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THOM POLYNOMIALS AND THE GREEN-GRIFFITHS CONJECTURE

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1. Introduction

The Green-Griffiths conjecture from 1979 ([24]) states that every projective algebraic variety X of general type contains a certain proper algebraic subvariety $Y \subset X$ such that all nonconstant entire holomorphic curves $f : \mathbb{C} \to X$ necessarily lie in Y.

This conjecture is related to the stronger concept of a hyperbolic variety in the sense of Kobayashi ([27]). A projective variety X is called Kobayashi-hyperbolic if there is no nonconstant entire holomorphic curve in X, i.e any holomorphic map $f : \mathbb{C} \to X$ must be constant. Hyperbolic algebraic varieties have attracted considerable attention, in part because of their conjectured diophantine properties. For instance, Lang ([28] has conjectured that any hyperbolic complex projective variety over a number field K can contain only finitely many rational points over K.

A positive answer to the Green-Griffiths conjecture has been given for surfaces by McQuillan [29] under the assumption that the second Segre number $c_1^2 - c_2$ is positive. More recently, Siu established in [40, 41] that there exists a integer d_n such that generic hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree greater than d_n are Kobayashi-hyperbolic. His estimates for d_n are, however, not effective. Following his strategy and using the algebraic Morse inequalities of Trapani, Diverio gave a short, elegant proof in [14] for the non-effective Green-Griffiths conjecture for projective hypersurfaces. The key idea in Diverio's work is that the Morse inequalities ensure the existence of global holomorphic invariant jet differentials on X if a certain intersection number of the n-th projectivized jet bundle over X—also called the Demailly-Semple tower—is positive.

The first effective lower bounds for the degree of the hypersurface in the Green-Griffiths conjecture was given recently by Diverio, Merker and Rousseau in [13]. They prove that for a projective hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $> 2^{n^5}$ the Green-Griffiths conjecture holds. Their proof follows the strategy of Siu and Diverio combined with a delicate, long calculation with the cohomology ring of the Demailly-Semple tower to prove the positivity of Diverio's intersection number.

In the first half of this paper we apply equivariant localization techniques on the Demailly-Semple tower to give a closed formula for Diverio's intersection number (Theorem 4.8). The key idea is to transform the Atiyah-Bott localization into an iterated

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residue formula, which allows us to give a better lower bound in the Green-Griffiths problem. We prove:

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a generic smooth projective hypersurface of degree* $\deg(X) \geq 2^{8n \log n}$*. Then there exists a proper algebraic subvariety $Y \subset X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ has image contained in Y.*

Let us give a short historical overview of the background and the proof.

Jet differentials of a complex manifold X are, roughly speaking, differential operators acting on germs of holomorphic curves in X. The fundamental idea, that global jet differentials vanishing on an ample divisor provide some algebraic differential equations that every entire holomorphic curve $f : \mathbb{C} \to X$ should satisfy, first appeared in the seminal paper of Bloch [8]. In [24] Green and Griffiths put these ideas in an algebraic context by defining the bundle $J_k = J_k(T_X)$ of k-jets at 0 of germs of holomorphic curves $f : \mathbb{C} \to X$ over X, and $E^{GG}_{k,m} = \mathbb{C}[J_k]$, the bundle of algebraic differential operators whose elements are polynomial functions $Q(f', \ldots, f^{(k)})$ of weighted degree m.

In [10] Demailly refined the theory by defining jet differentials that are invariant under reparametrization of the source $\mathbb{C}$, also called the Demailly subbundle $E_{k,m} \subset E^{GG}_{k,m}$. It is acted on fiberwise by the group $\mathbb{G}_k$ of k-jets of reparametrization germs $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ at the origin, and $\oplus_{m=1}^{\infty} E_{k,m} = \mathcal{O}(J_k)^{\mathbb{G}_k}$ is the algebra if invariant jet differentials. This bundle gives a better reflection of the geometry of entire curves, since it only takes care of the image of such curves and not of the way they are parametrized. However, it also comes with a technical difficulty, namely, the reparametrization group $\mathbb{G}_k$ is non-reductive, and the classical geometric invariant theory of Mumford is not applicable to describe the invariants and the quotient $J_k/\mathbb{G}_k$.

In [10] Demailly overcomes this problem by describing a smooth compactification of $J_k/\mathbb{G}_k$ as a tower of projectivized bundles on X—the Demailly-Semple tower—endowed with tautological bundles whose sections are $\mathbb{G}_k$-invariants. Diverio in [14] then applies the algebraic Morse inequalities to provide global invariant jet differentials on X by proving the positivity of a certain intersection number of the tautological bundle on the Demailly-Semple tower.

However, existence of global jet differentials is not enough: they provide constraints only on the jets of of holomorphic curves in X of a fix order. The final step of the strategy – which was established by Siu based on earlier works of Voisin, and then turned into a final form in [13] – is the deformation of such jet differentials by means of slanted vector fields having low pole order to produce, by plain differentiation, many new algebraically independent invariant jet differentials, which then force entire curves to lie in a proper closed subvariety $Y \subset X$.

Recently, in [33] Merker proves the existence of global jet differentials of order k sufficiently bigger than the dimension n of the hypersurface X with $\deg(X) \geq n + 3$. Demailly in [12] generalizes this result proving the existence of global jet differentials for projective varieties of general type, not just hypersurfaces. The results of the present paper, however, focus on the $k = n$ case, where Siu's deformation arguments give effective lower bounds for the Green-Griffiths conjecture when X is a hypersurface. Moreover, we work with invariant jet differentials instead of Green-Griffiths jet differentials (i.e non-invariant ones), in contrast to [33, 12].

In the second half of the present paper we describe a new compactification of $J_k/\mathbb{G}_k$ and the invariant jet differentials. This construction is motivated by the author's earlier work in global singularity theory, where jet differentials also play a very important role.

Consider a holomorphic map $f : N \to M$ between two complex manifolds, of dimensions $n \leq m$. We say that $p \in N$ is a *singular* point of f (or f has a singularity at p) if the rank of the differential $df_p : \mathrm{T}_pN \to \mathrm{T}_{f(p)}M$ is less than n. The topology of the situation often forces f to be singular at some points of N.

To introduce a finer classification of singular points, choose local coordinates near $p \in N$ and $f(p) \in M$, and consider the resulting map-germ $f_p : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$, which may be thought of as a sequence of m power series in n variables without constant terms. Let Diff_n denote the group of germs of local holomorphic reparametrisations $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Then $\mathrm{Diff}_n \times \mathrm{Diff}_m$ acts on the space $\mathcal{J}(n,m)$ of all such map-germs. We will call $\mathrm{Diff}_n \times \mathrm{Diff}_m$-orbits or, more generally, $\mathrm{Diff}_n \times \mathrm{Diff}_m$-invariant subsets $O \subset \mathcal{J}(n,m)$ *singularities*. For a singularity O and holomorphic $f : N \to M$, we can define the set

$$Z_O[f] = \{p \in N;\ \ f_p \in O\},$$

which is independent of any coordinate choices. Then, under some additional technical assumptions, for N compact, appropriate closed O, and f sufficiently generic, $Z_O[f]$ is an analytic subvariety of N. The computation of the Poincaré dual class $\alpha_O[f] \in H^*(N,\mathbb{Z})$ of this subvariety is one of the fundamental problems of global singularity theory. This is genuinely useful: for example, if we can prove that $\alpha_O[f]$ does not vanish, then we can guarantee that the singularity O occurs at some point of the map f.

This problem was first studied by René Thom (cf. [45, 25]) in the category of smooth varieties and smooth maps; in this case cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients is used. This study was later extended to the holomorphic category as well (cf. [26, 9, 37]. To describe the class $\alpha_O[f]$ in more concrete terms, denote by $\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_m}$ the space of those polynomials in the variables $(\lambda_1 \dots \lambda_n, \theta_1 \dots \theta_m)$ which are invariant under the permutations of the λ's and the permutations of the θ's. According to the structure theorem of symmetric polynomials, $\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_m}$ is itself a polynomial ring in the elementary symmetric polynomials:

$$\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_m} = \mathbb{C}[c_1(\boldsymbol{\lambda}) \dots c_n(\boldsymbol{\lambda}), c_1(\boldsymbol{\theta}) \dots c_m(\boldsymbol{\theta})].$$

Using the Chern-Weil map, any polynomial $b \in \mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_m}$, and every pair of bundles (E,F) over N of ranks n and m, respectively, produces a characteristic class $b(E,F) \in H^*(N,\mathbb{C})$. The following result is called Thom's principle in the literature:

For appropriate $\mathrm{Diff}_n \times \mathrm{Diff}_m$*-invariant O of codimension j in $\mathcal{J}(n,m)$, there exists a* *homogeneous polynomial* $\mathrm{Tp}_O \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{S_n \times S_m}$ *of degree* j *such that for an arbitrary, sufficiently generic map* $f : N \to M$*, the cycle* $Z_f[O] \subset N$ *is Poincaré dual to the characteristic class* $\mathrm{Tp}_O(TN, f^*TM)$*.*

A precise version of this statement is described in [4]. The polynomial Tp_O is called the *Thom polynomial* of O. The computation of these polynomials is a central problem of singularity theory.

There is an important class of orbits which is relevant to the Green-Griffiths problem. Define

$$O_k = \{f = (f^1, \dots f^m) \in \mathcal{J}(n,m) : \mathbb{C}[x_1, \dots, x_n]/\langle f^1, \dots, f^m\rangle \simeq \mathbb{C}[t]/t^{k+1}\}.$$

These are called Morin singularities, and in a recent paper ([4]) the author with A. Szenes gave a formula for their Thom polynomials $\mathrm{Tp}_k = \mathrm{Tp}_{O_k}$.

The relevance of Morin singularities becomes clear from the algebraic characterization of O_k. This is called the *test curve model* of T. Gaffney (see [23, 4]), saying that an element f of (an open dense subset of) $\mathcal{J}(n,m)$ lies in O_k if and only if there exist a test curve $\gamma \in \mathcal{J}_k(1,n)$ such that the k-jet of $f \circ \gamma$ is 0. Reparametrization of the test curve is again a test curve, and therefore we arrive the following observation
O_d *fibers over the quotient* $\mathcal{J}_k(1,n)/\mathbb{G}_k$ where $\mathbb{G}_k$ is the group of k-jets of reparametisations $(\mathbb{C},0) \to (\mathbb{C},0)$. (In other words $\mathbb{G}_k$ is the truncated Diff_1 at degree $k+1$.)

Note that for an open dense subset $J_k^{\mathrm{reg}}(1,n) \subset \mathcal{J}_k(1,n)$ the geometric quotient $J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k$ exists and the fiber of the Demailly-Semple tower over any $x \in X$ is a smooth compactification of this quotient. In [4] we developed a different construction of the quotient $J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k$, and applied equivariant localization on this quotient to integrate the so-called equivariant Thom class, and then we transformed the formula into an iterated residue. We proved a vanishing property of this iterated residue, saying that only one fixed point contributes to the sum, leaving us with a closed, short formula for the Thom polynomial:

$$\mathrm{Tp}_k^{m-n}(c_1, \dots) = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{(-1)^k \prod_{m<l}(z_m - z_l)\, Q_k(z_1 \dots z_k)}{\prod_{m+r \le l \le k}(z_m + z_r - z_l)} \prod_{l=1}^{k} C\left(\frac{1}{z_l}\right) z_l^{m-n}\, dz_l, \tag{1}$$

where

$$C\left(\frac{1}{z_l}\right) = 1 + \frac{c_1}{z_l} + \frac{c_2}{z_l^2} + \dots$$

is the total Chern class of $TN - f^*TM$, and $Q_k(z_1, \dots, z_k)$ is a homogeneous polynomial defined as the dual of a Borel orbit in [4].

The coefficients of the Thom-polynomials are therefore contained in the Thom generating function

$$\mathrm{Tp}_k(z_1, \dots, z_k) = \frac{\prod_{m<l}(z_m - z_l)\, Q_k(z_1 \dots z_k)}{\prod_{m+r \le l \le k}(z_m + z_r - z_l)} \tag{2}$$

The polynomial Q_k in the formula is known for $k \leq 6$, but only sporadic results are known in the generic case (see [2] for details). The precise definition of this homogeneous polynomial is given in Remark (6.8). The total degree of Tp_k is 0, that is, the homogeneous polynomials in its numerator and denominator have equal degree.

It is conjectured in [37], (Conjecture 5.5) that all coefficients of the Thom polynomials $\mathrm{Tp}_k(c_1, c_2, \ldots)$ expressed in terms of the relative Chern classes are nonnegative. In [4] we prove this for $k = 1, 2, 3$. The second part of the following conjecture is motivated in §8.

Conjecture 1.2. *For* $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{Z}^k$ *with* $i_1 + \ldots + i_k = 0$ *let* $\mathrm{Tp}_{\mathbf{i}}$ *denote the coefficient of* $z_1^{i_1} \ldots z_k^{i_k}$ *in the Laurent expansion of* $\mathrm{Tp}_k(z_1, \ldots, z_k)$ *in the domain* $|z_1| \ll \cdots \ll |z_d|$*. Then*

(1) *(Rimányi,* [37]*)* $\mathrm{Tp}_{\mathbf{i}} \geq 0$ *for any* $\mathbf{i}$*.*
(2) *Let* $1 \leq m, l \leq k$ *and* $i_1 + \ldots + i_k = 0$*, and* $e_j = (0, \ldots, 1^j, \ldots, 0)$*. Then* $\frac{\mathrm{Tp}_{\mathbf{i}+e_l-e_m}}{\mathrm{Tp}_{\mathbf{i}}} < k^2$

In the second part of the paper we generalize and extend the localization method developed in [4] to produce a closed iterated residue formula for intersection pairings on the quotient $J_k/\mathbb{G}_k$. Detailed analysis of the formula accompanied with the numerical bounds coming from Siu's deformation arguments leads us to

Theorem 1.3. *Let* $X \subset \mathbb{P}^{n+1}$ *be a generic smooth projective hypersurface of degree* $\deg(X) \geq n^6$*. Then Conjecture 1.2 for* $k = n$ *implies the existence of a proper algebraic subvariety* $Y \subset X$ *such that every nonconstant entire holomorphic curve* $f : \mathbb{C} \to X$ *has image* $f(\mathbb{C})$ *contained in* Y*.*

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2. Jet differentials

The central object of this paper is the algebra of invariant jet differentials under reparametrization of the source space $\mathbb{C}$. For more details see the survey paper [10].

2.1. Invariant jet differentials.

Let X be a complex n-dimensional manifold, and $V \subset T_X$ a holomorphic subbundle of the tangent bundle of X. Note that V is not necessarily integrable.

Green and Griffiths in [24] introduced a bundle $J_kV \to X$, the bundle of k-jets of germs of parametrized curves in X tangent to V; that is, the fibre over $x \in X$ is the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, such that $f'(t) \in V_{f(t)}$ for all t in a neighbourhood of 0, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ equal for $0 \le j \le k$. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$ on an open neighbourhood $\Omega \subset X$ around x, the elements of the fibre $J_{k,x}V$ are $\mathbb{C}^n$-valued maps

$$f = (f_1, f_2, \ldots, f_n) : (\mathbb{C}, 0) \to (\Omega, x),$$

and two maps represent the same jet if their Taylor expansions at $t = 0$

$$f(t) = x + tf'(t) + \frac{t^2}{2!} f''(0) + \ldots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1})$$

coincide up to order k. In these coordinates the fibre is

$$J_{k,x} = \left\{ (f'(0), \ldots, f^{(k)}(0)) \right\} = (\mathbb{C}^n)^k,$$

which we identify with $\mathbb{C}^{nk}$.

Note that J_kV is not a vector bundle, since the transition functions are polynomial, but not linear.

Let $\mathbb{G}_k$ be the group of k-jets of biholomorphism germs

$$(\mathbb{C}, 0) \to (\mathbb{C}, 0);$$

that is, the k-jets at the origin of local reparametrisations

$$t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C},$$

in which the composition law is taken modulo terms t^j for $j > k$. This group acts fibrewise on J_kV by substitution. A short computation shows that this is a linear action on the fibre:

$$\begin{aligned} f \circ \varphi(t) = \ & f'(0) \cdot (a_1 t + a_2 t^2 + \ldots + a_k t^k) + \frac{f''(0)}{2!} \cdot (a_1 t + a_2 t^2 + \ldots + a_k t^k)^2 + \ldots \\ & \ldots + \frac{f^{(k)}(0)}{k!} \cdot (a_1 t + a_2 t^2 + \ldots + a_k t^k)^k = \\ & (f' a_1) t + (f' a_2 + \frac{f''}{2!} a_1^2) t^2 + \ldots \end{aligned}$$

so the linear action of φ on the k-jet $(f'(0), \ldots, f^{(k)}(0))$ is given by the following matrix multiplication:

$$f \circ \varphi(0) = (f'(0), \ldots, \frac{f^{(k)}(0)}{k!}) \cdot \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_k \\ 0 & a_1^2 & 2a_1a_2 & \ldots & \\ 0 & 0 & a_1^3 & \ldots & \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & a_1^k \end{pmatrix},$$

where the $(i, j)th$ entry of the matrix is

$$\sum_{s_1 + \ldots + s_i = j} a_{s_1} \ldots a_{s_i}.$$

There is an exact sequence of groups:

$$0 \to \mathbb{G}_k' \to \mathbb{G}_k \to \mathbb{C}^* \to 0,$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the morphism $\varphi \to \varphi'(0) = a_1$ with the notation used above, and

$$\mathbb{G}_k = \mathbb{G}_k' \rtimes \mathbb{C}^*$$

is an extension of a unipotent group with $\mathbb{C}^*$. With the above identification, $\mathbb{C}^*$ is the subgroup of diagonal matrices satisfying $a_2 = \ldots = a_k = 0$ and $\mathbb{G}_k'$ is the unipotent radical of $\mathbb{G}_k$, i.e with $a_1 = 1$. The action of $\lambda \in \mathbb{C}^*$ on k-jets describes a grading by

$$\lambda \cdot (f'(0), f''(0), \ldots, f^{(k)}(0)) = (\lambda f'(0), \lambda^2 f''(0), \ldots, \lambda^k f^{(k)}(0)),$$

so the weight of $f^{(i)}$ is i.

The Green-Griffiths bundle, introduced by Green and Griffiths in [24], is the bundle $E^{GG}_{k,m}V^* \to X$, whose fibers are polynomial differential operators $Q(f', f'', \ldots, f^{(k)})$ on the fibers of J_kV of weighted degree m with respect to the fiberwise $\mathbb{C}^*$ action on J_kV. Note that this makes sense, since the transition functions are polynomial.

The action of $\mathbb{G}_k$ naturally induces an action on $E^{GG}_{k,m}V^*$ fiberwise.

Definition 2.1. *Following Demailly (see* [10]*), we define the vector bundle of invariant jet differentials of order k and degree m as the bundle $E_{k,m}V^* \subset E^{GG}_{k,m}V^*$ over X, whose fibers are invariant polynomial differential operators on J_kV, that is for any $\varphi \in \mathbb{G}_k$ they satisfy*

$$Q((f \circ \varphi)', (f \circ \varphi)'', \ldots, (f \circ \varphi)^{(k)}) = \varphi'(0)^m \cdot Q(f', f'', \ldots, f^{(k)}).$$

Let $E^n_k = \oplus_m E^n_{k,m}$ denote the graded algebra of invariants. This algebra has attracted considerable attention for long time. In [5] the authors give a geometric description of this invariant algebra.

2.2. **Compactification of $J_kV/\mathbb{G}_k$.** Given a space with a group action, intuitively we think of the ring of invariant polynomial functions on a space as polynomial functions on the quotient of the space by the group. Informally, we would like to think of the Demailly algebra of invariant jet differentials as sections of a line bundle over a GIT-like quotient $J_kV//\mathbb{G}_k$ w.r.t a line bundle on J_kV, that is,

$$\oplus_{m=1}^\infty E_{k,m}V = \mathbb{C}[J_kV]^{\mathbb{G}_k} = \mathbb{C}[J_kV//\mathbb{G}_k].$$

The question is, how can we interpret the quotient $J_kV//\mathbb{G}_k$ to realise this principle. Since $\mathbb{G}_k$ is not a reductive group, the arguments of Mumford‘s geometric invariant theory do not apply automatically here. However, we prove in [5] that the algebra of invariants $\mathbb{C}[J_kV]^{\mathbb{G}_k}$ is finitely generated, and therefore the categorical quotient

$$J_kV//\mathbb{G}_k = \mathrm{Spec}([\mathbb{C}[J_kV]]^{\mathbb{G}_k})$$

exists.

A more detailed study of the GIT-like quotient $J_kV//\mathbb{G}_k$ can be found in [5].

Demailly's "projectivized jet bundle" construction provides a smooth compactification of the geometric quotient $J_kV/\mathbb{G}_k$ of an open dense subset $J_k^{reg}(V) \subset J_k(V)$, constructed as an iterated tower of projectivized bundles over X. We will call this the Demailly-Semple tower. It is also called the Semple jet-bundle in the literature.

Let (X, V) be a directed manifold with $V \subset T_X$, $\dim(X) = n, rk(V) = r$. With (X, V), we associate another directed manifold $(\tilde{X}, \tilde{V})$ where $\tilde{X} = P(V)$ is the projectivized bundle, $\pi : \tilde{X} \to X$ is the natural projection and $\tilde{V}$ is the subbundle of $T\tilde{X}$ defined fiberwise as

$$\tilde{V}_{(x_0,[v_0])} = \{\xi \in T\tilde{X}_{(x_0,[v_0])} | \pi_*(\xi) \in \mathbb{C} \cdot v_0\}.$$

for any $x_0 \in X$ and $v_0 \in TX_{x_0} \setminus 0$. We also have a lifting operator which assigns to a germ of holomorphic curve $f : (\mathbb{C}, 0) \to X$ tangent to V a germ of holomorphic curve $\tilde{f} : (\mathbb{C}, 0) \to \tilde{X}$ tangent to $\tilde{V}$ in such a way that $\tilde{f}(t) = (f(t), [f'(t)])$.

Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface. Following Demailly [10], we define inductively the k-jet bundle $P_kT_X = X_k$ and the associated subbundle $V_k \subset T_{X_k}$ as follows.

$$(X_0, V_0) = (X, T_X),\ (X_k, V_k) = (\tilde{X}_{k-1}, \tilde{V}_{k-1}).$$

In other words, (X_k, V_k) is obtained from (X, T_X) by iterating k-times the lifting construction $(X, V) \mapsto (\tilde{X}, \tilde{V})$. Therefore,

$$\dim P_kT_X = n + k(n-1), \qquad \mathrm{rank} V_k = n - 1,$$

and the construction can be described inductively by the following exat sequences:

$$0 \longrightarrow T_{P_kT_X/P_{k-1}T_X} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{P_kT_X}(-1) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{P_kT_X} \longrightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{P_kV}(-1) \longrightarrow T_{P_kT_X/P_{k-1}T_X} \longrightarrow 0$$

where $\pi_k : P_kT_X \to P_{k-1}T_X$ is the natural projection and $(\pi_k)_*$ is its differential.

We also have natural projections

$$\pi_{j,k} = \pi_{j+1} \circ \ldots \circ \pi_{k-1} \circ \pi_k : P_kT_X \to P_jT_X, \tag{3}$$

and with this notation $\pi_{0,k} : P_kT_X \to X = P_0T_X$ is a locally trivial holomorphic fiber bundle over X, and the fibers $X_{k,x}(P_kT_X)_x = \pi_{0,k}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{n-1}$ bundles.

Theorem 2.2. *(*[10]*) Suppose that $n > 2$. The quotient $J_k^{reg}TX/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_k^{reg}TX/\mathbb{G}_k \hookrightarrow X_k$ which identifies $J_k^{reg}TX/\mathbb{G}_k$ with X_k^{reg}, that is the set of points in X_k of the form $f[k](0)$ for some non singular k-jet f. In other words X_k is a relative compactification of $J_k^{reg}TX/\mathbb{G}_k$ over X. Moreover, one has the direct image formula:*

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m}T_X^*).$$

3. Localization on the Demailly-Semple tower

Let $T = (\mathbb{C}^*)^n$ be the n-dimensional torus, and suppose we are given a local T-action on X near the point $x \in X$. By the definition, this action induces an action on the Semple jet bundle $P_k T_X$ around x. Indeed, we saw in the previous chapter that it induces an action on $\tilde{X}$ and $\tilde{V}$, so by induction it defines an action on $P_k T_X$.

Now fix a point $x \in X$. Take a local T-action on X near x such that x is a fixed point of the action. To do so, we can take an affine chart with origin at x, and define a linear T-action on this affine chart. This action induces a T-action on $V = T_{X,x}$, and since x is a fixed point, it also induces an action on the fiber $X_{k,x}$ and $V_k|_{X_{k,x}}$.

We aim to apply equvivariant localization on the fiber $X_{k,x}$ w.r.t this T-action. This is a k-stage tower of projective fibrations, and to understand the fixed point data and the weights of the action at the fixed points we use the exact sequences (2.2), restricted to the fibre over x. Note that (2.2) restricted to $X_{k,x}$ is T-equivariant.

For $k = 1$ we get

$$0 \longrightarrow T_{P(V)} \longrightarrow V_1|_{P(V)} \longrightarrow \mathcal{O}_{P(V)(-1)} \longrightarrow 0,$$

Let $\{e_1, \dots, e_n\}$ be an eigenbasis for the T-action on $V = T_{X,x}$ with weights $\lambda_1, \dots, \lambda_n$. Locally, $V_1|_{P(V)} = T_{P(V)} \oplus \mathcal{O}_{P(V)(-1)}$ as a T-module, and we can choose basis on the fibers of V_1, such that the induced action is diagonal.

In particular, at the fixed point $[e_j] = [0 : \dots 0 : 1 : 0 : \cdots : 0]$ the weights in a local diagonal basis on $(V_1)_{[e_j]}$ are $\lambda_i - \lambda_j, i \neq j, i = 1, \dots n$ (coming from $T_{P(V),[e_j]}$), and λ_j (the weight on $\mathcal{O}_{P(V)(-1),[e_j]}$). Therefore we say, that

$$\text{the weights on } V_1|_{[e_j]} \text{ are } \left\{\lambda_j, \lambda_i - \lambda_j, i \neq j\right\}. \tag{4}$$

Now (2.2) restricted to the fiber $X_{k,x}$ gives us:

$$0 \longrightarrow T_{X_{k,x}/X_{k-1,x}} \longrightarrow V_k|_{X_{k,x}} \longrightarrow \mathcal{O}_{X_{k,x}(-1)} \longrightarrow 0, .$$

Locally, again, V_k is the direct sum of the two bundles on the ends. Fix a point $y \in X_{k,x}$, and let $V_{k-1,\pi_* y}$ denote the fiber of V_{k-1} at the point $\pi_* y \in X_{k-1,x}$, where $\pi = \pi_{k,k-1}$. If y is a fixed point of the T-action on $X_{k,x}$, then $\pi_* y$ is a fixed point on $X_{k-1,x}$, and therefore $V_{k-1,\pi_* y}$ is T-invariant, acting on by T with weights $w_1, \dots, w_n \in \mathrm{Lin}(\lambda_1, \dots, \lambda_n)$ in the eigenbasis $e_1, \dots, e_n$. Then $X_{k,x} = P(V_{k-1})$, and since y is a fixed point, $y = [e_j]$ for some $1 \leq j \leq n$. The weight on $(T_y(X_{k,x}/X_{k-1,x}) = T_{\mathbb{P}(V_{k-1,\pi_* y})}$ at y are therefore

$$w_i - w_j \text{ for } i = 1, \dots n, i \neq j. \tag{5}$$

The weight on the tautologiacal bundle $\mathcal{O}_{X_{k,x}(-1)}$ at $y \in X_{k,x}$ is w_j, and by (3) the weights on $(V_{k,y})$ are

$$w_i - w_j \text{ for } i = 1, \dots n, i \neq j, \text{ and } w_j. \tag{6}$$

To make reference to the fixed points and the weights easier, we introduce the following notations. Recall that $X_{k,x}$ is a k-stage tower of projective fibrations

$$\begin{array}{ccc} & & V_k \\ & & \downarrow \\ \mathbb{P}^{n-1} & \longrightarrow & X_{k,x} = P(V_{k-1}) \\ & & \downarrow \\ \mathbb{P}^{n-1} & \longrightarrow & X_{k-1,x} = P(V_{k-2}) \\ & & \downarrow \\ & & \cdots \\ & & \downarrow \\ & & X_{1,x} = \mathbb{P}(T_{X,x}) \end{array}$$

If $y \in X_{k,x}$ is a fixed point, then $\pi_{k,0}(y) = x$, and $\pi_{k,1}(y) \in \mathbb{P}(T_{X,x})$ is a fixed point, corresponding to a weight on $T_{X,x}$, say λ_{i_1}. In general, $\pi_{k,s}(y) \in X_{s,x}$ is a fixed point, and $V_k|_{\pi_{k,s}(y)}$ is invariant under the T action and $X_{k-1,y}$ corresponds to one of the eigenbasis.

Therefore, a fixed point $y = P_{w_1,\dots,w_k}$ is characterised by a sequence of weights $w_i \in \mathrm{Lin}(\lambda_1, \dots, \lambda_n), i = 1, \dots, k$ with the following properties. For a set $S \subset \mathrm{Lin}(\lambda_1, \dots, \lambda_n)$ let $S^{\neq 0} \subset S$ denote the nonzero elements of S.

$$\begin{gathered} w_1 \in S_1 = \{\lambda_1, \dots, \lambda_n\} \\ w_2 \in S_2(w_1) = \{w_1, \lambda_i - w_1 : \lambda_i - w_1 \neq 0, 1 \leq i \leq n\} \\ w_3 \in S_3(w_1, w_2) = \{w_2, w - w_2 : w \in S_2(w_1)\}^{\neq 0} = \{\lambda_i - w_1 - w_2, w_1 - w_2, w_2 : \lambda_i - w_1 \neq 0\}^{\neq 0} \\ \dots\dots \\ w_k \in S_k(w_1, \dots, w_{k-1}) = \{w_{k-1}, w - w_{k-1} : w \in S_{k-1}(w_1, \dots, w_{k-2})\}^{\neq 0} = \\ = \{\lambda_i - w_1 - \dots - w_{k-1}, w_1 - w_2 - \dots - w_{k-1}, \dots, w_{k-2} - w_{k-1}, w_{k-1} : \\ \lambda_i - w_1 \neq 0, w_1 - w_2 \neq 0, w_2 - w_3 \neq 0, \dots, w_{k-2} - w_{k-1} \neq 0\}^{\neq 0} \end{gathered} \tag{7}$$

Note, that S_i contains n weights for $i = 1, \dots, k$. The weights in S_i are the weights of the T action on $V_{i-1}|_{\pi_{k,i-1}(y)}$. More precisely, $w_i \in S_i$ is the weight of $\mathcal{O}_{X_{i,x}(-1)}|_{\pi_{k,i}(y)}$, whereas $\{w - w_i : w \in S_{i-1}\}^{\neq 0}$ are the weights of the tangent space $T_{\pi_{k,i}(y)}\mathbb{P}(V_{i-1}|_{\pi_{k,i-1}(y)})$ at the the point $\pi_{k,i}(y)$.

For $n = k = 3$ the following table contains the sets S_i and the fixed points.

(8)

$(\lambda_1,\lambda_2,\lambda_3)$	$(\lambda_1,\lambda_2-\lambda_1,\lambda_3-\lambda_1)$	$(\lambda_1,\lambda_2-2\lambda_1,\lambda_3-2l_1)$
		$(2\lambda_1-\lambda_2,\lambda_2-\lambda_1,\lambda_3-\lambda_2)$
		$(2\lambda_1-\lambda_3,\lambda_2-\lambda_3,\lambda_3-\lambda_1)$
	$(\lambda_1-\lambda_2,\lambda_2,\lambda_3-\lambda_2)$	$(\lambda_1-\lambda_2,2\lambda_2-\lambda_1,\lambda_3-\lambda_1)$
		$(\lambda_1-2\lambda_2,\lambda_2,\lambda_3-2\lambda_2)$
		$(\lambda_1-\lambda_3,2\lambda_2-\lambda_3,\lambda_3-\lambda_2)$
	$(\lambda_3,\lambda_1-\lambda_3,\lambda_2-\lambda_3)$	$(\lambda_3,\lambda_1-2\lambda_3,\lambda_2-2\lambda_3)$
		$(2\lambda_3-\lambda_1,\lambda_1-\lambda_3,\lambda_2-\lambda_1)$
		$(2\lambda_3-\lambda_2,\lambda_1-\lambda_2,\lambda_2-\lambda_3)$

The Atiyah-Bott localization on the Demailly-Semple jet bundle reads as follows:

Proposition 3.1. *Let $X_{k,x}$ be the fiber of the Demailly-Semple k-jet bundle over X. Then* $\dim(X_{k,x})=k(n-1)$ *and for* $\alpha\in\Omega^{k(n-1)}(X_{k,x})$

$$\int_{X_{k,x}}\alpha=\sum_{p(w_1,\dots,w_k)\in\mathfrak{F}}\frac{\alpha|_{p(w_1,\dots,w_k)}}{\prod_{j=1}^{k}\prod_{\substack{w\in S_j(w_1,\dots,w_{j-1})\\ w\neq w_j}}(w-w_j)},\tag{9}$$

where $\mathfrak{F}$ denotes the set of fixed points on $X_{k,x}$, i.e the last column of our table (8)

Proof. The equivariant Euler class of the tangent bundle at the fixed point $p(w_1,\dots,w_k)$ is the product of the weights at that fixed point.

$$\mathrm{Euler}_T(T_{p(w_1,\dots,w_k)}X_{k,x}=\prod_{j=1}^{k}\mathrm{Euler}_T(T_{\pi_{k,j}p(w_1,\dots,w_k)}\mathbb{P}(V_{j-1}|_{\pi_{k,j-1}p(w_1,\dots,w_k)})$$

and the weights on $V_{j-1}|_{\pi_{k,j-1}p(w_1,\dots,w_k)}$ are collected in S_{j-1}. □

3.1. **Transforming the localization formula into iterated residue.** In this section we derive an iterated residue formula for the right hand side of the localization (9). The geometric meaning of this residue formula is not entirely clear, but its effectiveness enables us to avoid the lenghty and complicated computations with the cohomology ring of the Demailly-Semple tower and to handle the coefficients of the Green-Griffiths polynomial. This section can therefore be considered as the heart of our computations.

Assume that $\alpha|_{p(w_1,\dots,w_k)}$ is a homogeneous polyomial of the Chern classes $w_i=c_1(O_{P(V_i)}(-1))$, at the fixed point $p(w_1,\dots,w_k)$, that is,

$$\alpha|_{p(w_1,\dots,w_k)}=Q(w_1,\dots,w_k)$$

with some homogeneous polynomial Q of degree $\dim X_{k,x}=k(n-1)$. Note that the Diverio-form satisfies this condition, since it is the first Chern class of

$$\otimes_{i=1}^{k}\pi_{i,k}^{*}O_{P(V_i)}(a_i)$$

We explain the details in the next section.

We aim to find an iterated residue formula for the general Atiyah-Bott localization expression

$$AB(Q,X_k)=\sum_{p(w_1,\ldots,w_k)\in\mathfrak{F}_{\mathfrak{k}}}\frac{Q(w_1,\ldots,w_k)}{\prod_{j=1}^{k}\prod_{w\in S_j(w_1,\ldots,w_{j-1}),w\neq w_j}(w-w_j)}. \tag{10}$$

To describe this formula, we will need the notion of an *iterated residue* (cf. e.g. [43]) at infinity. Let $\omega_1,\ldots,\omega_N$ be affine linear forms on $\mathbb{C}^k$; denoting the coordinates by $z_1,\ldots,z_k$, this means that we can write $\omega_i = a_i^0 + a_i^1 z_1 + \ldots + a_i^k z_k$. We will use the shorthand $h(\mathbf{z})$ for a function $h(z_1\ldots z_k)$, and $d\mathbf{z}$ for the holomorphic n-form $dz_1\wedge\cdots\wedge dz_k$. Now, let $h(\mathbf{z})$ be an entire function, and define the *iterated residue at infinity* as follows:

$$\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\ldots\operatorname*{Res}_{z_k=\infty}\frac{h(\mathbf{z})\;d\mathbf{z}}{\prod_{i=1}^{N}\omega_i}\overset{\text{def}}{=}\left(\frac{1}{2\pi i}\right)^k\int_{|z_1|=R_1}\ldots\int_{|z_k|=R_k}\frac{h(\mathbf{z})\;d\mathbf{z}}{\prod_{i=1}^{N}\omega_i}, \tag{11}$$

where $1\ll R_1\ll\ldots\ll R_k$. The torus $\{|z_m|=R_m;\ m=1\ldots k\}$ is oriented in such a way that $\mathrm{Res}_{z_1=\infty}\ldots\mathrm{Res}_{z_k=\infty}\ d\mathbf{z}/(z_1\cdots z_k)=(-1)^k$.

We will also use the following simplified notation:

$$\operatorname*{Res}_{\mathbf{z}=\infty}\overset{\text{def}}{=}\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\ldots\operatorname*{Res}_{z_k=\infty}.$$

In practice, the iterated residue 11 may be computed using the following **algorithm**: for each i, use the expansion

$$\frac{1}{\omega_i}=\sum_{j=0}^{\infty}(-1)^j\frac{(a_i^0+a_i^1z_1+\ldots+a_i^{q(i)-1}z_{q(i)-1})^j}{(a_i^{q(i)}z_{q(i)})^{j+1}}, \tag{12}$$

where $q(i)$ is the largest value of m for which $a_i^m\neq 0$, then multiply the product of these expressions with $(-1)^k h(z_1\ldots z_k)$, and then take the coefficient of $z_1^{-1}\ldots z_k^{-1}$ in the resulting Laurent series.

3.2. **Warm-up: residues on projective spaces.** First we show how iterated residues arise when $k=1$. In this case $X_{1,x}=\mathbb{P}(TX)$, the projectivized tangent bundle of $X\in\mathbb{P}^{n+1}$, and the fiber over any $x\in X$ is $\mathbb{P}(T_xX)=\mathbb{P}(V)$.

Suppose we have a diagonal torus action of T^n on T_xX with weights $\lambda_1,\lambda_2,\ldots,\lambda_n$, as before.

We have the following residue theorem.

Proposition 3.2. *For a polynomial Q on $\mathbb{C}^n$, we have*

$$AB(Q,X_{1,n})=\sum_{i=1}^{n}\frac{Q(\lambda_i)}{\prod_{j\neq i}(\lambda_j-\lambda_i)}=\operatorname*{Res}_{z=\infty}\frac{Q(z)}{\prod_{j=1}^{n}(\lambda_j-z)}dz \tag{13}$$

Proof. We compute the residue (13) using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. This residue is a contour integral, whose value is minus the sum of the z-residues of the form in (13). These poles are at $z = \lambda_j$, $j = 1 \dots n$, and after canceling the signs that arise, we obtain the left hand side of (13). □

3.3. **Residues on a projective bundle.** For $k = 2$ the Demailly-Semple tower X_2 is a projective bundle over the projectivized bundle of T_X. The fiber $X_{2,x}$ over $x \in X$ is a $\mathbb{P}^{n-1}$ bundle over $\mathbb{P}(T_xX) = \mathbb{P}^{n-1}$. The fixed point data can be read off from the first two columns of table (8). The Atiyah-Bott localization formula (9) has the following form:

$$AB(Q, X_2) = \sum_{j_1=1}^{n} \sum_{j_2=1, j_2\neq j_1}^{n} \frac{Q(\lambda_{j_1}, \lambda_{j_2} - \lambda_{j_1})}{\prod_{i\neq j_1}(\lambda_i - \lambda_{j_1}) \prod_{i\neq j_1,j_2}((\lambda_i - \lambda_{j_2}) \cdot (2\lambda_{j_1} - \lambda_{j_2})} + \\ + \sum_{j_1=1}^{n} \frac{Q(\lambda_{j_1}, \lambda_{j_1})}{\prod_{i\neq j_1}(\lambda_i - \lambda_{j_1}) \prod_{i\neq j_1}(\lambda_i - 2\lambda_{j_1})}. \tag{14}$$

Here the first sum corresponds to the fixed points $w_1 = \lambda_{j_1}, w_2 = \lambda_{j_2} - \lambda_{j_1}$ whereas the second sum to the fixed points $w_1 = w_2 = \lambda_{j_1}$.

Proposition 3.3.

$$AB(Q, X_{2,n}) = \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \frac{-z_2}{z_1 - z_2} \frac{Q(z_1, z_2)dz_2dz_1}{\prod_{i=1}^{n}(\lambda_i - z_1) \prod_{i=1}^{n}(\lambda_i - z_1 - z_2)} \tag{15}$$

Proof. Again, as in the warm-up, we compute the iterated residue (15) using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. The first residue, which is taken with respect to z_2, is a contour integral, whose value is minus the sum of the z_2-residues of the form in (15). These poles are at $z_2 = \lambda_i - z_1$, $i = 1 \dots n$, and $z_2 = z_1$ After canceling the signs that arise, we obtain the following expression for the right hand side of (15):

$$\operatorname*{Res}_{z_1=\infty}\left(\sum_{j_1=1}^{n} \frac{z_1 - \lambda_{j_1}}{2z_1 - \lambda_{j_1}} \frac{Q(z_1, \lambda_{j_1} - z_1)}{\prod_{i=1}^{n}(\lambda_i - z_1) \prod_{i\neq j_1}(\lambda_i - \lambda_{j_1})} + \frac{-z_1 Q(z_1, z_1)}{\prod_{i=1}^{n}(\lambda_i - z_1) \prod_{i=1}^{n}(\lambda_i - 2z_1)}\right) dz_1 \tag{16}$$

After cancellation and exchanging the sum and the residue operation, at the next step, we have

$$\sum_{j_1=1}^{n} \operatorname*{Res}_{z_1=\infty} \frac{-Q(z_1, \lambda_{j_1} - z_1)dz_1}{(2z_1 - \lambda_{j_1}) \prod_{i\neq j_1}(\lambda_i - z_1) \prod_{i\neq j_1}(\lambda_i - \lambda_{j_1})} + \operatorname*{Res}_{z_1=\infty} \frac{-z_1 Q(z_1, z_1)dz_1}{\prod_{i=1}^{n}(\lambda_i - z_1) \prod_{i=1}^{n}(\lambda_i - 2z_1)} \tag{17}$$

The first sum corresponds to the fixed points $p(w_1, w_2)$ where $w_2 = \lambda_{j_1} - w_1$, and the second to the ones with $w_2 = w_1$. We denote these conributions by AB_1 and AB_2, respectively. Now we again apply the Residue Theorem separately to AB_1 and AB_2, transforming the residue into a sum over the poles. The poles of AB_1 are $z_1 = \lambda_i, i \neq j_1$ and $z_1 = \lambda_{j_1}/2$, giving us

$$AB_1 = \sum_{j_1=1}^{n} \sum_{j_2 \neq j_1} \underbrace{\frac{Q(\lambda_{j_2}, \lambda_{j_1} - \lambda_{j_2})}{(2\lambda_{j_2} - \lambda_{j_1}) \prod_{i \neq j_2}(\lambda_i - \lambda_{j_2}) \prod_{i \neq j_1, j_2}(\lambda_i - \lambda_{j_1})}}_{AB(w_1=\lambda_{j_2}, w_2=\lambda_{j_1})-w_1} + \tag{18}$$

$$+ \sum_{j_1=1}^{n} \frac{1}{2} \underbrace{\frac{Q(\lambda_{j_1}/2, \lambda_{j_1}/2)}{\prod_{i \neq j_1}(\lambda_i - \lambda_{j_1}/2) \prod_{i \neq j_1}(\lambda_i - \lambda_{j_1})}}_{AB(w=\lambda_{j_1}/2, w_2=\lambda_{j_1}-w_1)} \tag{19}$$

The poles of AB_2 are $z_1 = \lambda_i$ and $z_1 = \lambda_i/2$ giving us

$$AB_2 = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \underbrace{\frac{-\lambda_{j_2} Q(\lambda_{j_2}, \lambda_{j_2})}{\prod_{i \neq j_2}(\lambda_i - \lambda_{j_2}) \prod_{i=1}^{n}(\lambda_i - 2\lambda_{j_2})}}_{AB(w_1=\lambda_{j_2}, w_2=w_1)} + \tag{20}$$

$$+ \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \underbrace{\frac{-\lambda_{j_2}/2 Q(\lambda_{j_2}/2, \lambda_{j_2}/2)}{\prod_{i=1}^{n}(\lambda_i - \lambda_{j_2}/2) \prod_{i \neq j_2}(\lambda_i - \lambda_{j_2})}}_{AB(w_1=\lambda_{j_2}/2, w_2=w_1)} \tag{21}$$

Since $AB(w_1 = \lambda_{j_1}/2, w_2 = \lambda_{j_1} - w_1) = -AB(w_1 = \lambda_{j_1}/2, w_2 = w_1)$ (note that $\lambda_{j_2}/2$ in the numerator cancels with $\lambda_{j_2} - \lambda_{j_2}/2$ in the denominator) , we arrive at (14).

□

3.4. **General case: iterated residues on the Demailly-Semple tower.**

Proposition 3.4.

$$AB(Q, X_k) = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{2 \leq t_1 \leq t_2 \leq k} -(z_{t_1} + z_{t_1+1} + \ldots + z_{t_2})}{\prod_{1 \leq s_1 < s_2 \leq k}(z_{s_1} - z_{s_1+1} - \ldots z_{s_2})} \frac{Q(z_1, \ldots, z_k)}{\prod_{j=1}^{k} \prod_{i=1^n}(\lambda_i - z_1 - z_2 - \ldots - z_j)} d\mathbf{z} \tag{22}$$

Note that the denominator is the product of the tangent weights at a generic fixed point (see (7)), and the factors in the numrator serve the cancelation of the zero elements which we omitted when we formed the $\neq 0$ part of the sets at (7).

Proof. Recall that for the fixed point $y = p(w_1, \ldots, w_k) \in X_{k,x}$ we have defined in (7) the sets

$$S_i(w_1, \ldots w_{i-1}), S_i^{\neq 0}(w_1, \ldots, w_{i-1}) \subset \mathrm{Lin}(\lambda_1, \ldots, \lambda_n),$$

such that the equivariant Euler class of the tangent bundle at $p(w_1, \ldots, w_k)$, that is, the product of the weights is equal to

$$Euler^T(T_{p(w_1,\ldots,w_k)} X_{k,x}) = \prod_{j=1}^{k} \prod_{w \in S_j^{\neq 0}(w_1,\ldots,w_{j-1}), w \neq w_j} (w - w_j). \tag{23}$$

Note that by definition

$$\left\{w - w_j : w \in S_j^{\neq 0}(w_1, \dots, w_{j-1})\right\} =$$
$$= \left\{\lambda_i - w_1 - \dots, -w_j, w_1 - w_2 - \dots - w_j, w_2 - w_3 - \dots - w_j, \dots, w_{j-1} - w_j : 1 \le i \le n\right\} \setminus$$
$$\setminus \left\{-(w_t + w_{t+1} + \dots + w_j) : 2 \le t \le j\right\}.$$

Therefore,

(24)
$$\frac{1}{Euler^T(T_{p(w_1,\dots,w_k)}X_{k,x})} = \prod_{j=1}^{k} \frac{\prod_{2\le t\le j} -(w_t + \dots + w_j)}{\tilde{\prod}_{1\le s\le j}(w_s - w_{s+1} - \dots - w_j)\tilde{\prod}_{i=1}^{n}(\lambda_i - w_1 - w_2 - \dots - w_j)},$$

where
$$\tilde{\prod}_{\gamma\in\Gamma}\gamma = \prod_{\gamma\in\Gamma,\gamma\neq 0}\gamma$$

denotes the product of the nonzero elements of the set we sum over. The condition $w \neq w_j$ in (23) is equivalent to replacing $\prod$ by $\tilde{\prod}$. The Atiyah-Bott localization reads as

$$(25)\quad AB(Q, X_{k,n}) = \sum_{p(w_1,\dots,w_k)\in\mathfrak{F}_{t,n}} \frac{Q(w_1, \dots, w_k)}{\prod_{j=1}^{k}\prod_{w\in S_j(w_1,\dots,w_{j-1}), w\neq w_j}(w - w_j)} =$$
$$= \sum_{p(w_1,\dots,w_k)\in\mathfrak{F}_{t,n}} \frac{\prod_{2\le t_1\le t_2\le k} -(w_{t_1} + \dots + w_{t_2})Q(w_1, \dots, w_k)}{\tilde{\prod}_{1\le s_1<s_2\le k}(w_{s_1} - w_{s_1+1} - \dots w_{s_2})\prod_{j=1}^{k}\tilde{\prod}_{i=1}^{n}(\lambda_i - w_1 - w_2 - \dots - w_j)}.$$

The iterated residue on the right hand side in (22) can be computed using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. The first residue, which is taken with respect to z_k, is a contour integral, whose value is minus the sum of the z_k-residues at the finite poles. These poles are at

$$z_k \in \mathscr{P}_k = \left\{\lambda_i - z_1 - \dots - z_{k-1}, z_q - z_{q+1} - \dots - z_{k-1}, z_{k-1} : i = 1 \dots n, q = 1, \dots k-2\right\}.$$

Note that by definition
$$\mathscr{P}_k = S_k(z_1, \dots, z_{k-1}),$$

and therefore after canceling the signs that arise we obtain the following expression for the right hand side of (22):

(26)
$$\operatorname*{Res}_{z_1=\infty} \dots \operatorname*{Res}_{z_{k-1}=\infty} \frac{\prod_{2\le t_1\le t_2\le k-1} -(z_{t_1} + z_{t_1+1} + \dots + z_{t_2})}{\prod_{1\le s_1<s_2\le k-1}(z_{s_1} - z_{s_1+1} - \dots - z_{s_2})\prod_{j=1}^{k-1}\prod_{i=1}^{n}(\lambda_i - z_1 - z_2 - \dots - z_j)}\cdot$$
$$\cdot \sum_{w_k\in\mathscr{P}_k} \frac{\prod_{2\le t\le k-1} -(z_t + z_{t+1} + \dots + z_{k-1} + w_k)(-w_k)Q(z_1, \dots, z_{k-1}, w_k)}{\tilde{\prod}_{1\le s\le k-1}(z_s - z_{s+1} - \dots - z_{k-1} - w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i - z_1 - z_2 - \dots - z_{k-1} - w_k)}\, d\mathbf{z}$$

Now take the next residue, with respect to z_{k-1}. The poles of (26) coming from the first factor are

$$\mathscr{P}_{k-1} = S_{k-1}(z_1, \dots, z_{k-2}). \tag{27}$$

The second term formally has poles at

$$\mathscr{Q}_{k-1}(w_k) = \{\lambda_i - z_1 - \dots - z_{k-2} - w_k, z_s - z_{s+1} - \dots - z_{k-2} - w_k : 1 \le i \le n, 1 \le s \le k-2\}. \tag{28}$$

More precisely:

(1) If $w_k = \lambda_{i_0} - z_1 - \dots - z_{k-1}$ then the denominator is

$$\tilde{\prod}_{1\le s\le k-1}(z_s - z_{s+1} - \dots - z_{k-1} - w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i - z_1 - z_2 - \dots - z_{k-1} - w_k) = \tag{29}$$
$$= \prod_{i\neq i_0}(\lambda_i - \lambda_{i_0}) \prod_{1\le s\le k-1}(z_1 + \dots + z_{s-1} + 2z_s - \lambda_{i_0}),$$

and

$$\mathscr{Q}(w_k) = \lambda_{i_0} - z_1 - \dots - z_{k-1}) = \{(z_1 + \dots + z_{k-2} - \lambda_{i_0})/2\},$$

containing a single element.

(2) If $w_k = z_{q_0} - z_{q_0+1} - \dots - z_{k-1}$, $leq_0 \le k-2$, then the denominator is

$$\tilde{\prod}_{1\le q\le k-1}(z_q - z_{q+1} - \dots - z_{k-1} - w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i - z_1 - z_2 - \dots - z_{k-1} - w_k) = \tag{30}$$
$$= \prod_{1\le q<s_0}(z_q - z_{q+1} - \dots - z_{q_0-1} - 2z_{q_0}) \prod_{q_0<q\le k-2}(-z_{q_0} + z_{q_0+1} + \dots + z_{q-1} + 2z_q) \prod_{i=1}^{n}(\lambda_i - z_1 - z_2 - \dots - z_{q_0-1} - 2z_{q_0}) =$$

and

$$\mathscr{Q}(w_k = z_{q_0} - z_{q_0+1} - \dots - z_{k-1}) = \{(z_{q_0} - z_{q_0+1} - \dots - z_{k-2})/2\},$$

containing a single element again.

(3) Finally, for $w_k = z_{k-1}$ we have

$$\tilde{\prod}_{1\le q\le k-1}(z_q - z_{q+1} - \dots - z_{k-1} - w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i - z_1 - z_2 - \dots - z_{k-1} - w_k) = \tag{31}$$
$$= \prod_{1\le q\le k-2}(z_q - z_{q+1} - \dots - z_{k-2} - 2z_{k-1}) \prod_{i=1}^{n}(\lambda_i - z_1 - \dots - z_{k-2} - 2z_{k-1}),$$

and therefore

$$\mathscr{Q}(w_k = z_{k-1}) = \{(z_q - z_{q+1} - \dots - z_{k-2})/2, (\lambda_i - z_1 - \dots - z_{k-2})/2 : 1 \le q \le k-2, 1 \le i \le n\}.$$

Applying the Residue Theorem with respect to z_{k-1} in (26), we get a sum over these poles. We claim that the terms corresponding to the poles in $\mathcal{Q}_{k-1}$ add up to 0, that is,

(32)

$$\sum_{w_k\in\mathcal{P}_k}\sum_{w_{k-1}\in\mathcal{Q}(w_k)} c(w_k,w_{k-1})\cdot \frac{\prod_{2\le t_1\le t_2\le k-1} -(z_{t_1}+z_{t_1+1}+\ldots+z_{t_2})}{\prod_{1\le s_1<s_2\le k-1}(z_{s_1}-z_{s_1+1}-\ldots-z_{s_2})\prod_{j=1}^{k-1}\prod_{i=1}^{n}(\lambda_i-z_1-z_2-\ldots-z_j)}_{|z_{k-1}\leftrightarrow w_{k-1}}\cdot$$
$$\frac{-(w_{k-1}+w_k)\cdot(-w_k)\prod_{1\le t\le k-1}Q(z_1,\ldots,z_{k-2},w_{k-1},w_k)}{\tilde{\prod}_{1\le s\le k-1}(z_s-z_{s+1}-\ldots z_{k-2}-w_{k-1}-w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i-z_1-z_2-\ldots-z_{k-2}-w_{k-1}-w_k)}=0,$$

where $c(w_k,w_{k-1})$ is the coefficient of z_{k-1} in the corresponding term, see below.

We pair the terms in (32), such that each couple adds up to 0. We have described the sets $\mathcal{Q}(w_k)$ above, and get

(1)

$$(w_k,w_{k-1})=(\lambda_{i_0}-z_1-\ldots-z_{k-1},(z_1+\ldots+z_{k-2}-\lambda_{i_0})/2)=((z_1+\ldots+z_{k-2}-\lambda_{i_0})/2,(z_1+\ldots+z_{k-2}-\lambda_{i_0})/2)$$

Here $c(w_k,w_{k-1})=-1/2$.

(2)

$$(w_k,w_{k-1})=(z_{q_0}-z_{q_0+1}-\ldots-z_{k-1},z_{q_0}-z_{q_0+1}-\ldots-z_{k-2}/2)=(z_{q_0}-z_{q_0+1}-\ldots-z_{k-2}/2,z_{q_0}-z_{q_0+1}-\ldots-z_{k-2}/2)$$

Here $c(w_k,w_{k-1}=-1/2$.

(3)

$$(w_k,w_{k-1})=(z_{k-1},(z_q-z_{q+1}-\ldots-z_{k-2})/2)=((z_q-z_{q+1}-\ldots-z_{k-2})/2,(z_q-z_{q+1}-\ldots-z_{k-2})/2)$$

Here $c(w_k,w_{k-1})=-1/2$

(4)

$$(w_k,w_{k-1})=(z_{k-1},(\lambda_i-z_1-\ldots-z_{k-2})/2)=((\lambda_i-z_1-\ldots-z_{k-2})/2,(\lambda_i-z_1-\ldots-z_{k-2})/2)$$

Here $c(w_k,w_{k-1})=-1/2$

We see that in (32) the terms under (1) cancel with the terms under (3), and the terms under (2) cancel with the terms under (4).

Therefore taking the residue w.r.t z_{k-1} in (26), only the poles at $z_{k-1}\in\mathcal{P}_{k-1}$ contribute, and we get the following expression for the iterated residue:

(33)

$$\operatorname*{Res}_{z_1=\infty}\ldots\operatorname*{Res}_{z_{k-2}=\infty}\frac{\prod_{2\le t_1\le t_2\le k-2} -(z_{t_1}+z_{t_1+1}+\ldots+z_{t_2})}{\prod_{1\le s_1<s_2\le k-2}(z_{s_1}-z_{s_1+1}-\ldots-z_{s_2})\prod_{j=1}^{k-2}\prod_{i=1}^{n}(\lambda_i-z_1-z_2-\ldots-z_j)}\cdot$$
$$\cdot\sum_{\substack{w_k\in\mathcal{P}_k\\ w_{k-1}\in\mathcal{P}_{k-1}}}\frac{\prod_{2\le t\le k-2}-(z_t+z_{t+1}+\ldots+z_{k-2}+w_{k-1}+w_k)(-w_{k-1}-w_k)(-w_k)Q(z_1,\ldots,z_{k-2},w_{k-1},w_k)}{\tilde{\prod}_{1\le s\le k-2}(z_s-z_{s+1}-\ldots-z_{k-2}-w_{k-1}-w_k)(w_{k-1}-w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i-z_1-\ldots-z_{k-2}-w_{k-1}-w_k)}\,\boldsymbol{dz}$$

Now take the residue w.r.t z_{k-2} and play the same game again to prove that the contributions of the poles in the denominator of the second factor add up to 0. The poles in the second factor are formally at

$$\mathcal{Q}(w_k, w_{k-1}) =$$
$$\{\lambda_i - z_1 - \ldots - z_{k-3} - w_{k-1} - w_k, z_q - z_{q+1} - \ldots - z_{k-3} - w_{k-1} - w_k, w_{k-1} + w_k : 1 \le i \le n, 2 \le q \le k-3\}.$$

Note that w_{k-1}, w_k depends on z_{k-2}, but formally we have the following cases:

(1) $z_{k-2} = \lambda_{i_0} - z_1 - \ldots - z_{k-3} - w_{k-1} - w_k$, then the denominator of the second factor is

(34)
$$\tilde{\prod}_{1\le s\le k-2}(z_s - z_{s+1} - \ldots - z_{k-2} - w_{k-1} - w_k)(w_{k-1} - w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i - z_1 - \ldots - z_{k-2} - w_{k-1} - w_k) =$$
$$= \prod_{i\neq i_0}(\lambda_i - \lambda_{i_0}) \prod_{1\le s\le k-2}(z_1 + \ldots + z_{s-1} + 2z_s - \lambda_{i_0})$$

and therefore

$$\mathcal{Q}(w_{k-1}, w_k) = \{(\lambda_{i_0} - z_1 - \ldots - z_{k-3})/2\},$$

independent of w_{k-1}, w_k.

(2) If $z_{k-2} = z_{q_0} - \ldots - z_{k-3} - w_{k-1} - w_k$, we get similarly that the only pole is at

$$\mathcal{Q}(w_{k-1}, w_k) = \{(z_{q_0} - \ldots - z_{k-3})/2\},$$

independent of w_{k-1}, w_k.

(3) If $z_{k-2} = w_{k-1} + w_k$, then the denominator of the second factor is

(35)
$$\tilde{\prod}_{1\le s\le k-2}(z_s - z_{s+1} - \ldots - z_{k-2} - w_{k-1} - w_k)(w_{k-1} - w_k)\tilde{\prod}_{i=1}^{n}(\lambda_i - z_1 - \ldots - z_{k-2} - w_{k-1} - w_k) =$$
$$= \prod_{1\le q\le k-3}(z_q - z_{q+1} - \ldots - z_{k-3} - 2z_{k-2})(w_{k-1} - w_k)\prod_{i=1}^{n}(\lambda_i - z_1 - \ldots - z_{k-3} - 2z_{k-2}),$$

and the linear form $w_{k-1} - w_k$ does not contain z_{k-2} if $w_k \in \mathcal{P}_k, w_{k-1} \in \mathcal{P}_{k-1}$, therefore the only poles for z_{k-2} are at

$$\mathcal{Q}(w_{k-1}, w_k) = \{(z_q - z_{q+1} - \ldots - z_{k-3})/2, (\lambda_i - z_1 - \ldots - z_{k-3})/2 : 1 \le q \le k-3, 1 \le i \le n\}$$

Again, the terms under (1) and (2) cancel out with the terms in (3), leaving us with the poles for z_{k-2} in the first factor in (33)

Iterating this process we finally arive at (25), proving Proposition 3.4.

□

Remark 3.5. *Changing the order of the variables in iterated residues, usually, changes the result. In this case, however, because all the poles are normal crossing, formula* (3.4) *remains true no matter in what order we take the iterated residues.*

4. Proof of Theorem 1.1

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $\deg X = d$, and let X_k denote the k-level Demailly-Semple tower on X.

To start with, we recall the classical result of Demailly which connects jet differentials to the Green-Griffiths Conjecture:

Theorem 4.1. *(*[24, 10, 40]*) Assume that there exist integers $k, m > 0$ and ample line bundle $A \to X$ such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi^* A^{-1}) \simeq H^0(X, E_{k,m} T_X^* \otimes A^{-1}) \neq 0$$

has non zero sections $\sigma_1, \ldots, \sigma_N$, and let $Z \subset X_k$ be the base locus of these sections. Then every entire holomorphic curce $f : \mathbb{C} \to X$ necessarily satisfies $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant differential equation P vanishing on an ample divisor, every entire holomorphic curve f must satisfy the algebraic differential equation $P(j^k f(t)) \equiv 0$.

Note, that by Theorem 1. of [14],

$$H^0(X, E_{k,m} T_X^* \otimes A^{-1}) = 0$$

holds for all $m \geq 1$ if $k < n$, so we can restrict our attention to the range $k \geq n$.

To control the order of vanishing of these differential forms along the ample divisor we choose A to be –as in [13] – a proper twist of the canonical bundle of X. Recall that the canonical bundle of the smooth, degree d hypersurface X is

$$K_X = \mathcal{O}_X(d - n - 2),$$

which is ample as soon as $d \geq n + 3$.

The crucial step which connects algebraic degeneracy of germs with the degeneracy of the holomorphic curve is Siu's strategy of constructiung linealy independent jet differentials using the bundle of vertical jets. The following theorem is a reformulation of the results of §3 in [13].

Theorem 4.2. *(Algebraic degeneracy of entire curves,*[13], §*3)*
Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $D = D(n, \delta)$ such that

$$H^0(X_n, \mathcal{O}_{X_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{d,m} T_X^* \otimes K_X^{-\delta m}) \neq 0$$

whenever $\deg(X) > D(n, \delta)$ *for some $m \gg 0$. Then the Green-Griffiths conjecture holds for*

$$\deg(X) \geq \max(D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2).$$

Following [10], for $(a_1, \ldots, a_k) \in \mathbb{Z}^k$, we define the following line bundle on X_k:

$$\mathcal{O}_{X_k}(\mathbf{a}) = \pi_{1,k}^* \mathcal{O}_{X_1}(a_1) \otimes \pi_{2,k}^* \mathcal{O}_{X_2}(a_2) \otimes \cdots \otimes \mathcal{O}_{X_k}(a_k). \quad (36)$$

The following theorem is from [10], and [14]

Proposition 4.3. (1) *([10],Prop. 6.16)*
If $a_1 \geq 3a_2, \ldots, a_{k-2} \geq 3a_{k-1}$, and $a_{k-1} \geq 2a_k \geq 0$, the line bundle $\mathcal{O}_{X_k}(\mathbf{a})$ is relatively nef over X. If, moreover,

$$a_1 \geq 3a_2, \ldots, a_{k-2} \geq 3a_{k-1} \text{ and } a_{k-1} > 2a_k > 0 \tag{37}$$

holds, then $\mathcal{O}_{X_k}(\mathbf{a})$ is relatively ample over X.

(2) *([14], Prop. 3.2)*
Let $\mathcal{O}_X(1)$ denote the hyperplane divisor on X. If (37) holds, then $\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(l)$ is nef, provided that $l \geq 2|\mathbf{a}|$, where $|\mathbf{a}| = a_1 + \ldots + a_k$.

Theorem 4.2 accompanied with the following theorem gives us Theorem 1.1.

Theorem 4.4. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface with ample canonical bundle, that is $\deg X \geq n+3$. If $a_i = n^{8(n+1-i)}$,$\delta = \frac{1}{n^{8n}}$ and $d > D(n) = n^{8n}$ then*

$$H^0(X_n, \mathcal{O}_{X_n}(|\mathbf{a}|) \otimes \pi^* K_X^{-\delta|\mathbf{a}|}) \simeq H^0(X, E_{n,|\mathbf{a}|}T_X^* \otimes K_X^{-\delta|\mathbf{a}|}) \neq 0,$$

nonzero.

To prove Theorem 4.4 we follow [13] using the algebraic Morse inequalities of Demailly and Trapani. Let $L \to X$ be a holomorphic line bundle over a compact Kahler manifold of dimension n and $E \to X$ a holomorphic vector bundle of rank r. Suppose that L can be written as the difference of two nef line bundles, $L = F \otimes G^{-1}$, with F, G numerically effective. Demailly proved in [11] the following strong algebraic Morse inequalities.

Theorem 4.5. *([11, 44]) With this notation we have*

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes E) \leq r\frac{m^n}{n!} \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).$$

In particular, $q = 1$ asserts that $L^{\otimes m} \otimes E$ has a global section for m large as soon as

$$F^n - nF^{n-1}G > 0.$$

For $d > n+3$ the canonical bundle $K_X \simeq \mathcal{O}_X(d-n-2)$ is ample, and therefore we have the following expressions for $\mathcal{O}_{X_k}(\mathbf{a})$ and $\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* K_X^{-\delta\mathbf{a}}$ as a difference of two nef line bundles over X:

- $\mathcal{O}_{X_k}(\mathbf{a}) = (\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)) \otimes (\pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|))^{-1}$
- $\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* K_X^{-\delta\mathbf{a}} = (\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)) \otimes (\pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|) \otimes \pi_{0,k}^* K_X^{\delta\mathbf{a}})^{-1}$

Applying the Morse algebraic inequalities, we need to prove the positivity of the following intersection product:

$$I(X, n, k, \mathbf{a}, \delta) = (\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|))^{n+k(n-1)} - \\ ((k+1)(n-1))(\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|))^{(k+1)(n-1)} \cdot (\pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|) \otimes \pi_{0,k}^* K_X^{\delta\mathbf{a}}).$$

Let $h = c_1(\mathcal{O}_X(1))$ denote the first Chern class of the tautological line bundle $\mathcal{O}_X(1)$, $c_l = c_l(T_X)$ for $l = 1, \ldots, n$, and $u_s = c_1(\mathcal{O}_{X_s}(1))$ for $s = 1, \ldots, s$. Then $c_1(K_X) = -c_1 = (d - n - 2)h$, and the intersection product we have to evaluate becomes

$$\begin{aligned} & I(X, n, k, \mathbf{a}, \delta) = \\ &= (a_1u_1+\ldots+a_ku_k+2|\mathbf{a}|h)^{n+k(n-1)} - (k+1)(n-1)(a_1u_1+\ldots+a_ku_k+2|\mathbf{a}|h)^{(k+1)(n-1)}(2|\mathbf{a}|h - \delta|\mathbf{a}|c_1) = \\ &= (a_1u_1+\ldots+a_ku_k+2|\mathbf{a}|h)^{(k+1)(n-1)}\,(a_1u_1 + \ldots + a_ku_k + 2|\mathbf{a}|h - (k+1)(n-1)|\mathbf{a}|h(2 + \delta(d - n - 2))) \end{aligned}$$

We look at this intersection product as a polynomial in the variables $u_1, \ldots, u_k$, and also use the notation $I_{X,n,k,\mathbf{a},\delta}(u_1, \ldots, u_k)$.

The following proposition presents the first iterated residue formula for the Demailly intersection number.

Proposition 4.6.

(38)

$$\int_{X_k} I(X, n, k, \mathbf{a}, \delta) = (-1)^k \int_X \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{2\le t_1\le t_2\le k}(z_{t_1} + z_{t_1+1} + \ldots + z_{t_2})}{\prod_{1\le s_1<s_2\le k}(-z_{s_1} + z_{s_1+1} + \ldots + z_{s_2})} \frac{I_{X,n,k,\mathbf{a},\delta}(z_1, \ldots, z_k)}{\prod_{j=1}^k \prod_{i=1^n}(\lambda_i + z_1 + z_2 + \ldots + z_j)}\, d\mathbf{z}$$

Proof. Integrating first along the fibers we get

$$\int_{X_k} I(X, n, k, \mathbf{a}, \delta) = \int_X \int_{X_{k,x}} I(X, n, k, \mathbf{a}, \delta)$$

We apply the ABBV localization on the fiber $X_{k,x}$ with respect to the T^n action on $X_{k,x}$ described in the previous section, and get that (see (25))

$$\int_{X_k} I(X, n, k, \mathbf{a}, \delta) = AB(I_{X,n,k,\mathbf{a},\delta}(-u_1, \ldots, -u_k)).$$

To see this, recall that in formula (25), w_i stands for the weight of the T^n action on the tautological line bundle $\mathcal{O}_{X_i}(-1)$, which is by definition u_i. Now Proposition 3.4 gives the result, noticing that we made the substitution $z_i \leftrightarrow -z_i$, resulting the sign $(-1)^k$. □

We can get rid of the λ_i's on the right hand side of (38) with a small trick. To do it, we use the following well-known fact

Remark 4.7. *The Chern classes of X are expressible with d, h via the following identity:*

$$(1 + h)^{n+2} = (1 + \deg(X)h)c(X),$$

where $c(X) = c(T_X)$ is the total Chern class of X. After expansion we get for $1 \le j \le n$

$$c_j(X) = (-1)^j h^j \sum_{i=0}^{j} (-1)^i \binom{n+2}{i} d^{j-i} \qquad (39)$$

.

Therefore we have

$$\prod_{i=1}^{n}(\lambda_i + z_1 - \ldots + z_j) = (z_1 + \ldots + z_j)^n \prod_{i=1}^{n}(1 + \frac{\lambda_i}{z_1 + \ldots + z_j}) = \tag{40}$$

$$= (z_1 + \ldots + z_j)^n \cdot c\left(\frac{1}{z_1 + \ldots + z_j}\right) = (z_1 + \ldots + z_j)^n \frac{\left(1 + \frac{h}{z_1+\ldots+z_j}\right)^{n+2}}{1 + \frac{dh}{z_1+\ldots+z_j}}$$

and we arrive to the main formula of the first half of this paper.

Theorem 4.8.

$$\int_{X_k} I(X, n, k, \mathbf{a}, \delta) = \tag{41}$$

$$= (-1)^k \int_X \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{2\le t_1\le t_2\le k}(z_{t_1} + z_{t_1+1} + \ldots + z_{t_2}) I_{X,n,k,\mathbf{a},\delta}(z_1, \ldots, z_k)}{\prod_{1\le s_1<s_2\le k}(-z_{s_1} + z_{s_1+1} + \ldots + z_{s_2}) \prod_{j=1}^{k}(z_1 + \ldots + z_j)^n} \prod_{j=1}^{k} \frac{1 + \frac{dh}{z_1+\ldots+z_j}}{\left(1 + \frac{h}{z_1+\ldots+z_j}\right)^{n+2}} \, d\mathbf{z}$$

This formula has the pleasant feature that it expresses the aimed intersection number directly in terms of n, k, d, δ. Indeed, the result of the iterated residue is a polynomial in n, k, δ and h^n, and integrating over X simply means a substitution $d = h^n$.

4.1. **Computations with the iterated residue for $n = k$.** From now on we assume that $n = k$, focusing on Theorem 4.4. Before we start the explicit analysis of our formula (41), we carry out a simplification to get a formula which is easier to handle. Multiplying the terms in the denominator we get the following expression for the R.H.S of (41)

(42)

$$(-1)^n \int_X \operatorname*{Res}_{\mathbf{z}=\infty} \prod_{2\le t_1\le t_2\le n} \frac{z_{t_1} + z_{t_1+1} + \ldots + z_{t_2}}{-z_{t_1-1} + z_{t_1} + \ldots + z_{t_2}} \prod_{j=1}^{n} \frac{(z_1 + \ldots + z_j)(z_1 + \ldots + z_j + dh)}{(z_1 + \ldots + z_j + h)^{n+2}} I_{X,n,\mathbf{a},\delta}(z_1, \ldots, z_n) \, d\mathbf{z}$$

The iterated residue is formally a contour integral, but as we have explained in §3.1, it simply means an expansion of the fraction respecting the order $1 \ll |z_1| \ll \ldots \ll |z_k|$. The terms in (42) then have the following expansions

(1)

$$\frac{1}{z_1 + \ldots + z_j + h} = \frac{1}{z_j}\left(1 - \frac{z_1 + \ldots z_{j-1} + h}{z_j} + \left(\frac{z_1 + \ldots z_{j-1} + h}{z_j}\right)^2 - \ldots\right)$$

(2)

$$\frac{z_{t_1} + z_{t_1+1} + \ldots + z_{t_2}}{-z_{t_1-1} + z_{t_1} + \ldots + z_{t_2}} = 1 + \frac{z_{t_1-1}}{z_{t_2}}\left(1 + \frac{z_{t_1-1} - z_{t_1} - \ldots - z_{t_2-1}}{z_{t_2}} \left(\frac{z_{t_1-1} - z_{t_1} - \ldots - z_{t_2-1}}{z_{t_2}}\right)^2 + \ldots\right)$$

(3)

$$I_{X,n,d,\delta,\mathbf{a}} = (a_1z_1+\ldots+a_nz_n+2|\mathbf{a}|h)^{n^2-1}\left(a_1z_1+\ldots+a_nz_n+2|\mathbf{a}|h-(n^2-1)|\mathbf{a}|h(2+\delta(d-n-2))\right) =$$
$$= (a_1z_1+\ldots+a_nz_n+2|\mathbf{a}|h)^{n^2-1}\left(a_1z_1+\ldots+a_nz_n+|\mathbf{a}|h(4-2n^2+\delta(n+2))-\delta|\mathbf{a}|dh\right)$$

Substituting these into (41) we arrive at

$$\int_{X_n} I(X,n,d,\mathbf{a},\delta) =$$
$$= (-1)^n\int_X \operatorname*{Res}_{\mathbf{z}=\infty}\prod_{2\le t_1\le t_2\le n}\frac{z_{t_1}+z_{t_1+1}+\ldots+z_{t_2}}{-z_{t_1-1}+z_{t_1}+\ldots+z_{t_2}}\prod_{j=2}^n(1+\frac{z_1+\ldots+z_{j-1}}{z_j})\prod_{j=2}^n(1+\frac{z_1+\ldots+z_{j-1}+dh}{z_j})\cdot$$
$$\prod_{j=2}^n\left(1-\frac{z_1+\ldots z_{j-1}+h}{z_j}+\left(\frac{z_1+\ldots z_{j-1}+h}{z_j}\right)^2-\ldots\right)^{n+2}\cdot$$
$$\frac{(a_1z_1+\ldots+a_nz_n+2|\mathbf{a}|h)^{n^2-1}\left(a_1z_1+\ldots+a_nz_n+|\mathbf{a}|h(4-2n^2+\delta(n+2))-\delta|\mathbf{a}|dh\right)}{(z_1\ldots z_n)^n}d\mathbf{z}$$

Notation 4.1. (1) *For a monomial* $\mathbf{z}^{\mathbf{i}} = z_1^{i_1}\ldots z_n^{i_n}$, $i_s\in\mathbb{Z}$ *we call*

$$\mathrm{Defect}(\mathbf{i}) = ni_1+(n-1)i_2\ldots+i_n$$

the defect *of* $\mathbf{i}$. *Moreover, we define the set of positive roots as the semigroup generated by the simple positive roots*

$$\Lambda^+ = \mathbb{Z}^{\ge 0}\langle(0,\ldots,1^i,\ldots,-1^j,\ldots,0),(0,\ldots,-1^i,\ldots,0) : 1\le i<j\le n\rangle.$$

The negative roots are $\Lambda^- = -\Lambda^+$.

(2) *We say that* $\mathbf{a}\ge\mathbf{b}$ *if there is a* $\mathbf{c}\in\Lambda^+$ *with* $\mathbf{b}+\mathbf{c}=\mathbf{a}$.

Theorem 4.9. *With* $a_i = n^{8(n+1-i)}$ *and* $\delta=\frac{1}{n^{8n}}$ *we have* $\int_{X_n} I(X,n,d,\mathbf{a},\delta)>0$ *for* $d>n^{8n}$.

We devote the rest of this section to the proof of Theorem 4.9. For the sake of keeping our formulas under control, we introduce

$$(43)\quad A(\mathbf{z}) = \underbrace{\prod_{2\le t_1\le t_2\le n}\frac{z_{t_1}+z_{t_1+1}+\ldots+z_{t_2}}{-z_{t_1-1}+z_{t_1}+\ldots+z_{t_2}}}_{A^1}\underbrace{\prod_{j=2}^n(1+\frac{z_1+\ldots+z_{j-1}}{z_j})}_{A^2}\cdot$$
$$\underbrace{\prod_{j=2}^n\left(1-\frac{z_1+\ldots z_{j-1}+h}{z_j}+\left(\frac{z_1+\ldots z_{j-1}+h}{z_j}\right)^2-\ldots\right)^{n+2}}_{A^3}\underbrace{\prod_{j=2}^n(1+\frac{z_1+\ldots+z_{j-1}+dh}{z_j})}_{A^0},$$

and

$$(44)\quad B(\mathbf{z}) = \frac{(a_1z_1+\ldots+a_nz_n+2|\mathbf{a}|h)^{n^2-1}(a_1z_1+\ldots+a_nz_n+S(n,\delta)|\mathbf{a}|h-\delta|\mathbf{a}|dh)}{(z_1\ldots z_n)^n},$$

where $S(n,\delta) = (4 - 2n^2 + \delta(n+2))$.

Observation 4.10. $A_{\mathbf{z}^{\mathbf{i}}(dh)^m} = \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}}(dh)^m} A(\mathbf{z}) = 0$ *unless* $\mathbf{a} \in \Lambda^-$, *for any* $m \geq 0$.

Let's have a short break and step back a bit looking at formula (43). The residue is by definition the coefficient of $\frac{1}{z_1 \dots z_n}$ in the appropriate Laurent expansion of the big rational expression in $z_1, \dots, z_n, n, d, h$ and δ, multiplied by $(-1)^n$. We can therefore omit the $(-1)^n$ factor and simply compute the corresponding coefficient. The result is a polynomial in n, d, h, δ, and in fact, a relatively easy argument shows that it is a polynomial in n, d, δ multiplied by h^n

Indeed, giving degree 1 to $z_1, \dots, z_n, h$ and 0 to n, d, δ, the rational expression in the residue has total degree 0. Therefore the coefficient of $\frac{1}{z_1 \dots z_n}$ has degree n, so it has the form $h^n p(n, d, \delta)$ with a polynomial p. Since d appears only as a linear factor next to h, the degree of p in d is n.

Moreover, $\int_X h^n = d$, so the integration over X is simply a substitution $h^n = d$, resulting the equation

$$I(X, d, n, \mathbf{a}, \delta) = dp(n, d, \mathbf{a}, \delta),$$

where

$$p(n, d, \mathbf{a}, \delta) = p_n(n, \mathbf{a}, \delta)d^n + \dots + p_1(n, \mathbf{a}, \delta)d + p_0(n, \mathbf{a}, \delta)$$

is a polynomial in d of degree n.

4.2. **Estimation of the leading coefficent.** The next goal is to compute the leading coefficient $p_n(n, \mathbf{a}, \delta)$. From (43),(44)

$$p_n = \sum_{\Sigma\mathbf{i}=0} B_{\mathbf{z}^{\mathbf{i}}} A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}(dh)^n} - \delta|\mathbf{a}| \sum_{\Sigma\mathbf{i}=-1} B_{\mathbf{z}^{\mathbf{i}}(dh)} A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}(dh)^{n-1}} \tag{45}$$

where $\mathbf{1} = (1, \dots, 1)$. Note that – according to Observation 4.10 – some terms on the r.h.s is 0, since we have not made any restrictions on the relation of $\mathbf{i}$ to Λ^+.

There is a dominant term on the r.h.s, corresponding to $\mathbf{i} = (0, \dots, 0)$ in the first sum:

$$B_{\mathbf{0}} = B_{\mathbf{z}^{\mathbf{0}}} A_{\frac{(dh)^n}{\mathbf{z}^{\mathbf{1}}}} = (a_1 \dots a_n)^n \binom{n^2}{n, \dots, n}$$

We show that the absolute sum of the remaining terms is less than this dominant term, implying a lower bound for p_n when $\delta, \mathbf{a}$ as in Theorem 4.9.

First observe that for $\Sigma\mathbf{i} = 0$

$$B_{\mathbf{z}^{\mathbf{i}}} = \binom{n^2}{i_1 + n, \dots, i_n + n} a_1^{i_1+n} \dots a_n^{i_n+n} < a_1^{i_1} \dots a_n^{i_n} B_{\mathbf{0}} = n^{8\mathrm{Defect}(\mathbf{i})} B_{\mathbf{0}} \tag{46}$$

On the other hand

$$A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}(dh)^n} = \sum_{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3=-\mathbf{i}} A^1_{\mathbf{i}_1} A^2_{\mathbf{i}_2} A^3_{\mathbf{i}_3} < 3^{-\mathrm{Defect}(\mathbf{i})} n^{-3\mathrm{Defect}(\mathbf{i})}, \tag{47}$$

according to the following two lemmas, which will be repeteadly used:

Lemma 4.11. *Let* $\Sigma\mathbf{i} = 0$*,* $\mathbf{i} \in \Lambda^+$*. Then*

$$\sharp\{(\mathbf{i}_1, \ldots, \mathbf{i}_s) \in (\Lambda^+)^s : \mathbf{i}_1 + \ldots + \mathbf{i}_s = \mathbf{i}\} \le s^{\mathrm{Defect}(\mathbf{i})}$$

Proof. Let $\mathbf{i} = \mathbf{j}_1 + \ldots + \mathbf{j}_{\mathrm{Defect}(\mathbf{i})}$ be the sum of simple roots. This decomposition is unique, which can be seen eadily by induction. Then any term can be put into any of the s multiindices $\mathbf{i}_1, \ldots, \mathbf{i}_s$, and this gives us the desired upper bound. □

Lemma 4.12. *Let* $\Sigma\mathbf{i} = 0$*. Then*

$$A^1_{\mathbf{i}}, A^2_{\mathbf{i}}, A^3_{\mathbf{i}} < n^{3\mathrm{Defect}(\mathbf{i})}$$

Note that this means that $A^1_{\mathbf{i}} = A^2_{\mathbf{i}} = A^3_{\mathbf{i}} = 0$ for $\mathrm{Defect}(\mathbf{i}) > 0$, as expected, since all these coefficients are 0 unless $\mathbf{i} \in \Lambda^-$.

Proof.

(48)
$$|A^1_{\mathbf{i}}| = \sum_{\substack{1 \le t_1 < t_2 \le n \\ \sum_{t_1,t_2} \mathbf{i}(t_1,t_2) = \mathbf{i}}} \prod_{t_1,t_2} \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}(t_1,t_2)}} \left(1 + \frac{z_{t_1-1}}{z_{t_2}}\left(1 + \frac{z_{t_1-1} - z_{t_1} - \ldots - z_{t_2-1}}{z_{t_2}}\left(\frac{z_{t_1-1} - z_{t_1} - \ldots - z_{t_2-1}}{z_{t_2}}\right)^2 + \ldots\right)\right) <$$
$$< \sum_{\substack{\mathbf{i}(t_1,t_2) \\ \sum_{t_1,t_2} \mathbf{i}(t_1,t_2) = \mathbf{i}}} \prod_{t_1,t_2} \mathrm{comb}(\mathbf{i}^+(t_1,t_2)),$$

where $\mathbf{i}(t_1,t_2) = \mathbf{i}^+(t_1,t_2) - \mathbf{i}^-(t_1,t_2)$ for some $\mathbf{i}^+, \mathbf{i}^- > 0$, and $\mathrm{comb}(\mathbf{j}) = \binom{j_1+\ldots+j_n}{j_1,\ldots,j_n}$ is the number of different orders of the elements of $\mathbf{j} = (1^{j_1}, \ldots n^{j_n})$. Following the proof of Lemma 4.11,

$$\sum_{\substack{\mathbf{i}(t_1,t_2) \\ \sum_{t_1,t_2} \mathbf{i}(t_1,t_2) = \mathbf{i}}} 1 < \binom{n}{2}^{\mathrm{Defect}(\mathbf{i})}.$$

Moreover,

$$\mathrm{comb}(\mathbf{i}^+(t_1,t_2)) < n^{|\mathbf{i}^+(t_1,t_2)|} \le n^{\mathrm{Defect}(\mathbf{i}(t_1,t_2))} \text{ so } \prod_{t_1,t_2} \mathrm{comb}(\mathbf{i}^+(t_1,t_2)) < n^{\mathrm{Defect}(\mathbf{i})}$$

and therefore

$$|A^1_{\mathbf{i}}| < (n\binom{n}{2})^{\mathrm{Defect}(\mathbf{i})}.$$

Similarly, for $\Sigma\mathbf{i} = 0$

(49)
$$|A^2_{\mathbf{i}}| < 2^{\mathrm{Defect}(\mathbf{i})}$$

and finally, similarly as for A^1:

$$(50)\quad |A^3_{\mathbf{i}}| < \sum_{\sum_{j=2}^n \mathbf{i}(j)=\mathbf{i}} \prod_j \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}(j)}} \left(1 - \frac{z_1 + \ldots z_{j-1} + h}{z_j} + \left(\frac{z_1 + \ldots z_{j-1} + h}{z_j}\right)^2 - \ldots\right)^{n+2} <$$
$$< \sum_{\sum_{j=2}^n \mathbf{i}(j)=\mathbf{i}} \sum_{\mathbf{s}_1(j)+\ldots+\mathbf{s}_{n+2}(j)=\mathbf{i}(j)} \prod_{\substack{2\le j\le n,\\ 1\le m\le n+2}} \mathrm{comb}(\mathbf{s}_m^+(j))$$

and again

$$\sum_{\sum_{j=2}^n \mathbf{i}(j)=\mathbf{i}} \sum_{\mathbf{s}_1(j)+\ldots+\mathbf{s}_{n+2}(j)=\mathbf{i}(j)} 1 < ((n-1)(n+2))^{\mathrm{Defect}(\mathbf{i})},$$

and

$$\mathrm{comb}(\mathbf{s}_m^+(j)) < n^{\mathrm{Defect}(\mathbf{s}_m(j))}$$

giving us

$$|A^3_{\mathbf{i}}| < ((n-1)n(n+2))^{\mathrm{Defect}(\mathbf{i})},$$

which proves Lemma 4.12. □

Substituting inequalities (46) and (47) into (45) we get

(51)

$$\sum_{\substack{\mathbf{i}\neq 0\\ \Sigma\mathbf{i}=0}} B_{\mathbf{z}^{\mathbf{i}}} A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}(dh)^n} < \sum_{i=1}^{n^2} \sum_{\substack{\mathbf{i}\neq 0, \Sigma\mathbf{i}=0, \mathbf{i}\in\Lambda^-\\ \mathrm{Defect}(\mathbf{i})=-i}} \left(\frac{3}{n^5}\right)^i B_{\mathbf{0}} = \sum_{i=1}^{n^2} \left(\frac{3}{n^5}\right)^i \sum_{\substack{\mathbf{i}\neq 0, \Sigma\mathbf{i}=0, \mathbf{i}\in\Lambda^-\\ \mathrm{Defect}(\mathbf{i})=-i}} 1 < \sum_{i=1}^{n^2} \left(\frac{3}{n^5}\right)^i n^i B_{\mathbf{0}} < \frac{1}{4} B_{\mathbf{0}}$$

We can handle the second sum of the r.h.s in (46) in a similar fashion. For $\Sigma\mathbf{i} = -1$, and $e_j = (0, \ldots, 1^j, \ldots, 0)$

$$(52)\qquad A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}(dh)^{n-1}} = \sum_{j_2=2}^{n} \sum_{j_1\le j_2} \sum_{\substack{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3=-\mathbf{i}-e_{j_1}\\ \mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\in\Lambda^+}} A^1_{\mathbf{i}_1} A^2_{\mathbf{i}_2} A^3_{\mathbf{i}_3},$$

because we have to sum over all terms coming from A^0 in (43). So applying Lemma 4.12 again, we get

$$A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}(dh)^{n-1}} < \sum_{j_2=2}^{n} \sum_{\substack{j_1\le j_2\\ -\mathbf{i}-e_{j_1}\in\Lambda^+}} 3^{-\mathrm{Defect}(\mathbf{i})-n-1+j_1} n^{-3(\mathrm{Defect}(\mathbf{i})n+1-j_1)} <$$
$$< \sum_{\substack{1\le j\le n\\ -\mathbf{i}-e_j\in\Lambda^+}} 3^{-\mathrm{Defect}(\mathbf{i})-n-1+j} n^{-3(\mathrm{Defect}(\mathbf{i})+n+1-j)+1}$$

Similarly to (46), for $\Sigma\mathbf{i} = -1, \mathbf{i} + e_j \in \Lambda^-$

$$(53)\qquad B_{\mathbf{z}^{\mathbf{i}}(dh)} = -\delta|\mathbf{a}|\binom{n^2-1}{i_1+n, \ldots, i_n+n} a_1^{i_1+n} \ldots a_n^{i_n+n} < n^{8\mathrm{Defect}(\mathbf{i})} B_{\mathbf{0}}.$$

and therefore with $\delta = \frac{1}{n^{8n}}$

$$\sum_{\Sigma\mathbf{i}=-1} B_{\mathbf{z}^{\mathbf{i}}(dh)} A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}(dh)^{n-1}} < \delta|\mathbf{a}| \sum_{\substack{1\le j\le n \\ \mathbf{i}+e_j\in\Lambda^-}} 3^{-\mathrm{Defect}(\mathbf{i})-n-1+j} n^{5\mathrm{Defect}(\mathbf{i})-3n+3j+4} B_{\mathbf{0}} < \tag{54}$$

$$< \delta|\mathbf{a}| \sum_{\substack{\mathbf{i}\in\Lambda^- \\ \Sigma\mathbf{i}=0}} \frac{n^{5(\mathrm{Defect}(\mathbf{i})+1)}}{3^{\mathrm{Defect}(\mathbf{i})+1}} B_{\mathbf{0}} < \sum_{i=1}^{n^2} \sum_{\substack{\mathbf{i}\in\Lambda^- \\ \mathrm{Defect}(\mathbf{i})=-i}} n^{-4i} B_{\mathbf{0}} < B_{\mathbf{0}} \sum_{i=1}^{n^2} n^{-3i} < B_{\mathbf{0}} < \frac{1}{4} B_{\mathbf{0}}$$

4.3. **Estimation of the the coefficients $p_{n-1}(n,\delta)$.** In this section we give a formula and an upper bound for the coefficients $p_{n-l}(n,\delta) = \mathrm{coeff}_{d^{n+1-l}} I_X(n,d,\delta)$ in general. Using the notations introduced in (43),(44) we get

$$p_{n-l}(n,\delta) = \sum_{s=0}^{l} \sum_{\Sigma\mathbf{i}=-s} B_{\mathbf{z}^{\mathbf{i}}h^s} A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l}} - \delta|\mathbf{a}| \sum_{s=1}^{l+1} \sum_{\Sigma\mathbf{i}=-s-1} B_{\mathbf{z}^{\mathbf{i}}h^s(dh)} A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l-1}}. \tag{55}$$

where for $\Sigma\mathbf{i} = -s$

(56)

$$B_{\mathbf{z}^{\mathbf{i}}h^s} = \binom{n^2}{s, i_1+n, \ldots, i_n+n} (2|\mathbf{a}|)^s a_1^{i_1+n} \ldots a_n^{i_n+n} + (S(n,\delta)-2) \binom{n^2-1}{s-1, i_1+n, \ldots, i_n+n} (2|\mathbf{a}|)^s a_1^{i_1+n} \ldots a_n^{i_n+n}$$

and therefore

$$|B_{\mathbf{z}^{\mathbf{i}}h^s}| < 2n^2 \binom{n^2}{s, i_1+n, \ldots, i_n+n} (2|\mathbf{a}|)^s a_1^{i_1+n} \ldots a_n^{i_n+n} \tag{57}$$

With the choice $a_i = n^{8(n+1-i)}$, $\delta = \frac{1}{n^{8n}}$, the dominant term in (55) is (note that many of the terms vanish, because the $A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l}} = 0$ unless $-\mathbf{i}-\mathbf{1} \in \Lambda^-$.) the one with

$$s = l \text{ and } \mathbf{i}^l = (i_n = \ldots = i_{n-l+1} = -1, i_{n-l} = i_{n-l-1} = \ldots = i_1 = 0)$$

that is

$$|B_{\mathbf{z}^{\mathbf{i}^l}h^l} \underbrace{A_{\mathbf{z}^{-\mathbf{i}^l-\mathbf{1}}(dh)^{n-l}}}_{=1}| < 2n^2 \binom{n^2}{l, \underbrace{n-1,\ldots,n-1}_{l}, \underbrace{n,\ldots,n}_{n-l}} (2|\mathbf{a}|)^l a_1^n \ldots a_{n-l}^n a_{n-l+1}^{n-1} \ldots a_n^{n-1} <$$

$$< 2n^2(2|\mathbf{a}|)^l n^{-4l(l+1)} B_{\mathbf{0}} < n^{8ln} B_{\mathbf{0}} A_{\mathbf{z}^{-\mathbf{1}}(dh)^n}$$

The goal is to show that the sum of the rest of the terms in (55) is less than this dominant term, proving the following

Proposition 4.13. *For $a_i = n^{8(n+1-i)}$ and $\delta = \frac{1}{n^{8n}}$,*

$$|p_{n-l}| < n^{8ln}|p_n|$$

Theorem 4.9 is a straightforward consequence of this Proposition, applying the following elementary statement

Observation 4.14. *If* $p(d) = p_n d^n + p_{n-1}d^{n-1} + \ldots + p_1 d + p_0 \in \mathbb{R}[d]$ *satisfies the inequalities*

$$p_n > 0; \quad |p_{n-l}| < D^l |p_n| \text{ for } l = 0, \ldots n,$$

then $p(d) > 0$ *for* $d > D$.

We now prove that the term corresponding to $\mathbf{i}^l$ is indeed dominant. Note first that by (57)

$$|B_{\mathbf{z}^{\mathbf{i}}h^s}| < 2n^2 \binom{n^2}{s, i_1+n, \ldots, i_n+n} (2|\mathbf{a}|)^s a_1^{i_1+n} \ldots a_n^{i_n+n} < n^{8\mathrm{Defect}(\mathbf{i}-\mathbf{i}^l)} |\mathbf{a}|^{s-l} B_{\mathbf{i}^l h^l}. \tag{58}$$

Since $\mathrm{Defect}(\mathbf{i}^l) \geq \mathrm{Defect}(\mathbf{i})$ when $-\mathbf{i}-\mathbf{1} \in \Lambda^+$, the exponent of n is nonpositive, since the other term $A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l}} = 0$ if $-\mathbf{i}-\mathbf{1} \notin \Lambda^+$.

On the other hand, for $\Sigma\mathbf{i} = -s$

$$A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l}} = \sum_{1\le j_1<\ldots<j_l\le n} \sum_{m_1\le j_1,\ldots,m_l\le j_l} \sum_{\substack{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3=-\mathbf{i}-e_{m_1}-\ldots-e_{m_l}\\ \mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\in\Lambda^+}} A^1_{\mathbf{i}_1} A^2_{\mathbf{i}_2} A^3_{\mathbf{z}^{\mathbf{i}_3}h^{l-s}} \tag{59}$$

where in the summation we have $\Sigma\mathbf{i}_1 = \Sigma\mathbf{i}_2 = 0, \Sigma\mathbf{i}_3 = s-l$, otherwise the corresponding coefficients are zero.

Lemma 4.15. *Let* $\Sigma\mathbf{i} = -s$. *Then*

$$|A^3_{\mathbf{z}^{\mathbf{i}}h^s}| < n^{3\mathrm{Defect}(\mathbf{i})+s}$$

Proof. The proof is analogous to the proof of Lemma 4.12: we first choose s factors in the denominator of $\mathbf{z}^{\mathbf{i}}$ and pair them with h^s, and then repeat the same arguments. □

Applying Lemma 4.12 and Lemma 4.15 we get the following upper bound:

$$|A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l}}| < \sum_{1\le j_1<\ldots<j_l\le n} \sum_{m_1\le j_1,\ldots,m_l\le j_l} \sum_{\substack{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3=-\mathbf{i}-e_{m_1}-\ldots-e_{m_l}\\ \mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\in b\Lambda^+}} n^{-3(\mathrm{Defect}(\mathbf{i})+(n+1-m_1)+\ldots+(n+1-m_l))+l-s} <$$

$$< \sum_{1\le m_1<\ldots<m_l\le n} \sum_{\substack{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3=-\mathbf{i}-e_{m_1}-\ldots-e_{m_l}\\ \mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\in\Lambda^+}} (n+1-m_l)(n-m_{l-1})\cdot\ldots\cdot(n-l+2-m_1)n^{-3(\mathrm{Defect}(\mathbf{i})+(n+1-m_1)+\ldots+(n^1-m_l))+l-s} <$$

$$< \sum_{1\le m_1<\ldots<m_l\le n} \sum_{\substack{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3=-\mathbf{i}-e_{m_1}-\ldots-e_{m_l}\\ \mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\in\Lambda^+}} n^{-3\mathrm{Defect}(\mathbf{i}-\mathbf{i}^l)+l-s}$$

where we used that fact that for $1 \le m_1 < \ldots < m_l \le n$

$$(n+1-m_l)(n-m_{l-1})\ldots(n-l+2-m_1)n^{-(n+1-m_1)-\ldots-(n+1-m_l)} \le n^{-l-(l-1)-\ldots-1} = n^{\mathrm{Defect}(\mathbf{i}^l)}.$$

Applying Lemma 4.11 again we get

$$|A_{\mathbf{z}^{-\mathbf{i}-1}h^{l-s}(dh)^{n-l}}| < \sum_{\substack{1\le m_1<\ldots<m_l\le n\\ \mathbf{i}+e_{m_1}+\ldots+e_{m_l}\in\Lambda^-}} 3^{-\mathrm{Defect}(\mathbf{i})-(n+1-m_1)-\ldots-(n+1-m_l)} n^{3(\mathrm{Defect}(\mathbf{i}^l-\mathbf{i}))+l-s} \le \tag{60}$$

$$\le \sum_{\substack{1\le m_1<\ldots<m_l\le n\\ \mathbf{i}+e_{m_1}+\ldots+e_{m_l}\in\Lambda^-}} 3^{\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})} n^{3(\mathrm{Defect}(\mathbf{i}^l-\mathbf{i}))+l-s}$$

where, again, $n+1-m_1+\ldots+n+1-m_l \ge 1+2+\ldots+l = -\mathrm{Defect}(\mathbf{i}^l)$.

Summing up these for all the possible $(\mathbf{i}, s)$ we can estimate the first sum in (55) as follows.

$$|\sum_{s=0}^{l}\sum_{\Sigma\mathbf{i}=-s,\mathbf{i}\ne\mathbf{i}^l} B_{\mathbf{z}^{\mathbf{i}}h^s}A_{\mathbf{z}^{-\mathbf{i}-1}h^{l-s}(dh)^{n-l}}| < \sum_{s=0}^{l}\sum_{\substack{1\le m_1<\ldots<m_l\le n\\ \mathbf{i}+e_{m_1}+\ldots+e_{m_l}\in\Lambda^-\\ \Sigma\mathbf{i}=-s,\mathbf{i}\ne\mathbf{i}^l}} 3^{\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})} n^{-5\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})+l-s}|\mathbf{a}|^{s-l}B_{\mathbf{i}^l h^l} <$$

$$< \sum_{s=0}^{l}\sum_{\substack{1\le m_1<\ldots<m_l\le n\\ \mathbf{i}+e_{m_1}+\ldots+e_{m_l}\in\Lambda^-\\ \Sigma\mathbf{i}=-s,\mathbf{i}\ne\mathbf{i}^l}} n^{-4\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})} n^{(8n-1)(s-l)} B_{\mathbf{i}^l h^l} \tag{61}$$

Observe that for $\Sigma\mathbf{i} = -l$

$$\mathbf{i}+e_{m_1}+\ldots+e_{m_l} \in \lambda^- \Rightarrow \mathrm{Defect}(\mathbf{i})+\mathrm{Defect}(e_{m_1}+\ldots+e_{m_l}) = \mathrm{Defect}(\mathbf{i})+(n+1-m_1)+\ldots+(n+1-m_l) \le 0$$

$$\Rightarrow \mathrm{Defect}(\mathbf{i})+(n+1)l \le m_1+\ldots+m_l \Rightarrow \mathrm{Defect}(\mathbf{i}-\mathbf{i}^l) \le (m_1+l-n-1)+(m_2+l-n-2)+\ldots+(m_l-n)$$

Therefore using the temporary notation $r_i = m_i + l - n - i \le 0$, we get

$$\sharp\{1 \le m_1 < \ldots < m_l : \mathbf{i}+e_{m_1}+\ldots+e_{m_l} \in \Lambda^-\} < \sharp\{r_1,\ldots,r_l \le 0 : r_1+\ldots+r_l > \mathrm{Defect}(\mathbf{i}-\mathbf{i}^l)\} < l^{\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})} \tag{62}$$

For $\Sigma\mathbf{i} = -s > -l$, clearly

$$\sharp\{1 \le m_1 < \ldots < m_l \le n : \mathbf{i}+e_{m_1}+\ldots+e_{m_l} \in \Lambda^-\} \le$$

$$< \sharp\{1 \le m_1 < \ldots < m_l \le n : \underbrace{(\mathbf{i}-e_n-\ldots-e_{n-l+s+1})}_{\Sigma=-l}+e_{m_1}+\ldots+e_{m_l} \in \Lambda^-\} < l^{\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})+1+\ldots+l-s}$$

Substituting this into (61) we get

$$|\sum_{s=0}^{l}\sum_{\Sigma\mathbf{i}=-s,\mathbf{i}\neq\mathbf{i}^l} B_{\mathbf{z}^{\mathbf{i}}h^s}A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l}}| < \sum_{s=0}^{l}\sum_{\Sigma\mathbf{i}=-s,\mathbf{i}\neq\mathbf{i}^l} l^{\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})+1+\ldots+(l-s)}n^{-4\mathrm{Defect}(\mathbf{i}^l-\mathbf{i})}n^{(8n-1)(s-l)} <$$

$$< \sum_{s=0}^{l}\sum_{m=1}^{\infty}\sum_{\substack{\Sigma\mathbf{i}=-s\\ \mathrm{Defect}(\mathbf{i}^l-\mathbf{i})=m}} n^{-3m+(7n-1)(s-l)}B_{\mathbf{i}^l h^l} < \sum_{s=0}^{l}\sum_{m=1}^{\infty} n^{-2m+(7n-1)(s-l)}B_{\mathbf{i}^l h^l} < \sum_{s=0}^{l}\frac{1}{8}n^{(7n-1)(s-l)}B_{\mathbf{i}^l h^l} < \frac{1}{4}B_{\mathbf{i}^l h^l}$$

To summarize our results, since $A_{\mathbf{z}^{-\mathbf{i}^l-\mathbf{1}}(dh)^{n-l}} = 1$, we got

$$|\sum_{s=0}^{l}\sum_{\Sigma\mathbf{i}=-s,\mathbf{i}\neq\mathbf{i}^l} B_{\mathbf{z}^{\mathbf{i}}h^s}A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l}}| < \frac{1}{4}B_{\mathbf{i}^l h^l}A_{\mathbf{z}^{-\mathbf{i}^l-\mathbf{1}}(dh)^{n-l}} \tag{63}$$

The analogous computation for the second sum in (55) shows that for $\delta = \frac{1}{n^{8n}}$, $a_i = n^{8i}$ we have

$$\delta|\mathbf{a}|\sum_{s=1}^{l+1}\sum_{\Sigma\mathbf{i}=-s-1} B_{\mathbf{z}^{\mathbf{i}}h^s(dh)}A_{\mathbf{z}^{-\mathbf{i}-\mathbf{1}}h^{l-s}(dh)^{n-l-1}}| < \frac{1}{4}B_{\mathbf{i}^l h^l}A_{\mathbf{z}^{-\mathbf{i}^l-\mathbf{1}}(dh)^{n-l}} \tag{64}$$

Then (63),(64) and (58) gives the desired Proposition 4.13:

$$|p_{n-l}| < \frac{3}{2}|B_{\mathbf{i}^l h^l}A_{\mathbf{z}^{-\mathbf{i}^l-\mathbf{1}}(dh)^{n-l}}| < \frac{3}{2}n^{8ln}|B_{\mathbf{0}}A_{\mathbf{z}^{-\mathbf{1}}(dh)^n}| < \frac{9}{4}n^{8ln}|p_n|,$$

and Theorem 4.9 is proved. This proves Theorem 4.4 applying the Morse inequalities. Theorem 4.2 and Theorem 4.4 together give Theorem 1.1.

5. An other compactification of $J_kT_X/\mathbb{G}_k$

In this section we give a detailed description of a new compactification of $J_kT_X\mathbb{G}_k$, as a singular subvariety of some Grassmannian manifold. Since $\mathbb{G}_k$ acts on J_kT_X fiber-wise, we construct the quotient $(J_kT_X)_x/\mathbb{G}_k$ first. The idea comes from global singularity theory, and originally was presented in [4].

If u, v are positive integers, let $J_k(u, v)$ denote the vector space of k-jets of holomorphic maps $(\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$ at the origin; that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \dots, k$.

With this notation, the fibres of J_k are isomorphic to $J_k(1, n)$, and the group $\mathbb{G}_k$ is simply $J_k(1, 1)$ with the composition action on itself.

If we fix local coordinates $z_1, \dots, z_u$ at $0 \in \mathbb{C}^u$ we can again identify the k-jet of f with the set of derivatives at the origin, that is $(f'(0), f''(0), \dots, f^{(k)}(0))$, where $f^{(j)}(0) \in \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v)$. This way we get the equality

$$J_k(u, v) = \oplus_{j=1}^k \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v)$$

One can compose map-jets via substitution and elimination of terms of degree greater than k; this leads to the composition maps

$$(65)\quad J_k(v, w) \times J_k(u, v) \to J_k(u, w), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1 \text{modulo terms of degree} > k .$$

When $k = 1$, $J_1(u, v)$ may be identified with u-by-v matrices, and (65) reduces to multiplication of matrices.

The k-jet of a curve $(\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ is simply an element of $J_k(1, n)$. We call such a curve γ *regular*, if $\gamma'(0) \neq 0$; introduce the notation $J_k^{\mathrm{reg}}(1, n)$ for the set of regular curves:

$$J_k^{\mathrm{reg}}(1, n) = \{\gamma \in J_k(1, n); \gamma'(0) \neq 0\}$$

Let $N \geq n$ be any integer and define

$$\Theta_k = \left\{\Psi \in J_k(n, N) : \exists \gamma \in J_k^{\mathrm{reg}}(1, n) : \Psi \circ \gamma = 0\right\}$$

In words: Θ_k is the set of those k-jets of maps, which take at least one regular curve to zero. By definition, Θ_k is the image of the closed subvariety of $J_k(n, N) \times J_k^{\mathrm{reg}}(1, n)$ defined by the algebraic equations $\Psi \circ \gamma = 0$, under the projection to the first factor. If $\Psi \circ \gamma = 0$, we call γ a *test curve* of Θ. This term originally comes from global singularity theory as explained below.

A basic but crucial observation is the following. If γ is a test curve of $\Psi \in \Theta_k$, and $\varphi \in J_k^{\mathrm{reg}}(1, 1) = \mathbb{G}_k$ is a holomorphic reparametrisation of $\mathbb{C}$, then $\gamma \circ \varphi$ is, again, a test curve of Ψ:

$$\mathbb{C} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\gamma} \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^N$$

$$\Psi \circ \gamma = 0 \quad \Rightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0$$

In fact, we get all test curves of Ψ in this way if the following open dense property holds: the linear part of Ψ has 1-dimensional kernel. Before stating this in Proposition

5.2 below, let us write down the equation $\Psi\circ\gamma=0$ in coordinates in an illustrative case. Let $\gamma=(\gamma',\gamma'',\ldots,\gamma^{(k)})\in J_k^{\mathrm{reg}}(1,n)$ and $\Psi=(\Psi',\Psi'',\ldots,\Psi^{(k)})\in J_k(n,N)$ be the k-jets. Using the chain rule and the notation $v_i=\gamma^{(i)}/i!$, the equation $\Psi\circ\gamma=0$ reads as follows for $k=4$:

$$\begin{gathered}\Psi'(v_1)=0,\\ \Psi'(v_2)+\Psi''(v_1,v_1)=0,\\ \Psi'(v_3)+2\Psi''(v_1,v_2)+\Psi'''(v_1,v_1,v_1)=0,\\ \Psi'(v_4)+2\Psi''(v_1,v_3)+\Psi''(v_2,v_2)+3\Psi'''(v_1,v_1,v_2)+\Psi''''(v_1,v_1,v_1,v_1)=0.\end{gathered}\tag{66}$$

To simplify our formulas we introduce the following notations for a partition $\tau=[i_1\ldots i_l]$ of the integer $i_1+\ldots+i_l$:

- the *length*: $|\tau|=l$,
- the *sum*: $\sum\tau=i_1+\ldots+i_l$,
- *number of permutations*: $\mathrm{perm}(\tau)$, which is the number of different sequences consisting of the numbers $i_1,\ldots,i_l$; e.g. $\mathrm{perm}([1,1,1,3])=4$.
- $\gamma_\tau=\prod_{j=1}^{l}\gamma^{(i_j)}\in\mathrm{Sym}^l\mathbb{C}^n$ and $\Psi(\gamma_\tau)=\Psi^l(\gamma^{(i_1)},\ldots,\gamma^{(i_l)})\in\mathbb{C}^N$.

Lemma 5.1. *Let $\gamma=(\gamma',\gamma'',\ldots,\gamma^{(k)})\in J_k^{\mathrm{reg}}(1,n)$ and $\Psi=(\Psi',\Psi'',\ldots,\Psi^{(k)})\in J_k(n,N)$ be k-jets. Then substituting $v_i=\gamma^{(i)}/i!$, the equation $\Psi\circ\gamma$ is equivalent to the following system of k linear equations with values in $\mathbb{C}^N$:*

$$\sum_{\tau\in\Pi[m]}\mathrm{perm}(\tau)\Psi(\mathbf{v}_\tau)=0,\quad m=1,2,\ldots,k,\tag{67}$$

where $\Pi[m]$ denotes the set of all partitions of m.

For a given $\gamma\in J_k^{\mathrm{reg}}(1,n)$ let $\mathcal{S}_\gamma$ denote the set of solutions of (67), that is,

$$\mathcal{S}_\gamma=\{\Psi\in J_k(n,N);\Psi\circ\gamma=0\}$$

The equations (67) are linear in Ψ, hence

$$\mathcal{S}_\gamma\subset J_k(n,N)$$

is a linear subspace of codimension kN. Moreover, the following holds:

Proposition 5.2. *(*[4]*, Proposition 4.4)*

1. *For $\gamma\in J_k^{\mathrm{reg}}(1,n)$, the set of solutions $\mathcal{S}_\gamma\subset J_k(n,N)$ is a linear subspace of codimension kN.*
2. *Set*
$$J_k^o(n,N)=\{\Psi\in J_k(n,N)|\dim\ker(\Psi')=1\}.$$
For any $\gamma\in J_k^{\mathrm{reg}}(1,n)$, the subset $\mathcal{S}_\gamma\cap J_k^o(n,N)$ of $\mathcal{S}_\gamma$ is dense.
3. *If $\Psi\in J_k^o(n,N)$, then Ψ belongs to at most one of the spaces $\mathcal{S}_\gamma$. More precisely,*
$$\text{if } \gamma_1,\gamma_2\in J_k^{\mathrm{reg}}(1,n),\ \ \Psi\in J_k^o(n,N) \text{ and } \Psi\circ\gamma_1=\Psi\circ\gamma_2=0,$$
then there exists $\varphi\in J_k^{\mathrm{reg}}(1,1)$ such that $\gamma_1=\gamma_2\circ\varphi$.

(4) *Given* $\gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(1,n)$*, we have* $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ *if and only if there is some* $\varphi \in J_k^{\mathrm{reg}}(1,1)$ *such that* $\gamma_1 = \gamma_2 \circ \varphi$*.*

By the second part of Proposition 5.2 we have a well-defined map $\phi : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}(\mathrm{codim} = kN, J_k(n,N))$, $\gamma \mapsto \mathcal{S}_\gamma$ to the Grassmannian of codimension-kN subspaces in $J_k(n,N)$. From the last part of Proposition 5.2 it follows that:

Corollary 5.3. *(*[4]*)* ϕ *is invariant on the* $J_k^{\mathrm{reg}}(1,1)$*-orbits, and the induced map on the orbits is injective:*

$$\phi : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \mathrm{Grass}(\mathrm{codim} = kN, J_k(n,N)) \tag{68}$$

Let us rewrite the linear system $\Psi \circ \gamma = 0$ associated to $\gamma \in J_k^{\mathrm{reg}}(1,n)$ in a dual form. The system is based on the standard composition map (65):

$$J_k(n,N) \times J_k(1,n) \longrightarrow J_k(1,N),$$

which, via the identification $J_k(n,N) = J_k(n,1) \otimes \mathbb{C}^N$, is derived from the map

$$J_k(n,1) \times J_k(1,n) \longrightarrow J_k(1,1)$$

via tensoring with $\mathbb{C}^N$. Observing that composition is linear in its first argument, and passing to linear duals, we may rewrite this correspondence in the form

$$\phi : J_k(1,n) \longrightarrow \mathrm{Hom}\,(J_k(1,1)^*, J_k(n,1)^*). \tag{69}$$

If $\gamma = (\gamma', \gamma'', \ldots, \gamma^{(k)}) \in J_k(1,n) = (\mathbb{C}^n)^k$ is the k-jet of a curve, we can put $v_i = \gamma^{(i)}/i! \in \mathbb{C}^n$ into the jth column of an $n \times k$ matrix, and

- identify $J_k(1,n)$ with $\mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n)$;
- identify $J_k(n,1)^*$ with $\mathrm{Sym}^{\le k}\mathbb{C}^n = \oplus_{l=1}^k \mathrm{Sym}^{\,l}\mathbb{C}^n$;
- identify $J_k(1,1)^*$ with $\mathbb{C}^k$;

Using these identifications, we can recast the map ϕ in (69) as

$$\phi_k : \mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n) \longrightarrow \mathrm{Hom}\,(\mathbb{C}^k, \mathrm{Sym}^{\le k}\mathbb{C}^n), \tag{70}$$

which may be written out explicitly as follows

$$(v_1, v_2, \ldots, v_k) \longmapsto \left(v_1, v_2 + v_1^2, \ldots, \sum_{\tau \in \Pi[d]} \mathrm{perm}(\tau)\mathbf{v}_\tau \right).$$

The set of solutions $\mathcal{S}_\gamma$ is the linear subspace orthogonal to the image of $\phi_k(\gamma', \ldots, \gamma^{(k)}/k!)$ tensored by $\mathbb{C}^N$, that is

$$\mathcal{S}_\gamma = \mathrm{im}(\phi_k(\gamma', \ldots, \gamma^{(k)}/k!))^\perp \otimes \mathbb{C}^N \subset J_k(n,N)$$

Consequently, it is straightforward to take $N = 1$ and define

$$\mathcal{S}_\gamma = \mathrm{im}(\phi_k(\gamma', \ldots, \gamma^{(k)}/k!)) \in \mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n) \tag{71}$$

Moreover, let $B_k \subset GL(k)$ denote the maximal Borel of upper triangulars and

$$\mathrm{Flag}_k(\mathbb{C}^n) = \mathrm{Hom}\,(\mathbb{C}^k, \mathrm{Sym}^{\le k}\mathbb{C}^n)/B_k = \{0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset \mathbb{C}^n,\ \dim F_l = l\}\,.$$

denote the full flag of k-dimensional subspaces of $\mathrm{Sym}^{\le k}\mathbb{C}^n$.

In addition to (71) we can analogously define

(72)
$$\mathcal{F}_\gamma = (\mathrm{im}(\phi_1(\gamma')) \subset \mathrm{im}(\phi_2(\gamma', \gamma''/2)) \subset \ldots \subset \mathrm{im}(\phi_k(\gamma', \ldots, \gamma^{(k)})/k!)) \in \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

where we think of $\mathrm{Sym}^{\le i}\mathbb{C}^n \subset \mathrm{Sym}^{\le j}\mathbb{C}^n$ as a subspace for $i < j$.

Using these definitions Proposition 5.2 implies the following version of Corollary 5.3, which does not contain the parameter N.

Proposition 5.4. *The map* ϕ *in* (70) *is a* $\mathbb{G}_k$*-invariant algebraic morphism*

$$\phi : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^{\le k}\mathbb{C}^n),$$

which induces

(1) *an injective map on the* $\mathbb{G}_k$*-orbits to the Grassmannian:*
$$\phi^{Gr} : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n)$$
defined by $\phi^{Gr}(\gamma) = \mathcal{S}_\gamma$.

(2) *an injective map on the* $\mathbb{G}_k$*-orbits to the Flag manifold:*
$$\phi^{Flag} : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$
defined by $\phi^{Flag}(\gamma) = \mathcal{F}_\gamma$.

(3) *In addition,*
$$\phi^{Gr} = \phi^{Flag} \circ \pi_k$$
where $\pi_k : \mathrm{Flag}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n) \to \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ *is the projection to the k-dimensional subspace.*

Moreover, all these maps are $GL(n)$*-equivariant with respect to the standard action of* $GL(n)$ *on* $J_k^{\mathrm{reg}}(1,n) \subset \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ *and the induced action on* $\mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$.

Composing ϕ^{Gr} with the Plucker embedding

$$\mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \mathrm{Sym}^{\le k}\mathbb{C}^n)$$

we get an embedding

(73)
$$\phi^{\mathrm{Proj}} : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$$

Since $\phi^{\mathrm{Grass}}, \phi^{\mathrm{Flag}}$ are $GL(n)$-equivariant, for $k \le n$ the image $\phi^{Gr}(J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k) \subset \mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n)$ is a $GL(n)$-orbit in $\mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n)$, and therefore a nonsingular quasiprojective variety. Its closure, however, is highly singular, and a finite union of $GL(n)$ orbits, with a nice orbit structure. The boundary orbits have codimension at least two, which allows us to describe a generating set for Demailly's algebra of invariant jet differentials as the Plucker coordinates on this Grassmannian. For the details see [5].

In this paper, however, we are rather interested in the image of $\overline{\phi^{\mathrm{Flag}}}$ in $\mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$. This will substitute the Demailly-Semple tower in our computation in the next section.

We introduce the following notations

$$X_k = \overline{\phi^{\mathrm{Flag}}(J_k^{\mathrm{reg}}(1,n))} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

In the next section, following [4], we develop a double-localization method on X_k allowing us to compute its intersection numbers. The process leads us to an iterated residue formula.

6. Localization on X_k- the Snowman Model

6.1. **Equivariant Poincaré duals, Multidegrees.** Denote the weight lattice of $T = (\mathbb{C}^*)^r$ by Λ; this is the lattice in $\mathrm{Lie}(T)^* = \mathbb{C}^r$ generated by the standard weights (the coordinate vectors) $\lambda_1, \dots, \lambda_r$. Let W be an N-dimensional complex vector space endowed with an action of T. This action is diagonalizable, hence one can choose coordinates $y_1, \dots, y_N$ on W in such a way that the action in the dual basis is diagonal; denote the respective weights by $\eta_1 \dots \eta_N$.

Let Σ be a closed T-invariant algebraic subvariety of W, and denote by $I(\Sigma) \subset S$ the ideal of polynomials vanishing on Σ. This ideal is *reduced*, i.e. has the property that $f^n \in I(\Sigma) \Rightarrow f \in I(\Sigma)$. Our plan is to define an extended invariant: $I \mapsto \mathrm{mdeg}[I, S]$, called the *multidegree* of I, where I is an arbitrary T-invariant ideal in $S = \mathbb{C}[y_1 \dots y_N]$. Then we can simply define the equivariant Poincaré of a variety as the the multidegree of the corresponding ideal (cf. Definition 6.1 below). Now we sketch an explicit and an axiomatic definition of the multidegree.

For the construction, let D be the codimension of the variety defined by the ideal $I \subset S$, and consider a finite, T-graded resolution of S/I by free S-modules:

$$\oplus_{i=1}^{j[M]} S\, w_i[M] \to \cdots \to \oplus_{i=1}^{j[m]} S\, w_i[m] \to \cdots \to \oplus_{i=1}^{j[1]} S\, w_i[1] \to S \to S/I \to 0;$$

where $w_i[m]$ is a free generator of degree $\eta_i[m] \in \Lambda$ for $i = 1, \dots j[m]$, $m = 1 \dots M$. Then

$$\mathrm{mdeg}[I, S] = \frac{1}{D!} \sum_{m=1}^{M} \sum_{i=1}^{j[m]} (-1)^{D-m} \eta_i[m]^D. \tag{74}$$

Definition 6.1. *Let $\Sigma \subset W$ be T-invariant closed subvariety as in §6.1. Then we define the T*-equivariant Poincaré dual of Σ in W *by*

$$\mathrm{eP}[\Sigma, W]_T = \mathrm{mdeg}[I(\Sigma), \mathbb{C}[y_1 \dots y_N]].$$

We will usually omit the lower index T when this does not cause confusion. Note that the multidegree, and hence the equivariant Poincaré dual, is manifestly a homogeneous polynomial of degree D.

While (74) is explicit, its meaning is not transparent, and we note that, usually, it is rather difficult to write down free resolutions of ideals. Hence we turn to an axiomatic description, which is more intuitive, and provides us with a more algorithmic understanding of the invariant as well.

6.2. **Axiomatic definition.** We follow the treatment of [34] to give the axiomatic definition: we describe 3 characterizing properties of the multidegree, and then we prove that these properties indeed determine the polynomial.

The monomials $\mathbf{y}^{\mathbf{a}} = \prod_{i=1}^{N} y_i^{a_i} \in S = \mathbb{C}[y_1, \dots, y_N]$ are parametrized by the integer vectors $\mathbf{a} = (a_1 \dots a_N) \in \mathbb{Z}_+^N$. A *monomial order* $<$ on S is a total order of the monomials in S such that for any three monomials m_1, m_2, n satisfying $m_1 > m_2$, we have $nm_1 > nm_2 > m_2$ (see [17, §15.2]).

An ordering of the coordinates $y_1, \dots, y_N$ induces the so-called *lexicographic* monomial order of the monomials, that is, $\mathbf{y}^{\mathbf{a}} > \mathbf{y}^{\mathbf{b}}$ if and only if $a_i > b_i$ for the first index i with $a_i \neq b_i$. We will use this lexicographic monomial order throughout this paper.

Now let $I \subset S$ be a T-invariant ideal. Define the *initial ideal* $\mathrm{in}_<(I) \subset S$ to be the ideal generated by the monomials $\{\mathrm{in}_<(p) : p \in I\}$, where $\mathrm{in}_<(p)$ is the largest monomial of p w.r.t $<$. There is a flat deformation of I into $\mathrm{in}_<(I)$ ([17], Theorem 15.17.), and the first axiom says that mdeg$[I]$ does not change under this deformation:

1. Deformation invariance: $\mathrm{mdeg}[I, S] = \mathrm{mdeg}[\mathrm{in}_<(I), S]$.

To describe the second axiom, we define the multiplicity of a maximal-dimensional component of a non-reduced variety. Let $I \subset S$ be an ideal, and denote $\Sigma(I)$ the variety of common zeros of the polynomials in I:

$$\Sigma(I) = \{p \in W;\ f(p) = 0\ \forall f \in I\}.$$

Denote by $\Sigma_1, \Sigma_2, \dots, \Sigma_m$ the maximal-dimensional irreducible components of $\Sigma(I)$. Then each Σ_i corresponds to a prime ideal $\mathfrak{p}_i \subset S$, and one can define a positive integer $\mathrm{mult}(\mathfrak{p}_i, I)$, the *multiplicity of* Σ_i with respect to I, as the length of the largest finite-length $S_{\mathfrak{p}_i}$-submodule in $(S/I)_{\mathfrak{p}_i}$, where $S_{\mathfrak{p}_i}$ (resp. $(S/I)_{\mathfrak{p}_i}$) is the localization of S (resp. S/I) at $\mathfrak{p}_i$ (see section II.3.3 in [18]). Then we have

2. Additivity:

$$\mathrm{mdeg}[I, S] = \sum_{i=1}^{m} \mathrm{mult}(\mathfrak{p}_i, I) \cdot \mathrm{mdeg}[\mathfrak{p}_i, S]. \tag{75}$$

The last axiom describes the multidegree for the case of coordinate subspaces:

3. Normalization: for every subset $\mathbf{i} \subset \{1 \dots N\}$ we have

$$\mathrm{mdeg}\left[\langle y_i,\ i \in \mathbf{i}\rangle, S\right] = \prod_{i \in \mathbf{i}} \eta_i, \tag{76}$$

where $\langle \cdot \rangle$ stands for the ideal generated by the polynomials listed in the angle brackets.

A special case of the normalization axiom is the case $\Sigma = \{0\}$. We will often use the notation $\mathrm{euler}^T(W)$ for $\mathrm{eP}[\{0\}, W]$, since, indeed, this is the equivariant Euler class of W thought of as a T-vector bundle over a point. We have thus

$$\mathrm{eP}[\{0\}, W]_T = \mathrm{euler}^T(W) = \prod_{i=1}^{N} \eta_i. \tag{77}$$

Remark 6.2. *Using this notation, the normalization axiom may be recast in a geometric form as follows: given a surjective equivariant linear map $\gamma : W \to E$ from W to another T-module E, we have*

$$\mathrm{eP}[\gamma^{-1}(0), W] = \mathrm{euler}^T(E). \tag{78}$$

Consider the following three examples:

(1) Set $N = 4$, and consider the ideal $I = \langle y_1^2, y_2^3, y_3\rangle$ in $S = \mathbb{C}[y_1, y_2, y_3, y_4]$. This is the line $\{y_1 = y_2 = y_3 = 0\}$ with multiplicity 6, so its multidegree is

$$\mathrm{mdeg}[I, S] = 6\eta_1\eta_2\eta_3.$$

(2) The ideal $I = \langle y_1^2 y_2^3 y_3\rangle$ in $S = \mathbb{C}[y_1, y_2, y_3]$ corresponds to the union of the hyperplanes $y_1 = 0, y_2 = 0, y_3 = 0$ with multiplicities $2, 3, 1$, respectively. By the normalization and additivity properties

$$\mathrm{mdeg}[I, S] = 2\eta_1 + 3\eta_2 + \eta_3$$

(3) The ideal $I = \langle y_1y_2, y_2y_3, y_1y_3\rangle = \langle y_1, y_2\rangle \cap \langle y_2, y_3\rangle \cap \langle y_1, y_3\rangle$ in $S = \mathbb{C}[y_1, y_2, y_3]$ has three components with multiplicity 1, corresponding to the given decomposition, so

$$\mathrm{mdeg}[I, S] = \eta_1\eta_2 + \eta_2\eta_3 + \eta_1\eta_3$$

Following [34] §8.5, now we sketch an algorithm for computing $\mathrm{mdeg}[I, S]$, proving that the axioms determine this invariant.

An ideal $M \subset S$ generated by a set of monomials in $y_1, \ldots, y_N$ is called a *monomial ideal*. Since $\mathrm{in}_<(I)$ is such an ideal, by the deformation invariance it is enough to compute $\mathrm{mdeg}[M]$ for monomial ideals M. If the codimension of $\Sigma(M)$ in W is s, then the maximal dimensional components of $\Sigma(M)$ are codimension-s coordinate subspaces of W. Such subspaces are indexed by subsets $\mathbf{i} \in \{1 \ldots N\}$ of cardinality s; the corresponding associated primes $\mathfrak{p}[\mathbf{i}] = \langle y_i : i \in \mathbf{i}\rangle$.

It is not difficult to check that

$$\mathrm{mult}(\mathfrak{p}[\mathbf{i}], M) = \left|\left\{\mathbf{a} \in \mathbb{Z}_+^{[\mathbf{i}]};\ \mathbf{y}^{\mathbf{a}+\mathbf{b}} \notin M \text{ for all } \mathbf{b} \in \mathbb{Z}_+^{[\hat{\mathbf{i}}]}\right\}\right|, \tag{79}$$

where $\mathbb{Z}_+^{[\mathbf{i}]} = \{\mathbf{a} \in \mathbb{Z}_+^N; a_i = 0 \text{ for } i \notin \mathbf{i}\}$, $\hat{\mathbf{i}} = \{1 \ldots N\} \setminus \mathbf{i}$, and $|\cdot|$, as usual, stands for the number of elements of a finite set.

Then by the normalization and additivity axiom we have

$$\mathrm{mdeg}[M, S] = \sum_{|\mathbf{i}|=s} \mathrm{mult}(\mathfrak{p}[\mathbf{i}], M) \prod_{i \in \mathbf{i}} \eta_i. \tag{80}$$

By definition, the weights $\eta_1, \ldots \eta_N$ on W are linear forms of $\lambda_1, \ldots \lambda_r$, the basis of $(\mathbb{C}^*)^r$, and we denote the coefficient of λ_j in η_i by $\mathrm{coeff}(\eta_i, j,\ 1 \le i \le N, 1 \le j \le r$. Introduce also the following notation:

$$\deg(\eta_1, \ldots, \eta_N; m) = \#\{i;\ \mathrm{coeff}(\eta_i, m) \neq 0\}\}.$$

It is clear from the formula (80) that

$$\deg_{\lambda_m} \mathrm{mdeg}[I, S] \leq \deg(\eta_1, \ldots, \eta_N; m) \tag{81}$$

holds for any $1 \leq m \leq r$. We need a slightly stronger result in the next section which we formulate and prove here.

Proposition 6.3. *Let W be an N-dimensional complex vector space endowed with an diagonal action of $(\mathbb{C}^*)^r$, with coordinates $y_1, \ldots, y_N$ and respective weights $\eta_1 \ldots \eta_N$. Let $I \subset S$ be a $(\mathbb{C}^*)^r$-invariant ideal. Then*

(1) $\mathrm{mdeg}[I, S] \in \mathbb{C}[\lambda_1, \ldots, \lambda_r]$ *is a polynomial of* $\eta_1, \ldots, \eta_N$.
(2)

$$\deg_{\lambda_m} \mathrm{mdeg}[I, S] \leq \deg(\eta_1, \ldots, \eta_N; m) - 1 \tag{82}$$

Proof. The first part is obvious from (80). Let

$$\mathrm{coeff}(\eta_i, m) \neq 0 \text{ for } 1 \leq i \leq s; \ \mathrm{coeff}(\eta_{s+1}, m) = \ldots = \mathrm{coeff}(\eta_N, m) = 0.$$

The idea of the proof is to choose an appropriate monomial order on the polynomial ring $S = \mathbb{C}[y_1, \ldots, y_N]$ to ensure that y_1 does not appear in the corresponding initial ideal.

To that end recall, that a weight function is a linear map $\rho : \mathbb{Z}^N \to \mathbb{Z}$. This defines a partial order $>_\rho$ on the monomials of S, called the weight order associated to ρ, by the rule $m = y^a >_\rho n = y^b$ iff $\rho(a) > \rho(b)$. Here $a = (a_1, \ldots, a_N), b = (b_1, \ldots, b_N)$ are arbitrary multiindices. Any weight order can be extended to a compatible monomial order $>$ (see [17], Ch 15.2), which means that $m >_\rho n$ implies $m > n$.

For our purposes define

$$\rho(y_1) = -1, \rho(y_2) = \ldots = \rho(y_N) = 0 \tag{83}$$

and let $>$ denote arbitrary compatible monomial order on S. By definition for a monomial $m \in S$

$$\rho(m) < 0 \Longleftrightarrow y_1 | m \tag{84}$$

Let $p \in I$, and assume that not all monomials of p are divisible by y_1. If they all did, $y_1 | p$, and therefore $in_>(p/y_1) | in_>(p)$ would hold, and therefore p would not be among the generators of the $in_>(I)$. Therefore y_1 does not divide p.

Then there is a monomial of p not containing y_1, and by (84) the weight of this monomial is strictly bigger to the weight of any other containing y_1. Consequently, y_1 does not divide any of the generators of $in_>(I)$, and by (80) $\mathrm{mdeg}[I, S]$ does not depend on η_1. The only possible variables containing λ_m are therefore $\eta_2, \ldots, \eta_s$, giving a maximum total degre $s - 1$. □

6.3. **Localization on X_k.** In this subsection we sketch the localization procedure developed in [4] on X_k. We also refer this later as the Snowman Model, due to the figure in §6 of [4], which summarizes the process.

Let $J_k^{\mathrm{nondeg}}(1,n) \subset J_k^{\mathrm{reg}}(1,n)$ be the set of test curves with $\gamma', \ldots, \gamma^{(k)}$ linearly independent. These correspond to the regular $n \times k$ matrices under the identification of § 5.

According to our construction, we have the following picture:

$$
\begin{array}{ccccc}
J_k^{\mathrm{nondeg}}(1,n)/\mathbb{G}_k & \xrightarrow{\phi^{\mathrm{Flag}}} & \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) & \xrightarrow{Pluck} & \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n)) \\
\downarrow \pi & & & & \\
J_k^{\mathrm{nondeg}}(1,n)/B_k = \mathrm{Flag}_k(\mathbb{C}^n) & & & &
\end{array} \tag{85}
$$

Let τ be the restriction of the tautological line bundle on $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$ to X_k. To proceed a similar computation as on the Demailly-Semple tower, we have to compute the intersection number

$$\int_{X_k} c_1(\tau)^{k(n-1)}$$

We will explain this in more detail in the next section.

Note that the fibration π and the embedding ϕ^{Flag} are GL_n-equivariant with respect to the induced $GL(n)$ action from the standard action on $\mathbb{C}^n$. Let $\lambda_1, \ldots, \lambda_n$ be the weights of this action with eigenbasis $e_1, \ldots, e_n \in \mathbb{C}^n$. Let

$$\mathbf{f} = \{\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \ldots \subset \mathbb{C}^n\}$$

denote the standard flag in $\mathbb{C}^n$ Applying the ABBV localization formula on $\mathrm{Flag}_k(\mathbb{C}^n)$ we get

$$\int_{X_k} c_1(\tau)^{k(n-1)} = \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-k}} \frac{Q_{\mathbf{f}}(\lambda_{\sigma\cdot 1} \ldots \lambda_{\sigma\cdot k})}{\prod_{1\le m\le k}\prod_{i=m+1}^{n}(\lambda_{\sigma\cdot i} - \lambda_{\sigma\cdot m})}, \tag{86}$$

where

$$Q_{\mathbf{f}}(\lambda_1, \ldots, \lambda_n) = \int_{\overline{\phi^{\mathrm{Flag}}(\pi^{-1}(\mathbf{f}))}} (\lambda_1 + \ldots + \lambda_n)^{k(n-1)} \tag{87}$$

Similarly to the localization on the Demailly-Semple tower, we can derive an efficient residue formula for the right hand side of (86). While the geometric meaning of this formula is not entirely clear, our summation procedure yields an effective, "truly" localized formula; by this we mean that for its evaluation one only needs to know the behavior of a certain function at a single point, rather than at a large, albeit finite number of points.

Proposition 6.4. ([4],*Proposition 6.4) For a polynomial $Q(\mathbf{z})$ on $\mathbb{C}^k$, we have*

$$\sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-k}} \frac{Q(\lambda_{\sigma\cdot 1} \ldots \lambda_{\sigma\cdot k})}{\prod_{1\le m\le k}\prod_{i=m+1}^{n}(\lambda_{\sigma\cdot i} - \lambda_{\sigma\cdot m})} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{1\le m<l\le k}(z_m - z_l)\, Q(\mathbf{z})\, d\mathbf{z}}{\prod_{l=1}^{k}\prod_{i=1}^{n}(\lambda_i - z_l)} \tag{88}$$

Following [4], we proceed a second localization on the fiber $X_{\mathbf{f}} = \overline{\phi^{flag}(\pi^{-1}(\mathbf{f}))}$ to compute $Q_{\mathbf{f}}(\mathbf{z})$. Since $X_{\mathbf{f}}$ is invariant under the $(\mathbb{C}^*)^n \subset GL_n$ action on $\mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$, we want apply Rossmann's integration formula, which is explained in §3.3 of [4]. The rough idea is the following.

Let Z be a complex manifold with a holomorphic T-action, and let $M \subset Z$ be a T-invariant analytic subvariety with an isolated fixed point $p \in M^T$. Then one can find local analytic coordinates near p, in which the action is linear and diagonal. Using these coordinates, one can identify a neighborhood of the origin in T_pZ with a neighborhood of p in Z. We denote by $\hat{T}_pM$ the part of T_pZ which corresponds to M under this identification; informally, we will call $\hat{T}_pM$ the T-invariant *tangent cone* of M at p. This tangent cone is not quite canonical: it depends on the choice of coordinates; the multidegree of $\Sigma = \hat{T}_pM$ in $W = \mathrm{T}_pZ$, however, does not. Rossmann named this the *equivariant multiplicity of M in Z at p*:

$$\mathrm{emult}_p[M, Z] \stackrel{\mathrm{def}}{=} \mathrm{mdeg}[\hat{T}_pM, \mathrm{T}_pZ]. \tag{89}$$

Remark 6.5. *In the algebraic framework one might need to pass to the* tangent scheme *of M at p (cf.* [22]*). This is canonically defined, but we will not use this notion.*

Rossmann's localization formula [38] reads then as follows.

Let $\mu \in H_T^*(Z)$ be an equivariant class represented by a holomorphic equivariant map $\mathrm{Lie}(T) \to \Omega^\bullet(Z)$. Then

$$\int_M \mu = \sum_{p \in M^T} \frac{\mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_pZ)} \cdot \mu^{[0]}(p), \tag{90}$$

where $\mu^{[0]}(p)$ is the differential-form-degree-zero component of μ evaluated at p.

In [4] we apply this formula with $M = X_{\mathbf{f}}, Z = \mathrm{Flag}_k^*(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ and $\alpha = \mathrm{Thom}(\mathrm{Flag}_k^*)$, the equivariant Thom class of Flag_k^* where

$$\mathrm{Flag}_k^* = \{V_1 \subset \ldots V_k \subset \mathrm{Sym}^{\leq k}\mathbb{C}^n : \dim(V_i) = i, V_i \subset \mathbb{C}\langle e_\tau : \mathrm{sum}(\tau) \leq i\rangle\} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n).$$

The fixed points on $\mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ are parametrized by admissible sequences of partitions $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$. We call a sequence of partitions $\boldsymbol{\pi} = (\pi_1 \ldots \pi_k) \in \Pi^{\times d}$ admissible if

(1) $\mathrm{sum}(\pi_l) \leq l$
(2) $\pi_l \neq \pi_m$ for $1 \leq l \neq m \leq k$.

We will denote the set of admissible sequences of length k by $\mathbf{\Pi}_k$.

Following [4] we arrive to the following formula with $k \leq n$

Proposition 6.6.

$$\int_{X_k} c_1(\tau)^{k(n-1)} = \sum_{\boldsymbol{\pi}\in\mathbf{\Pi}_k} \operatorname*{Res}_{\mathbf{z}=\infty} \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod_{m<l}(z_m - z_l)(z_{\pi_1} + \ldots + z_{\pi_k})^{k(n-1)}}{\prod_{l=1}^{k} \prod_{\tau\le l}^{\tau\neq\pi_1\ldots\pi_l} (z_\tau - z_{\pi_l}) \prod_{l=1}^{k}\prod_{i=1}^{n}(\lambda_i - z_l)} \, d\mathbf{z}. \tag{91}$$

where $Q_{\boldsymbol{\pi}}(\mathbf{z}) = \mathrm{emult}_{\boldsymbol{\pi}}[X_{\mathbf{f}}, \mathrm{Flag}_k^*]$ *and* $z_\pi = \sum_{i\in\pi} z_i$.

The total degree of the rational expression in (91) is $-k$, and the iterated residue gives a number.

The following theorem is a stronger version of the vanishing theorem in [4]. We devote the next subsection to the proof.

Theorem 6.7. ***The Residue Vanishing Theorem***

(1) *All terms but the one corresponding to* $\boldsymbol{\pi} = (1, 2, \ldots, k)$ *vanish in* (91) *leaving us with*

$$\int_{X_k} c_1(\tau)^{k(n-1)} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{Q_{[1],\ldots,[k]}(\mathbf{z}) \prod_{m<l}(z_m - z_l)(z_1 + \ldots + z_k)^{k(n-1)}}{\prod_{\mathrm{sum}(\tau)\le l\le k} (z_\tau - z_l) \prod_{l=1}^{k}\prod_{i=1}^{n}(\lambda_i - z_l)} \, d\mathbf{z}. \tag{92}$$

(2) *If* $|\tau| \geq 3$ *then* $Q_{[1],\ldots,[k]}(\mathbf{z})$ *is divisible by* $z_\tau - z_l$ *for all* $l \geq \mathrm{sum}(\tau)$*, so we arrive at the simplified formula*

$$\int_{X_k} c_1(\tau)^{k(n-1)} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{Q(\mathbf{z}) \prod_{m<l}(z_m - z_l)(z_1 + \ldots + z_k)^{k(n-1)}}{\prod_{m+r\le l\le k} (z_m + z_r - z_l) \prod_{l=1}^{k}\prod_{i=1}^{n}(\lambda_i - z_l)} \, d\mathbf{z}. \tag{93}$$

Remark 6.8. *The geometric meaning of* $Q(\mathbf{z})$ *in* (93) *is the following, see* [4]*. Let* $T_k \subset B_k \subset GL(k)$ *be the subgroups of invertible diagonal and upper-triangular matrices, respectively; denote the diagonal weights of* T_k *by* $z_1, \ldots, z_k$*. Consider the* $GL(k)$*-module of 3-tensors* $\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k)$*; identifying the weight-*$(z_m + z_r - z_l)$ *symbols* q_l^{mr} *and* q_l^{rm}*, we can write a basis for this space as follows:*

$$\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k) = \bigoplus \mathbb{C}q_l^{mr}, \ 1 \leq m, r, l \leq k.$$

Consider the reference element

$$\epsilon = \sum_{m=1}^{k}\sum_{r=1}^{k-m} q_{mr}^{m+r},$$

in the B_k-invariant subspace

$$N_k = \bigoplus_{1\le m+r\le l\le k} \mathbb{C}q_l^{mr} \subset \mathrm{Hom}\,(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k). \tag{94}$$

Set the notation O_k for the orbit closure $\overline{B_k\epsilon} \subset N_k$, then $Q(\mathbf{z})$ is the T_k-equivariant Poincaré dual

$$Q(z_1,\dots,z_k) = \mathrm{eP}[O_k, N_k]_{T_k},$$

which is a homogeneous polynomial of degree $\dim(N_k) - \dim(O_k)$.

6.4. **The vanishing of residues.** In this paragraph, we describe the conditions under which iterated residues of the type appearing in the sum in (91) vanish.

We start with the 1-dimensional case, where the residue at infinity is defined by (11) with $d = 1$. By bounding the integral representation along a contour $|z| = R$ with R large, one can easily prove

Lemma 6.9. *Let $p(z), q(z)$ be polynomials of one variable. Then*

$$\operatorname*{Res}_{z=\infty} \frac{p(z)\,dz}{q(z)} = 0 \quad \textit{if } \deg(p(z)) + 1 < \deg(q).$$

Consider now the multidimensional situation. Let $p(\mathbf{z}), q(\mathbf{z})$ be polynomials in the d variables $z_1 \dots z_d$, and assume that $q(\mathbf{z})$ is the product of linear factors $q = \prod_{i=1}^N L_i$, as in (91). We continue to use the notation $d\mathbf{z} = dz_1 \dots dz_k$. We would like to formulate conditions under which the iterated residue

$$\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\dots\operatorname*{Res}_{z_k=\infty} \frac{p(\mathbf{z})\ d\mathbf{z}}{q(\mathbf{z})} \tag{95}$$

vanishes. Introduce the following notation:

- For a set of indices $S \subset \{1\dots k\}$, denote by $\deg(p(\mathbf{z}); S)$ the degree of the one-variable polynomial $p_S(t)$ obtained from p via the substitution $z_m \to \begin{cases} t \text{ if } m \in S, \\ 1 \text{ if } m \notin S. \end{cases}$
- For a nonzero linear function $L = a_0 + a_1z_1 + \dots + a_kz_k$, denote by $\mathrm{coeff}(L, z_l)$ the coefficient a_l;
- finally, for $1 \le m \le k$, set

  $$\mathrm{lead}(q(\mathbf{z}); m) = \#\{i;\ \max\{l;\ \mathrm{coeff}(L_i, z_l) \neq 0\} = m\},$$

  which is the number of those factors L_i in which the coefficient of z_m does not vanish, but the coefficients of $z_{m+1}, \dots, z_k$ are 0.

Thus we group the N linear factors of $q(\mathbf{z})$ according to the nonvanishing coefficient with the largest index; in particular, for $1 \le m \le k$ we have

$$\deg(q(\mathbf{z}); m) \ge \mathrm{lead}(q(\mathbf{z}); m), \text{ and } \sum_{m=1}^{k} \mathrm{lead}(q(\mathbf{z}); m) = N.$$

Now applying Lemma 6.9 to the first residue in (95), we see that

$$\operatorname*{Res}_{z_k=\infty} \frac{p(z_1,\dots,z_{k-1},z_k)\,d\mathbf{z}}{q(z_1,\dots,z_{k-1},z_k)} = 0$$

whenever $\deg(p(\mathbf{z});k)+1 < \deg(q(\mathbf{z}),k)$; in this case, of course, the entire iterated residue (95) vanishes.

Now we suppose the residue with respect to z_k does not vanish, and we look for conditions of vanishing of the next residue:

$$\operatorname*{Res}_{z_{k-1}=\infty}\operatorname*{Res}_{z_k=\infty} \frac{p(z_1,\dots,z_{k-2},z_{k-1},z_k)\,d\mathbf{z}}{q(z_1,\dots,z_{k-2},z_{k-1},z_k)}. \tag{96}$$

Now the condition $\deg(p(\mathbf{z});k-1)+1 < \deg(q(\mathbf{z}),k-1)$ will *insufficient*; for example,

$$\operatorname*{Res}_{z_{k-1}=\infty}\operatorname*{Res}_{z_k=\infty} \frac{kz_{k-1}kz_k}{z_{k-1}(z_{k-1}+z_k)} = \operatorname*{Res}_{z_{k-1}=\infty}\operatorname*{Res}_{z_k=\infty} \frac{kz_{k-1}kz_k}{z_{k-1}z_k}\left(1-\frac{z_{k-1}}{z_k}+\dots\right) = 1. \tag{97}$$

After performing the expansions (12) to $1/q(\mathbf{z})$, we obtain a Laurent series with terms $z_1^{-i_1}\dots z_k^{-i_k}$ such that $i_{k-1}+i_k \geq \deg(q(z);k-1,k)$, hence the condition

$$\deg(p(\mathbf{z});k-1,k)+2 < \deg(q(\mathbf{z});k-1,k) \tag{98}$$

will suffice for the vanishing of (96).

There is another way to ensure the vanishing of (96): suppose that for $i=1\dots N$, every time we have $\operatorname{coeff}(L_i,z_{k-1})\neq 0$, we also have $\operatorname{coeff}(L_i,z_k)=0$, which is equivalent to the condition $\deg(q(\mathbf{z}),k-1)=\operatorname{lead}(q(\mathbf{z});k-1)$. Now the Laurent series expansion of $1/q(\mathbf{z})$ will have terms $z_1^{-i_1}\dots z_k^{-i_k}$ satisfying $i_{k-1}\geq \deg(q(\mathbf{z}),k-1)=\operatorname{lead}(q(\mathbf{z});k-1)$, hence, in this case the vanishing of (96) is guaranteed by $\deg(p(\mathbf{z}),k-1)+1 < \deg(q(\mathbf{z}),k-1)$. This argument easily generalizes to the following statement.

Proposition 6.10. *Let $p(\mathbf{z})$ and $q(\mathbf{z})$ be polynomials in the variables $z_1\dots z_k$, and assume that $q(\mathbf{z})$ is a product of linear factors: $q(\mathbf{z})=\prod_{i=1}^N L_i$; set $d\mathbf{z}=dz_1\dots dz_k$. Then*

$$\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\dots\operatorname*{Res}_{z_k=\infty}\frac{p(\mathbf{z})\,d\mathbf{z}}{q(\mathbf{z})} = 0$$

if for some $l\leq k$, either of the following two options hold:

- $\deg(p(\mathbf{z});k,k-1,\dots,l)+k-l+1 < \deg(q(\mathbf{z});k,k-1,\dots,l)$,
  or
- $\deg(p(\mathbf{z});l)+1 < \deg(q(\mathbf{z});l)=\operatorname{lead}(q(\mathbf{z});l)$.

Note that for the second option, the equality $\deg(q(\mathbf{z});l)=\operatorname{lead}(q(\mathbf{z});l)$ means that

$$\text{for each } i=1\dots N \text{ and } m>l,\ \operatorname{coeff}(L_i,z_l)\neq 0 \text{ implies } \operatorname{coeff}(L_i,z_m)=0. \tag{99}$$

Recall that our goal is to show that all the terms of the sum in (91) vanish except for the one corresponding to $\boldsymbol{\pi}_{\mathrm{dst}}=([1]\dots[k])$. Let us apply our new-found tool, Proposition 6.10, to the terms of this sum, and see what happens.

Fix a sequence $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k) \in \boldsymbol{\Pi}_k$, and consider the iterated residue corresponding to it on the right hand side of (91). The expression under the residue is the product of two fractions:

$$\frac{p(\mathbf{z})}{q(\mathbf{z})} = \frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} \cdot \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})},$$

where

$$\frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} = \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod_{m<l}(z_m - z_l)}{\prod_{l=1}^{k} \prod_{\substack{\tau \neq \pi_1 \ldots \pi_l \\ \mathrm{sum}(\tau) \leq l}} (z_\tau - z_{\pi_l})} \text{ and } \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})} = \frac{(z_{\pi_1} + \ldots + z_{\pi_k})^{k(n-1)}}{\prod_{l=1}^{k} \prod_{i=1}^{n} (\lambda_i - z_l)}. \tag{100}$$

Note that $p(\mathbf{z})$ is a polynomial, while $q(\mathbf{z})$ is a product of linear forms.

Proof of Theorem 6.10

As a warm-up, we show that if the last element of the sequence is not the trivial partition, i.e. if $\pi_k \neq [k]$, then already the first residue in the corresponding term on the right hand side of (91) – the one with respect to z_k – vanishes. Indeed, if $\pi_k \neq [k]$, then $\deg(q_2(\mathbf{z}); k) = n$, while z_k does not appear in $p_2(\mathbf{z})$.

On the other hand, $\deg(q_1(\mathbf{z}); k) = 1$, because the only term with z_k is the one corresponding to $l = k, \tau = [k] \neq \pi_k$. If $\deg(Q_{\boldsymbol{\pi}}(\mathbf{z}), k) = 0$ held, we would be ready, as

$$\deg(p(\mathbf{z}); k) = k - 1 \text{ and } \deg(q(\mathbf{z}); k) = n + 1 \tag{101}$$

and $k \leq n$.

Lemma 6.11. *For* $\boldsymbol{\pi} \neq ([1], [2], \ldots, [k])$

$$\deg(Q_{\boldsymbol{\pi}}(\mathbf{z}); k) = 0. \tag{102}$$

Proof. Recall from Proposition 6.6 that $Q_{\boldsymbol{\pi}}(\mathbf{z})$ is the multidegree of a $(\mathbb{C}^*)^k$-invariant cone $X_{\mathbf{f}}$ in the tangent space of the flag manifold Flag_k^* at the fixed point $\boldsymbol{\pi}$. The weights of the $(\mathbb{C}^*)^k$-action on this tangent space are exactly the factors of q_1, namely

$$z_\tau - z_{\pi_l} \ : \ \tau \neq \pi_1, \pi_2, \ldots \pi_k; \sum \tau \leq l, |\tau| \leq 2$$

and therefore the only weight containing z_k is

$$z_{\pi_k} - z_k$$

Applying Proposition 6.3 with $m = k$ we arrive at (102). □

We can thus assume that $\pi_k = [k]$, and proceed to the study of the next residue, the one taken with respect to z_{k-1}. Again, assume that $\pi_{k-1} \neq [k-1]$. As in the case of z_k above,

$$\deg(q_2(\mathbf{z}), k-1) = n, \deg(p_2(\mathbf{z}); k-1) = 0.$$

In q_1 the linear terms containing z_{k-1} are

$$z_{k-1} - z_k, z_1 + z_{k-1} - z_k, z_{k-1} - z_{\pi_{k-1}} \tag{103}$$

The first term here cancels with the identical term in the Vandermonde in p_1. The second term divides $Q_{\boldsymbol{\pi}}$, according to the following proposition from [4] applied with $l = k-1$:

Proposition 6.12. *(*[4]*, Proposition 7.4)*
Let $l \geq 1$, and let $\boldsymbol{\pi}$ be an admissible sequence of partitions of the form $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_l, [l+1], \ldots, [k])$, where $\pi_l \neq [l]$. Then for $m > l$, and every partition τ such that $l \in \tau$, $\mathrm{sum}(\tau) \leq m$, and $|\tau| > 1$, we have

$$(z_\tau - z_m) | Q_{\boldsymbol{\pi}}. \tag{104}$$

Therefore, after cancellation, all linear factors from $q_1(\mathbf{z})$ which have nonzero coefficients in front of both z_{k-1} and z_k vanish, and we can apply the second option in Proposition 6.10, leaving us with checking the degrees of z_k in the new numerator and denominator of the fraction $\frac{p'(\mathbf{z})}{q'(\mathbf{z})}$.

Note that $\frac{Q_\pi(\mathbf{z})}{z_1+z_{k-1}-z_k}$ is the multidegree of the same cone in a smaller vector space, namely, the cone sits in the subspace

$$S = \{y_{z_1+z_{k-1}-z_k} = 0\} \subset T_{p_{\boldsymbol{\pi}}}\mathrm{Flag}_k^*,$$

where $y_{z_1+z_{k-1}-z_k}$ is eigencoordinate corresponding to the weight $z_1+z_{k-1}-z_k$. The weights with nonzero coefficient of z_{k-1} in S are

$$z_{k-1} - z_{\pi_{k-1}}, z_{k-1} - z_k,$$

and by Lemma 6.3

$$\deg(p'(\mathbf{z}); k-1) \leq k-2+1 = k-1$$

On the other hand,

$$\deg(q'(\mathbf{z}); k-1) = n+1,$$

so we can apply the second part of Proposition 6.10.

In general, assume that

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_l, [l+1], \ldots, [k]), \pi_l \neq [l],$$

and embark to the study of the residue with repect to z_l. The weights containing z_l in q_1 are

$$z_l - z_k, z_l - z_{k-1}, \ldots, z_l - z_{l+1} \tag{105}$$

$$z_\tau - z_s \text{ with } l \in \tau, \tau \neq l, l+1 \leq s \leq k, \mathrm{sum}(\tau) \leq s \tag{106}$$

$$z_l - z_{\pi_l} \tag{107}$$

The weights in (105) cancel out with the identical terms in $p_1(\mathbf{z})$. By Propostition 6.12, the cone, whose multidegree is $Q_{\boldsymbol{\pi}}(\mathbf{z})$ sits in the subspace S, orthogonal to the coordinates corresponding to the weights in (106), and therefore $Q_{\boldsymbol{\pi}}$ is divisible by these. Using Lemma 6.3, after cancellation we are left with

$$\deg(p'(\mathbf{z}); l) = l-1+\deg(Q'(\mathbf{z}), l) \leq l-1+k-l = k-1; \deg(q'(\mathbf{z})) = n+1,$$

again. Since $k \leq n$, by applying the second option of Proposition 6.10 we arrive to the vanishing of the residue, forcing π_l to be $[l]$.

7. Proof of Theorem 1.3

7.1. **The Flag Manifold Model for the Jet differentials.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d. Using the embedding ϕ^{Proj} fiberwise, we get the following analog of Theorem 2.1 in [13]. Note, that the Plucker coordinate functions in the Plucker embedding $\mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \mathrm{Sym}^{\leq k}\mathbb{C}^n)$ have weighted degree

$$1 + 2 + \ldots + k = \binom{k+1}{2}.$$

Proposition 7.1. *The quotient $J_k(T_X^*)/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding*

$$\phi^{\mathrm{Proj}} : J_k(T_X^*)/\mathbb{G}_k \hookrightarrow \mathbb{P}(\wedge^k(T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))$$

into the projectivisation of $\wedge^k(T_X^* \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*))$ over X. The fiberwise closure $\mathcal{X}_k = \overline{\phi^{\mathrm{Proj}}(J_k(T_X^*))}$ of the image is a relative compactification of $J_k(T_X^*)/\Gamma\mathbb{G}_k$ over X.

Following our notations in §5, we introduce

$$\mathrm{Sym}^{\leq k} T_X^* = T_X^* \oplus \mathrm{Sym}^2(T_X^*) \oplus \ldots \oplus \mathrm{Sym}^k(T_X^*).$$

Note that $\mathcal{X}_k \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} T_X^*))$ is a closed subvariety, and we can define the tautological line bundle over $\mathcal{X}_k$ as the restriction

$$\mathcal{O}_{\mathcal{X}_k}(1) = \mathcal{O}_{\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} T_X^*))}(1)|_{\mathcal{X}_k}.$$

Proposition 7.2. *The following direct image formula holds:*

$$\pi_* \mathcal{O}_{\mathcal{X}_k}(m) \subset \mathcal{O}(E_{k,m\binom{k+1}{2}} T_X^*) \tag{108}$$

where $\pi : \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} T_X^*) \to X$ is the projection.

Proof. By definition, the sections of the tautological bundle pull back to $\mathbb{G}_k$-invariant functions on $J_k T_X^*$. □

Remark 7.3. *Note that*

$$\pi_* \mathcal{O}_{\mathcal{X}_k}(m) \subset \mathcal{O}(E_{k,m\binom{k+1}{2}} T_X^*)$$

is enough to proceed with the strategy of [13]. *We produce a nonzero global section of the smaller sheaf, which gives a global section of the Demailly jet bundle. A more detailed study of $\mathcal{X}_k$ shows in* [5] *that its boundary components have codimension at least two, and therefore all the invariants are stored in the tautological bundle, and in* (108) *equality holds.*

We now replace the Demailly-Semple tower X_k constructed in [10] as a tower of k projective fibrations with our new contruction $\mathcal{X}_k$, and follow the strategy of [13] using this. We define and give an iterated residue formula for the analog of Diverio's intersection number in [13], and prove the positivity to get the stronger result Theorem 1.3.

The starting point is, again, Theorem 4.1 which connects jet differentials to the Green-Griffiths Conjecture.

Note, that by Theorem 1 of [14],

$$H^0(X, E_{k,m\binom{k+1}{2}} T_X^* \otimes A^{-1}) = 0$$

holds for $k < n$, so we can again restrict our attention to the range $k \geq n$. However, for $k > n$, the flag manifold $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathbb{C}^n)/GL(k)$ is not defined in the snowman-model, and therefore our residue formula does not hold.

This forces us to study the $k = n$ case.

Similarly to §4, to control the order of vanishing of these differential forms along the ample divisor we choose A to be –as in [13] – a proper twist of the canonical bundle of X, which is ample as soon as $d \geq n + 3$.

Theorem 4.2 on degeneracy of entire curves with Proposition 7.2 ensures that we have to prove the existence of nonzero global sections of

$$\mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}m}$$

for some $\delta > 0, m \gg 0$ and $d \gg 0$. The precise statement is the following

Theorem 7.4. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface with ample canonical bundle, that is* $\deg X \geq n + 3$. *If $\delta = \frac{1}{n^3(n+1)}$ and $d > D(n) = n^6$ then*

$$H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}(m) \otimes \pi^* K_X^{-\delta\binom{n+1}{2}m}) \simeq H^0(X, E_{n,m\binom{n+1}{2}} T_X^* \otimes K_X^{-\delta\binom{n+1}{2}m}) \neq 0,$$

nonzero, provided that $\delta\binom{n+1}{2}m$ is integer and Conjecture 1.2 holds.

Theorem 1.3 follows from Theorem 4.2 and Theorem 7.4.

The technical tool for proving Theorem 7.4 is again the Morse inequalities of Trapani given in Theorem 4.5. In order to apply this, we have to express $\mathcal{O}_{\mathcal{X}_n}(1)$ as a difference of nef bundles.

Proposition 7.5. *Let $d \geq n + 3$ and therefore K_X ample. The following line bundles are nef on X:*

(1) $\mathcal{O}_{\mathcal{X}_n}(1) \otimes \pi^* \mathcal{O}_X(2n^2)$

(2) $\pi^* \mathcal{O}_X(2n^2) \otimes \pi^* K_X^{\delta\binom{n+1}{2}}$ *for any $\delta > 0$ and $\delta\binom{n+1}{2}$ integer.*

Proof. Let $\mathcal{O}(m)$ denote the m-twisted tautological bundle on $\mathbb{P}^{n+1}$. Then $T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}(2)$ is globally generated, and there is a surjective bundle map

$$(T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}(2))|_X^{\otimes m} \to T_X^* \otimes \mathcal{O}_X(2)^{\otimes m},$$

therefore $T_X^*\otimes \mathcal{O}_X(2)$ is globally generated. Consequently, the left hand side of the following surjective bundle map is globally generated,

$$\wedge^n\left(T_X^*\otimes\mathcal{O}_X(2)\oplus \mathrm{Sym}^2T_X^*\otimes\mathcal{O}_X(4)\oplus\ldots\oplus\mathrm{Sym}^nT_X^*\otimes\mathcal{O}_X(2n)\right)\to \tag{109}$$
$$\wedge^n\left((T_X^*\oplus\mathrm{Sym}^2T_X^*\oplus\ldots\oplus\mathrm{Sym}^nT_X^*)\otimes\mathcal{O}_X(2n)\right)=\wedge^n(T_X^*\oplus\ldots\oplus\mathrm{Sym}^nT_X^*)\otimes\mathcal{O}_X(2n^2),$$

and therefore the right hand side is also globally generated. So

$$\mathcal{O}_{\mathbb{P}(\wedge^n(\mathrm{Sym}^{\le n}T_X^*))}(1)\otimes\pi^*\mathcal{O}_X(2n^2)$$

is nef, the first part of Proposition 7.5 is proved. The second part follows from the standard fact that the pull-back of an ample line bundle is nef. □

Consequently, we can express $\mathcal{O}_{\mathcal{X}_n}(1)\otimes\pi^*K_X^{-\delta\binom{n+1}{2}}$ as the following difference of two nef line bundles:

$$\mathcal{O}_{\mathcal{X}_n}(1)\otimes\pi^*K_X^{-\delta\binom{n+1}{2}}=(\mathcal{O}_{\mathcal{X}_n}(1)\otimes\pi^*\mathcal{O}_X(2n^2))\otimes(\pi^*\mathcal{O}_X(2n^2)\otimes\pi^*K_X^{\delta\binom{n+1}{2}})^{-1}. \tag{110}$$

In order to prove Theorem 4.4, by the Morse inequalities we need to evaluate the intersection product

(111)
$$I_X(d,n,\delta)=(\mathcal{O}_{\mathcal{X}_n}(1)\otimes\pi^*\mathcal{O}_X(2n^2))^{n^2)}-(n^2)(\mathcal{O}_{\mathcal{X}_n}(1)\otimes\pi^*\mathcal{O}_X(2n^2))^{(n^2-1)}(\pi^*\mathcal{O}_X(2n^2)\otimes\pi^*K_X^{\delta\binom{n+1}{2}}),$$

and to prove that it is positive if $d>n^6$.

7.2. **Plan of the computation.** Let us introduce the cohomological classes $h=c_1(\mathcal{O}_X(1)), u=c_1(\mathcal{O}_{\mathcal{X}_n}(1))$, and $c_1=c_1(X)$. Then $c_1(K_X)=-c_1$, and the intersection form (111) is the integral of the top form $R_X(d,n,\delta)\in H^{n^2}(\mathcal{X}_n)$

$$I_X(d,n,\delta)=\int_{\mathcal{X}_n}R_X(d,n,\delta),$$

where

$$R_X(d,n,\delta)=(u+2n^2\pi^*h)^{n^2}-n^2(u+2n^2\pi^*h)^{n^2-1}(2n^2\pi^*h-\delta\binom{n+1}{2}\pi^*c_1) \tag{112}$$

The Chern classes of X are expressible with d,h via the following identity:

$$(1+h)^{n+2}=(1+dh)c(X),$$

where $c(X)=c(TX)$ is the total Chern class of X. After expansion we get the identities (39).

In particular, this gives us

$$c_1=-(d-n-2)h.$$

To apply the iterated residue formula of Theorem 6.7, we assume that $n=k$ and simplify our notation by using h instead of π^*h. We define

(113) $$R(d,n,\delta,z_1,\ldots,z_n)=(-z_1-\ldots-z_n+2n^2h)^{n^2}-\\ -n^2(-z_1-\ldots-z_n+2n^2h)^{n^2-1}(2n^2h+\delta\binom{n+1}{2}(d-n-2)h)$$

Theorem 6.7 gives the desired formula for $n=k$ as follows

Proposition 7.6.

(114) $$I_X(d,n,\delta)=\int_X \operatorname*{Res}_{\mathbf{z}=\infty}\frac{Q(\mathbf{z})\prod_{m<l}(z_m-z_l)R(d,n,\delta,z_1\ldots z_n)}{\prod_{m+r\le l\le n}(z_m+z_r-z_l)\prod_{l=1}^n\prod_{i=1}^n(\lambda_i-z_l)}\,d\mathbf{z}.$$

Changing the coordinates $z_i\to -z_i$ and applying the identities of Remark 4.7 we have

(115) $$\frac{1}{\prod_{l=1}^n\prod_{i=1}^n(\lambda_i+z_l)}=\frac{1}{(z_1\ldots z_n)^n}\frac{1}{\prod_{i,l=1}^n(1+\frac{\lambda_i}{z_l})}=\frac{1}{(z_1\ldots z_n)^n}\frac{1}{\prod_{l=1}^n C(X)(1/z_l)}=\\ =\frac{1}{(z_1\ldots z_n)^n}\prod_{l=1}^n\frac{1+\frac{dh}{z_l}}{(1+\frac{h}{z_l})^{n+2}}=\frac{1}{(z_1\ldots z_n)^n}\prod_{l=1}^n\left(1+\frac{dh}{z_l}\right)\prod_{l=1}^n\left(1-\frac{h}{z_l}+\frac{h^2}{z_l^2}-\ldots\right)^{n+2}$$

and therefore after substituting (113) into (114) we arrive to

(116)
$$I_X(d,n,\delta)=\int_X \operatorname*{Res}_{\mathbf{z}=\infty}\frac{(-1)^nQ(\mathbf{z})\prod_{m<l}(z_m-z_l)}{\prod_{m+r\le l\le n}(z_m+z_r-z_l)(z_1\ldots z_n)^n}\prod_{l=1}^n\left(1+\frac{dh}{z_l}\right)\prod_{l=1}^n\left(1-\frac{h}{z_l}+\frac{h^2}{z_l^2}-\ldots\right)^{n+2}\cdot\\ (z_1+\ldots+z_n+2n^2h)^{n^2-1}\left(z_1+\ldots+z_n-\delta n^2\binom{n+1}{2}dh-\left(2n^4-n^2\delta(n+2)\binom{n+1}{2}-2n^2\right)h\right)\,d\mathbf{z}.$$

Let's take a short break again, and step back a bit looking at this formula. The residue is by definition the coefficient of $\frac{1}{z_1\ldots z_n}$ in the appropriate Laurent expansion of the big rational expression in $z_1,\ldots,z_n,n,d,h$ and δ, multiplied by $(-1)^n$. We can therefore omit the $(-1)^n$ factor from the numerator and simply compute the corresponding coefficient. The result is therefore a polynomial in n,d,h,δ, and in fact, a relatively easy argument shows that it is a polynomial in n,d,δ multiplied by h^n

Indeed, giving degree 1 to $z_1,\ldots,z_n,h$ and 0 to n,d,δ, the rational expression in the residue has total degree 0. Therefore the coefficient of $\frac{1}{z_1\ldots z_n}$ has degree n, so it has the form $h^np(n,d,\delta)$ with a polynomial p.

Since $\int_X h^n = d$, integration over X is simply a substitution $h^n = d$, resulting the equation

$$I_X(d,n,\delta) = dp(n,d,\delta).$$

To overcome the difficulties in handling the rational expression, we introduce a useful notation.

Notation 7.1. *For* $\mathbf{i} = (i_1, \dots i_n) \in \mathbb{Z}^n$

$$\theta(\mathbf{i}) = \theta(z_1^{i_1} \dots z_n^{i_n}) = \mathrm{coeff}_{z_1^{i_1} \dots z_n^{i_n}} \frac{Q(\mathbf{z}) \prod_{m<l} (z_m - z_l)(z_1 + \dots + z_n)^{n^2+i_1+\dots+i_n}}{\prod_{m+r \le l \le n} (z_m + z_r - z_l)(z_1 \dots z_n)^n} \tag{117}$$

and

$$\Theta(i_1, \dots, i_n) = \sum_{\sigma \in \mathrm{perm}(i_1,\dots,i_n)} \theta(\sigma(i_1), \dots, \sigma(i_n)) \tag{118}$$

Note that we will usually omit the zero components after Θ *to shorten the notation, for instance write* $\Theta(1) = \Theta(1, 0, \dots, 0)$.

The total degree of the rational expression on the r.h.s of (117) is $i_1 + \dots + i_n$ and therefore the coefficient of $z_1^{i_1} \dots z_n^{i_n}$ can be nonzero.

The following proposition describes $I(n,d,\delta)$ in more detail.

Proposition 7.7. (1) $I(n,d,\delta)$ *is a polynomial in* d *of degree* $n+1$ *without constant term:*

$$I(n,d,\delta) = a_{n+1}(n,\delta)d^{n+1} + a_n(n,\delta)d^n + \dots + a_1(n,\delta)d$$

where the coefficients are linear in δ*, polynomial in* n*.*

(2) *The leading coefficient of* $I(n,d,\delta)$ *is*

$$a_{n+1}(n,\delta) = \left(1 - n^2 \binom{n+1}{2} \delta\right) \Theta(0, \dots, 0).$$

Proof. The first part follows from the previous remarks. The second equation comes from (116), and the fact that in order to get d^{n+1} we either have to choose all the $\frac{dh}{z_l}$ terms in the product $\prod_{l=1}^n \left(1 + \frac{dh}{z_l}\right)$, or we need to pick up the $\frac{dh}{z_s}$ term in

$$\frac{Q(\mathbf{z}) \prod_{m<l} (z_m - z_l) R(z_1, \dots, z_n, \delta, h)}{\prod_{m+r \le l \le n} (z_m + z_r - z_l)(z_1 \dots z_n)^n},$$

and pair up with the terms $\frac{dh}{z_l}$, $l \neq s$ in the product $\prod_{l=1}^n \left(1 + \frac{dh}{z_l}\right)$. This argument gives us

$$a_{n+1}(n,\delta) = \Theta(0, \dots, 0) - n^2 \binom{n+1}{2} \delta \sum_{s=1}^n \theta(0, \dots, -1^s, \dots, 0) \tag{119}$$

By definiton,

(120)

$$\sum_{s=1}^{n}\theta(0,\dots,-1^s,\dots,0)=\Theta(-1)=\sum_{s=1}^{n}\mathrm{coeff}_{\frac{1}{z_s}}\frac{Q(\mathbf{z})\prod_{m<l}(z_m-z_l)(z_1+\ldots+z_n)^{n^2-1}}{\prod_{m+r\le l\le n}(z_m+z_r-z_l)(z_1\ldots z_n)^n}=$$

$$=\sum_{s=1}^{n}\mathrm{coeff}_1\frac{Q(\mathbf{z})\prod_{m<l}(z_m-z_l)z_s(z_1+\ldots+z_n)^{n^2-1}}{\prod_{m+r\le l\le n}(z_m+z_r-z_l)(z_1\ldots z_n)^n}=\Theta(0,\dots,0).$$

□

Lemma 7.8.

$$\Theta(0,\dots,0)>0.$$

Proof. This is the leading coefficient of the intersection number

$$(\mathcal{O}_{X_n}(1)\otimes\pi^*\mathcal{O}_X(2n^2))^{n^2}.$$

Since this is a nef line bundle, this is positive, and therefore the leading coefficient is also positive. □

Corollary 7.9. *For $\delta<\frac{2}{n^3(n+1)}$ the leading coefficient of $I(n,d,\delta)$ is positive, and therefore $I(n,d,\delta)>0$ for $d\gg 0$.*

According to Proposition 7.7, we cannot expect better than polynomial bound for the Green-Griffiths conjecture from this model.

7.3. **Estimation of the coefficients.** Before we proceed, and fling ourselves into the computation of the further coefficients, let's have a look at (116) again. Introduce

$$\Delta(z_1,\dots,z_n,\delta,n)=\frac{Q(\mathbf{z})\prod_{m<l}(z_m-z_l)R(z_1,\dots,z_n,\delta,h)}{\prod_{m+r\le l\le n}(z_m+z_r-z_l)(z_1\ldots z_n)^n},\tag{121}$$

and with abuse of this notation we often omit the parameters and write Δ.

If the degree of $z_1,\dots,z_n$ and h is 1, then the denominator and numerator of Δ are homogeneous polynomials of the same degree, and therefore in the Laurent expansion we have terms

$$\frac{(dh)^\epsilon h^m z_1^{a_1}\ldots z_n^{a_n}}{z_1^{b_1}\ldots z_n^{b_n}},$$

with $\epsilon=0$ or 1, $\epsilon+m+a_1+\ldots+a_n=b_1+\ldots+b_n$, and $a_ib_i=0$. Let

$$\eta(\epsilon,m,\mathbf{a},\mathbf{b})$$

denote the coefficient of this term in Δ. Using our previously introduced notations we have

$$\eta(1,m,\mathbf{a},\mathbf{b}) = -\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\delta n^2\binom{n+1}{2}\binom{n^2-1}{m}(2n^2)^m \tag{122}$$

and

$$\eta(0,m{+}1,\mathbf{a},\mathbf{b}) = \theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\left(\binom{n^2-1}{m+1}(2n^2)^{m+1} - \left(2n^4 - n^2\delta(n+2)\binom{n+1}{2} - 2n^2\right)\binom{n^2-1}{m}(2n^2)^m\right)$$

From these expressions we clearly have
For $\delta = \frac{1}{n^3(n+1)}$

$$|\eta(0,m+1,\mathbf{a},\mathbf{b})| < n^4|\eta(1,m,\mathbf{a},\mathbf{b})| \tag{123}$$

If $\eta(m,\mathbf{a},\mathbf{b})$ denotes the coefficient of $h^m\mathbf{z}^{\mathbf{a}-\mathbf{b}}$ in Δ, then

$$\eta(m,\mathbf{a},\mathbf{b}) = d\eta(1,m-1,\mathbf{a},\mathbf{b}) + \eta(0,m,\mathbf{a},\mathbf{b}),$$

and we arrive at

Lemma 7.10. *For $\delta = \frac{1}{n^3(n+1)}$ and $d > n^5$*

$$|\eta(0,m,\mathbf{a},\mathbf{b})| < \frac{1}{n}|\eta(m,\mathbf{a},\mathbf{b})|,$$

and therefore for $m \geq 1$

$$\eta(m,\mathbf{a},\mathbf{b}) = C_{m,\mathbf{a},\mathbf{b}}d\eta(1,m-1,\mathbf{a},\mathbf{b}) \text{ with } 1-\frac{1}{n} < C_{m,\mathbf{a},\mathbf{b}} < 1+\frac{1}{n} \tag{124}$$

Next, to handle the remaining part of (116) we introduce

$$\Lambda = \prod_{l=1}^n\left(1+\frac{dh}{z_l}\right)\prod_{l=1}^n\left(1-\frac{h}{z_l}+\frac{h^2}{z_l^2}-\ldots\right)^{n+2}$$

and for $\mathbf{a} = (a_1,\ldots,a_n)$ with $a_1 \geq 0$

$$\nu(\mathbf{a}) = \mathrm{coeff}_{\frac{h^{\Sigma\mathbf{a}}}{\mathbf{z}^{\mathbf{a}}}}\Lambda,$$

Let $\mathbf{1} = (1,\ldots,1)$ denote the multiindex of all 1s. Then

$$I_X(d,n,\delta) = d\sum_{\mathbf{b}\in\{0,1\}^n,m,\mathbf{a}}\eta(m,\mathbf{a},\mathbf{b})\nu(\mathbf{1}+\mathbf{b}-\mathbf{a}) \tag{125}$$

Indeed, the left hand side is by definition is the coefficient of $\frac{1}{z_1\ldots z_n}$ modified by the substitution $h^n = d$. $\mathbf{b} \in [0,1]^n$ means that $z^{\mathbf{b}}$ is square-free, which is necessary to get $z_1\ldots z_n$ in the denominator.

If $\delta < \frac{1}{n^3(n+1)}$ and $d > n^5$, then Lemma 7.10 leads us to

$$(126)\quad I_X(d,n,\delta) = d \sum_{\mathbf{b}\in\{0,1\}^n, m, \Sigma\mathbf{a}=\Sigma\mathbf{b}-m} \eta(m,\mathbf{a},\mathbf{b})\nu(\mathbf{1}-\mathbf{b}+\mathbf{a}) =$$

$$= d^2 \sum_{\mathbf{b}\in\{0,1\}^n, m>1, \Sigma\mathbf{a}=\Sigma\mathbf{b}-m} |C_{m,\mathbf{a},\mathbf{b}}\eta(1,m-1,\mathbf{a},\mathbf{b})\nu(\mathbf{1}-\mathbf{b}+\mathbf{a})| + d \sum_{\mathbf{b}\in\{0,1\}^n, \Sigma\mathbf{a}=\Sigma\mathbf{b}} |\eta(0,0,\mathbf{a},\mathbf{b})\nu(\mathbf{1}-\mathbf{b}+\mathbf{a})|$$

From now on to simplify our formulas, we fix the value of δ to be

$$\delta = \frac{1}{n^3(n+1)}.$$

Then (122) can be rewritten for $m \geq 1$ as

$$\eta(1,m-1,\mathbf{a},\mathbf{b}) = -\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\frac{1}{2}\binom{n^2-1}{m-1}(2n^2)^{m-1}$$

On the other hand, by definition

$$\eta(0,0,\mathbf{a},\mathbf{b}) = \theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}}).$$

A short computation shows that using the notation

$$\Lambda_1 = \prod_{l=1}^{n}\left(1 - \frac{h}{z_l} + \frac{h^2}{z_l^2} - \ldots\right)^{n+2},$$

and assuming that $\mathbf{b} \subset \{0,1\}^n$ (and therefore $\mathbf{1}-\mathbf{b}+\mathbf{a} \geq 0$) we have

$$(127)\quad \nu(\mathbf{1}-\mathbf{b}+\mathbf{a}) = h^{n+\Sigma\mathbf{a}-\Sigma\mathbf{b}}\Big(d^{n-\Sigma\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}}}}\Lambda_1 + d^{n-1-\Sigma\mathbf{b}}\sum_{s\in\mathbf{1}-\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+s}}}\Lambda_1 + $$
$$+ d^{n-2-\Sigma\mathbf{b}}\sum_{s_1,s_2\in\mathbf{1}-\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+s_1+s_2}}}\Lambda_1 + \ldots +\Big)$$

where $\mathbf{a}+s = \mathbf{a}+(0,\ldots,0,1^s,0\ldots,0)$. Putting these together we arrive at the following formulas:

$$(128)\quad I_X(d,n,\delta) = -d^2 \sum_{\substack{\mathbf{b}\in\{0,1\}^n, m\geq 1\\ \Sigma\mathbf{a}=\Sigma\mathbf{b}-m}} C_{m,\mathbf{a},\mathbf{b}}\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\frac{1}{2}\binom{n^2-1}{m-1}(2n^2)^{m-1}\cdot$$

$$\left(d^{n-\Sigma\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}}}}\Lambda_1 + d^{n-1-\Sigma\mathbf{b}}\sum_{s\in\mathbf{1}-\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+s}}}\Lambda_1 + d^{n-2-\Sigma\mathbf{b}}\sum_{s_1,s_2\in\mathbf{1}-\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+s_1+s_2}}}\Lambda_1 + \ldots\right) +$$

$$+d\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=\Sigma\mathbf{b}}}\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\left(d^{n-\Sigma\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}}}}\Lambda_1 + d^{n-1-\Sigma\mathbf{b}}\sum_{s\in\mathbf{1}-\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+s}}}\Lambda_1 + d^{n-2-\Sigma\mathbf{b}}\sum_{s_1,s_2\in\mathbf{1}-\mathbf{b}}\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+s_1+s_2}}}\Lambda_1 + \ldots\right).$$

After arranging this expression as a polynomial in d we get

$$\left(\theta(1)\mathrm{coeff}_1\Lambda_1 - \frac{1}{2}\sum_{s=1}^{n} C_{1,0,e_s}\theta(z_s^{-1})\Lambda_1\right) d^{n+1} + \tag{129}$$

$$+(\theta(1)\sum_{s=1}^{n}\mathrm{coeff}_{z_s^{-1}}\Lambda_1 + \sum_{s_1\neq s_2}\theta\left(\frac{z_{s_1}}{z_{s_2}}\right)\mathrm{coeff}_{z_{s_1}^{-1}}\Lambda_1 - \frac{n-1}{2n}\sum_{s_1\neq s_2}\theta(z_{s_1}^{-1})\mathrm{coeff}_{z_{s_2}^{-1}}\Lambda_1 -$$

$$-\binom{n^2-1}{1}2n\sum_{s_1\neq s_2} C_{2,0,e_{s_1}+e_{s_2}}\theta((z_{s_1}z_{s_2})^{-1})\mathrm{coeff}_1\Lambda_1 - \frac{1}{2}\sum_{\substack{s_1,s_2,s_3\\ s_1\neq s_2,s_3}} C_{1,e_{s_1},e_{s_1}+e_{s_2}+e_{s_3}}\theta\left(\frac{z_{s_1}}{z_{s_2}z_{s_3}}\right)\mathrm{coeff}_{z_{s_1}^{-1}}\Lambda_1)d^n +$$

$$+(...)d^{n-1} + \dots \tag{130}$$

The coefficient of d^{n+1-l} is

$$\sum_{r=0}^{l}\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=\Sigma\mathbf{b}=r,\mathbf{ab}=0}}\sum_{\substack{\mathbf{s}\subset\mathbf{1}-\mathbf{b}\\ \Sigma\mathbf{s}=l-r}}\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+\mathbf{s}}}}\Lambda_1 -$$

$$-\sum_{r=1}^{l+1}\sum_{m=1}^{r}\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}}\frac{1}{2}\binom{n^2-1}{m-1}(2n^2)^{m-1}\sum_{\substack{\mathbf{s}\subset\mathbf{1}-\mathbf{b}\\ \Sigma\mathbf{s}=l-r}} C_{m,\mathbf{a},\mathbf{b}}\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+\mathbf{s}}}}\Lambda_1 \tag{131}$$

Lemma 7.11. *For* $i_1,\dots,i_n\in\mathbb{N}$

$$\mathrm{coeff}_{\frac{1}{z_1^{i_1}\dots z_n^{i_n}}}\Lambda_1 = (-1)^{i_1+\dots+i_n}\binom{n+i_1+1}{i_1}\binom{n+i_2+1}{i_2}\cdots\binom{n+i_n+1}{i_n}$$

Proof. By definition

$$\mathrm{coeff}_{\frac{1}{z_s^{i_s}}}\Lambda_1 = \mathrm{coeff}_{\frac{1}{z_s^{i_s}}}\left(1-\frac{h}{z_s}+\frac{h^2}{z_s^2}-\dots\right)^{n+2} = (-1)^s\sum_{m_1+\dots m_{n+2}=s}1 = (-1)^s\binom{n+s+1}{s}$$

and the lemma follows. □

Corollary 7.12. *For* $i_1,\dots,i_n\in\mathbb{N}$

$$\mathrm{coeff}_{\frac{1}{z_1^{i_1}\dots z_n^{i_n}}}\Lambda_1 \le (-1)^{i_1+\dots+i_n}(n+2)^{i_1+\dots+i_n}$$

Lemma 7.13.

$$\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}}\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}}) < n^{4(r-m)}\theta(1)$$

Proof. By definiton we have
(132)

$$\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}} \theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}}) = \mathrm{coeff}_{\mathbf{z}^{\mathbf{1}}} \frac{Q(\mathbf{z})\prod_{m<l}(z_m-z_l)}{\prod_{m+r\le l\le n}(z_m+z_r-z_l)(z_1\dots z_n)^n} \cdot (z_1+\dots+z_n)^{n^2-m}\Big(\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}} \mathbf{z}^{\mathbf{b}-\mathbf{a}}\Big)$$

For $i_1+\dots+i_n=n^2$ we have

$$\mathrm{coeff}_{z_1^{i_1}\dots z_n^{i_n}}(z_1+\dots+z_n)^{n^2-m}\Big(\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}} \mathbf{z}^{\mathbf{b}-\mathbf{a}}\Big) = \sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0,\mathbf{i}+\mathbf{a}-\mathbf{b}\ge 0}} \frac{(n^2-m)!}{(i_1+a_1-b_1)!\cdots(i_n+a_n-b_n)!} <$$

$$< \sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0,\mathbf{i}+\mathbf{a}-\mathbf{b}\ge 0}} \frac{(n^2)!}{\prod_{a_s>0}(i_s+a_s)!\prod_{s\in\mathbf{b}^0}(i_s-1)\prod_{s\in[n]\setminus(\mathbf{a}\cup\mathbf{b})} i_s} <$$

$$= \sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r-m,\\ \mathbf{ab}=0,\mathbf{i}+\mathbf{a}-\mathbf{b}\ge 0}} \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}+\mathbf{a}-\mathbf{b}}}(z_1+\dots+z_n)^{n^2}$$

Introduce the notation

$$\mathrm{Tp}_n(\mathbf{z}) = \frac{Q(\mathbf{z})\prod_{m<l}(z_m-z_l)}{\prod_{m+r\le l\le n}(z_m+z_r-z_l)(z_1\dots z_n)^n}$$

Here Tp stands for Thom polynomial, since the coefficients of the Laurent expansion are the coefficients of the Thom polynomials, see [4]. By (132) and Rimányi's conjecture (see the first part of Conjecture 1.2) we arrive at

$$\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}} \theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}}) = \sum_{i_1+\ldots+i_n=n^2} \mathrm{Tp}_{z_1^{-i_1}\ldots z_n^{-i_n}} \cdot \mathrm{coeff}_{z_1^{i_1}\ldots z_n^{i_n}}(z_1+\ldots+z_n)^{n^2-m}\Big(\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}} \mathbf{z}^{\mathbf{b}-\mathbf{a}}\Big) <$$

$$< \sum_{\Sigma\mathbf{i}=n^2} \mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}}} \cdot \sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=\Sigma\mathbf{b}=r-m,\\ \mathbf{ab}=0,\mathbf{i}+\mathbf{a}-\mathbf{b}\geq 0}} \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}+\mathbf{a}-\mathbf{b}}}(z_1+\ldots+z_n)^{n^2} =$$

$$= \sum_{\substack{\Sigma\mathbf{i}=n^2\\ \mathbf{i}\geq\mathbf{0}}} \sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=\Sigma\mathbf{b}=r-m,\\ \mathbf{ab}=0,\mathbf{i}+\mathbf{a}-\mathbf{b}\geq 0}} \cdot \mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}}} \cdot \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}}}(z_1+\ldots+z_n)^{n^2} \cdot \frac{\mathrm{coeff}_{\mathbf{z}^{\mathbf{i}+\mathbf{a}-\mathbf{b}}}(z_1+\ldots+z_n)^{n^2}}{\mathrm{coeff}_{\mathbf{z}^{\mathbf{i}}}(z_1+\ldots+z_n)^{n^2}} +$$

$$+ \sum_{\substack{\Sigma\mathbf{i}=n^2\\ \exists s, i_s<0}} \sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=\Sigma\mathbf{b}=r-m,\\ \mathbf{ab}=0,\mathbf{i}+\mathbf{a}-\mathbf{b}\geq 0}} \cdot \mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}+\mathbf{b}-\mathbf{a}}} \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}+\mathbf{a}-\mathbf{b}}}(z_1+\ldots+z_n)^{n^2} \cdot \frac{\mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}}}}{\mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}+\mathbf{b}-\mathbf{a}}}}$$

To estimate the first sum, notice that

$$\frac{\mathrm{coeff}_{\mathbf{z}^{\mathbf{i}+\mathbf{a}-\mathbf{b}}}(z_1+\ldots+z_n)^{n^2}}{\mathrm{coeff}_{\mathbf{z}^{\mathbf{i}}}(z_1+\ldots+z_n)^{n^2}} < n^{2\Sigma\mathbf{b}}$$

Moreover,

$$\sharp\{\mathbf{b}\in\{0,1\}^n \Sigma\mathbf{a}=\Sigma\mathbf{b}=r-m, \mathbf{ab}=0, \mathbf{i}+\mathbf{a}-\mathbf{b}\geq 0\} \leq \binom{n}{\Sigma\mathbf{b}} \cdot \binom{n-\Sigma\mathbf{b}+\Sigma\mathbf{a}}{\Sigma\mathbf{a}} < n^{2\Sigma\mathbf{b}} \tag{133}$$

As for the second sum, we need to estimate the ratio $\frac{\mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}}}}{\mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}+\mathbf{a}-\mathbf{b}}}}$. We devote the next subsection to explain the following

Conjecture 7.14. *Let* $\mathbf{i} \ngeq \mathbf{0}$ *be a multiindex with negative elements, and* $\Sigma\mathbf{i} = n^2$*. Then for* $\mathbf{a},\mathbf{b} \geq \mathbf{0}$, $\Sigma\mathbf{a} = \Sigma\mathbf{b}$, $\mathbf{ab} = 0$ *we have*

$$\frac{\mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}}}}{\mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}+\mathbf{a}-\mathbf{b}}}} < n^{\Sigma\mathbf{b}} \tag{134}$$

Finally, similarly to (133), the number of ways we can get a given multiindex $\mathbf{j} \geq 0, \Sigma\mathbf{j} = n^2$ in the form $\mathbf{i}+\mathbf{a}-\mathbf{b}$ is less than $\binom{n}{\Sigma\mathbf{b}} \cdot \binom{n-\Sigma\mathbf{b}+\Sigma\mathbf{a}}{\Sigma\mathbf{a}} < n^{2\Sigma\mathbf{b}}$

Applying this conjecture in our case results

$$\sum_{\substack{\mathbf{b}\in\{0,1\}^n\\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r,\\ \mathbf{ab}=0}} \theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}}) < n^{4(r-m)} \sum_{\Sigma\mathbf{i}=n^2,\mathbf{i}\geq 0} \mathrm{Tp}_{\mathbf{z}^{-\mathbf{i}}} \mathrm{coeff}_{\mathbf{z}^{\mathbf{i}}}(z_1+\ldots+z_n)^{n^2} = n^{4(r-m)}\theta(1) \tag{135}$$

and Lemma 7.13 is proved. □

Using (124), the same proof gives us

Lemma 7.15.

$$\sum_{\substack{\mathbf{b}\in\{0,1\}^n \\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r, \\ \mathbf{ab}=0}} C_{m,\mathbf{a},\mathbf{b}}\theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}}) < n^{4(r-m)}\theta(1)$$

Next, we substitute Lemma 7.13 and Corollary 7.12 in the expression (131) for the coefficient of d^{n+1-l}. The first term in (131) can be estimated as

$$\sum_{r=0}^{l} \sum_{\substack{\mathbf{b}\in\{0,1\}^n \\ \Sigma\mathbf{a}=\Sigma\mathbf{b}=r,\mathbf{ab}=0}} \sum_{\substack{\mathbf{s}\subset\mathbf{1}-\mathbf{b} \\ \Sigma\mathbf{s}=l-r}} \theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+\mathbf{s}}}}\Lambda_1 < \sum_{r=0}^{l}\binom{n-r}{n-l}(n+2)^l n^{4r}\theta(1), \tag{136}$$

and the second term as

$$\sum_{r=1}^{l+1}\sum_{m=1}^{r} \sum_{\substack{\mathbf{b}\in\{0,1\}^n \\ \Sigma\mathbf{a}=r-m\Sigma\mathbf{b}=r, \\ \mathbf{ab}=0}} \frac{1}{2}\binom{n^2-1}{m-1}(2n^2)^{m-1} \sum_{\substack{\mathbf{s}\subset\mathbf{1}-\mathbf{b} \\ \Sigma\mathbf{s}=l-r}} \theta(\mathbf{z}^{\mathbf{a}-\mathbf{b}})\mathrm{coeff}_{\frac{1}{\mathbf{z}^{\mathbf{a}+\mathbf{s}}}}\Lambda_1 < \tag{137}$$

$$< \sum_{r=1}^{l+1}\sum_{m=1}^{r}\frac{1}{2}\binom{n^2-1}{m-1}(2n^2)^{m-1}\binom{n-r}{n-l}(n+2)^{l-m}n^{4(r-m)}\theta(1)$$

Recall, that the leading coefficient is $\mathrm{coeff}_{d^{n+1}}I_X(n,d) = \frac{1}{2}\theta(1)$. Therefore we arrive at

$$\mathrm{coeff}_{d^{n+1-l}}I_X(d,n) < \left(\sum_{r=0}^{l}\binom{n-r}{n-l}2(n+2)^l n^{4r} + \sum_{r=1}^{l+1}\sum_{m=1}^{r}\binom{n^2-1}{m-1}(2n^2)^{m-1}\binom{n-r}{n-l}(n+2)^{l-m}n^{4(r-m)}\right)\cdot$$
$$\cdot\,\mathrm{coeff}_{d^{n+1}}I_X(n,d) < n^{6l}\cdot\mathrm{coeff}_{d^{n+1}}I_X(n,d)$$

So we have proved the following

Proposition 7.16. *For* $\delta = \frac{1}{n^3(n+1)}$

$$|\mathrm{coeff}_{d^{n+1-l}}I_X(d,n)| < n^{6l}\cdot\mathrm{coeff}_{d^{n+1}}|I_X(n,d)|$$

Theorem 7.4 now follows from Proposition 7.16 and Observation 4.10.

8. On Conjecture 1.2

Finally, we motivate Conjecture 1.2 with some observations.

8.1. **The convergence of** Tp_k**.** The Laurent expansion of $\mathrm{Tp}_k(z_1,\ldots,z_k)$ is convergent when $z_i+z_j<z_l$ for $i+j\le l\le k$. Indeed, in this case

$$\frac{1}{z_i+z_j-z_l} = \frac{-1}{z_l}\left(1+\frac{z_i+z_j}{z_l}+\frac{(z_i+z_j)^2}{z_l^2}\right)$$

is convergent. The coefficients $\mathrm{Tp}_{\mathbf{i}}$ are positive by Rimányi's conjecture, so for $\mathbf{i} = (i_1, \dots, i_k), i_1 + \dots + i_k = 0$ and $1 \le l, m \le k$ the series

$$\sum_{s=0}^{\infty} \mathrm{Tp}_{\mathbf{i}+s(e_l-e_m)} \mathbf{z}^{\mathbf{i}+s(e_l-e_m)}$$

is convergent with the substitution $z_j = j^2$, that is

$$\mathrm{Tp}_{\mathbf{i}} \cdot (1^{i_1} 2^{2i_2} \dots k^{2i_k}) \sum_{s=0}^{\infty} \frac{\mathrm{Tp}_{\mathbf{i}+s(e_l-e_m)}}{\mathrm{Tp}_{\mathbf{i}}} \left(\frac{l}{m}\right)^{2s} < \infty$$

But $\frac{l}{m} \ge \frac{1}{n}$, so

$$\frac{\mathrm{Tp}_{\mathbf{i}+s(e_l-e_m)}}{\mathrm{Tp}_{\mathbf{i}}} < n^{2s}$$

"in average", suggesting the second part of Conjecture 1.2.

8.2. **Checking the known cases $n = m$, $k \le 7$.** In [37] (Theorem 5.1) Rimányi computes the Thom polynomials $\mathrm{Tp}_k^{m-n}(c_1, c_2, \dots)$ in (1) for $m = n, k \le 8$. The list is as follows:

$\mathrm{Tp}_1^0 = c_1$
$\mathrm{Tp}_2^0 = c_1^2 + c_2$
$\mathrm{Tp}_3^0 = c_1^3 + 3c_1c_2 + 2c_3$
$\mathrm{Tp}_4^0 = c_1^4 + 6c_1^2c_2 + 2c_2^2 + 9c_1c_3 + 6c_4$
$\mathrm{Tp}_5^0 = c_1^5 + 10c_1^3c_2 + 25c_1^2c_3 + 10c_1c_2^2 + 38c_1c_4 + 12c_2c_3 + 24c_5$
$\mathrm{Tp}_6^0 = c_1^6 + 15c_1^4c_2 + 55c_1^3c_3 + 30c_1^2c_2^2 + 141c_1^2c_4 + 79c_1c_2c_3 + 5c_2^3 + 202c_1c_5 + 55c_2c_4$ $+17c_3^2 + 120c_6$
$\mathrm{Tp}_7^0 = c_1^7 + 21c_1^5c_2 + 105c_1^4c_3 + 70c_1^3c_2^2 + 399c_1^3c_4 + 301c_1^2c_2c_3 + 35c_1c_2^3 + 960c_1^2c_5+$ $+467c_1c_2c_4 + 139c_1c_3^2 + 58c_2^2c_3 + 1284c_1c_6 + 326c_2c_5 + 154c_3c_4 + 720c_7$
$\mathrm{Tp}_8^0 = c_1^8 + 28c_1^6c_2 + 140c_1^4c_2^2 + 140c_1^2c_2^3 + 14c_2^4 + 182c_1^5c_3 + 868c_1^3c_2c_3 + 501c_1c_2^2c_3+$ $+642c_1^2c_3^2 + 202c_2c_3^2 + 952c_1^4c_4 + 2229c_1^2c_2c_4 + 364c_2^2c_4 + 1559c_1c_3c_4 + 332c_4^2+$ $+3383c_1^3c_5 + 3455c_1c_2c_5 + 954c_3c_5 + 7552c_1^2c_6 + 2314c_2c_6 + 9468c_1c_7 + 5040c_8$

All coefficients are positive in the table, suggesting Rimányi's conjecture. Moreover the residue formula (1) for $m = n$ tells us that for $1 \le i_1 \le i_2 \le \dots \le i_s \le k, i_1 + \dots + i_s = k$

$$\mathrm{coeff}_{c_{i_1}c_{i_2}\dots c_{i_s}} \mathrm{Tp}_k^0 = \sum_{\sigma \in S_{s \to k}} \mathrm{coeff}_{(z_1\dots z_k)^{-1} z_{\sigma(1)}^{-i_1} z_{\sigma(2)}^{-i_2} \dots z_{\sigma(s)}^{-i_s}} \mathrm{Tp}_k(z_1, \dots, z_k), \tag{138}$$

where the right hand side is the sum of the coefficients in the Laurent expansion on the contour $z_1 \ll \dots \ll z_k$. Here $S_{s\to k}$ is the set of injective maps $\{1, \dots, s\} \to \{1, \dots, k\}$. In particular, using the notation of Conjecture 1.2:

$$\mathrm{coeff}_{c_1^k} \mathrm{Tp}_k^0 = \mathrm{Tp}_{(0,\dots,0)} = \theta(0); \quad \mathrm{coeff}_{c_1^{k-2}c_2} \mathrm{Tp}_k^0 = \sum_{1 \le a < b \le k} \mathrm{Tp}_{e_a - e_b}$$

$$\text{coeff}_{c_1^{k-3}c_3}\text{Tp}_k^0 = \sum_{1\le a,b,c\le k, b<c} \text{Tp}_{2e_a-e_b-e_c}; \quad \text{coeff}_{c_1^{k-4}c_2^2}\text{Tp}_k^0 = \sum_{1\le a,b,c,d\le k, b<c} \text{Tp}_{e_a+e_b-e_c-e_d};$$

etc. So some of the quotients in Conjecture 1.2 can be detected using the table above. For example,

$$\frac{\text{Tp}_{e_a-a_b}}{\text{Tp}_{\mathbf{0}}} < \frac{\text{coeff}_{c_1^{k-2}c_2}\text{Tp}_k^0}{\text{coeff}_{c_1^k}\text{Tp}_k^0},$$

and from the table we see that these are $\le k^2$ for $k \le 8$. In general

$$\frac{\text{Tp}_{\mathbf{i}}}{\text{Tp}_{\mathbf{0}}} < \frac{\text{coeff}_{c_{i_1+1}c_{i_2+1}\dots c_{i_k+1}}\text{Tp}_k^0}{\text{coeff}_{c_1^k}\text{Tp}_k^0},$$

which is, again, less then $k^{2\sharp\{s:i_s>0\}}$ in the listed cases.

8.3. **Checking $k = 3$ for any value of $m - n$.** Since $Q_3(z_1, z_2, z_3) = 1$, the Thom series for $k = 3$ is given as

(139)

$$\text{Tp}_3(z_1,z_2,z_3) = \frac{(z_2-z_1)(z_3-z_2)(z_3-z_1)}{(z_2-2z_1)(z_3-z_2-z_1)(z_3-2z_1)} = \frac{z_2-z_1}{z_2-2z_1}\cdot\frac{z_3-z_2}{z_3-z_2-z_1}\cdot\frac{z_3-z_1}{z_3-2z_1} =$$

$$\frac{z_1}{z_2}\left(1+\frac{2z_1}{z_2}+\frac{(2z_1)^2}{z_2^2}+\dots\right)\cdot\frac{z_1}{z_3}\left(1+\frac{z_1+z_2}{z_3}+\frac{(z_1+z_2)^2}{z_3^2}+\dots\right)\cdot\frac{z_1}{z_3}\left(1+\frac{2z_1}{z_3}+\frac{(2z_1)^2}{z_3^2}+\dots\right)$$

We leave as an exercise to the reader to show that in this case, indeed, $\frac{\text{Tp}_{\mathbf{i}}}{\text{Tp}_{\mathbf{i}+s(e_l-e_m)}} < 3^{2s}$ holds.

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