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# CR GEOMETRY AND CONFORMAL FOLIATIONS

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Abstract. We use the CR geometry of the standard hyperquadric in $\mathbb{CP}_3$ to give a detailed twistor description of conformal foliations in Euclidean 3-space.

## 1. Introduction

A foliation of a Riemannian manifold is said to be conformal if and only if the corresponding locally defined submersion is conformal on the orthogonal spaces to the leaves (precise definitions are given in §3). Locally a conformal foliation induces $\mathbb{R}^3 \supseteq {}^{\text{open}}\Omega \xrightarrow{h} \mathbb{C}$, defined up to composition $\alpha \circ h$ for $\alpha$ conformal, with the property that the gradient of $h$ is null: $(\nabla h)^2 = 0$. Such a mapping is said to be horizontally conformal [5]. In [7] Nurowski showed how real-analytic horizontally conformal mappings in $\mathbb{R}^3$ may be constructed from a holomorphic function of two complex variables. The mappings obtained in this way are certainly real-analytic. Otherwise they are generic but, as we shall see, not completely general.

In this article we construct conformal foliations themselves starting with a holomorphic function of two complex variables. The foliations constructed in this way are real-analytic but are otherwise general, at least locally. Our construction is one of twistor geometry, a viewpoint which also allows us to identify the smooth conformal foliations with local CR hypersurfaces in the standard Levi-indefinite hyperquadric in $\mathbb{CP}_3$. Real-analyticity is a familiar distinguishing feature in CR geometry. Our construction parallels the twistor interpretation of the Kerr Theorem in special relativity as explained in [8] and, following [4], the precise link is explained at the end of our paper in §10. Another intimately related construction appears in [3] where twistor geometry is used directly to describe horizontally conformal mappings in $\mathbb{R}^3$.

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## 2. CR geometry

The book [2] by Baouendi, Ebenfelt, and Rothschild provides a good reference for CR geometry. Here, we provide an outline of the specific results we shall need, referring to [2] for proofs and further detail.

We start with some linear algebra, presented in the dimensions where it will be needed. Let us consider a real linear subspace $T \subseteq \mathbb{C}^3$ of real codimension $d$. Let us denote by $J$ the real linear endomorphism of $\mathbb{C}^3$ given by multiplication by $i$ and let $H = T \cap JT$. It is the maximal complex-linear subspace of $T$.

- Case $d = 1$: it follows that $\dim_{\mathbb{C}} H = 2$ and all such $T$ are on an equal footing. More precisely, $\mathrm{GL}(3, \mathbb{C})$ acts transitively on $\mathrm{Gr}_5(\mathbb{R}^6)$, the Grassmannian of real hyperplanes in $\mathbb{R}^6$.
- Case $d = 2$: there are two cases according to whether $T$ is a complex subspace. If not, then $\dim_{\mathbb{C}} H = 1$ and we shall refer to $T$ as *generic*. There are two orbits for the action of $\mathrm{GL}(3, \mathbb{C})$ on $\mathrm{Gr}_4(\mathbb{R}^6)$, namely $\mathrm{Gr}_2(\mathbb{C}^3)$ and its complement.
- Case $d = 3$: again there are two orbits for the action of $\mathrm{GL}(3, \mathbb{C})$ on the relevant Grassmannian. Generically $H = 0$ and we shall refer to $T$ as *totally real*. Otherwise $\dim_{\mathbb{C}} H = 1$.

Now suppose $Z$ is a complex manifold with $\dim_{\mathbb{C}} Z = 3$ and $M \subset Z$ is smooth real submanifold of real codimension $d$. When $d = 1$, we may apply the construction above in each tangent space, obtaining a smooth subbundle $H \subset TM$ equipped with an endomorphism $J : H \to H$ with $J^2 = -\mathrm{Id}$. Additionally, the bundle

$$H^{0,1}M = \{X \in \mathbb{C}H \text{ s.t. } JX = -iX\} \tag{1}$$

is closed under Lie bracket. In abstraction, such a structure is called a *CR structure of hypersurface type*.

When $d = 2$, we shall say that $M$ is a *CR submanifold* if and only if $T_xM \cap JT_xM$ is of constant dimension as $x \in M$ varies. Again, we obtain a subbundle $H \subset TM$ equipped with a complex structure $J : H \to H$. When $\mathrm{rank}_{\mathbb{C}} H = 1$ we shall say that $M \subset Z$ is *generic*. If $\mathrm{rank}_{\mathbb{C}} H = 2$, then $M$ is simply a complex submanifold of $Z$.

Finally, when $d = 3$ there are two cases corresponding infinitesimally to the two orbits of $\mathrm{GL}(3, \mathbb{C})$ on $\mathrm{Gr}_3(\mathbb{R}^6)$ identified above. When $H = 0$ we shall say that $M$ is *totally real*. Otherwise $\mathrm{rank}_{\mathbb{C}} H = 1$. In general, $\mathrm{rank}_{\mathbb{C}} H$ is called the *CR dimension* of $M$.

An abstract CR structure on a smooth manifold $M$ of CR dimension $m$ is defined by a smooth subbundle $H \subset TM$ of real rank $2m$ equipped with an endomorphism $J$ with $J^2 = -\mathrm{Id}$ and such that the bundle $H^{0,1}$ defined by (1) is closed under Lie bracket. If the CR structure on $M$ is induced by an embedding $M \hookrightarrow Z$ in a complex manifold $Z$ and $f : Z \to \mathbb{C}$ is a holomorphic function on $Z$, then $f|_M$ satisfies the partial differential equations

$$Xf = 0 \ \text{ for all } \ X \in \Gamma(M, H^{0,1}).$$

These are the remnants on $M$ of the Cauchy-Riemann equations on $Z$ and solutions of these equations on $M$ are called *CR functions* (whether or not they arise by restriction of holomorphic functions on $Z$).

Suppose $Q$ is a smooth CR manifold of hypersurface type and real dimension 5 (hence of CR dimension 2). Suppose that $f : Q \to \mathbb{C}$ is a smooth CR function without critical points. Of course,

$$M \equiv \{p \in Q \text{ s.t. } f(p) = 0\}$$

is a smooth submanifold and it is easy to check that the CR equations on $f$ imply that intrinsically $M$ is a CR manifold of CR dimension 1. This is the CR analogue of the statement that holomorphic functions without critical points on complex manifolds implicitly define complex submanifolds. Unlike the holomorphic case, however, there is no CR implicit function theorem. As we shall see, it is not necessarily the case that a CR submanifold of $Q$ of real codimension 2 and CR dimension 1 need be locally defined as the zeroes of a CR function, even when

$$Q = \{[Z_1, Z_2, Z_3, Z_4] \in \mathbb{CP}_3 \text{ s.t. } |Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2\} \tag{2}$$

as we shall suppose henceforth. This particular CR manifold is known as the hyperquadric (of CR dimension 2 and indefinite signature). The following theorem follows from the classical Lewy extension theorem as, for example, proved in [6, Theorem 2.6.13].

**Theorem 1.** *Suppose $\Omega^{\mathrm{open}} \subseteq Q \subset \mathbb{CP}_3$ and $f : \Omega \to \mathbb{C}$ is a smooth CR function. Then $f$ automatically extends as a holomorphic function to a neighbourhood of $\Omega$ in $\mathbb{CP}_3$. This extension is germ-unique.*

It is immediate that the CR functions on the indefinite hyperquadric $Q$ are necessarily real-analytic.

**Theorem 2.** *Suppose $M \subset \Omega^{\mathrm{open}} \subseteq Q \subset \mathbb{CP}_3$ is a real-analytic CR submanifold of real dimension 3 and CR dimension 1. Then $M$ extends into $\mathbb{CP}_3$ as a complex submanifold. This extension is germ-unique. In particular, $M$ can be locally defined by a CR function.*

*Proof.* Immediate from [2, Corollary 1.8.10]. An alternative argument may be constructed from the holomorphic implicit function theorem in the complexification. □

Later in this article, we shall find smooth 3-dimensional local CR submanifolds of $Q$ of CR dimension 1 that are not real-analytic.

## 3. Conformal foliations in $\mathbb{R}^3$

A unit vector field $U$ on $\Omega^{\mathrm{open}} \subseteq \mathbb{R}^3$ induces a 1-dimensional foliation of $\Omega$ as its integral curves. Conversely, all 1-dimensional foliations arise in this way. We shall say that $U$ is *transversally conformal* if the Lie derivative $\mathcal{L}_U$ preserves the conformal metric orthogonal to its integral curves. In case the foliation is locally defined by a submersion $\pi : \Omega \to \Sigma$ then this is equivalent to saying that $\pi$ is horizontally conformal [5] (whence $\Sigma$ is naturally a Riemann surface).

By writing the Lie derivative in terms of the flat connection $\nabla$ on $\mathbb{R}^3$ it follows that a unit vector field $U$ on $\mathbb{R}^3$ defines a conformal foliation if and only if

$$\langle U, \nabla_X Y + \nabla_Y X\rangle = 0$$

for all vector fields $X$ and $Y$ with

$$\langle U, X\rangle = 0 \qquad \langle U, Y\rangle = 0 \qquad \langle X, Y\rangle = 0 \qquad \|X\| = \|Y\|.$$

As detailed in [5], it is equivalent to check for a particular non-zero pair of such fields $X, Y$, that

$$\langle U, \nabla_X Y + \nabla_Y X\rangle = 0 \quad \text{and} \quad \langle U, \nabla_X X - \nabla_Y Y\rangle = 0.$$

We can write these conditions as partial differential equations on the components of $U$. Specifically, let us write $(q, r, s)$ for the standard Euclidean coördinates on $\mathbb{R}^3$ and write

$$U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u(q,r,s) \\ v(q,r,s) \\ w(q,r,s) \end{pmatrix}. \tag{3}$$

We may suppose without loss of generality that

$$U = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad X = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

at the origin and take

$$X = \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -uw \\ -vw \\ u^2 + v^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - w \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

nearby. Since $u^2+v^2+w^2=1$, it follows that all partial derivatives of $u$ vanish at the origin and then we compute

$$\langle U,\nabla_X Y+\nabla_Y X\rangle+\frac{\partial w}{\partial r}+\frac{\partial v}{\partial s}=0=\langle U,\nabla_X X-\nabla_Y Y\rangle+\frac{\partial v}{\partial r}-\frac{\partial w}{\partial s}.$$

It follows that the conformality of $U$ is captured by the equations

$$\frac{\partial v}{\partial r}=\frac{\partial w}{\partial s}\quad\text{and}\quad\frac{\partial v}{\partial s}=-\frac{\partial w}{\partial r}\tag{4}$$

at points where $(u,v,w)=(1,0,0)$.

## 4. Twistor fibrations

If we choose an identification $\mathbb{C}^4=\mathbb{H}^2$, where $\mathbb{H}$ is the space of quaternions, then we obtain a submersion

$$\tau:\mathbb{CP}_3\longrightarrow\mathbb{HP}_1=S^4\tag{5}$$

by taking the quaternionic span. See [1] for details, where this fibration is also realised as the bundle of orthogonal complex structures over the round 4-sphere. In coördinates we have

$$\mathbb{CP}_3\ni[Z]\overset{\tau}{\mapsto}\frac{1}{\|Z\|^2}\begin{pmatrix}|Z_1|^2+|Z_2|^2-|Z_3|^2-|Z_4|^2\\ 2(Z_2\bar Z_3+Z_4\bar Z_1)\\ 2(Z_1\bar Z_3-Z_4\bar Z_2)\end{pmatrix}\in\begin{matrix}\mathbb{R}\\ \oplus\\ \mathbb{C}^2\end{matrix}=\mathbb{R}^5\tag{6}$$

and we observe that, when restricted to the hyperquadric $Q$, this gives a submersion

$$\tau:Q\to S^3.\tag{7}$$

We shall refer to both (5) and (7) as *twistor fibrations*. Both have fibres intrinsically isomorphic to the Riemann sphere $\mathbb{CP}_1$ with its usual complex structure.

As in [1], it is useful to view these fibrations as intrinsically attached to the manifolds $S^4$ and $S^3$, regarded as flat conformal manifolds in the usual manner. More specifically, we may use stereographic projection $S^n\setminus\{\infty\}\xrightarrow{\simeq}\mathbb{R}^n$ to restrict these fibrations

$$\begin{array}{ccc}\mathbb{CP}_3\setminus\mathbb{I}&\xrightarrow{\tau}&\mathbb{R}^4\\ \cup&&\cup\\ Q\setminus\mathbb{I}&\xrightarrow{\tau}&\mathbb{R}^3\end{array}$$

where $\mathbb{I}=\tau^{-1}(\infty)=\{[*,*,0,0]\}$ and rewrite them as follows.

Suppose $J \in \mathrm{SO}(4)$ satisfies $J^2 = -\mathrm{Id}$. It follows that

$$J = \begin{pmatrix} 0 & -u & -v & -w \\ u & 0 & -w & v \\ v & w & 0 & -u \\ w & -v & u & 0 \end{pmatrix} \tag{8}$$

for some $(u, v, w)$ s.t. $u^2 + v^2 + w^2 = 1$. In other words, the complex structures on the vector space $\mathbb{R}^4$ preserving the standard metric and orientation are parameterised by $(u, v, w) \in S^2$. As in [1], we identify

$$\begin{array}{ccc} \mathbb{CP}_3 \setminus \mathbb{I} & \xrightarrow{\tau} & \mathbb{R}^4 \\ \| & & \| \\ (p,q,r,s,u,v,w) \in \mathbb{R}^4 \times S^2 & \xrightarrow{\pi} & \mathbb{R}^4 \end{array} \tag{9}$$

where $\pi$ is projection onto the first factor and we may rewrite the complex structure on $\mathbb{CP}_3$ as induced by the action of

$$\mathbb{J} = \begin{pmatrix} 0 & -u & -v & -w & 0 & 0 & 0 \\ u & 0 & -w & v & 0 & 0 & 0 \\ v & w & 0 & -u & 0 & 0 & 0 \\ w & -v & u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -w & v \\ 0 & 0 & 0 & 0 & -v & 0 & -u \\ 0 & 0 & 0 & 0 & -v & u & 0 \end{pmatrix} \tag{10}$$

on $\mathbb{R}^4 \times \mathbb{R}^3$. Similarly, the CR structure on $Q$ when viewed as

$$\begin{array}{ccc} Q \setminus \mathbb{I} & \xrightarrow{\tau} & \mathbb{R}^3 \\ \| & & \| \\ (q,r,s,u,v,w) \in \mathbb{R}^3 \times S^2 & \xrightarrow{\pi} & \mathbb{R}^3 \end{array} \tag{11}$$

comprises

- the contact structure defined by $\theta \equiv u\,dq + v\,dr + w\,ds$,
- the endomorphism of $H \equiv \ker \theta$ induced by $\mathbb{J}$.

## 5. Integrable Hermitian structures

In §4 we saw that the almost Hermitian structures for the standard Euclidean metric $\Omega^{\mathrm{open}} \subseteq \mathbb{R}^4$ are parameterised by smooth functions $U : \mathbb{R}^4 \to S^2$ specifying matrices of the form (8). More generally, for $\Omega^{\mathrm{open}} \subseteq S^4$, the almost complex structures compatible with the flat conformal structure correspond to sections of the twistor fibration. Let us see what it means for such complex structures to be integrable.

**Lemma 1.** *At a point where* $(u,v,w)=(1,0,0)$*, integrability of* (8) *is captured by the equations*

$$\frac{\partial v}{\partial p}=\frac{\partial w}{\partial q}\qquad \frac{\partial v}{\partial q}=-\frac{\partial w}{\partial p}\qquad \frac{\partial v}{\partial r}=\frac{\partial w}{\partial s}\qquad \frac{\partial v}{\partial s}=-\frac{\partial w}{\partial r}$$

*Proof.* If we compute the Nijenhuis tensor

$$N(X,Y)\equiv[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY]$$

at such a point we find that

$$N\left(\tfrac{\partial}{\partial p},\tfrac{\partial}{\partial q}\right)=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad N\left(\tfrac{\partial}{\partial p},\tfrac{\partial}{\partial r}\right)=\begin{pmatrix}\partial w/\partial p+\partial v/\partial q\\ \partial w/\partial q-\partial v/\partial p\\ \partial w/\partial r+\partial v/\partial s\\ \partial w/\partial s-\partial v/\partial r\end{pmatrix}=N\left(\tfrac{\partial}{\partial s},\tfrac{\partial}{\partial q}\right)$$

$$N\left(\tfrac{\partial}{\partial s},\tfrac{\partial}{\partial p}\right)=\begin{pmatrix}\partial v/\partial p-\partial w/\partial q\\ \partial v/\partial q+\partial w/\partial p\\ \partial v/\partial r-\partial w/\partial s\\ \partial v/\partial s+\partial w/\partial r\end{pmatrix}=N\left(\tfrac{\partial}{\partial r},\tfrac{\partial}{\partial q}\right)\quad N\left(\tfrac{\partial}{\partial r},\tfrac{\partial}{\partial s}\right)=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

and the proof is complete. □

**Theorem 3.** *If* $J$ *is an integrable Hermitian structure on* $\Omega^{\mathrm{open}}\subseteq\mathbb{R}^4$*, then* $U=J(\partial/\partial p)$ *defines a conformal foliation on*

$$\Omega\cap\mathbb{R}^3=\{(p,q,r,s)\in\Omega\text{ s.t. }p=0\}.$$

*Proof.* From (8) we see that $U=(u,v,w)$ and we are obliged to show that the equations (4) hold at points where $(u,v,w)=(1,0,0)$. But these equations are just two of the four equations from Lemma 1. □

Of course, the conformal foliations obtained in this way are obliged to be real-analytic. The following theorem is a well-known result in twistor theory (see, e.g. [5, Proposition 7.1.3(iii)]). For completeness we include a proof here based on our standard normalisation. We regard an almost Hermitian structure $J$ on $\Omega^{open}\subseteq S^4$ as a section of the twistor fibration (5) and write $S$ for its range.

**Theorem 4.** *A smooth section* $J$ *of* $\pi:\mathbb{CP}_3\to S^4$ *defined on* $\Omega$ *is integrable if and only if* $S\equiv J(\Omega)$ *is a complex submanifold of* $\mathbb{CP}_3$*.*

*Proof.* Without loss of generality we may suppose that the value of $J$ at some point in $\Omega$ is given by (8) for $(u,v,w)=(1,0,0)$ and check the statement of the theorem above that point. The tangent space to $S$ is given there in the local coördinates (9) by

$$\mathrm{span}_{\mathbb{R}}\left\{\left(\begin{array}{c}1\\0\\0\\0\\0\\\partial v/\partial p\\\partial w/\partial p\end{array}\right),\left(\begin{array}{c}0\\1\\0\\0\\0\\\partial v/\partial q\\\partial w/\partial q\end{array}\right),\left(\begin{array}{c}0\\0\\1\\0\\0\\\partial v/\partial r\\\partial w/\partial r\end{array}\right),\left(\begin{array}{c}0\\0\\0\\1\\0\\\partial v/\partial s\\\partial w/\partial s\end{array}\right)\right\}$$

and this is preserved by $\mathbb{J}$ (as in (10) with $(u,v,w)=(1,0,0)$) if and only if the equations of Lemma 1 are satisfied. $\square$

## 6. The twistor theory of conformal foliations

The following result is the direct analogue of Theorem 4 for conformal foliations of $\Omega^{\mathrm{open}}\subseteq S^3$ in which we regard a unit vector field on $\Omega$ as a section of the twistor fibration (7).

**Theorem 5.** *A smooth section $U$ of $\pi:Q\to S^3$ over $\Omega^{\mathrm{open}}\subseteq S^3$ defines a conformal foliation if and only if its range $M\equiv U(\Omega)$ is a CR submanifold of $Q$ of CR dimension* 1.

*Proof.* Without loss of generality we may suppose that the value of $U$ at some point in $\Omega$ is given by (3) for $(u,v,w)=(1,0,0)$ and check the statement of the theorem above that point. The tangent space to $M$ is given there in the local coördinates (11) by

$$\mathrm{span}_{\mathbb{R}}\left\{\left(\begin{array}{c}1\\0\\0\\0\\\partial v/\partial q\\\partial w/\partial q\end{array}\right),\left(\begin{array}{c}0\\1\\0\\0\\\partial v/\partial r\\\partial w/\partial r\end{array}\right),\left(\begin{array}{c}0\\0\\1\\0\\\partial v/\partial s\\\partial w/\partial s\end{array}\right)\right\}.$$

The contact form $\theta$ is simply $dq$ when $(u,v,w)=(1,0,0)$ whence $H$ intersects the tangent space to $M$ as the span of the second and third of these three vectors and then this span is preserved by $\mathbb{J}$ if and only if the equations (4) characterising conformal foliations are satisfied. $\square$

**Corollary 1.** *Let $U$ be a real-analytic unit vector field on $\Omega^{\mathrm{open}}\subseteq S^3$. The following conditions are equivalent.*

- *$U$ defines a conformal foliation;*
- *$U(\Omega)\subset Q$ is locally defined by a CR function;*
- *$U(\Omega)=S\cap Q$ for some complex hypersurface $S$ in $\mathbb{CP}_3$;*
- *$U=J(\partial/\partial p)$ for some orthogonal complex structure $J$ on $S^4$,*

*where the complex hypersurface $S$ need only be defined and non-singular near $U(\Omega) \subset Q$ and, similarly, the integrable Hermitian structure $J$ need only be defined near $U \subseteq S^3 \subset S^4$. In the fourth condition, $S^4$ is equipped with its round metric, an orientation induced by stereographic projection, and a great subsphere $S^3 \subset S^4$ to which $\partial/\partial p$ denotes the unit normal field.*

*Proof.* In the real-analytic case, we may employ Theorem 2 to extend CR data from $Q$ to holomorphic data on $\mathbb{CP}_3$. All other equivalences have already been discussed. $\square$

## 7. An example—the Hopf fibration

To employ Corollary 1 in practise we need to be more specific about the identifications (9) and (11). For (9) the convenient choice is

$$\begin{pmatrix} p+iq \\ r+is \\ u \\ v+iw \end{pmatrix} = \frac{1}{|Z_3|^2+|Z_4|^2}\begin{pmatrix} Z_2\bar{Z}_3 + Z_4\bar{Z}_1 \\ Z_1\bar{Z}_3 - Z_4\bar{Z}_2 \\ |Z_3|^2 - |Z_4|^2 \\ 2iZ_4\bar{Z}_3 \end{pmatrix} \tag{12}$$

for which is it evident, from (6) and the formula

$$\mathbb{R}^5 = \begin{array}{c} \mathbb{R} \\ \oplus \\ \mathbb{C}^2 \end{array} \supset S^5 \ni \begin{pmatrix} t \\ \zeta_1 \\ \zeta_2 \end{pmatrix} \xmapsto{\sigma} \frac{1}{1-t}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in \mathbb{C}^2 = \mathbb{R}^4 \tag{13}$$

for stereographic projection, that (9) commutes. It is also routine to check that (12) is holomorphic for the complex structure $\mathbb{J}$ given by (10) on $\mathbb{R}^4 \times S^2$. More specifically, if we use

$$(x_1+iy_1, x_2+iy_2, x_3+iy_3) = (z_1, z_2, z_3) \mapsto [z_1, z_2, z_3, 1]$$

as local coördinates on $\mathbb{CP}_3$ then (12) becomes

$$\begin{array}{rcl} p &=& (x_2x_3+y_2y_3+x_1)/({x_3}^2+{y_3}^2+1) \\ q &=& (x_3y_2-x_2y_3-y_1)/({x_3}^2+{y_3}^2+1) \\ r &=& (x_1x_3+y_1y_3-x_2)/({x_3}^2+{y_3}^2+1) \\ s &=& (x_3y_1-x_1y_3+y_2)/({x_3}^2+{y_3}^2+1) \\ u &=& ({x_3}^2+{y_3}^2-1)/({x_3}^2+{y_3}^2+1) \\ v &=& 2y_3/({x_3}^2+{y_3}^2+1) \\ w &=& 2x_3/({x_3}^2+{y_3}^2+1) \end{array} \tag{14}$$

and it is straightforward to check that

$$\mathbb{J}\,\mathbf{Jac} = \mathbf{Jac}\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

where **Jac** is the Jacobian of the transformation (14).

For (11) it is convenient to change coördinates on $\mathbb{CP}_3$, writing

$$Q = \{[Z_1, Z_2, Z_3, Z_4] \in \mathbb{CP}_3 \text{ s.t. } Z_1\bar{Z}_4 + Z_2\bar{Z}_3 + Z_3\bar{Z}_2 + Z_4\bar{Z}_1 = 0\}$$

instead of (2), for then it is clear that $Q \setminus \mathbb{I}$ is identified as $\{p = 0\}$ under (12).

To write out the conclusions of Corollary 1 more explicitly, let us observe that (14) implies that

$$\begin{array}{rcl} z_1 & = & (r+is)z_3 + (p-iq) \\ z_2 & = & (p+iq)z_3 - (r-is) \end{array} \quad \text{and} \quad \begin{pmatrix} u \\ v+iw \end{pmatrix} = \frac{1}{|z_3|^2+1}\begin{pmatrix} |z_3|^2 - 1 \\ 2i\bar{z}_3 \end{pmatrix}$$

and so if $S$ is locally written as $\{z \in \mathbb{C}^3 \text{ s.t. } f(z_1, z_2, z_3) = 0\}$ for some holomorphic function $f$, then $U = (u, v, w)$ is defined by

$$\begin{pmatrix} u \\ v+iw \end{pmatrix} = \frac{1}{|z_3|^2+1}\begin{pmatrix} |z_3|^2 - 1 \\ 2i\bar{z}_3 \end{pmatrix} \tag{15}$$

where $z_3$ is the smooth function of $(q, r, s)$ implicitly defined by

$$f((r+is)z_3 - iq, iqz_3 - (r-is), z_3) = 0. \tag{16}$$

Corollary 1 says that the unit vector fields $U(q, r, s)$ obtained in this way define real-analytic conformal foliations and that locally all real-analytic conformal foliations arise in this way.

As an example, if we take $f(z_1, z_2, z_3) = z_1 - 1$, then (16) reads

$$(r+is)z_3 - iq - 1 = 0 \qquad \text{whence} \qquad z_3 = \frac{(1+iq)(r-is)}{r^2+s^2}$$

and so

$$U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{1+q^2+r^2+s^2}\begin{pmatrix} 1+q^2-r^2-s^2 \\ 2(qr-s) \\ 2(qs+r) \end{pmatrix}. \tag{17}$$

Integrating this vector field gives the mapping

$$\begin{pmatrix} q \\ r \\ s \end{pmatrix} \longmapsto \frac{1}{r^2+s^2}\begin{pmatrix} (1-q^2-r^2-s^2)r + 2qs \\ (1-q^2-r^2-s^2)s - 2qr \end{pmatrix},$$

which is the Hopf fibration $S^3 \to S^2$, when viewed in stereographic coördinates.

In this particular case there is also a global viewpoint as follows. The zero locus of the function $f(z_1, z_2, z_3) = z_1 - 1$ has a non-singular closure in $\mathbb{CP}_3$, namely

$$S \equiv \{[Z_1, Z_2, Z_3, Z_4] \in \mathbb{CP}_3 \text{ s.t. } Z_1 = Z_4\}, \tag{18}$$

which we may identify as $\mathbb{CP}_2$ via the mapping

$$\mathbb{CP}_2 \ni [W_1, W_2, W_3] \mapsto [W_1, W_2 + W_3, W_2 - W_3, W_1] \in \mathbb{CP}_3.$$

If we restrict $\tau$ to $S$ viewed in this way, then (6) yields

$$[W] \overset{\tau}{\mapsto} \frac{1}{\|W\|^2}\begin{pmatrix} W_2\bar{W}_3 + W_3\bar{W}_2 \\ |W_1|^2 + |W_2|^2 - |W_3|^2 - W_2\bar{W}_3 + W_3\bar{W}_2 \\ -2W_1\bar{W}_3 \end{pmatrix} \in \begin{matrix} \mathbb{R} \\ \oplus \\ \mathbb{C}^2 \end{matrix},$$

which we may, by an elementary SO(5) change of coördinates, rewrite as

$$[W] \overset{\tau}{\mapsto} \frac{1}{\|W\|^2}\begin{pmatrix} |W_1|^2 + |W_2|^2 - |W_3|^2 \\ 2W_1\bar{W}_3 \\ 2W_2\bar{W}_3 \end{pmatrix} \in \begin{matrix} \mathbb{R} \\ \oplus \\ \mathbb{C}^2 \end{matrix} = \mathbb{R}^5. \tag{19}$$

From this viewpoint we see that there is a particular line

$$\mathbb{CP}_1 \cong \{[W_1, W_2, W_3] \in \mathbb{CP}_2 \text{ s.t. } W_3 = 0\},$$

which is sent to $(1, 0, 0, 0, 0) \in S^4$, whilst on the affine coördinate chart complementary to this line, following $\tau$ with stereographic projection (13) gives

$$[w_1, w_2, 1] \overset{\tau}{\mapsto} \frac{1}{|w_1|^2 + |w_2|^2 + 1}\begin{pmatrix} |w_1|^2 + |w_2|^2 - 1 \\ 2w_1 \\ 2w_2 \end{pmatrix} \overset{\sigma}{\mapsto} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

In accordance with Theorem 4, we see that the holomorphic structure on $\mathbb{R}^4$ defined by (18) is just the standard identification $\mathbb{R}^4 = \mathbb{C}^2$. To see the consequences of Corollary 1 in this case, note that

$$\begin{aligned} S \cap Q &= \left\{[Z] \in \mathbb{CP}_3 \text{ s.t. } \begin{array}{l} Z_1\bar{Z}_4 + Z_2\bar{Z}_3 + Z_3\bar{Z}_2 + Z_4\bar{Z}_1 = 0 \\ Z_1 = Z_4 \end{array}\right\}, \\ &\cong \left\{[W] \in \mathbb{CP}_2 \text{ s.t. } |W_1|^2 + |W_2|^2 = |W_3|^2\right\} \end{aligned}$$

namely the standard hyperquadric in $\mathbb{CP}_2$, whereupon (19) reduces to

$$[W] \overset{\tau}{\mapsto} \frac{1}{2|W_3|^2}\begin{pmatrix} 2W_1\bar{W}_3 \\ 2W_2\bar{W}_3 \end{pmatrix} = \begin{pmatrix} W_1/W_3 \\ W_2/W_3 \end{pmatrix} \in S^3 \subset \mathbb{C}^2,$$

which induces the usual identification of this hyperquadric with the round three-sphere. Since $\tau : S \cap Q \to S^3$ is an isomorphism in this case, Corollary 1 yields a global conformal foliation on $S^3$. We have already seen in local coördinates (17) that it is the Clifford foliation whose integral curves define the Hopf fibration $S^3 \to S^2$.

## 8. Explicit constructions and comparisons

The conclusions of Corollary 1 may be written out more explicitly as follows. In §7 we saw that the real-analytic conformal foliations were generated from a holomorphic function $f(z_1, z_2, z_3)$ by solving (16) for $z_3$ as a smooth function of $(q, r, s)$ and then using (15) to define the unit vector field $U(q, r, s)$. Noting that the $z_3$-axis $\{(0, 0, z_3)\}$ is the fibre of the mapping

$$\begin{array}{ccccccc} \mathbb{C}^3 & \hookrightarrow & \mathbb{CP}_3 \setminus \mathbb{I} & \xrightarrow{\tau} & \mathbb{C}^2 & = & \mathbb{R}^4 \\ (z_1, z_2, z_3) & \mapsto & [z_1, z_2, z_3, 1] & \mapsto & \begin{pmatrix} \dfrac{z_2\bar{z}_3 + \bar{z}_1}{z_3\bar{z}_3 + 1} \\ \dfrac{z_1\bar{z}_3 - \bar{z}_2}{z_3\bar{z}_3 + 1} \end{pmatrix} & = & \begin{pmatrix} p + iq \\ r + is \end{pmatrix} \end{array}$$

over the origin, if we seek conformal foliations near the origin in $\mathbb{R}^3$, then we may take the complex surface $S$ in Corollary 1 to be the graph $\{z_3 = \Phi(z_1, z_2)\}$ for a holomorphic function $\Phi = \Phi(z_1, z_2)$ defined near the origin in $\mathbb{C}^2$. In other words, we may take

$$f(z_1, z_2, z_3) = z_3 - \Phi(z_1, z_2).$$

Notice from (15) that precisely $(1, 0, 0)$ is excluded from the possible values of $U$ at the origin (since in our formula (13) for stereographic projection, we have chosen to project from the north pole). Instead, we may insist that $U(0, 0, 0) = (-1, 0, 0)$ and obtain the following recasting of the equivalence of the first and third conditions in Corollary 1.

**Theorem 6.** *There is a 1–1 correspondence between*

- *germs of holomorphic functions $\Phi = \Phi(z_1, z_2)$ defined near and vanishing at the origin in $\mathbb{C}^2$;*
- *germs of real-analytic unit vector fields $U = U(q, r, s)$ defined near and taking on the value $(-1, 0, 0)$ at the origin in $\mathbb{R}^3$ with the property that the foliation defined by $U$ is conformal.*

*This correspondence is induced by the equations*

$$\begin{pmatrix} u \\ v + iw \end{pmatrix} = \frac{1}{|z|^2 + 1} \begin{pmatrix} |z|^2 - 1 \\ 2i\bar{z} \end{pmatrix} \tag{20}$$

*and*

$$z = \Phi((r + is)z - iq, iqz - (r - is)), \tag{21}$$

*where $U = (u, v, w) = (u(q, r, s), v(q, r, s), w(q, r, s))$.*

An alternative proof in one direction may be obtained by implicit differentiation of the equation (21). Since $\Phi$ is holomorphic, the chain rule gives

$$\begin{array}{rcl} \partial z/\partial q &=& \big(-i+(r+is)\partial z/\partial q\big)\partial\Phi/\partial z_1+\big(iz+iq\partial z/\partial q\big)\partial\Phi/\partial z_2 \\ \partial z/\partial r &=& \big(z+(r+is)\partial z/\partial r\big)\partial\Phi/\partial z_1+\big(-1+iq\partial z/\partial r\big)\partial\Phi/\partial z_2 \\ \partial z/\partial s &=& \big(iz+(r+is)\partial z/\partial s\big)\partial\Phi/\partial z_1+\big(i+iq\partial z/\partial s\big)\partial\Phi/\partial z_2 \end{array}$$

from which $\partial\Phi/\partial z_1$ and $\partial\Phi/\partial z_2$ may be eliminated to obtain

$$2z\frac{\partial z}{\partial q}+i(1+z^2)\frac{\partial z}{\partial r}+(1-z^2)\frac{\partial z}{\partial s}=0 \tag{22}$$

and it may be verified that this equation is precisely the condition that the unit vector field $U=(u,v,w)$ defined by (20) generate a conformal foliation (e.g., at $z=0$ we obtain (4) as expected). It is interesting to note that the unit vector field defined by (20) is characterised, up to sign, as orthogonal to the real and imaginary parts of the complex-valued field

$$Z \equiv \begin{pmatrix} 2z \\ i(1+z^2) \\ (1-z^2) \end{pmatrix}.$$

Letting $\omega$ denote the equivalent 1-form

$$\omega \equiv 2z\,dq+i(1+z^2)\,dr+(1-z^2)\,ds, \tag{23}$$

we find that

$$\omega\wedge d\omega = 2i\left(2z\frac{\partial z}{\partial q}+i(1+z^2)\frac{\partial z}{\partial r}+(1-z^2)\frac{\partial z}{\partial s}\right)dq\wedge dr\wedge ds \tag{24}$$

and hence obtain a compact way of rewriting the equations (22). It leads to an alternative geometric interpretation of (22) as follows.

**Lemma 2.** *A non-zero complex-valued real-analytic* 1*-form* $\omega$ *satisfies* $\omega\wedge d\omega=0$ *if and only if* $\omega$ *can be rescaled to be closed, i.e. if and only if there is a smooth function* $\psi$ *such that* $d(e^{\psi}\omega)=0$.

*Proof.* If $\omega$ is real-valued (and then need only be smooth), this criterion is well-known. Specifically, $\omega\wedge d\omega$ is the obstruction to Frobenius integrability of the two-dimensional distribution defined by $\omega$ and the conclusion is evident. In the complex-valued case it is clear from

$$\omega\wedge d(e^{\psi}\omega)=\omega\wedge e^{\psi}(d\psi\wedge\omega+d\omega)=e^{\psi}\omega\wedge d\omega$$

that the vanishing of $\omega\wedge d\omega$ is necessary. When $\omega$ is real-analytic, the argument via Frobenius integrability applies in the complexification (and the function $\psi$ is necessarily real-analytic back on $\mathbb{R}^3$). $\square$

Having chosen a real-analytic $\psi$ such that the rescaled 1-form $e^\psi\omega$ is closed, the Poincaré lemma implies that locally we may find a real-analytic function $\mathbb{R}^3 \supseteq {}^{\text{open}}\Omega \xrightarrow{h} \mathbb{C}$ with $dh = e^\psi\omega$. In particular, since $\omega$ is null, the same is true of $dh$, i.e. $(dh)^2 = 0$. Writing $f$ and $g$ for the real and imaginary parts of $h$, we conclude that

$$\|\nabla f\| = \|\nabla g\| \quad \text{and} \quad \langle \nabla f, \nabla g\rangle = 0.$$

In other words, since in addition $dh$ is nowhere vanishing, we see that $h : \Omega \to \mathbb{C}$ is a *horizontally conformal mapping* [5] and we have shown in Theorem 6 and Lemma 2 that locally all real-analytic horizontally conformal mappings arise in this way.

Inspired by these formulæ, we now present a direct construction of closed null complex-valued 1-forms on $\mathbb{R}^3$ avoiding the use of Frobenius integrability. Firstly, let us observe that the implicit derivation leading to (22) and (24) may be easily accomplished as follows. On the space $\mathbb{C} \times \mathbb{R}^3$ with coördinates $(z, q, r, s)$ let us write

$$z_1 \equiv (r + is)z - iq \quad \text{and} \quad z_2 = iqz - (r - is)$$

and consider the 1-form $\omega$ defined by (23). One easily verifies that

$$\omega \wedge d\omega = 2\, dz \wedge dz_1 \wedge dz_2.$$

Therefore, if $z = z(q, r, s)$ is defined implicitly by

$$z = \Phi(z_1, z_2)$$

for some holomorphic function $\Phi$ of two variables, then

$$dz = d\Phi = \frac{\partial\Phi}{\partial z_1}\, dz_1 + \frac{\partial\Phi}{\partial z_2}\, dz_2$$

and it follows immediately that $\omega \wedge d\omega = 0$, as required.

Now on $\mathbb{C}^2 \times \mathbb{R}^3$ with coördinates $(w, z, q, r, s)$ let us write

$$z_1 \equiv (r + is)z - iqw \quad \text{and} \quad z_2 \equiv iqz - (r - is)w \tag{25}$$

and consider the 1-form

$$\omega \equiv 2wz\, dq + i(w^2 + z^2)\, dr + (w^2 - z^2)\, ds. \tag{26}$$

One easily verifies that

$$d\omega = 2i(dz \wedge dz_1 - dw \wedge dz_2) = 2i\, d(z\, dz_1 - w\, dz_2). \tag{27}$$

Therefore, if $z = z(q, r, s)$ and $w = w(q, r, s)$ are defined implicitly by

$$z\, dz_1 - w\, dz_2 = d(\Xi(z_1, z_2)) = \frac{\partial\Xi}{\partial z_1}\, dz_1 + \frac{\partial\Xi}{\partial z_2}\, dz_2 \tag{28}$$

for some holomorphic function $\Xi$ of two variables, then $d\omega = 0$ is manifest. The formula (26) for $\omega$ ensures that $\omega^2 = 0$ (and also that $\omega$ is non-zero wherever $w$ is non-zero).

Postponing for the moment the precise relationship between these two constructions, we note that this second construction is extremely similar to Nurowski's method [7], which is as follows. With the same notation he rewrites (27) as

$$d\omega = 2i(dz \wedge dz_1 - dw \wedge dz_2) = 2i\,d(z_2\,dw - z_1\,dz),$$

concluding that if $w$ and $z$ are implicitly defined by

$$z_2\,dw - z_1\,dz = d(F(z,w)) = \frac{\partial F}{\partial z}\,dz + \frac{\partial F}{\partial w}\,dw \tag{29}$$

for some holomorphic function $F$ of two variables, then $d\omega = 0$.

The relationship between these two methods is clear. In (28) we give $(z, -w)$ in terms of $(z_1, z_2)$ whereas in (29) it is the other way round:

$$\begin{array}{rcl} z & = & \frac{\partial\Xi}{\partial z_1}(z_1,z_2) \\ \tilde{w} & = & \frac{\partial\Xi}{\partial z_2}(z_1,z_2) \end{array} \quad \text{versus} \quad \begin{array}{rcccl} z_1 & = & -\frac{\partial F}{\partial z}(z,w) & = & \frac{\partial\tilde{F}}{\partial z}(z,\tilde{w}) \\ z_2 & = & \frac{\partial F}{\partial w}(z,w) & = & \frac{\partial\tilde{F}}{\partial\tilde{w}}(z,\tilde{w}), \end{array}$$

where $\tilde{w} \equiv -w$ and $\tilde{F}(z,\tilde{w}) \equiv -F(z,-\tilde{w}) = -F(z,w)$. In either case, the key point is that the Jacobian of the transformation is symmetric

$$\frac{\partial(z,\tilde{w})}{\partial(z_1,z_2)} = \begin{pmatrix} \Xi_{z_1z_1} & \Xi_{z_1z_2} \\ \Xi_{z_1z_2} & \Xi_{z_2z_2} \end{pmatrix} \quad \text{v} \quad \frac{\partial(z_1,z_2)}{\partial(z,\tilde{w})} = \begin{pmatrix} \tilde{F}_{zz} & \tilde{F}_{z\tilde{w}} \\ \tilde{F}_{z\tilde{w}} & \tilde{F}_{\tilde{w}\tilde{w}} \end{pmatrix}. \tag{30}$$

Since the inverse of an invertible symmetric $2 \times 2$ matrix is necessarily symmetric, we may obtain one prescription from the other by inverting the relationship between $(z, -w)$ and $(z_1, z_2)$. This assumes, of course, that this relationship is indeed invertible: we shall come back to this point shortly.

Firstly, we shall explain the precise link between Theorem 6 and the construction determined by formulæ (25), (26), and (28). Recall that Theorem 6 may be viewed as generating all real-analytic null 1-forms $\omega$ near the origin in $\mathbb{R}^3$ satisfying

$$\omega|_{(0,0,0)} = i\,dr + ds \quad \text{and} \quad \omega \wedge d\omega = 0.$$

Hence, to compare with (26) we should suppose that $w$ and $z$ are given as general holomorphic functions of $(z_1, z_2)$, say

$$z = \Xi_1(z_1,z_2) \qquad w = -\Xi_2(z_1,z_2) \tag{31}$$

but insist that $\Xi_1(0,0) = 0$ and $\Xi_2(0,0) = -1$ for then, assuming that our construction makes sense, it will certainly create a real-analytic null 1-form near the origin in $\mathbb{R}^3$ with $\omega|_{(0,0,0)} = i\,dr + ds$ and it remains to explain the geometric origin of the equation $d\omega = 0$ and the precise link with Theorem 6. Regard $(z_1, z_2, z, w)$ as coördinates on $\mathbb{C}^4$ and let

$$\eta \equiv dz \wedge dz_1 - dw \wedge dz_2,$$

be a non-degenerate closed symplectic form on $\mathbb{C}^4 \setminus \{0\}$. Let $\tilde{S}$ denote the complex surface through the point $(0,0,0,1) \in \mathbb{C}^4$ defined by (31).

**Lemma 3.** *There is a holomorphic function $\Xi(z_1, z_2)$ such that locally $\Xi_j = \partial\Xi/\partial z_j$ if and only if $\eta|_{\tilde{S}} = 0$, i.e. if and only if $\tilde{S}$ is Lagrangian.*

*Proof.* We compute

$$\begin{aligned} \eta|_{\tilde{S}} &= \left(\frac{\partial \Xi_1}{\partial z_1}\, dz_1 + \frac{\partial \Xi_1}{\partial z_2}\, dz_2\right) \wedge dz_1 + \left(\frac{\partial \Xi_2}{\partial z_1}\, dz_1 + \frac{\partial \Xi_2}{\partial z_2}\, dz_2\right) \wedge dz_2 \\ &= \frac{\partial \Xi_1}{\partial z_2}\, dz_2 \wedge dz_1 + \frac{\partial \Xi_2}{\partial z_1}\, dz_1 \wedge dz_2 = \left(\frac{\partial \Xi_2}{\partial z_1} - \frac{\partial \Xi_1}{\partial z_2}\right) dz_1 \wedge dz_2, \end{aligned}$$

which vanishes if and only if the holomorphic 1-form $\Xi_1\, dz_1 + \Xi_2\, dz_2$ is closed. This is the case if and only if this 1-form is locally of the form $d\Xi$ for some holomorphic function $\Xi$, as required. □

By construction (31), the surface $\tilde{S}$ passes through the point $(0,0,0,1)$ and is transverse to the $(z,w)$-plane there. It follows that the image of a sufficiently small open subset of $\tilde{S}$ around $(0,0,0,1)$ under the natural projection $\mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}_3$ is a complex surface $S \subset \mathbb{CP}_3$ containing the point $[0,0,0,1]$ and in the local coördinates

$$\mathbb{C}^3 \ni (z_1, z_2, z) \mapsto [z_1, z_2, z, 1] \in \mathbb{CP}_3$$

is transverse to the $z$-axis. We may write such a surface $S$ as a graph

$$S = \{(z_1, z_2, z) \in \mathbb{C}^3 \text{ s.t. } z = \Phi(z_1, z_2)\}$$

for some uniquely determined holomorphic function $\Phi(z_1, z_2)$ defined near and vanishing at the origin. More explicitly, there is a holomorphic function $\Phi(z_1, z_2)$ so that

$$\left.\begin{array}{rcl} z &=& (\partial\Xi/\partial z_1)(z_1, z_2) \\ -w &=& (\partial\Xi/\partial z_2)(z_1, z_2) \end{array}\right\} \implies \frac{z}{w} = \Phi\left(\frac{z_1}{w}, \frac{z_2}{w}\right).$$

If we now substitute for $(z_1, z_2)$ according to (25) and write $z/w$ as $\zeta$, then we find that

$$(28) \implies \zeta = \Phi\left((r + is)\zeta - iq, iq\zeta - (r - is)\right),$$

which coincides with (21) with $\zeta$ substituted for $z$. Moreover, we may rescale the 1-form (26) as

$$\hat{\omega} \equiv \frac{1}{w^2}\omega = 2\zeta\, dq + i(1 + \zeta^2)\, dr + (1 - \zeta^2)\, ds,$$

which coincides with (23) save that again $\zeta$ is substituted for $z$. This is exactly as expected from Theorem 6 and Lemma 2 with $e^{\psi} = w^2$. In other words, we have found the precise geometric link between

- the equivalence of the first and third conditions of Corollary 1, as recast in Theorem 6, and then rewritten in terms of a null 1-form $\omega$ via equations (23) and (24);
- the direct construction of real-analytic closed null 1-forms given by formulæ (25), (26), and (28).

Specifically, the complex surface $S$ appearing in the third condition of Corollary 1 is obtained as the image under the canonical projection $\mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}_3$ of the Lagrangian surface $\tilde{S}$ defined by (28), namely

$$\tilde{S} = \left\{ (z_1, z_2, z, w) \in \mathbb{C}^4 \text{ s.t. } z = \frac{\partial \Xi}{\partial z_1}(z_1, z_2) \text{ and } w = -\frac{\partial \Xi}{\partial z_2}(z_1, z_2) \right\}.$$

In fact, all real-analytic closed null 1-forms are locally so obtained as follows.

**Theorem 7.** *There is a 1–1 correspondence between*

- *germs of complex Lagrangian submanifolds $\tilde{S} \subset \mathbb{C}^4$ with respect to the symplectic form $\eta \equiv dz \wedge dz_1 - dw \wedge dz_2$ passing through $(0, 0, 0, 1)$ and transverse to the $(z, w)$-plane there;*
- *germs of real-analytic closed null 1-forms $\omega$ at the origin in $\mathbb{R}^3$ taking on the value $i\,dr + ds$ there.*

*This correspondence is induced by writing $\tilde{S}$ locally as a graph*

$$z = \Xi_1(z_1, z_2) \qquad w = -\Xi_2(z_1, z_2)$$

*for holomorphic functions $\Xi_1(z_1, z_2)$ and $\Xi_2(z_1, z_2)$, using the equations*

$$\begin{aligned} z &= \Xi_1((r + is)z + (p - iq)w, (p + iq)z - (r - is)w) \\ w &= -\Xi_2((r + is)z + (p - iq)w, (p + iq)z - (r - is)w) \end{aligned}$$

*implicitly to define $z = z(p, q, r, s)$ and $w = w(p, q, r, s)$, restricting the real-analytic functions $z$ and $w$ to $\{p = 0\}$, and finally setting*

$$\omega = 2wz\,dq + i(w^2 + z^2)\,dr + (w^2 - z^2)\,ds. \tag{32}$$

*Proof.* Let us firstly establish a 1–1 correspondence, induced by exactly the same procedure between

- germs of complex submanifolds $\tilde{S} \subset \mathbb{C}^4$ through $(0, 0, 0, 1)$ and transverse to the $(z, w)$-plane there;
- germs of real-analytic null 1-forms $\omega$ at the origin in $\mathbb{R}^3$ taking on the value $i\,dr + ds$ there and satisfying

$$\sigma \wedge d\omega = 0 \quad \forall \text{ 1-forms } \sigma \text{ s.t. } \sigma\omega = 0. \tag{33}$$

For this, let us note that (32) is the general form of a null 1-form on $\mathbb{R}^3$ and that near the origin $z = z(q, r, s)$ and $w = w(q, r, s)$ are uniquely determined by $z(0, 0, 0) = 0$, $w(0, 0, 0) = 1$, and continuity. Then, the 1-forms complex-orthogonal to $\omega$ are spanned by

$$\sigma_1 \equiv z\,dq + iw\,dr + w\,ds \quad \text{and} \quad \sigma_2 \equiv w\,dq + iz\,dr - z\,ds$$

so one easily computes that (33) holds if and only if the operator

$$2wz\frac{\partial}{\partial q} + i(w^2 + z^2)\frac{\partial}{\partial r} + (w^2 - z^2)\frac{\partial}{\partial s} \tag{34}$$

annihilates both $z(q, r, s)$ and $w(q, r, s)$. Now let us ask what it means for a surface $\tilde{S}$ through $(0, 0, 0, 1)$ in $\mathbb{C}^4$ to be complex in terms of the local coördinates $(\alpha, \beta, z, w)$ defined in terms of $(z_1, z_2, z, w)$ by the relations

$$z_1 = \alpha z + \bar{\beta} w \quad \text{and} \quad z_2 = \beta z - \bar{\alpha} w,$$

which are obtained by substituting

$$\alpha \equiv r + is \quad \beta \equiv p + iq \quad z_1 \equiv Z_1 \quad z_2 \equiv Z_2 \quad z \equiv Z_3 \quad w \equiv Z_4$$

into (12). Certainly, we may write $\tilde{S}$ locally as a smooth graph

$$z = z(\alpha, \beta) \qquad w = w(\alpha, \beta).$$

We may then verify, using the chain rule to change coördinates, that $\tilde{S}$ is complex if and only if $z(\alpha, \beta)$ and $w(\alpha, \beta)$ are annihilated by the operators

$$w\frac{\partial}{\partial \alpha} - z\frac{\partial}{\partial \bar{\beta}} \quad \text{and} \quad w\frac{\partial}{\partial \beta} + z\frac{\partial}{\partial \bar{\alpha}}. \tag{35}$$

Recall that, with the conventions of §7, the hyperquadric $Q \subset \mathbb{CP}_3$ is covered by

$$\tilde{Q} \equiv \{(z_1, z_2, z, w) \in \mathbb{C}^4 \text{ s.t. } z_1\bar{w} + z_2\bar{z} + z\bar{z}_2 + w\bar{z}_1 = 0\},$$

a CR hypersurface in $\mathbb{C}^4 \setminus \{0\}$ of Levi-signature $(+, 0, -)$. Sitting over Theorem 2 and similarly proved, suppose $\tilde{M} \subset \tilde{\Omega}^{\text{open}} \subseteq \tilde{Q} \subset \mathbb{C}^4$ is a real-analytic submanifold of real dimension 3 and CR dimension 1. Then $\tilde{M}$ extends into $\mathbb{C}^4$ as a complex submanifold $\tilde{S}$ and this extension is germ-unique. Since,

$$z_1\bar{w} + z_2\bar{z} + z\bar{z}_2 + w\bar{z}_1 = 2(|z|^2 + |w|^2)p$$

we conclude that $\tilde{Q}$ is defined as the zero locus of $p$ in a neighbourhood of $(0, 0, 0, 1)$. Writing (35) more fully, we see that $\tilde{S}$ is complex if and only if the operators

$$w\frac{\partial}{\partial r} - iw\frac{\partial}{\partial s} - z\frac{\partial}{\partial p} - iz\frac{\partial}{\partial q} \quad \text{and} \quad w\frac{\partial}{\partial p} - iw\frac{\partial}{\partial q} + z\frac{\partial}{\partial r} + iz\frac{\partial}{\partial s}$$

annihilate both $z(p,q,r,s)$ and $w(p,q,r,s)$. On $\tilde{Q} = \{p = 0\}$ it follows that the operator (34) annihilates both $z(0,q,r,s)$ and $w(0,q,r,s)$. We have shown that the complex submanifold $\tilde{S}$ gives rise to a real-analytic null 1-form $\omega$ satisfying (33). Conversely, it is readily verified that being annihilated by (34) is exactly the condition that the functions $z(q,r,s)$ and $w(q,r,s)$ define a CR submanifold $\tilde{M}$ of $\tilde{Q}$. (Warning: this is not to say that the defining functions

$$(q,r,s,z,w) \mapsto z - z(q,r,s) \quad \text{and} \quad (q,r,s,z,w) \mapsto w - w(q,r,s)$$

are CR functions on $\tilde{Q}$ but only that the tangent space defined by them intersects the contact distribution defined by

$$(|w|^2 - |z|^2)\, dq + i(z\bar{w} - w\bar{z})\, dr - (z\bar{w} + w\bar{z})\, ds$$

in a complex subspace.) When $z(q,r,s)$ and $z(q,r,s)$ are real-analytic this CR submanifold $\tilde{M}$ extends germ-uniquely into $\mathbb{C}^4$ as $\tilde{S}$, a complex surface: we have now shown the equivalence of the two entities claimed to be equivalent at the beginning of this proof.

To finish the proof it remains to show that $d\omega = 0$ if and only if $\tilde{S}$ is Lagrangian. Since $\eta$ is a holomorphic form of type $(2,0)$ its pullback $\eta|_{\tilde{S}}$ is a holomorphic section of the canonical bundle of $\tilde{S}$. Hence $\eta|_{\tilde{S}}$ vanishes near $\tilde{M}$ if and only if $\eta|_{\tilde{M}} = 0$. However, as we have already implicitly noticed in (27), writing $\tau : \tilde{M} \to \mathbb{R}^3$ for the canonical projection,

$$\tau^*(d\omega) = 2i\eta|_{\tilde{M}}$$

whence $\eta|_{\tilde{M}} = 0$ if and only if $\omega$ is closed. □

As an example of Theorem 7 in action, let us consider the 1-form

$$\omega \equiv (1 + ir + s)(i\, dr + ds) \tag{36}$$

on $\mathbb{R}^3$. Evidently, it is closed, null, and real-analytic. Near the origin, it is of the form (32) for $z(q,r,s) \equiv 0$ and $w(q,r,s) = \sqrt{1 + ir + s}$ where $\sqrt{\ }$ is a branch of square root with $\sqrt{1} = 1$. Theorem 7 says firstly that there are holomorphic functions $\Xi_1(z_1,z_2)$ and $\Xi_2(z_1,z_2)$ such that

$$\begin{aligned} 0 &= \Xi_1((p - iq)\sqrt{1 + ir + s}, -(r - is)\sqrt{1 + ir + s}) \\ \sqrt{1 + ir + s} &= -\Xi_2((p - iq)\sqrt{1 + ir + s}, -(r - is)\sqrt{1 + ir + s}) \end{aligned}$$

and this is clear by taking $\Xi_1 \equiv 0$ and $\Xi_2(z_1,z_2) = -g(z_2)$ where $g(z_2)$ is implicitly defined near $z_2 = 0$ by

$$\zeta = g(i(\zeta^2 - 1)\zeta) \ \text{ near } \zeta = 1.$$

Additionally, in conjunction with Lemma 3, Theorem 7 says that we can find a function $\Xi(z_1,z_2)$ such that $\Xi_j = \partial\Xi/\partial z_j$ and in this example we may take $\Xi(z_1,z_2) = -f(z_2)$ where $f$ is any primitive for $g$.

Returning now to the relationship between our construction, now formulated in Theorem 7, and Nurowski's construction in [7], we see that [7] provides an alternative way of writing a real-analytic closed null 1-form $\omega$ near the origin if and only if the left hand matrix from (30) is invertible. From (26) and (28), however, one easily computes that

$$dz \wedge dw = \omega \wedge (\Xi_{11}\Xi_{22} - \Xi_{12}{}^2)dq \quad \text{at the origin.}$$

It follows that [7] pertains if and only if $dz \wedge dw$ is non-vanishing at the origin (a requirement independent of choice of coördinates on $\mathbb{R}^3$). Generically, this is true but not so for (36) nor for

$$\omega = \frac{i\,dr + ds}{(1 + (ir + s))^2},$$

(which gives the Hopf fibration in another guise). To repair Nurowski's construction one can treat the case $dz \wedge dw = 0$ separately or one can view his construction as giving all real-analytic null 1-forms in a neighbourhood of $\infty \in S^3$.

In [3, Example 2.6] it is explained how to use special holomorphic coördinates on the surface $S \hookrightarrow \mathbb{CP}_3$ from Corollary 1 to produce the general real-analytic horizontally conformal submersion on $\mathbb{R}^3$. These special coördinates are adapted to the contact structure on $\mathbb{CP}_3$ induced by our symplectic form $\eta$. It is unclear whether one can use the lifted Lagrangian surface $\tilde{S} \hookrightarrow \mathbb{C}^4$ from Theorem 7 to generate such adapted coördinates directly.

## 9. A counterexample—the eikonal equation

At the end of §2 we mentioned that there are smooth 3-dimensional CR submanifolds $M \subset Q$ of CR dimension 1 that are not real-analytic. In view of Theorem 5, to find such an $M$ it suffices to find a smooth conformal foliation of $\Omega^{\text{open}} \subseteq \mathbb{R}^3$ that is not real-analytic. To construct such a foliation, let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be any smooth function, let

$$\Gamma \equiv \{(r,s) \in \mathbb{R}^2 \text{ s.t. } s = \phi(r)\}$$

denote its graph, and define

$$\rho(r,s) = \begin{cases} \phantom{-}\text{distance from } (r,s) \text{ to } \Gamma, \text{ if } s \geq \phi(r) \\ -\text{distance from } (r,s) \text{ to } \Gamma, \text{ if } s \leq \phi(r). \end{cases}$$

Evidently, the function $\rho$ is a smooth solution of the eikonal equation $\|\nabla\rho\| = 1$ in a suitable neighbourhood $\mathcal{N}$ of $\Gamma$. It follows that

$$\Omega = \mathbb{R} \times \mathcal{N} \xrightarrow{\pi} \mathbb{R}^2 \quad \text{given by} \quad \pi(q,r,s) = (q, \rho(r,s))$$

is a horizontally conformal submersion. The corresponding conformal foliation is real-analytic if and only if the same is true for our original function $\phi$.

## 10. Further equivalences—the Kerr Theorem

In this section we relate the notions of conformal foliation and shear-free ray congruence from relativity. In [4] this relationship was used to derive the twistor description of conformal foliations, i.e. Corollary 1.

We may regard Euclidean space $\mathbb{R}^3$ as the slice $\{t = 0\}$ in Minkowski space $\mathbb{R}^{3,1}$ equipped with coördinates $(q, r, s, t)$ and pseudo-metric

$$dq^2 + dr^2 + ds^2 - dt^2$$

in the usual way. If $U$ is a smooth unit vector field on $\Omega^{\text{open}} \subseteq \mathbb{R}^3$, then $U + \partial/\partial t$ is a null direction in $\mathbb{R}^{3,1}$ defined and smoothly varying along $U \subset \mathbb{R}^{3,1}$. We consider the region swept out in $\mathbb{R}^{3,1}$ by the null rays emanating from $\Omega$ in the direction given by $U + \partial/\partial t$. For a suitable neighbourhood $\tilde{\Omega}$ of $\Omega$ in $\mathbb{R}^{3,1}$, this construction gives what is called a 'ray congruence' in the relativity literature. It is a smooth family of null geodesics with one such geodesic passing through each point. Locally all such congruences near the slice $\{t = 0\}$ arise in this way and the vector field $U$ defines a conformal foliation if and only if the corresponding ray congruence is 'shear-free' [8] (the 'shear' being a measure of the distortion of circles to ellipses in the normal bundle to the foliation of $\Omega$ defined by $U$).

From Corollary 1 we conclude that the real-analytic shear-free ray congruences defined near $\{t = 0\}$ are locally in 1–1 correspondence with complex hypersurfaces $S \subset \mathbb{CP}_3$ meeting $Q \subset \mathbb{CP}_3$ as discussed in §6. This is the Kerr Theorem [8, Theorem 7.4.8]. Furthermore, the smooth shear-free ray congruences correspond to CR submanifolds of $Q$ of CR dimension 1 as in Theorem 5. This fact is also observed on pages 220–222 of [8]. As detailed in [8], the Kerr Theorem was highly instrumental in Penrose's development of twistor theory.

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