Homotopy normal maps

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Abstract

Normal maps between discrete groups $N \to G$ were characterized[FS] as those which induce a compatible topological group structure on $G//N \equiv EN \times_N G$. Here we deal with topological group maps $N \to G$ being normal in the same sense as above and hence forming a homotopical analogue to the inclusion of a normal subgroup in a reasonable way.

We characterize these maps by a compatible simplicial loop space structure on $Bar_\bullet(N, G)$, invariant under homotopy monoidal functors, e.g. Localizations and Completions. In the course of characterizing homotopy normality, we define a notion of a *homotopy action* similar to an A_∞ action on a space(as in [Now]), but phrased in terms of Segal's 'special $\Delta-$spaces'.

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1.1 Introduction

Following a usual path, a group property made homotopical is a property of the corresponding classifying space. An inclusion of topological groups $N \hookrightarrow G$ is the inclusion of a normal subgroup iff it is the kernel inclusion of some group map $G \to H$. Since any map is up to homotopy an inclusion, one need to consider all group maps $N \to G$. Such a map should then be 'homotopy normal' if $BN \to BG$ is the inclusion of the homotopy fiber for some map $BG \to W$. There is another angle from which this notion makes sense. To every group map $N \to G$, one can associate the Borel construction $EN \times_N G \equiv G//N$ which is the 'correct' quotient in the homotopical world. We note that such an extension $BG \dashrightarrow W$ induces a loop space structure $G//N$ and a loop map structure(up to map equivalence) on $G \to G//N$, providing a second analogy to the group theoretic notion: a group inclusion $N \hookrightarrow G$ is the inclusion of a normal subgroup iff G/N is a group and the natural quotient map $G \to G/N$ is a group map w.r.t this group structure.

Note also that $G//N \simeq hfib(BN \to BG)$ making it simultaneously a homotopy limit and a homotopy colimit. Let $X \xrightarrow{f} Y$ be a pointed map of connected spaces. Consider the Nomura sequence[Nom]

$$\Omega X \to \Omega Y \to \Omega Y//\Omega X \to X \to Y$$

where we denote $\Omega Y//\Omega X \equiv hfib(f)$.
The following is essentially taken from ([FS], §5).

Definition 1.1. A loop map $\Omega X \overset{\Omega f}{\to} \Omega Y$ is *homotopy normal* if there exist a connected space W with a map $Y \xrightarrow{\pi} W$ so that $X \xrightarrow{f} Y \xrightarrow{\pi} W$ is a homotopy fibration. The map $Y \overset{\pi}{\to} W$ is called a *normal structure.*

Remarks 1.2. (a) We see that a loop map $\Omega X \overset{\Omega f}{\to} \Omega Y$ is homotopy normal if and only if $X \overset{f}{\to} Y$ is a homotopy principal fibration i.e. equivalent to a principal fibration.

(b) When ΩX, ΩY are homotopicaly discrete, homotopy normality of Ωf is the same as being part of a crossed module structure on the corresponding groups. Whitehead showed (see. [WH])that crossed modules correspond to connected 2-types and we note that in case $\Omega X \xrightarrow{\Omega f} \Omega Y$ is homotopy normal with $Y \to W$ its normal structure W is the corresponding connected 2-type.

Example 1. *If $F \to E \to B$ is a fibration sequence, the map $\pi_1 F \to \pi_1 E$ is a homotopy normal map of discrete groups. It is also true that any homotopy normal map of discrete groups is of this form.*

Example 2. *Any double loop map $\Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y$ is homotopy normal. This is because $\Omega X \overset{\Omega\varphi}{\to} \Omega Y$ is a principal fibration. We see that homotopy normality is stronger than 1-fold loop maps but weaker than 2-fold loop maps.*

1.2 Main results

Given a group map $N \to G$, each level of the Bar Construction $Bar_\bullet(N, G) = \{N^k \times G\}_{k \geq 0}$ admits an action of G namely the one induced from the group inclusions $s_0 : G \to N \times G$, $s_1 s_0 : G \to N^2 \times G$ etc. More generally, in any simplicial group $\Gamma_\bullet$, Γ_0 acts on each level via degeneracies and endows a structure of $\Gamma_0 - \Delta-$space on $\Gamma_\bullet$.
The following is the main theorem in ([FS], §4), rephrased.

Theorem 1.3. *A map of discrete groups $N \xrightarrow{f} G$ is homotopy normal iff there exists a simplicial group $\Gamma_\bullet$ with an isomorphism $\Gamma_0 \cong G$ which extends to a G-equivariant isomorphism of simplicial sets*

$$Bar_\bullet(N, G) \to \Gamma_\bullet$$

.

Thus, the Bar Construction is a convenient resolution for $G//N$ while treating homotopy normal maps.

In §3 we define a homotopical analogue to the Bar Construction in the case of loop maps $\Omega X \to \Omega Y$, $Bar_\bullet(\Omega X, \Omega Y)$. In the degenerate case of $\Omega Y \simeq *$,

$Bar_\bullet(\Omega X, *) \equiv Bar_\bullet(\Omega X)$ is a "special Δ-space" for ΩX.
The first step in characterizing homotopy normality is the assertion that for homotopy normal map $\Omega X \to \Omega Y$, $Bar_\bullet(\Omega X, \Omega Y)$ is equivalent to a simplicial loop space $\Gamma_\bullet$ where the later means a pointed $\Delta-$space $X_\bullet$ composed with the loop functor Ω. The second step in the characterization is the following. For every $X \xrightarrow{f} Y$, the homotopy class of Ωf is determined by the homotopy principal fibration sequence

$$(1) \quad \Omega Y \to \Omega Y//\Omega X \to X$$

which in turn can be resolved by a homotopy fibration sequence of $\Delta-$spaces

$$(2) \quad \Omega Y \to Bar_\bullet(\Omega X, \Omega Y) \to B_\bullet(\Omega Y)$$

The resolution is level-wise trivial $\Omega Y \to (\Omega X)^n \times \Omega Y \to (\Omega X)^n$ which means that the group action related to (1) can be encoded via products and homotopy equivalences.
The sequence (2), induced by the loop map $\Omega X \xrightarrow{\Omega f} \Omega Y$, is called a *homotopy action* of ΩY on ΩX. Since the maps $s_n ... s_1 s_0$ $(n \geq 0)$ in $Bar_\bullet(\Omega X, \Omega Y)$ are loop maps, they induce a homotopy action of ΩY on $Bar_n(\Omega X, \Omega Y)$ and thus define a homotopy action of ΩY on $Bar_\bullet(\Omega X, \Omega Y)$[The precise notion is defined in §5]]. If we require that the homotopy action of ΩY on $Bar_\bullet(\Omega X, \Omega Y)$ is equivalent to that of Γ_0 on $\Gamma_\bullet$ we get a complete characterization of homotopy normality. More precisely:

Theorem 1.4. *Let $\Omega X \xrightarrow{\Omega f} \Omega Y$ be a loop map. Then Ωf is homotopy normal iff there exist a simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 \simeq \Omega Y$ and such that the canonical homotopy actions(defined in* §6*) of ΩY on $\Gamma_\bullet$ and $Bar_\bullet(\Omega X, \Omega Y)$ are naturally equivalent.*

This is proved in §6. We devote §5 to define the meaning of a homotopy action of a loop space on a space, study its basic properties and characterize it 'in terms of products' i.e. invariant under *homotopy monoidal* functors, where

Definition 1.5. A functor $L : Top \to Top$ is called a *homotopy monoidal* (HM) functor if it preserves homotopy equivalences and for every $X, Y \in Top$, $L(X \times Y) \simeq LX \times LY$ by the canonical map.

Let L be a HM functor and $\Omega f : \Omega X \to \Omega Y$ a loop map. It is implicit in the works of [Bo] and [Far](using the delooping theorem of [Seg]) that $L(\Omega X)$ is always of the homotopy type of a loop space and $L(\Omega f)$ is always equivalent to a loop map. In addition theorem 1.4 implies

Corollary 1.6. *If in the above Ωf is homotopy normal, then $L(\Omega f) : L(\Omega X) \to L(\Omega Y)$ is a homotopy normal map.*

This in turn gives a simple proof of a theorem due to Dwyer and Farjoun ([DF], §3), namely:

Theorem 1.7. *Let $X \xrightarrow{f} Y$ be a map of pointed connected spaces. If $E \xrightarrow{p} B$ is a homotopy principle fibration of connected spaces. If $L_{\Sigma f}$ is the localization functor by $\Sigma f : \Sigma X \to \Sigma Y$ then $L_{\Sigma f} E \xrightarrow{L_{\Sigma f}(p)} L_{\Sigma f} B$ is a homotopy principle fibration.*

Remark 1.8. In what follows, we use L to denote an arbitrary HM functor. Thus L reflects the case of localization by a map.

To sum up, the main thrust of the present paper is to construct a Segal-like recognition principle, made of products of the spaces involved that is easily seen to be invariant under homotopy monoidal functors.

2 Preliminaries

Throughout this paper, *topological spaces* or simply *spaces* will mean topological spaces of the homotopy type of CW complexes. We denote the corresponding category by Top or $\{spaces\}$.
A *loop space* is understood to be a space ΩX where X is a pointed connected space. We call two maps $X \to Y$, $Z \to W$ *equivalent* if there exist a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

with vertical arrows being homotopy equivalences.
A map $E \to B$ is a *fibration* if it has the usual homotopy lifting property. A sequence of the form $F \to E \xrightarrow{p} B$, where $E \xrightarrow{p} B$ is a fibration, $F = p^{-1}(b_0)$ and either (B, b_0) is a pointed space or $b_0 \in B$ and B is connected is called a *fibration sequence*. A sequence $X \to Y \to Z$ is called a *homotopy fibration sequence* if there is a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with vertical arrows being homotopy equivalences and the top being a fibration sequence. Similarly, a sequence $X \to Y \to Z$ is called a *homotopy principal fibration sequence* if there is a principal fibration sequence $G \to E \to E/G$ and a cummutative diagram

$$\begin{array}{ccccc} G & \longrightarrow & E & \longrightarrow & E/G \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

As usual we denote by Δ the category of finite non-empty ordered sets $[n] = (0, ..., n)$ with all non-decreasing maps between them.
Given a category $\mathcal{C}$, we denote a simplicial object in $\mathcal{C}$ by $X_\bullet$ and write X_n for its value on $[n]$
Of special importance to this note are simplicial objects in $\{Spaces\}$, namely *simplicial spaces*. We abbreviate these by *'$\Delta-$spaces'* and also use the term *'$\Delta-$maps'* for simplicial maps. If X is a space, we shall denote the constant $\Delta-$space on it by X when there is no place for confusion. An *equivalence of $\Delta-$spaces* $X_\bullet, Y_\bullet$ is a $\Delta-$map $X_\bullet \xrightarrow{f} Y_\bullet$ such that $X_n \xrightarrow{f_n} Y_n$ is a homotopy equivalence for each n. Similarly, a *(resp. homotopy)fibration sequence of $\Delta-$spaces* is a diagram of $\Delta-$spaces $F_\bullet \to E_\bullet \to B_\bullet$ which is level-wise (resp. homotopy)fibration sequence.
We will often use a particular class of simplicial spaces introduced in [Seg](we omit the word 'group-like' since we don't deal with other cases)

Definition 2.1. A *special $\Delta-$spaces* is a Δ-space $A_\bullet$ such that

1. $A_0 \simeq *$
2. For each $n \geq 1$, the maps $p_n : A_n \to A_1 \times \cdots \times A_1 (n - times)$ induced by the maps
$$i_k : [1] \to [n] \;\; 0 \mapsto k-1 \;\; 1 \mapsto k$$
are homotopy equivalences.
3. $\pi_0(A_1)$ is a group.

We say that $A_\bullet$ is a special Δ space for X if $X \simeq A_1$.

If a group G acts on a space X we denote by $Bar_\bullet(G, X)$ the Bar Construction with X being a right G-space([May], §7). In the special case where $N \xrightarrow{f} G$ is a group map, we denote by $Bar_\bullet(f) \equiv Bar_\bullet(N, G)$ the Bar Construction for the action $n \cdot g := f(n)g$ induced by f.

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3 The Homotopy Power of a Map

Given a fibration $E \xrightarrow{p} B$, one can define a $\Delta-$space called the *power of p* $Pow_\bullet(E, B)$ by $Pow_n(E, B) = E \times_B E \cdots \times_B E$ $(n + 1$ *times*$)$ with face and degeneracies being the obvious projections and diagonals. In [Lod], it is shown that for(E non-empty and) B connected, $|Pow_\bullet(E, B)| \simeq B$. We note that for a general space B, $|Pow_\bullet(E, B)|$ is homotopy equivalent to the connected components of B of the image of p.

Here, we wish to construct such a space for an arbitrary map $X \xrightarrow{f} B$ by means of homotopy pullbacks, thus turning it to a homotopically invariant construction.

We define the *n-th homotopy power* of $X \to B$ to be

$$Powh_n(X,B) = \; map\left(\begin{array}{c} \Delta[n]_0 \\ \iota\downarrow \\ \Delta[n] \end{array} , \begin{array}{c} X \\ \downarrow f \\ B \end{array} \right) = \text{holim} \left(\begin{array}{ccc} X \; X \cdots X \\ \searrow \; \searrow \; \swarrow \\ B \end{array} \right)$$

with $\Delta[n]_0 \stackrel{\iota}{\to} \Delta[n]$ being the inclusion of the 0−skeleton to the topological $n-$simplex.

This clearly yields a functorial construction over Δ^{op} and we define

Definition 3.1. Given a map $X \to B$, its *homotopy power*, denoted $Powh_\bullet(X,B)$, is the $\Delta-$space with $Powh_n(X,B)$ on level n and face and degeneracies given by the above functorial construction.

Note that for a fibration $E \stackrel{p}{\to} B$ one gets equivalence of $\Delta-$spaces $Powh_\bullet(E,B) \simeq Pow_\bullet(E,B)$.

We shall use homotopy powers to replace 'rigid' constructions.
Consider a topological group G acting on a space X and the corresponding (homotopy)principal fibration $G \to X \to X//G$. One has the 'usual' bar construction $Bar_\bullet(G,X) = \{G^k \times X\}_{k\geq 0}$ and $|Bar_\bullet(G,X)| = X//G$. On the other hand, we can resolve $X//G$ by taking homotopy powers of the map $X \stackrel{q}{\to} X//G$.

Proposition 3.2. Let G act on X as above. Then there are simplicial equivalences $Bar_\bullet(G,X) \rightleftarrows Powh_\bullet(X,X//G)$

Proof. Replacing $X \stackrel{q}{\to} EG\times_G X$ by the fibration $EG\times X \stackrel{p}{\to} EG\times_G X$ and taking pullback we get $Powh_1(X,X//G) = (EG\times X)\times_{X//G}(EG\times X) \cong EG\times G\times X$ since $EG\times X$ is a free $G-$space. In general,

$$Powh_n(X,X//G) = (EG\times X)\times_{X//G}\cdots(EG\times X) \cong EG\times G^n\times X$$

and the obvious map $EG\times G^n\times X \to G^n\times X$ defines a simplicial equivalence $Powh_\bullet(X,X//G) \to Bar_\bullet(G,X)$. Taking (for example) Milnor's join construction for EG we have a natural base point for EG and hence a canonical map $G^n\times X \to EG\times G^n\times X$ which in turn defines another simplicial equivalence. □

In light of the last proposition, we define

Definition 3.3. Given a (homotopy) principal fibration sequence, $\Omega Y \to X \stackrel{q}{\to} Q$ the *homotopy bar construction*, $Bar_\bullet(\Omega Y,X)$ is the homotopy power $Powh_\bullet(X,Q)$.

Remark 3.4. In the case of a loop map $\Omega f : \Omega Y \to \Omega Z$, $Bar_\bullet(\Omega Y,\Omega Z)$ is the homotopy power of the map $\Omega Z \stackrel{q}{\to} \Omega Z//\Omega Y \equiv hfib(f)$. If $\Omega Z =\simeq *$, $Bar_\bullet(\Omega Y,*)$ becomes the power of the map $PY \to Y$ which a is special $\Delta-$space for ΩY. Put differently, one can recover Segal's loop machine by using homotopy powers.

It is also useful to have the following property

Proposition 3.5. Let $X \xrightarrow{f} B$ be any pointed map and denote by $\Omega(Powh_\bullet(X,B))$ the composition of $\Omega : \{Spaces\}_* \to \{Spaces\}$ and $Powh_\bullet(X,B)$. Then the canonical map induces an equivalence of $\Delta-$spaces $\Omega(Powh_\bullet(X,B)) \simeq Powh_\bullet(\Omega X, \Omega B)$

The proof is based on the fact that given a pointed diagram, $A \to X \leftarrow Y$, $\Omega holim(A \to X \leftarrow Y) \simeq holim(\Omega A \to \Omega X \leftarrow \Omega Y)$.

4 From homotopy normality to a simplicial loop space structure on the (homotopy)Bar construction

Let $\Omega X \xrightarrow{\Omega f} \Omega Y$ be a homotopy normal map. We form the Nomura sequence:

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{q} \Omega Y//\Omega X \longrightarrow X \longrightarrow Y \xrightarrow{\pi} W$$

Then by [Nom] there is a commutative triangle in which the vertical arrow is a homotopy equivalence

$$\begin{array}{ccc} \Omega Y & \xrightarrow{q} & \Omega Y//\Omega X \\ & \searrow^{\Omega\pi} & \downarrow \\ & & \Omega W \end{array}$$

Passing to (homotopy)powers, we get an equivalence of $\Delta-$spaces $Powh_\bullet(\Omega Y, \Omega W) \simeq Powh_\bullet(\Omega Y, \Omega Y//\Omega X)$ and by proposition 3.5 an equivalence of $\Delta-$spaces $\Omega(Powh_\bullet(Y,W)) \simeq Powh_\bullet(\Omega Y, \Omega Y//\Omega X)$. More precisely, we have just shown

Theorem 4.1. *Let* $\Omega X \xrightarrow{\Omega f} \Omega Y$ *be homotopy normal. Then there are natural simplicial equivalences* $Bar_\bullet(\Omega X, \Omega Y) \rightleftarrows \Omega(Powh_\bullet(Y,W))$

Notation 4.2. 1. For a homotopy normal map $\Omega f : \Omega X \to \Omega Y$ and a given normal structure $Y \xrightarrow{\pi} W$, we denote by $\Gamma_\bullet$ the simplicial loop space $\Omega(Powh_\bullet(Y,W))$.

2. The equivalences given in theorem 4.1 will be denoted as

$$\epsilon : Bar_\bullet(\Omega X, \Omega Y) \rightleftarrows \Gamma_\bullet : \eta$$

Using the machinery of special $\Delta-$spaces one can easily see that applying an HM functor on a simplicial loop space in every level yields a $\Delta-$space *simplicially equivalent* to a simplicial loop space.

5 Homotopy action

By remark 1 in 1.2 a homotopy normal map is a loop map with its underlying map being a principal fibration. Hence, invariance of homotopy normal maps under HM functors, should include invariance of group actions to some extent. Given an action of a topological group G on a space X and an HM functor $L : Top \to Top$, one would like to construct an 'action' of LG (not a group, not a loop space) on LX. In other words, we would like to have a homotopical notion of an action of (a space of the homotopy type of) a loop space on a space, invariant under HM functors. One approach we wish to refer the reader to is that of A_∞ actions[Now]. The main difference between the two approaches is that the one we develop here is based on Segal's work while the one in [Now] is based on Stasheff's work.

In [DFK] Dwyer, Farjoun and Kan proved the following:

Theorem 5.1. *Let G be a topological group. Denote by Top_{BG} the category of fibrations over BG with maps being the usual commuting triangles.*
Call a map in Top_{Bg} a homotopy equivalence if its underlying map in Top is a homotopy equivalence. Denote by Top^G the category of $G-$ spaces and equivariant maps. Call a map $X \to X'$ in Top^G a homotopy equivalence if the corresponding map in Top is a homotopy equivalence.
Then there is an equivalence of categories:

$$Ho(Top_{BG}) \simeq Ho(Top^G)$$

Remark 5.2. The equivalence of the homotopy categories comes form a Quillen equivalence of appropriate model structures. See [Hol] for a general treatment.

The above theorem treats the homotopy category of $G-spaces$ so in a way, actions up to homotopy. Another treatment can be found in [Cooke].

5.1 Construction

If a topological group G acts on a space X, one has a simplicial fibration sequence of the form $X \to Bar_\bullet(G, X) \to B_\bullet G$ where the maps $X \to Bar_n(G, X)$ and $Bar_n(G, X) \to B_n G$ are given by $s_n...s_0$ and projection respectively.

Under realization this becomes a (homotopy)fibration sequence $X \to X//G \to BG$ with a connected base space i.e. an 'action up to homotopy' in the sense of [DFK]. The above simplicial fibration sequence is trivial in each level $X \to G^n \times X \to G^n$ and hence constitutes a useful resolution. We note also that for all n the map $d_1 d_2 ... d_n : Bar(G, X)_n \to Bar(G, X)_0$ is the projection on X. As we saw, the $\Delta-$spaces $Bar_\bullet(G, X)$, $B_\bullet G$ can be relaxed to their 'homotopy versions' namely $Bar_\bullet(\Omega Y, X)$ and $Bar_\bullet(\Omega Y, *)$ (which is a special Δ-space for ΩY when $BG \simeq Y$).

Definition 5.3. We say that a space S of the homotopy type of a loop space *homotopy acts* on a space X if there exist a $\Delta-$map

$$A_\bullet \xrightarrow{\pi} B_\bullet$$

such that:

1. $A_0 \simeq X$
2. $B_\bullet$ is a special Δ-space for S.
3. For every n, the map $A_n \xrightarrow{\pi_n \times d_1 \ldots d_n} B_n \times A_0$ is a homotopy equivalence.

Definition 5.4. Given a homotopy action as above, the map $d_0 : A_1 \to A_0$ will be called the *homotopy action map*. The terminology should be understood from theorem 5.12.

Remark 5.5. If S and S' are of the homotopy type of ΩY and S homotopy acts on X then S' homotopy acts on X since a special Δ-space for S is also a special Δ-space for S'(see definition 2.1).

We will need a generalization of the last definition as follows

Definition 5.6. A homotopy action of ΩY on a $\Delta-$space $X_\bullet$ is a map of bisimplicial spaces $A_{\bullet\bullet} \to B_{\bullet\bullet}$ such that for each n, $A_{\bullet n} \to B_{\bullet n}$ is a homotopy action of ΩY on X_n.

Maps and equivalences are defined in the usual way

Definition 5.7.

1. Given two homotopy actions of ΩY on X and of $\Omega(Y')$ on X', represented by $A_\bullet \longrightarrow B_\bullet$ and $A'_\bullet \longrightarrow B'_\bullet$ a map between them is a commutative square of simplicial spaces

$$\begin{array}{ccc} A_\bullet & \longrightarrow & B_\bullet \\ \downarrow & & \downarrow \\ A'_\bullet & \longrightarrow & B'_\bullet \end{array}$$

   Such a map will be called an *equivalence*, if both of the vertical maps are simplicial equivalences.

2. Given two homotopy actions of ΩY on $X_\bullet$ and of $\Omega(Y')$ on $X'_\bullet$, represented by $A_{\bullet\bullet} \longrightarrow B_{\bullet\bullet}$ and $A'_{\bullet\bullet} \longrightarrow B'_{\bullet\bullet}$ a map between them is a commutative square of bisimplicial spaces

$$\begin{array}{ccc} A_{\bullet\bullet} & \longrightarrow & B_{\bullet\bullet} \\ \downarrow & & \downarrow \\ A'_{\bullet\bullet} & \longrightarrow & B'_{\bullet\bullet} \end{array}$$

   Such a map will be called an *equivalence* if there is a square of bisimplicial spaces with vertical arrows being bisimplicial equivalences.

It is commonly said that in every fibration sequence, the loop space of the base 'acts' on the fiber. We wish to demonstrate how a homotopy action interprets this case.

Theorem 5.8. *Given a fibration sequence* $F \stackrel{i}{\to} E \stackrel{p}{\to} B$ *with* B *pointed connected, there is a homotopy action of* ΩB *on* F*, represented by* $A_\bullet \stackrel{\pi}{\to} B_\bullet$ *such that the map* $|\pi| : |A_\bullet| \to |B_\bullet|$ *is equivalent to* $p : E \to B$.

Proof. Consider the commutative square

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \longrightarrow & B \end{array}$$

Taking homotopy powers in each row produces a simplicial map

$$\pi : A_\bullet \equiv Powh_\bullet(F \to E) \to Powh_\bullet(* \to B) \equiv B_\bullet$$

By remark 3.4, $B_\bullet$ is a special $\Delta-$space and thus $|B_\bullet| \simeq B$. Since B is connected, it follows from § 3.1, that $|A_\bullet| \simeq E$. To see that $\pi : A_\bullet \to B_\bullet$ is a homotopy action, we first replace $i : F \to E$ and $* \to B$ by equivalent fibrations $ev_1 : F_i \to E$ and $ev_1 : PB \to B$ where PB is the path space and $F_i \subseteq F \times B^I$ is the space $\{(f,\alpha)|\alpha(0) = i(f)\}$. Taking $\pi_1 : F_i \to PB$ to be $\pi_1(f,\alpha) = p \circ \alpha$ we obtain the commutative square

$$\begin{array}{ccc} F_i & \xrightarrow{ev_1} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ PB & \xrightarrow{ev_1} & B \end{array} \qquad (*)$$

and taking powers(i.e. fiber products) of the rows we obtain a simplicial map which by abuse of notation we denote as $\pi : A_\bullet \to B_\bullet$. Let us show that $\pi_1 \times d_1 : A_1 \to B_1 \times A_0$ is a homotopy equivalence. We have $B_1 = \{(\beta, \beta')|\beta, \beta' : I \to B, \beta(0) = * = \beta'(0) \; and \; \beta(1) = \beta'(1)\} \simeq \Omega B$ and $A_1 = \{(f, \alpha, f', \alpha')|\alpha(0) = i(f), \alpha'(0) = i(f') \; and \; \alpha(1) = \alpha'(1)\}$. Thus, the map $\pi_1 \times d_1$ sends (f, α, f', α') to $(p \circ \alpha, p \circ \alpha', f, \alpha)$ and since $p : E \to B$ is a fibration, one can define $w : B_1 \times A_0 \to A_1$ being a homotopy inverse to $\pi_1 \times d_1$ by applying the path lifting property on the path $\beta * \beta'$ and the point $i(f)$ coming from an arbitrary triple $(\beta, \beta', \alpha, f) \in B_1 \times A_0$. The fact that in general $\pi_n \times d_1 ... d_n : A_n \to B_n \times A_0$ follows by a similar argument. Thus, $\pi : A_\bullet \to B_\bullet$ is a homotopy action.
Lastly since the equivalences $|Pow_\bullet(F_i \to E)| \simeq E$ and $|Pow_\bullet(PB \to B)| \simeq B$ are natural and in view of $(*)$ the map $|\pi| : |A_\bullet| \to |B_\bullet|$ is equivalent to $p : E \to B$. □

5.2 Connection with group actions and invariance

We start with the following well-known fact, which is essentially contained in [Kan] and [Mil].

Theorem 5.9. *Let X be a pointed connected space. Then there is a topological group G, with $X \overset{\simeq}{\to} BG$. Moreover, one can construct G functorially in X, i.e. if $\Omega X \overset{\Omega f}{\to} \Omega Y$ is a loop map, there is a commutative diagram*

$$\begin{array}{ccc} \Omega X & \longrightarrow & \Omega Y \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

with vertical arrows homotopy equivalences, and the bottom arrow being a topological group map.

To get a sense of where homotopy actions arise in our case, we start with a simple

Lemma 5.10. *If $\Omega X \xrightarrow{\Omega f} \Omega Y$ is a loop map, then there is a canonical homotopy action of ΩX on ΩY, natural in f.*

Proof. This follows from theorem 5.8 if we consider the homotopy fibration sequence $\Omega Y \to \Omega Y//\Omega X \to X$. Alternatively, if we (functorially) rigidify $\Omega X \xrightarrow{\Omega f} \Omega Y$ to a topological group map $G \to H$ as in 5.9, then as we saw, $Bar_\bullet(G, H) \to B_\bullet G$ is a homotopy action.

□

The following theorem establishes the precise way in which homotopy actions and group actions interact. We need

Definition 5.11. Maps $X \overset{f}{\to} Y$ and $X' \overset{f'}{\to} Y'$ are called *weakly equivalent* if there is a zig-zag of commutative squares with all horizontal arrows being homotopy equivalences

$$\begin{array}{ccccccccc} X & \xrightarrow{\simeq} & X_1 & \xleftarrow{\simeq} & \cdots & \xrightarrow{\simeq} & X_n & \xleftarrow{\simeq} & X' \\ {\scriptstyle f}\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\simeq} & Y_1 & \xleftarrow{\simeq} & \cdots & \xrightarrow{\simeq} & Y_n & \xleftarrow{\simeq} & Y' \end{array}$$

The number of squares involved in such a zig-zag is said to be its *length*. In particular, maps are called *equivalent* if they are weakly equivalent via a zig-zag of length 1.

Theorem 5.12. *Let ΩY be a loop space and G a topological group with $Y \simeq BG$.*

1. *If G acts strictly on X, then G homotopy acts on X with homotopy action map being equal to the action map $G \times X \to X$.*

   In particular, $|A_\bullet| \simeq X//G$

2. *If ΩY homotopy acts on X' then there exist a space $X \simeq X'$ and a strict action of G on X such that*

$$\begin{array}{ccccc} A_0 & \longrightarrow & |A_\bullet| & \longrightarrow & |B_\bullet| \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ X & \longrightarrow & X//G & \longrightarrow & BG \end{array}$$

commutes and the map $A_1 \stackrel{d_0}{\to} A_0$ is weakly equivalent to the action map $G \times X \stackrel{a}{\to} X$.

Proof. 1. A group action give rise to a simplicial map $Bar_\bullet(G, X) \stackrel{\pi}{\to} B_\bullet G$ (defined for each n by the projection $G^n \times X \to G^n$). To see that π is indeed a homotopy action, note first that $B_\bullet G$ is a special $\Delta-$space(with Segal maps being the identity $1_{G^n} : G^n \to G^n$). As we discussed in 5.1, for each n, the map $d_1 \cdots d_n : Bar_n(G, X) \to Bar_0(G, X)$ is the projection on X and hence the map $\pi_n \times (d_1 \cdots d_n) : Bar_n(G, X) \to B_n \times Bar_0(G, X)$ is the identity as well. We see also that the map d_0 equals to the action map.

2. Let $A_\bullet \stackrel{\pi}{\to} B_\bullet$ be a homotopy action of ΩY on X'. Define a $\Delta-$map $A_0 \stackrel{i}{\to} A_\bullet$ by $i_n = s_{n-1}...s_0$. Choose $b_0 \in B_0$ or and endow B_n with a base point $s_{n-1}...s_0(b_0)$. By definition, the map $\pi_n \times (d_1 \cdots d_n)A_n \to B_n \times A_0$ is a homotopy equivalence and hence the map $A_n \stackrel{\pi_n}{\to} B_n$ is equivalent to the trivial fibration $B_n \times A_0 \to B_n$. We now claim that $A_0 \stackrel{i_n}{\to} A_n \stackrel{\pi_n}{\to} B_n$ is a homotopy fibration sequence. To see this, note that by simplicial identities, the composition $A_0 \stackrel{\pi \circ i_n}{\to} B_n \times A_0$ equals $(\pi_n \circ i_n) \times 1_{A_0}$ and since B_0 is contractible $\pi_n \circ i_n = s_{n-1}...s_0 \circ \pi_0$ is null-homotopic. Hence, i_n is equivalent to the inclusion of the fiber $A_0 \to B_n \times A_0$.

   We thus have a homotopy fibration sequence of simplicial spaces $A_0 \to A_\bullet \to B_\bullet$ and by [Puppe] we have a homotopy fibration sequence upon realization

$$A_0 \to |A_\bullet| \stackrel{\sigma}{\to} |B_\bullet| \quad (1)$$

   . By [DFK] there exist a topological group G and a space X such that $|B_\bullet| \simeq BG$, G acts freely on X, say by $G \times X \stackrel{a}{\to} X$ and there exist an equivalence of homotopy fibration sequences

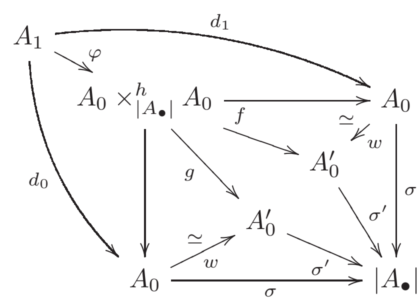
$$\begin{array}{ccccc} A_0 & \longrightarrow & |A_\bullet| & \longrightarrow & |B_\bullet| \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ X & \longrightarrow & X/G & \longrightarrow & BG \end{array}$$

   .

   From the homotopy commutative square

$$
\begin{array}{ccc}
A_1 & \xrightarrow{d_0} & A_0 \\
{\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle \sigma} \\
A_0 & \xrightarrow{\sigma} & |A_\bullet|
\end{array} \qquad (2)
$$

. one derives the diagram



(3)

In which the (not unique)map $A_1 \stackrel{\varphi}{\to} A_0 \times^h_{|A_\bullet|} A_0$ comes from the defining property of homotopy pullbacks and $A_0 \to A'_0 \stackrel{\sigma'}{\to} |A_\bullet|$ a factorization of σ into a homotopy equivalence followed by a fibration. Thus, the inner rhombus is a strict pullback(and hence commutative) the two triangles representing the factorization of σ are commutative as well but all other triangles and squares are homotopy commutative. Precomposing f and g with φ produces a commutative square

$$
\begin{array}{ccc}
A_1 & \longrightarrow & A'_0 \\
\downarrow & & \downarrow \\
A'_0 & \longrightarrow & |A_\bullet|
\end{array} \qquad (4)
$$

It follows from (1) that $hfib(\sigma) = \Omega|B_1|$ and since $A_\bullet \to B_\bullet$ is a homotopy action, $hfib(d_1) = B_1$. Since $B_\bullet$ is a special Δ-space, the map $B_1 \to \Omega|B_\bullet|$ is a homotopy equivalence. Now, $f \circ \varphi \simeq w \circ d_1$ which imply $hfib(f \circ \varphi) \simeq B_1$ and we conclude that (4) is homotopy cartesian. Thus, the map $A_1 \stackrel{\varphi}{\to} A_0 \times^h_{|A_\bullet|} A_0$ is a homotopy equivalence. The (homotopy) pullback square

$$
\begin{array}{ccc}
G \times X & \xrightarrow{a} & X \\
{\scriptstyle pr}\downarrow & & \downarrow \\
X & \longrightarrow & X/G
\end{array}
$$

is equivalent via a commutative cube to the rhombus of diagram (3) and therefore there is a homotopy commutative square with vertical arrows being homotopy equivalences

$$\begin{array}{ccc} A_1 & \xrightarrow{\simeq} & G \times X \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle a} \\ A_0 & \xrightarrow{\simeq} & X \end{array} \qquad (5)$$

To the end, one can break (5) to a zig-zag (of length 3) of commutative squares with horizontal arrows being homotopy equivalences. In view of definition 5.11, $A_1 \overset{d_0}{\to} A_0$ is weakly equivalent to $G \times X \overset{a}{\to} X$.

□

Part [2] of the last theorem establishes a 'rigidification theorem' which we wish to state separately

Theorem 5.13. *A map $\Omega Y \times X' \overset{\mu}{\to} X'$ is weakly equivalent to an action map $G \times X \to X$ iff there is a homotopy action $A_\bullet \to B_\bullet$ of ΩY on X' such that μ is equivalent to $A_1 \overset{d_0}{\to} A_0$.*

Since a homotopy action is essentially a level-wise trivial fibration we naturally get:

Proposition 5.14. *If $A_\bullet \to B_\bullet$ is a homotopy action of ΩY on X, and $L : Top \to Top$ is a HM functor then $LA_\bullet \to LB_\bullet$ is a homotopy action of $L\Omega Y$ on LX.*

Proof. $LB_\bullet$ is a special $\Delta-$space for LB_1. In particular, LB_1 is of the homotopy type of a loop space. Applying L on the structure maps of the homotopy action yields the structure maps for $LA_\bullet \to LB_\bullet$ and L preserves homotopy equivalences. □

6 An invariant characterization of Normality

Theorem 1.3 characterize homotopy normal maps of discrete groups in terms of a simplicial group, equivariantly equivalent to the bar construction. By analogy, the mere fact that the bar construction $Bar_\bullet(\Omega X, \Omega Y)$ is simplicialy equivalent to a simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 = \Omega Y$ is a necessary but not sufficient condition for a loop map $\Omega f : \Omega X \to \Omega Y$ to be homotopy normal.

In both $\Delta-$spaces $Bar_\bullet(\Omega X, \Omega Y), \Gamma_\bullet$(see notation 4.2), the map $s_{n-1}...s_0$ is a loop map and therefore induces a homotopy action of ΩY on Γ_n and $Bar_n(\Omega X, \Omega Y)$(see lemma 5.10).
We begin with

Proposition 6.1. *Let $\Omega f : \Omega X \to \Omega Y$ be a homotopy normal map. For each n, the homotopy actions induced by the loop maps $\Gamma_0 \to \Gamma_n$ and $\Omega Y \to Bar_n(\Omega X, \Omega Y)$ are equivalent via the map $\eta : \Gamma_\bullet \to Bar_\bullet(\Omega X, \Omega Y)$ defined in 4.2.*

Proof. We only do the case $n = 1$ since other cases are similar. Denote $\sigma \equiv s_0 : \Gamma_0 \to \Gamma_1$ and $s \equiv s_0 : Bar_0(\Omega X, \Omega Y) \hookrightarrow Bar_1(\Omega X, \Omega Y)$. The simplicial equivalence $\eta : \Gamma_\bullet \to Bar_\bullet(\Omega X, \Omega Y)$(see 4.2) induces a commutative square with vertical arrows being homotopy equivalences and with the left vertical arrow being a loop map

$$\begin{array}{ccc} \Gamma_0 & \xrightarrow{\sigma} & \Gamma_1 \\ {\scriptstyle \eta_0}\downarrow & & \downarrow{\scriptstyle \eta_1} \\ \Omega Y & \xrightarrow{s} & \Omega X \times \Omega Y \end{array}$$

Both σ and s are homotopy principal fibrations and hence have classifying maps $\gamma,\ c$(resp.). Now finding the dashed arrow

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\gamma} & \Gamma_1//\Gamma_0 \\ {\scriptstyle \eta_1}\downarrow & & \vdots\,{\scriptstyle d_1} \\ \Omega X \times \Omega Y & \xrightarrow[c]{} & \Omega X \end{array}$$

will end the proof because the first(resp. second) homotopy action is built out of homotopy powers of γ(resp. c) with itself. Since σ and s are (homotopy)principal fibrations, finding the dashed arrow amounts showing the equivalence between their homotopy fibers $F \equiv hfib(\sigma) \to hfib(s) \simeq \Omega^2 X$ is a loop map. To prove the last statement we use the path-space to model the homotopy fiber. On the one hand, we have the pullback square

$$\begin{array}{ccc} \Omega^2 X & \longrightarrow & P(\Omega X \times \Omega Y) \\ \downarrow & & \downarrow \\ \Omega Y & \xrightarrow{s} & \Omega X \times \Omega Y \end{array}$$

and on the other hand the pullback square

$$\begin{array}{ccc} F & \longrightarrow & P(\Gamma_1) \\ \downarrow & & \downarrow \\ \Gamma_0 & \xrightarrow{\sigma} & \Gamma_1 \end{array}$$

with all maps being of the homotopy type of loop maps. The compositions $F \to \Gamma_0 \to \Omega Y$ and $F \to P(\Gamma_1) \to P(\Omega X \times \Omega Y)$ are both of the homotopy type of loop maps and thus the universal map they induce $F \to \Omega^2 X$ is itself (of the homotopy type of) a loop map. □

Starting with a homotopy normal map $\Omega X \to \Omega Y$ we first got(4.1) a simplicial loop space $\Gamma_\bullet$, equivalent to the Bar construction which is a necessary condition for homotopy normality. The loop maps $s_{n-1}...s_0 : \Gamma_0 \to \Gamma_n$ ($n = 0$

understood as the identity map) induce homotopy actions of Γ_0 on Γ_n. We can pack all the maps in one simplicial map $\Gamma_0 \to \Gamma_\bullet$ which will then induce a simplicial object in the category of homotopy actions. Recalling definition 5.6 this is a homotopy action of Γ_0 on $\Gamma_\bullet$. Similarly, one has a homotopy action of ΩY on $Bar_\bullet(\Omega X, \Omega Y)$ and the equivalence $\Gamma_0 \simeq \Omega Y$ as loop spaces, makes the first homotopy action to be one of ΩY on $\Gamma_\bullet$(see 5.10). We call these actions the *canonical homotopy actions* of ΩY on $\Gamma_\bullet$ and $Bar_\bullet(\Omega X, \Omega Y)$. The additional condition for a characterization of normality is that the two are equivalent.

Theorem 6.2. *Let* $\Omega X \xrightarrow{\Omega f} \Omega Y$ *be a loop map. Then* Ωf *is homotopy normal iff there exist a simplicial loop space* $\Gamma_\bullet$ *with* $\Gamma_0 \simeq \Omega Y$ *and such that the canonical homotopy actions of* ΩY *on* $\Gamma_\bullet$ *and* $Bar_\bullet(\Omega X, \Omega Y)$ *are equivalent.*

Proof. Assume Ωf is homotopy normal. We have a commutative square of $\Delta-$spaces

$$\begin{array}{ccccc}
\Omega Y & \xrightarrow{\sigma} & \Gamma_\bullet & \longrightarrow & \Gamma_\bullet//\Gamma_0 \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle d} \\
\Omega Y & \xrightarrow{s} & Bar_\bullet(\Omega X, \Omega Y) & \longrightarrow & Bar_\bullet(\Omega X, \Omega Y)//\Omega Y
\end{array}$$

with φ the simplicial equivalence of theorem **??** and the dashed arrow d with d_1(of proposition 6.1) as its first component and with the analogues d_n as its n-th component. This gives the desired equivalence of the canonical actions. On the other hand, if we have an equivalence of homotopy actions(see 5.7)

$$\begin{array}{ccc}
\Gamma_\bullet & \longrightarrow & B_\bullet \\
\downarrow & & \downarrow \\
Bar_\bullet(\Omega X, \Omega Y) & \longrightarrow & B_\bullet(\Omega X)
\end{array}$$

then we have equivalent principal fibrations upon realization

$$\begin{array}{ccccc}
\Gamma_0 & \longrightarrow & |\Gamma_\bullet| & \longrightarrow & |B_\bullet| \\
\downarrow & & \downarrow & & \downarrow \\
\Omega Y & \longrightarrow & \Omega Y//\Omega X & \longrightarrow & X
\end{array}$$

but the operation of taking loop commutes with that of realization and hence $|\Gamma_\bullet| \simeq \Omega W$ for some connected space W. The map $\Gamma_0 \to |\Gamma_\bullet|$ is the realization of a simplicial loop map $\Gamma_0 \to \Gamma_\bullet$ hence a loop map itself and delooping it gives the desired extension $Y \dashrightarrow W$. $\square$

Let $A_\bullet \to B_\bullet$ be a homotopy action. From the proof of theorem 5.2, [2], it follows that there is a homotopy fibration sequence $A_0 \xrightarrow{\sigma} |A_\bullet| \to |B_\bullet|$ where σ is the realization of the simplicial map $A_0 \to A_\bullet$ which has its nth component the map $s_{n-1}...s_0$. Since $B_\bullet$ is a special $\Delta-$space, $\Omega|B_\bullet| \simeq B_1$. We denote by $B_1 \overset{\psi}{\to} A_0$ the inclusion of the homotopy fiber of $A_0 \overset{\sigma}{\to} |A_\bullet|$ and endow A_0 with a base-point via ψ. Denote by $B_1 \overset{i}{\to} B_1 \times A_0$ the natural inclusion. We shall need the following technical lemma.

Lemma 6.3. *For any choice of homotopy inverse, $B_1 \times A_0 \overset{e}{\to} A_1$ for $A_1 \overset{\pi_1 \times d_1}{\to} B_1 \times A_0$, the composition $B_1 \overset{i}{\to} B_1 \times A_0 \overset{e}{\to} A_1 \overset{d_0}{\to} A_0$ is homotopic to ψ.*

Proof. As was noted in the proof of theorem 5.2 [2], the square

$$(*)\qquad \begin{array}{ccc} A_1 & \xrightarrow{d_1} & A_0 \\ {\scriptstyle d_0}\big\downarrow & & \big\downarrow \\ A_0 & \longrightarrow & |A_\bullet| \end{array}$$

is homotopy commutative. We thus obtain a homotopy commutative diagram of solid arrows

$$\begin{array}{ccccc} B_1 & \xrightarrow{i} & B_1 \times A_0 & \xrightarrow{pr} & A_0 \\ {\scriptstyle c_2}\big\uparrow \simeq & & {\scriptstyle \pi_1 \times d_1}\big\uparrow \big\downarrow {\scriptstyle e} & & {\scriptstyle 1}\big\uparrow \\ B_1 & \longrightarrow & A_1 & \xrightarrow{d_1} & A_0 \\ {\scriptstyle c_1}\big\downarrow \simeq & & \big\downarrow {\scriptstyle d_0} & & \big\downarrow {\scriptstyle \sigma} \\ B_1 & \xrightarrow{\psi} & A_0 & \xrightarrow{\sigma} & |A_\bullet| \end{array}$$

where the map $B_1 \to A_1$ is the inclusion of homotopy fiber, the map c_1 is the comparison map between homotopy fibers of d_1 and σ which is a homotopy equivalence as was mentioned in 5.2 [2] and the map c_2 is the comparison map between homotopy fibers of d_1 and *pr* which is again a a homotopy equivalence. The lemma now follows from inverting c_2.

□

We can now prove:

Theorem 6.4. *Let $\Omega X \xrightarrow{\Omega f} \Omega Y$ be a homotopy normal map.*

If $L : Top \to Top$ is an HM functor, then $L\Omega X \xrightarrow{Lf} L\Omega Y$ is a homotopy normal map.

Proof. We shall construct an extension $BL\Omega Y \to W$ with W a connected space, thus providing a normal structure for $L\Omega X \to L\Omega Y$(see 1.1). Given homotopy normality of Ωf we have from theorem 1.4 a simplicial loop space $\Gamma_\bullet$ and an equivalence of homotopy actions of ΩX on ΩY (horizontal arrows below)

$$\begin{array}{ccc} \Gamma_\bullet & \longrightarrow & Bar_\bullet(\Omega X, *) \\ \big\downarrow & & \big\downarrow \\ Bar_\bullet(\Omega X, \Omega Y) & \xrightarrow{\pi} & Bar_\bullet(\Omega X, *) \end{array}$$

Apply L on each term to get a homotopy action $L\Gamma_\bullet \to LB_\bullet$ of $L\Omega X$ on $L\Omega Y$. Since $L\Omega Y \to |LBar_\bullet(\Omega X, \Omega Y)| \to |LBar_\bullet(\Omega X, *)|$ is a homotopy fibration sequence(being the realization of a simplicial fibration sequence), and since $|LBar_\bullet(\Omega X, *)| \simeq B(L\Omega X)$ ($LBar_\bullet(\Omega X, *)$ is a special $\Delta-$space for $L\Omega X$), there is a map $L\Omega X \xrightarrow{\varphi} L\Omega Y$ which is the inclusion of homotopy fiber of $L\Omega Y \to |LBar_\bullet(\Omega X, \Omega Y)|$.
Abbreviate $A_\bullet \equiv Bar_\bullet(\Omega X, \Omega Y)$ and $B_\bullet \equiv Bar(\Omega X, *)$. If $B_1 \times A_0 \xrightarrow{e}$ is a homotopy inverse to $\pi_1 \times d_1$ then Le is a homotopy inverse to $L(\pi_1 \times d_1)$ which is equivalent to $L(\pi_1) \times L(d_1)$. By lemma 6.3, Ωf is homotopic to the composition $B_1 \xrightarrow{i} B_1 \times A_0 \xrightarrow{e} A_1 \xrightarrow{d_0} A_0$ and so $L\Omega f$ is homotopic to the composition $Ld_0 \circ Le \circ Li$. The last composition is equivalent to the composition $LB_1 \hookrightarrow LB_1 \times LA_0 \xrightarrow{L\pi_1 \times Ld_0} LA_1 \xrightarrow{Ld_0} A_0$ which is homotopic to φ by lemma 6.3. It follows that $L\Omega f$ is equivalent to φ

Upon realization, we have equivalent homotopy fibration sequences

$$\begin{array}{ccccc} |L\Omega Y| & \longrightarrow & |L\Gamma_\bullet| & \longrightarrow & |LB_\bullet| \\ \downarrow & & \downarrow & & \downarrow \\ |L\Omega Y| & \longrightarrow & |LBar_\bullet(\Omega X, \Omega Y)| & \longrightarrow & |LBar_\bullet(\Omega X, *)| \end{array}$$

in which the bottom one is equivalent to $L\Omega Y \to L\Omega Y//L\Omega X \to BL\Omega X$ (gotten from $L\Omega f : L\Omega X \to L\Omega Y$).
But now, $\Omega Y \to \Gamma_\bullet$ is a simplicial loop map and so $L\Omega Y \to L\Gamma_\bullet$ is a simplicial loop map making

$$L\Omega Y \to |L\Gamma_\bullet|$$

a loop map and delooping it gives the desired extension $BL\Omega Y \to W$, proving homotopy normality of $L\Omega f$ □

Let us demonstrate a use of theorem 6.4 by proving with it the following theorem, originally appeared in [DF].

Theorem 6.5. *Let $E \xrightarrow{p} B$ be a principal fibration with B connected, $f : X \to Y$ a map of pointed connected spaces and $L_{\Sigma f}$ the localization with respect to its suspension. Then $L_{\Sigma f}E \to L_{\Sigma f}B$ is equivalent to a principal fibration.*

Remark 6.6. Note that if G is the structure group of $E \to B$, $L_{\Sigma f}G$ need not be the structure group of $L_{\Sigma f}E \to L_{\Sigma f}B$

Proof. Note that $\Omega E \to \Omega B$ is homotopy normal. Hence, $L_f\Omega E \to L_f\Omega B$ is homotopy normal. Since for any pointed space A, there is a natural equivalence $L_f\Omega A \simeq \Omega L_{\Sigma f}A$ we get that $\Omega L_{\Sigma f}E \to \Omega L_{\Sigma f}B$ is homotopy normal and thus $L_{\Sigma f}E \to L_{\Sigma f}B$ is equivalent to a principal fibration. □

7 k-Normality

As mentioned in example 2 , any double loop map is automatically homotopy normal. However, it is natural to ask when the homotopy quotient admits a natural 2-fold loop space structure.

Definition 7.1. A $0-$homotopy normal map is a principal fibration of connected spaces. For $k \geq 1$, define inductively a $k-$fold loop map $\Omega^k f : \Omega^k X \to \Omega^k Y$ to be *k-homotopy normal* if $\Omega^{k-1} f$ is $k-1$-homotopy normal.

Remark 7.2. One may wonder about the definition of '∞-homotopy normality'. However, any $\infty-$loop map induces a principal fibration sequence of $\infty-$loop spaces $X \to Y \to Y//X$.

We begin with an extension of 1.4

Theorem 7.3. *A k-fold loop map* $\Omega^k X \xrightarrow{\Omega^k f} \Omega^k Y$ *is* $k-$*homotopy normal iff there exists a* $k-$*fold simplicial loop space* $\Gamma_\bullet$ *with* $\Gamma_\bullet = \Omega^k Y$ *and such that the canonical homotopy actions of* $\Omega^k Y$ *on* $Bar_\bullet(\Omega^k X, \Omega^k Y)$ *and* $\Gamma_\bullet$ *are naturally equivalent.*

Proof. This is analogous to the proof of 1.4. If $\Omega^k f$ is $k-$homotopy normal, then Ωf is homotopy normal and looping down its extension $Y \to W$ $k-$times gives a $k-$fold loop map equivalent to $\Omega^k Y \to \Omega^k Y//\Omega^k X$. Taking (homotopy)power of that map gives the desired $k-$fold loop space. Conversely, such a $k-$fold loop space gives a (homotopy)principal fibration sequence of $k-$fold loop spaces $\Omega^k X \to \Omega^k Y \to |\Gamma_\bullet|$ equivalent to the Borel construction, and hence $k-$homotopy normality. □

We wish to use the same methods as before to prove invariance of $k-$homotopy normal maps under HM functors. For that, we need to know that $k-$fold loop spaces are invariant under these functors. A slight generalization of 'grouplike special $\Delta-$spaces' is the tool needed. The following is taken from [BFSV]

Definition 7.4. A $(\Delta)^k-$space $X_{\bullet \dots \bullet}$ is *grouplike special* if

1. $X_{0,\dots,0} \simeq *$.
2. $\pi_0(X_{1,\dots,1})$ is a group.
3. The Segal maps induces homotopy equivalences $X_{p_1,\dots,p_k} \xrightarrow{\simeq} (X_{1,\dots,1})^{p_1 \cdots p_k}$.

There is then an analogous theorem for $k-$fold loop spaces

Theorem 7.5. *A space* X *is of the homotopy type of a* $k-$*fold loop space iff there exist a grouplike special* $(\Delta)^k-$*space* $X_{\bullet,\dots,\bullet}$ *with* $X_{1,\dots,1} \simeq X$.

Corollary 7.6. *Homotopy monoidal functors preserve* $k-$*fold loop spaces.*

Using exactly the same arguments of theorem 6.4, we deduce from theorem 7.3

Theorem 7.7. *If* $\Omega^k f : \Omega^k X \to \Omega^k Y$ *is* $k-$*homotopy normal and* $L : Top \to Top$ *an HM functor, then* $L(\Omega^k f)$ *is* $k-$*homotopy normal.*

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