On the heterochromatic number of hypergraphs associated to geometric graphs and to matroids*

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Abstract

The heterochromatic number $h_c(H)$ of a non-empty hypergraph H is the smallest integer k such that for every colouring of the vertices of H with exactly k colours, there is a hyperedge of H all of whose vertices have different colours. We denote by $\nu(H)$ the number of vertices of H and by $\tau(H)$ the size of the smallest set containing at least two vertices of each hyperedge of H. For a complete geometric graph G with $n \geq 3$ vertices let $H = H(G)$ be the hypergraph whose vertices are the edges of G and whose hyperedges are the edge sets of plane spanning trees of G. We prove that if G has at most one interior vertex, then $h_c(H) = \nu(H) - \tau(H) + 2$. We also show that $h_c(H) = \nu(H) - \tau(H) + 2$ whenever H is a hypergraph with vertex set and hyperedge set given by the ground set and the bases of a matroid, respectively.

Keywords: Heterochromatic; Geometric Graph; Matroid.

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1 Introduction

Let k be a positive integer and H be a hypergraph with at least k vertices. A *k-colouring* of H is an assigment of colours to the vertices of H that uses exactly k colours. Given a colouring c of a hypergraph H, a *heterochromatic hyperedge* is a hyperedge e of H such that c assigns different colours to different vertices of e.

The *heterochromatic number* $h_c(H)$ of a non-empty hypergraph is the smallest integer k such that H contains a heterochromatic hyperedge for each k-colouring of H. The heterochromatic number was defined by Arocha *et al* [1] and is closely related to *anti-Ramsey numbers* [6] and to the *upper chromatic number* of mixed hypergraphs [8].

A *double transversal of hyperedges* of a loopless hypergraph H is a set T of vertices of H such that each hyperedge of H contains at least two vertices in T. We denote by $\nu(H)$ and $\tau(H)$, the number of vertices and the size of the smallest double transversal of hyperedges of H, respectively.

A general lower bound for the heterochromatic number of loopless non-empty hypergraphs is obtained as follows: Consider a double transversal T of hyperedges of H with $|T| = \tau(H)$. Assign colour 1 to every vertex in T and a different colour to each of the remaining $\nu(H) - \tau(H)$ vertices of H. Since T is a double transversal of hyperedges of H, there are no heterochromatic edges of H for this $(\nu(H) - \tau(H) + 1)$-colouring of H and therefore $h_c(H) \geq \nu(H) - \tau(H) + 2$.

Jiang and West [4] proved that $h_c(H_n) = \binom{n-2}{2} + 2$ if H_n is the hypergraph whose vertices are the edges of a complete graph G with n vertices and whose hyperedges are the the edge sets of all spanning trees of G. Notice that, in this case, $\nu(H_n) = \binom{n}{2}$, $\tau(H_n) = 2n - 3$ and $h_c(H_n) = \nu(H_n) - \tau(H_n) + 2$.

In this article we study hypergraphs H associated to complete geometric graphs and to matroids for which the general lower bound also give the exact value of $h_c(H)$. For a complete geometric graph G, we associate a hypergraph $H(G)$ with vertex set given by the edges of G and whose hyperedges are the sets of edges of non-self intersecting spanning trees of G. For a matroid M we consider the hypergraph $H(M)$ whose vertices are the elements of the ground set of M and whose hyperedges are the bases of M.

We show that $h_c(H) = \nu(H) - \tau(H) + 2$ if H is the hypergraph associated as above to a complete geometric graph with at most one interior vertex or to a matroid.

2 Geometric graphs

Let P be a set of points in general position in the plane, a *geometric graph* on P is a graph G with vertex set P drawn in such away that each edge is a straight line segment with ends in P.

The *complement* of a geometric graph G is the geometric graph G^c with the same vertex set P whose edges are all the line segments with ends in P which are not edges of G.

A *plane spanning tree* of a geometric graph G is a non-selfintersecting subtree that contains every vertex of G.

A tree T is a *caterpillar* if the removal of the endpoints of T leaves a path called the *body* of T. Let P be a set of points in general position in the plane. A plane geometric tree R with vertex set P is a *geometric caterpillar* if R is a caterpillar such that the entire body of R lies in the boundary of the convex hull $CH(P)$ of P and for each leg e of R, the straight line containing e does not intersect R at any point not in e.

The following results, due to Károlyi *et al* [5] and to Urrutia-Galicia [7], respectively, give sufficient conditions for the complement of a geometric graph to contain a plane spanning tree.

Theorem 1. *If the edge set of a complete geometic graph G is partitioned into two sets, then there exists a plane spanning tree of G, all of whose edges lie in the same part.*

Theorem 2. *Let R be a plane spanning tree of a complete geometric graph G. The complementary geometric graph R^c contains a plane spanning tree if and only if R is neither a star or a geometric caterpillar.*

For any set P of points in general position in the plane, we denote by $i(P)$ the number of points of P not lying in the boundary of $CH(P)$.

Lemma 3 (due to Garcia *et al* [3]) and Lemma 4 provide double transversals of plane spanning trees in complete geometric graphs G with $i(V(G)) \leq 1$.

Lemma 3. *Let P be a set of $n \geq 3$ points in convex position in the plane. If R is a plane spanning tree of the complete geometric graph with vertex set P, then at least two edges of R lie in the boundary of $CH(P)$.*

Lemma 4. *Let P be a set of n points in general position in the plane such that $i(P) = 1$ and w be the unique point in P not lying in the boundary of $CH(P)$. Let Q be the set of edges of the boundary of $CH(P)$ together with two edges uw and vw such that the angle $\angle uwv$ is maximal. If R is a plane spanning tree of the complete geometric graph with vertex set P, then at least two edges of R lie in Q.*

Proof. Let G be the complete geometric graph with vertex set P. If $n = 4$, then each spanning tree of G has 3 edges and Q contains all but one of the edges of G. We proceed by induction assuming $n \geq 5$ and that the result holds for every subset P' of P with $i(P') = 1$.

Let R be a plane spanning tree of G. Either R contains two edges ux and vy in Q which are incident in u and v, respectively, or R contains an edge e incident in u or in v which is a diagonal of the boundary of $CH(P)$.

Supose R contains such a diagonal edge e. As $\angle uwv$ is maximal, then e cannot intersect the edges uw and wv in a point other than u or v. Let P^- be the set of points in P lying on or to the left of e and P^+ be the set of points in P lying on or to the right of e, notice that w is an interior point of P^- or an interior point of P^+. Again without loss of generality, assume $i(P^-) = 0$ and $i(P^+) = 1$.

Let Q^- be the set of edges in the boundary of $CH(P^-)$ and Q^+ be the set of edges of the boundary of $CH(P^+)$ together with the two edges uw and vw. By Lemma 3, the subtree R^- of R with vertex set P^- contains two edges in Q^- and by inducction, the subtree R^+ of R with vertex set P^+ contains two edges in Q^+. Since $Q^- \cup Q^+ = Q$ and $Q^- \cap Q^+ = \{e\}$, at least two edges of R lie in Q.

□

We can now prove the main results of this section.

Theorem 5. *Let G be a complete geometric graph with $n \geq 3$ vertices. If c is a colouring of the edges of G with exactly $\binom{n}{2} - n + 2$ colours, then G has a heterochromatic plane spanning tree.*

Proof. Let X be a heterochromatic set with $\binom{n}{2} - n + 2$ edges of G and let $Y = E(G) \setminus X$. Since $|Y| = n - 2$, no spanning tree of G has all edges in Y. By Theorem 1, graph G has a plane spanning tree R all of whose edges lie in $X = E(G) \setminus Y$. Clearly R is a heterochromatic tree.

□

Theorem 6. *Let G be a complete geometric graph with $n \geq 3$ vertices. If $i(V(G)) = 1$ and c is a colouring of the edges of G with exactly $\binom{n}{2} - n + 1$ colours, then G has a heterochromatic plane spanning tree.*

Proof. Let X be a heterochromatic set with $\binom{n}{2} - n + 1$ edges of G and let $Y = E(G) \setminus X$. Since $|Y| = n - 1$, either Y is the set of edges of a plane spanning tree of G or no spanning tree of G has all edges in Y. As in the proof of Theorem 5, in the later case, G has a heterochromatic plane spanning tree R.

Assume Y is the set of edges of a plane spanning tree S of G. By Theorem 2, either S is a geometric caterpillar, S is a star or S^c contains a plane spanning tree R. In the latter case, R is a heterochromatic plane spanning tree of G.

For the case where S is a geometric caterpillar but not a star let $y = uv$ be an edge in the body of S with $d_S(u) \geq 2$ and $d_S(v) \geq 2$. There is an edge $x \in X$ with $c(x) = c(y)$ since each colour is assigned to an edge in X.

Let $S' = (S - y) + x$ and notice that S' is not a star by the choice of y. Suppose S' is a geometric caterpillar in which case x must lie in the body of S' since y lies in the body of S. This implies that both S and S' are paths whose union is the boundary of $CH(P)$ which is not possible since $i(V(G)) = 1$. Therefore S' is neither a star or a geometric caterpillar.

By Theorem 2, the geometric graph $(S')^c$ contains a plane spanning tree, that is a plane spanning tree R' of G all of whose edges lie in $X' = E(G) \setminus E(S') = (X \setminus \{x\}) \cup \{y\}$. Since X is heterochromatic and $c(x) = c(y)$, the set X' and the tree R' are also heterochromatic.

In an analogous way, one may find a heterochromatic plane spanning tree of G in the case where S is a star.

□

Theorem 7. *Let P be a set of $n \geq 3$ points in general position in the plane and let G be the complete geometric graph with vertex set P. If $i(P) \leq 1$ and $H = H(G)$ is the hypergraph with vertex set $E(G)$ whose hyperedges are the sets of edges of plane spanning trees of G, then $h_c(H) = \nu(H) - \tau(H) + 2$.*

Proof. By Theorem 5, $h_c(H) \leq \binom{n}{2} - n + 2$. For the case where $i(V(G)) = 0$, Lemma 3 asserts that the boundary of $CH(P)$ is a double transversal of hyperedges of H. Hence $\tau(H) \leq n$, and by the general lower bound, $h_c(H) \geq \nu(H) - \tau(H) + 2 = \binom{n}{2} - n + 2$.

If $i(V(G)) = 1$, by Theorem 6, $h_c(H) \leq \binom{n}{2} - n + 1$. By Lemma 4, G contains a double transversal of plane spanning trees with $n+1$ edges; therefore $\tau(H) \leq n + 1$. By the general lower bound, $h_c(H) \geq \nu(H) - \tau(H) + 2 = \binom{n}{2} - (n+1) + 2 = \binom{n}{2} - n + 1$.

□

3 Matroids

For a simple connected graph G with at least three vertices, we denote by $\gamma(G)$ the smallest integer k for which there is a set of k edges of G whose removal brakes G into three connected components.

J. Arocha and V. Neumann-Lara [2] proved that if G is a simple connected graph with $m \geq 2$ edges and c is a colouring of the edges of G with exactly $m - \gamma(G) + 2$ colours, then G has an heterochromatic spanning tree. We generalise this result for matroids.

Theorem 8. *Let M be a matroid with m elements and rank at least 2 and let τ denote the size of a smallest double transversal of bases of M. If c is a colouring of M with exactly $m - \tau + 2$ colours, then M has a heterochromatic basis.*

Proof. Denote by E the ground set of M and let $X \subset E$ be a heterochromatic set with $m - \tau + 2$ elements. The complementary set $Y = E \setminus X$ cannot be a double transversal of bases of M since $|Y| = |E \setminus X| = \tau - 2$. Hence, there is a basis R of M that meets Y in at most one element.

If R is not heterochromatic, then R contains an element $x \in X$ such that $c(x) = c(y)$, where $y \neq x$ is the unique element in $R \cap Y$.

Let $Z = Y \cup \{x\}$, as $|Z| = |Y| + 1 = \tau - 1$, set Z is not a double transversal of bases of M either. This implies that there is a basis S of M that intersects Z in at most one element.

Assume S is not heterochromatic in which case $|S \cap Z| = 1$ and $|S \setminus Z| = |S| - 1$. Since $|R \setminus \{x, y\}| = |R| - 2 = |S| - 2 = |S \setminus Z| - 1$, there is an element $z \in S \setminus Z \subset X$ such that $(R \setminus \{x, y\}) \cup \{z\}$ is an independent set of M. This implies that either x or y must lie in the unique circuit contained in $R \cup \{z\}$ and therefore either $(R \cup \{z\}) \setminus \{x\}$ or $(R \cup \{z\}) \setminus \{y\}$ is a basis of M, a heterochromatic basis of M.

□

As an immediate consequence we obtain the following:

Corollary 9. *Let M be a matroid with rank $r \geq 2$. If $H = H(M)$ is the hypergraph whose vertices and hyperedges are the elements of the ground set and the bases of M, respectively, then $h_c(H) = \nu(H) - \tau(H) + 2$.*

4 Final remarks and acknowledgements

We conjecture that if H is the hypergraph associated as in section 2 to any complete geometric graph G with $n \geq 3$ vertices in general position, then $h_c(H) = \nu(H) - \tau(H) + 2$. Moreover, we conjecture that if G is a complete geometric graph with $n \geq 3$ vertices in general position, then there is a double transversal of plane spanning trees of G with $n + i(V(G))$ edges and that if c is a colouring of the edges of G with $\binom{n}{2} - (n + i(V(G))) + 2$ colours, then G has a heterochromatic plane spanning tree.

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