)-(\rho_{12}-\rho_{13}\rho_{23})^2}{(1-\rho_{13}^2)(1-\rho_{23}^2)}}$$
$$\times\sum_{s=0}^{\infty}\frac{(\rho_{12}-\rho_{13}\rho_{23})^s}{s!(1-\rho_{13}^2)^{s/2}(1-\rho_{23}^2)^{s/2}}H_s\left(\frac{x_1-\rho_{13}x_3}{\sqrt{1-\rho_{13}^2}}\right)H_s\left(\frac{x_2-\rho_{23}x_3}{\sqrt{1-\rho_{23}^2}}\right)$$
$$=\exp\left(-\frac{1}{2}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\left[\begin{bmatrix}1 & \rho_{12} & \rho_{13}\\ \rho_{12} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1}-\begin{bmatrix}1 & \rho_{13}\rho_{23} & \rho_{13}\\ \rho_{13}\rho_{23} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\right)$$

Proof. Follows (3.2) and the following facts: $P_n(x|y,\rho,1)/\left(\rho^2|1\right)_n=H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)/\left(1-\rho^2\right)^{n/2}$, $\det\begin{bmatrix}1 & \rho_{12} & \rho_{13}\\ \rho_{12} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}=(1-\rho_{13}^2)(1-\rho_{23}^2)-(\rho_{12}-\rho_{13}\rho_{23})^2$ and $f_{CN}(x_1|x_3,\rho_{12},1)\,f_{CN}(x_3|x_2,\rho_{23},1)\,f_N(x_2|1)=\frac{1}{\sqrt{2\pi(1-\rho_{13}^2)(1-\rho_{23}^2)}}\times$ $\exp(-\frac{1}{2}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}1 & \rho_{13}\rho_{23} & \rho_{13}\\ \rho_{13}\rho_{23} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix})$. □

We have also the following observation concerning Askey-Wilson density.

Remark 3. Following observation (2.14) and assertion vii) of Lemma 1 we deduce that $\frac{f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_3|x_2,\rho_{23},q)}{f_{CN}(x_1|x_2,\rho_{13}\rho_{23},q)}=f_{AW}(x_3|x_1,\rho_{13},x_2,\rho_{23},q)$ Askey-Wilson density. Hence we have the following expansion of AW density:

$$f_{AW}(x_3|x_1,\rho_{13},x_2,\rho_{23})=f_{CN}(x_3|x_2,\rho_{23},q)\sum_{s\geq 0}\frac{\rho_{13}^3}{[s]_q!\left(\rho_{13}^2\rho_{23}^2\right)_s}P_s(x_1|x_2,\rho_{13}\rho_{23},q)\,P_s(x_3|x_2,\rho_{23},q),$$

which is an analogue of Poisson-Mehler expansion formula interpreted as one dimensional expansion given in vii) of Lemma 1. This result has been obtained by other methods in [17].

As far as general properties of function f_{3D} we have the following formula that expresses function C_n in terms of $q-$Hermite polynomials.

Proposition 2. $\forall n\geq 1:$

$$C_n(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q)=\frac{1}{(\rho_{13}^2)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^kq^{\binom{k}{2}}\begin{bmatrix}n\\ 2k\end{bmatrix}_q\begin{bmatrix}2k\\ k\end{bmatrix}_q[k]_q!\rho_{12}^k\rho_{13}^k\rho_{23}^k\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_k\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}\right)_k$$
$$\sum_{i=0}^{n-2k}\begin{bmatrix}n-2k\\ i\end{bmatrix}_q\rho_{23}^i\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^k\right)_i\rho_{12}^{n-i-2k}\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}q^k\right)_{n-2k-i}H_i(x_1|q)\,H_{n-2k-i}(x_3|q)$$

Proof. Lengthy proof is shifted to section 5 □

Let us remark that when $\rho_{13}=\rho_{12}\rho_{23}$ then

$$C_n\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{12}\rho_{23},q\right)=\frac{1}{\left(\rho_{12}^2\rho_{23}^2\right)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\2k\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q[k]_q!\rho_{12}^{2k}\rho_{23}^{2k}\left(\rho_{12}^2\right)_k\left(\rho_{23}^2\right)_k$$
$$\sum_{i=0}^{n-2k}\begin{bmatrix}n-2k\\i\end{bmatrix}_q\rho_{23}^i\left(\rho_{12}^2q^k\right)_i\rho_{12}^{n-i-2k}\left(\rho_{23}^2q^k\right)_{n-2k-i}H_i\left(x_1|q\right)H_{n-2k-i}\left(x_3|q\right)$$

the formula obtained in [16] in the context of Askey-Wilson polynomials.

As stated in the introduction the problem of non-negativity of the function f_{3D} for all allowed values of q and $\rho's$ has negative solution. Above we indicated that if $\rho_{12}=\rho_{13}\rho_{23}$ (or similarly for some other pair of indices) then f_{3D} is positive for all $-1<q\leq 1$ and $x_1,x_2,x_3\in S\left(q\right)$. Now we will indicate some another relationship between $\rho's$ and q so that for some values of $x_1,x_2,x_3\in S\left(q\right)$ f_{3D} is negative.

We have the following Theorem

Theorem 1. *Assume $q\neq 0$. Let $\rho_{12}=q\rho_{13}\rho_{23}$, then there exists a set $S\subset S\left(q\right)\times S\left(q\right)\times S\left(q\right)$ of positive Lebesgue measure such that for all $\left(x_1,x_2,x_3\right)\in S$, f_{3D} is negative.*

Proof. Let us take $\rho_{12}=q\rho_{13}\rho_{23}$ and consider (3.2), then $\rho_{12}^s\left(\rho_{13}\rho_{23}/\rho_{12}\right)_s=\begin{cases}1 & if\quad s=0\\ -(1-q)\rho_{13}\rho_{23} & if\quad s=1\\ 0 & if\quad s>1\end{cases}$.
Hence $f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)=f_{CN}\left(x_1|x_3,\rho_{13},q\right)f_{CN}\left(x_3|x_2,\rho_{23},q\right)f_N\left(x_2|q\right)(1-(1-q)\rho_{13}\rho_{23}(x_1-\rho_{13}x_3)(x_2-\rho_{23}x_3)/((1-\rho_{13}^2)(1-\rho_{23}^2)))$. Thus the sign of f_{3D} is the same as the sign of $(1-(1-q)\rho_{13}\rho_{23}(x_1-\rho_{13}x_3)(x_2-\rho_{23}x_3)/((1-\rho_{13}^2)(1-\rho_{23}^2)))$ or equivalently if for all $x_1,x_2,x_3\in S\left(q\right)$ and $\left|\rho_{13}\right|,\left|\rho_{23}\right|<1$ and

$$(1-\rho_{13}^2)(1-\rho_{23}^2)\geq(1-q)^2\rho_{13}^2\rho_{23}^2$$

which comes from positivity in (2.17) we have

$$(1-\rho_{13}^2)(1-\rho_{23}^2)\geq(1-q)\rho_{13}\rho_{23}(x_1-\rho_{13}x_3)(x_2-\rho_{23}x_3).$$

Since $y_i\overset{df}{=}\sqrt{1-q}x_i\in[-2,2]$ this inequality reduces to the following :

$$(1-\rho_{13}^2)(1-\rho_{23}^2)\geq\rho_{13}\rho_{23}(y_1-\rho_{13}y_3)(y_2-\rho_{23}y_3)\tag{3.5}$$

Let us select $y_1=2\rho_{13}$ and $y_2=-2\rho_{23}$. The inequality now takes a form

$$(1-\rho_{13}^2)(1-\rho_{23}^2)\geq\rho_{13}^2\rho_{23}^2(4-y_3^2).$$

Now it suffices take $\rho_{13}^2,\rho_{23}^2=0.6$, $1>q>1/3$ and $y_3^2<4-\frac{4}{9}$ on one hand get

$$(1-\rho_{13}^2)(1-\rho_{23}^2)=.16>\left(1-\frac{1}{3}\right)^2.6^2$$

and on the other to get

$$(1-\rho_{13}^2)(1-\rho_{23}^2)=(1-.6)^2=<.6^2(4-(4-\frac{4}{9}))\leq\rho_{13}^2\rho_{23}^2(4-y_3^2).$$

Since function is continuous in x_1,x_2,x_3 thus there exists a neighborhood of points $\left(2\rho_{13}/(\sqrt{1-q},2\rho_{23}/\sqrt{1-q},2y_3/\sqrt{1-q}\right)$ such that this function is negative. □

As a corollary we get the following fact.

Corollary 4.

$$\frac{f_{CN}\left(x_1|x_3,\rho_{13},q\right)}{f_{CN}\left(x_1|x_2,q\rho_{13}\rho_{23},q\right)}\left(1-\frac{(1-q)\rho_{13}\rho_{23}(x_1-\rho_{13}x_3)(x_2-\rho_{23}x_3)}{(1-\rho_{13}^2)(1-\rho_{23}^2)}\right)$$
$$=\sum_{s=0}^{\infty}\frac{\rho_{13}^s\left(1-q^s\rho_{23}^2\right)}{[s]_q!\left(q^2\rho_{13}^2\rho_{23}^2\right)_s\left(1-\rho_{23}^2\right)}P_s\left(x_1|x_2,q\rho_{13}\rho_{23},q\right)P_s\left(x_3|x_2,\rho_{23},q\right). \tag{3.6}$$

Proof. Follows the (3.2) and the fact that for $\rho_{12}=q\rho_{13}\rho_{23}$ $\rho_{13}^s\left(\rho_{12}\rho_{23}/\rho_{13}\right)_s=\rho_{13}^s\left(q\rho_{23}^2\right)_s$ and further that $\frac{\left(q\rho_{23}^2\right)_s}{\left(\rho_{23}^2\right)_s}=\frac{1-q^s\rho_{23}^2}{1-\rho_{23}^2}$. □

Remark 4. Notice that we can take $q\rho_{13}\rho_{23}$ instead $\rho_{13}\rho_{23}$ in (3.3) getting $\frac{f_{CN}\left(x_1|x_3,\rho_{13},q\right)}{f_{CN}\left(x_1|x_2,q\rho_{13}\rho_{23},q\right)}$ $=\sum_{s=0}^{\infty}\frac{\rho_{13}^s}{[s]_q!\left(q^2\rho_{13}^2\rho_{23}^2\right)_s}P_s\left(x_1|x_2,q\rho_{13}\rho_{23},q\right)P_s\left(x_3|x_2,\rho_{23},q\right)$. Hence the positivity conditions for kernels of the form $\sum_{s=0}^{\infty}a_s\left(\rho_1,\rho_2\right)P_s\left(x_1|x_2,\rho_1,q\right)P_s\left(x_3|x_2,\rho_2,q\right)$ are quite subtle. Besides comparison of (3.3) and (3.6) can be the source of many interesting kernels involving Al-Salam-Chihara polynomials. In particular for ρ_{12} $=q^k\rho_{13}\rho_{23}$ we have

$$\frac{f_{CN}\left(x_1|x_3,\rho_{13},q\right)}{f_{CN}\left(x_1|x_2,q^k\rho_{13}\rho_{23},q\right)}\sum_{j=0}^{k}\frac{q^{kj}\rho_{13}^j\rho_{23}^j\left(q^{-k}\right)_j}{[j]_q!\left(\rho_{13}^2\right)_j\left(\rho_{23}^2\right)_j}P_j\left(x_1|x_3,\rho_{13},q\right)P_j\left(x_2|x_3,\rho_{23},q\right)$$
$$=\sum_{s=0}^{\infty}\frac{\rho_{13}^s\left(q^k\rho_{23}^2\right)_s}{[s]_q!\left(q^{2k}\rho_{13}^2\rho_{23}^2\right)_s\left(\rho_{23}^2\right)_s}P_s\left(x_1|x_2,q^k\rho_{13}\rho_{23},q\right)P_s\left(x_3|x_2,\rho_{23},q\right).$$

4. Open Problems

(1) Consider K-S formulae for higher (than 3) dimensions. Can it be nonnegative for all allowed values of parameters ρ and all values of variables form $S\left(q\right)$?
(2) Our counterexample does not work for $q=0$. Hence is K-S formula nonnegative for all allowed values of parameters ρ all values of variables form $[-2,2]$?
(3) What about cases $\rho_{12}=q^k\rho_{13}\rho_{23}$ for $k>1$. Do we get non-positivity for all $k>1$ or there are some for which it is nonnegative?
(4) Numerical simulations suggest that for say $\rho_{12}=0$ one can find q, ρ_{13},ρ_{23}, $x_1,x_2,x_3\in S\left(q\right)$ such that one of the kernels (and consequently both) given in (3.4) is negative. What would be the proof of this fact? More precisely what would be range of parameters q, ρ_{13},ρ_{23} and subset $A\subset S\left(q\right)^3$ of positive Lebesgue measure such that for $x_1,x_2,x_3\in A$ the kernel is negative.
(5) What about cases $|\rho_{12}|>\rho_{13}\rho_{23}$?

5. Proofs

Proof of Proposition 1. i) We apply assertion i) of the Lemma 2 to (2.15) and (2.16) and remembering that $\gamma_{0,0}\left(x,y|\rho,q\right)=f_{CN}\left(x|y,\rho,q\right)/f_{N}\left(x|q\right)$ we get:

$$f_{3D}\left(x_{1},x_{2},x_{3}|\rho_{12},\rho_{13},\rho_{23},q\right)=\prod_{i=1}^{3}f_{N}\left(x_{i}|q\right)\sum_{j,k\geq0}\frac{\rho_{12}^{j}\rho_{23}^{k}}{\left[j\right]_{q}!\left[k\right]_{q}!}H_{j+k}\left(x_{2}|q\right)\sum_{i\geq0}\frac{\rho_{13}^{i}}{\left[i\right]_{q}!}H_{i+j}\left(x_{1}|q\right)H_{i+k}\left(x_{3}|q\right)$$

$$=f_{CN}\left(x_{1}|x_{3},\rho_{13},q\right)f_{N}\left(x_{3}|q\right)f_{N}\left(x_{2}|q\right)\sum_{j,k\geq0}\frac{\rho_{12}^{j}\rho_{23}^{k}}{\left[j\right]_{q}!\left[k\right]_{q}!}H_{j+k}\left(x_{2}|q\right)Q_{j,k}(x_{1},x_{3}|\rho_{13},q).$$

Now changing order of summation and substituting $j+k->s$, $j->k$ we get

$$f_{3D}\left(x_{1},x_{2},x_{3}|\rho_{12},\rho_{13},\rho_{23},q\right)=f_{CN}\left(x_{3}|x_{1},\rho_{13},q\right)f_{N}\left(x_{1}|q\right)f_{N}\left(x_{2}|q\right)$$
$$\times\sum_{s\geq0}\frac{1}{\left[s\right]_{q}!}H_{s}\left(x_{2}|q\right)C_{s}\left(x_{1},x_{3}|\rho_{12},\rho_{23},\rho_{13},q\right).$$

ii) We apply (2.5). Then

$$f_{3D}(x_{1},x_{2},x_{3}|\rho_{12},\rho_{13},\rho_{23},q)=\prod_{i=1}^{3}f_{N}\left(x_{i}|q\right)\sum_{i,j,k\geq0}\frac{\rho_{12}^{j}\rho_{23}^{k}\rho_{13}^{i}}{\left[i\right]_{q}!\left[j\right]_{q}!\left[k\right]_{q}!}H_{i+j}\left(x_{1}|q\right)H_{j+k}\left(x_{2}|q\right)$$
$$\times\sum_{n\geq0}\begin{bmatrix}i\\n\end{bmatrix}_{q}\begin{bmatrix}k\\n\end{bmatrix}_{q}\left[n\right]_{q}!\left(-1\right)^{n}q^{\binom{n}{2}}H_{i-n}\left(x_{3}|q\right)H_{k-n}\left(x_{3}|q\right)=$$
$$\prod_{i=1}^{3}f_{N}\left(x_{i}|q\right)\sum_{n=0}^{\infty}\left(-1\right)^{n}q^{\binom{n}{2}}\frac{\rho_{23}^{n}\rho_{13}^{n}}{\left[n\right]_{q}!}\sum_{j=0}^{\infty}\frac{\rho_{12}^{j}}{\left[j\right]_{q}!}$$
$$\times\sum_{i=n}^{\infty}\sum_{k=n}^{\infty}\frac{\rho_{23}^{k-n}\rho_{13}^{i-n}}{\left[i-n\right]_{q}!\left[k-n\right]_{q}!}H_{i-n}\left(x_{3}|q\right)H_{k-n}\left(x_{3}|q\right)H_{i+j}\left(x_{1}|q\right)H_{j+k}\left(x_{2}|q\right)$$

$$=\prod_{i=1}^{3}f_{N}\left(x_{i}|q\right)\sum_{n=0}^{\infty}\left(-1\right)^{n}q^{\binom{n}{2}}\frac{\rho_{23}^{n}\rho_{13}^{n}}{\left[n\right]_{q}!}\sum_{j=0}^{\infty}\frac{\rho_{12}^{j}}{\left[j\right]_{q}!}$$
$$\times\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\rho_{23}^{k}\rho_{13}^{i}}{\left[i\right]_{q}!\left[k\right]_{q}!}H_{i}\left(x_{3}|q\right)H_{i+n+j}\left(x_{1}|q\right)H_{k}\left(x_{3}|q\right)H_{j+n+k}\left(x_{2}|q\right).$$

Now we use quantities: $\gamma_{m,k}\left(x,y|\rho,q\right)=\sum_{i=0}^{\infty}\frac{\rho^{i}}{\left[i\right]_{q}!}H_{i+m}\left(x|q\right)H_{i+k}\left(y|q\right)$ and apply (2.18) and assertions iii) of Lemma 2. We get then

$$f_{3D}(x_{1},x_{2},x_{3}|\rho_{12},\rho_{13},\rho_{23},q)=\prod_{i=1}^{3}f_{N}\left(x_{i}|q\right)\sum_{n=0}^{\infty}\left(-1\right)^{n}q^{\binom{n}{2}}\frac{\rho_{23}^{n}\rho_{13}^{n}}{\left[n\right]_{q}!}$$
$$\times\sum_{j=0}^{\infty}\frac{\rho_{12}^{j}}{\left[j\right]_{q}!}\gamma_{0,n+j}\left(x_{3},x_{1}|\rho_{13},q\right)\gamma_{0,n+j}\left(x_{3},x_{2}|\rho_{23},q\right)=f_{CN}\left(x_{1}|x_{3},\rho_{13},q\right)f_{CN}\left(x_{2}|x_{3},\rho_{23},q\right)f_{N}\left(x_{3}|q\right)$$
$$\times\sum_{n=0}^{\infty}\left(-1\right)^{n}q^{\binom{n}{2}}\frac{\rho_{23}^{n}\rho_{13}^{n}}{\left[n\right]_{q}!}\sum_{j=0}^{\infty}\frac{\rho_{12}^{j}}{\left[j\right]_{q}!\left(\rho_{13}^{2}\right)_{n+j}\left(\rho_{23}^{2}\right)_{n+j}}P_{n+j}\left(x_{1}|x_{3},\rho_{13},q\right)P_{n+j}\left(x_{2}|x_{3},\rho_{23},q\right)=$$

$$f_{CN}\left(x_1|x_3,\rho_{13},q\right)f_{CN}\left(x_2|x_3,\rho_{23},q\right)f_N\left(x_3|q\right)$$
$$\times\sum_{s=0}^{\infty}\frac{1}{[s]_q!\left(\rho_{13}^2\right)_s\left(\rho_{23}^2\right)_s}P_s\left(x_1|x_3,\rho_{13},q\right)P_s\left(x_2|x_3,\rho_{23},q\right)$$
$$\times\sum_{n=0}^{s}\begin{bmatrix}s\\n\end{bmatrix}_q(-1)q^{\binom{n}{2}}\left(\rho_{13}\rho_{23}\right)^n\rho_{12}^{s-n}=f_{CN}\left(x_1|x_3,\rho_{13},q\right)f_{CN}\left(x_2|x_3,\rho_{23},q\right)f_N\left(x_3|q\right)$$
$$\times\sum_{s=0}^{\infty}\frac{\rho_{12}^s\left(\rho_{13}\rho_{23}/\rho_{12}\right)_s}{[s]_q!\left(\rho_{13}^2\right)_s\left(\rho_{23}^2\right)_s}P_s\left(x_1|x_3,\rho_{13},q\right)P_s\left(x_2|x_3,\rho_{23},q\right).$$

□

Proof of Proposition 2. We start with $C_n\left(x,y|\rho_1,\rho_2,\rho_3,q\right)=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\rho_1^{n-i}\rho_2^iQ_{n-i,i}\left(x,y|\rho_{3,}q\right)$ $=\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_qH_{n-s}\left(y|q\right)P_s\left(x|y,\rho_3,q\right)\rho_1^{n-s}\rho_2^s\left(\rho_1\rho_3/\rho_2\right)_s/\left(\rho_3^2\right)_s$ and apply assertion ii) of Lemma 1.

$$C_n\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right)=\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}\left(x_3|q\right)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_sP_s\left(x_1|x_3,\rho_{13},q\right)/\left(\rho_{13}^2\right)_s$$
$$=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}\left(x_3|q\right)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_sP_s\left(x_1|x_3,\rho_{13},q\right)\left(\rho_{13}^2q^s\right)_{n-s}$$
$$=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}\left(x_3|q\right)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_s\left(\rho_{12}^2q^s\right)_{n-s}\sum_{i=0}^{s}\begin{bmatrix}s\\i\end{bmatrix}_q\rho_{13}^{s-i}B_{s-i}\left(x_3|q\right)H_i\left(x_1|q\right)$$
$$=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_qH_i\left(x_1|q\right)\sum_{s=i}^{n}\begin{bmatrix}n-i\\s-i\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}\left(x_3|q\right)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_s\left(\rho_{13}^2q^s\right)_{n-s}\rho_{13}^{s-i}B_{s-i}\left(x_3|q\right)$$

$$=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_qH_i\left(x_1|q\right)\times\sum_{j=0}^{n-i}\begin{bmatrix}n-i\\j\end{bmatrix}_q\rho_{12}^{n-i-j}H_{n-i-j}\left(x_3|q\right)\rho_{23}^{i+j}\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j}\left(\rho_{13}^2q^{i+j}\right)_{n-i-j}\rho_{13}^jB_j\left(x_3|q\right)$$

And further using formula

$$H_m\left(x|q\right)B_n\left(x|q\right)=(-1)^nq^{\binom{n}{2}}\sum_{i=0}^{\lfloor(n+m)/2\rfloor}\begin{bmatrix}n\\i\end{bmatrix}_q\begin{bmatrix}n+m-i\\i\end{bmatrix}_q[i]_q!q^{-i(n-i)}H_{n+m-2i}\left(x|q\right),$$

proved in [16] (assertion i) of Lemma 2) we get

$$C_n\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right)=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_qH_i\left(x_1|q\right)\times$$
$$\sum_{j=0}^{n-i}\begin{bmatrix}n-i\\j\end{bmatrix}_q\rho_{12}^{n-i-j}\rho_{23}^{i+j}\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j}\left(\rho_{13}^2q^{i+j}\right)_{n-i-j}\rho_{13}^j(-1)^jq^{\binom{j}{2}}$$
$$\times\sum_{k=0}^{\lfloor(n-i)/2\rfloor}\begin{bmatrix}j\\k\end{bmatrix}_q\begin{bmatrix}n-i-k\\k\end{bmatrix}_q[k]_q!q^{-k(j-k)}H_{n-i-2k}\left(x_3|q\right)$$

Now keeping in mind that $\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]_q = 0$ for $j < k$ we split first internal sum into two sums : one form 0 to $\lfloor (n-i)/2 \rfloor$ and second from $\lfloor (n-i)/2 \rfloor + 1$ to $(n-i)$.

$$= \frac{1}{(\rho_{13}^2)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q H_i(x_1|q) \times (\sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \begin{bmatrix} n-i \\ j \end{bmatrix}_q \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}} \right)_{i+j} \left(\rho_{13}^2 q^{i+j} \right)_{n-i-j} \rho_{13}^{j} (-1)^j q^{\binom{j}{2}}$$
$$\times \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-i-k \\ k \end{bmatrix}_q [k]_q! q^{-k(j-k)} H_{n-i-2k}(x_3|q)$$
$$+ \sum_{j=\lfloor (n-i)/2 \rfloor + 1}^{(n-i)} \begin{bmatrix} n-i \\ j \end{bmatrix}_q \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}} \right)_{i+j} \left(\rho_{13}^2 q^{i+j} \right)_{n-i-j} \rho_{13}^{j} (-1)^j q^{\binom{j}{2}}$$
$$\times \sum_{k=0}^{\lfloor (n-i)/2 \rfloor} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-i-k \\ k \end{bmatrix}_q [k]_q! q^{-k(j-k)} H_{n-i-2k}(x_3|q))$$

We have further after changing the order of summation.

$$C_n(x_1, x_3 | \rho_{12}, \rho_{23}, \rho_{13}, q) = \frac{1}{(\rho_{13}^2)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q H_i(x_1|q)$$
$$\times (\sum_{k=0}^{\lfloor (n-i)/2 \rfloor} \frac{[n-i]_q!}{[k]_q! [n-i-2k]_q!} H_{n-i-2k}(x_3|q)$$
$$\times \sum_{j=k}^{\lfloor (n-i)/2 \rfloor} q^{-k(j-k)} \rho_{13}^{j} (-1)^j q^{\binom{j}{2}} \frac{[n-i-k]_q!}{[j-k]_q! [n-i-j]_q!} \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}} \right)_{i+j} \left(\rho_{13}^2 q^{i+j} \right)_{n-i-j}$$
$$+ \sum_{k=0}^{\lfloor (n-i)/2 \rfloor} \frac{[n-i]_q!}{[k]_q! [n-i-2k]_q!} H_{n-i-2k}(x_3|q)$$
$$\times \sum_{j=\lfloor (n-i)/2 \rfloor + 1}^{(n-i)} \frac{[n-i-k]_q!}{[j-k]_q! [n-i-j]_q!} (-1)^j q^{-k(j-k)} q^{\binom{j}{2}} \rho_{13}^{j} \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}} \right)_{i+j} \left(\rho_{13}^2 q^{i+j} \right)_{n-i-j})$$

$$= \frac{1}{(\rho_{13}^2)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q H_i(x_1|q)$$
$$\times (\sum_{k=0}^{\lfloor (n-i)/2 \rfloor} \frac{[n-i]_q!}{[k]_q! [n-i-2k]_q!} H_{n-i-2k}(x_3|q)$$
$$\times \sum_{j=k}^{(n-i)} \begin{bmatrix} n-i-k \\ j-k \end{bmatrix}_q (-1)^j q^{-k(j-k)} q^{\binom{j}{2}} \rho_{13}^{j} \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}} \right)_{i+j} \left(\rho_{13}^2 q^{i+j} \right)_{n-i-j})$$

Hence

$$C_n\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right)=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i\left(x_1|q\right)$$

$$\times\sum_{k=0}^{\lfloor (n-i)/2\rfloor}(-1)^k q^{\binom{k}{2}}\frac{\left[n-i\right]_q!}{\left[k\right]_q!\left[n-i-2k\right]_q!}\rho_{13}^k H_{n-i-2k}\left(x_3|q\right)$$

$$\times\sum_{s=0}^{n-i-k}\begin{bmatrix}n-i-k\\s\end{bmatrix}_q(-1)^s q^{\binom{s}{2}}\rho_{13}^s\rho_{12}^{n-i-k-s}\rho_{23}^{i+k+s}\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+k+s}\left(\rho_{13}^2 q^{i+k+s}\right)_{n-i-k-s}$$

So

$$C_n\left(x_1,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i\left(x_1|q\right)$$

$$\times\sum_{k=0}^{\lfloor (n-i)/2\rfloor}(-1)^k q^{\binom{k}{2}}\frac{\left[n-i\right]_q!}{\left[k\right]_q!\left[n-i-2k\right]_q!}\rho_{13}^k\rho_{23}^{k+i}H_{n-i-2k}\left(x_3|q\right)\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{k+i}$$

$$\times\sum_{s=0}^{n-i-k}\begin{bmatrix}n-i-k\\s\end{bmatrix}_q(-1)^s q^{\binom{s}{2}}\rho_{13}^s\rho_{12}^{n-i-k-s}\rho_{23}^{s}\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^{i+k}\right)_{s}\left(\rho_{13}^2 q^{i+k+s}\right)_{n-i-k-s}.$$

Now we use formula $\sum_{i=0}^{n}(-1)^i q^{\binom{i}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q(a)_i b^i\left(abq^i\right)_{n-i}=(b)_n$ proved in ([16]) (assertion ii) of Lemma 1) with $a=\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^{i+k}$, $b=\rho_{13}\rho_{23}/\rho_{12}$ getting

$$C_n\left(x_1,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i\left(x_1|q\right)$$

$$\times\sum_{k=0}^{\lfloor (n-i)/2\rfloor}(-1)^k q^{\binom{k}{2}}\frac{\left[n-i\right]_q!}{\left[k\right]_q!\left[n-i-2k\right]_q!}\rho_{13}^k\rho_{23}^{k+i}H_{n-i-2k}\left(x_3|q\right)\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{k+i}\rho_{12}^{n-i-k}\left(\frac{\rho_{12}\rho_{23}}{\rho_{12}}\right)_{n-k-i}$$

Finally change the order of summation and using on the way an obvious property of $(a)_n=(a)_j\left(aq^j\right)_{n-j}$ for every $0\leq j\leq n$

$$C_n\left(x_1,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\2k\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q[k]_q!\rho_{12}^k\rho_{23}^k\rho_{13}^k\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_k\left(\frac{\rho_{12}\rho_{23}}{\rho_{12}}\right)_k$$

$$\times\sum_{i=0}^{n-2k}\frac{\left[n-2k\right]_q!}{\left[n-i-2k\right]_q!\left[i\right]_q!}\rho_{23}^i\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^k\right)_i\rho_{12}^{n-i-2k}\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}q^k\right)_{n-i-2k}H_i\left(x_1|q\right)H_{n-2k-i}\left(x_3|q\right)$$

□

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