POSITIVE SCALAR CURVATURE, K-AREA AND ESSENTIALNESS

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ABSTRACT. The Lichnerowicz formula yields an index theoretic obstruction to positive scalar curvature metrics on closed spin manifolds. The most general form of this obstruction is due to Rosenberg and takes values in the K-theory of the group C^*-algebra of the fundamental group of the underlying manifold.

We give an overview of recent results clarifying the relation of the Rosenberg index to notions from large scale geometry like enlargeability and essentialness. One central topic will be the concept of K-homology classes of infinite K-area. This notion, which in its original form is due to Gromov, will be systematically used as a link between geometrically defined large scale properties and index theoretic considerations. In particular, we prove essentialness and the non-vanishing of the Rosenberg index for manifolds of infinite K-area.

1. INTRODUCTION AND SUMMARY

One of the fundamental problems in Riemannian geometry is to investigate the types of Riemannian metrics that exist on a given closed smooth manifold. It turns out that the signs of the associated curvature invariants distinguish classes of Riemannian manifolds with considerably different geometric and topological properties. Usually the class of manifolds admitting metrics with negative curvature is "big" and the one with positive curvature is "small". The general existence theorems for negative Ricci curvature metrics [29] and negative scalar curvature metrics [45], the classical theorem of Bonnet-Myers on the finiteness of the fundamental group for closed Riemannian manifolds with positive Ricci curvature, Gromov's Betti number theorem for closed manifolds of non-negative sectional curvature [17], the recent classification of manifolds with positive curvature operators [4] and the proof of the differentiable sphere theorem [5, 6] are prominent illustrations of this empirical fact.

In this context one may formulate two goals: The first is to develop methods to construct Riemannian metrics with distinguished properties on general classes of smooth manifolds. Important examples are the powerful tools in the theory of geometric partial differential equations, the surgery method due to Gromov-Lawson [15] and Schoen-Yau [40] for the construction of positive scalar curvature metrics, and methods based on geometric flow equations. The second deals with the formulation of (computable) obstructions to the existence of Riemannian metrics with specific properties. Often this happens in connection with topological invariants associated to the given manifold like homology and homotopy groups and related data. These two goals are usually not completely seperate from each other in that they can result in overlapping questions, concepts and methods. For example the Ricci flow is used to produce metrics with special properties, which a posteriori determine the topological type of the underlying manifold.

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Here we shall concentrate on the most elementary curvature invariant associated to a Riemannian manifold (M, g), the scalar curvature $\mathrm{scal}_g : M \to \mathbb{R}$. This is usually defined by a twofold contraction of the Riemannian curvature tensor of (M, g), but also has a geometric interpretation in terms of the deviation of the volume growth of geodesic balls in M to geodesic balls in Euclidean space:

$$\mathrm{scal}_g(p) = \frac{\mathrm{vol}_{(M^n,g)}(B_p(\epsilon))}{\mathrm{vol}_{(\mathbb{R}^n,g_{eucl})}(B_0(\epsilon))} = 1 - \frac{\mathrm{scal}_g(p)}{6(n+2)} \cdot \epsilon^2 + O(\epsilon^4)\,.$$

Given a closed smooth manifold M, we shall study whether M admits a Riemannian metric g of positive scalar curvature, i.e. satisfying $\mathrm{scal}_g(p) > 0$ for all $p \in M$. In view of the preceding description and the previous remarks it is on the one hand plausible that the resulting "inside bending of M" at every point might put topological restrictions on M. On the other hand the scalar curvature involves an averaging process over sectional curvatures of M so that a certain variability of the precise geometric shape and the topological properties of M can be expected.

In connection with the positive scalar curvature question both aspects, the obstructive and constructive side, play important roles and have led to a rich body of mathematical insight with connections to index theory, geometric analysis, non-commutative geometry, surgery theory, bordism theory and stable homotopy theory. The paper [37] gives a comprehensive survey of the subject. As such it represents not only an interesting geometric field of its own, but serves as a unifying link between several well established areas in geometry, topology and analysis.

For metrics of positive scalar curvature there are two important obstructions, whose relation to each other is still not completely understood. One is based on the method of minimal hypersurfaces [40] and the second one on the analysis of the Dirac operator and index theory [27].

In some sense the first obstruction is more elementary than the second as it can be shown by a direct calculation [40] that a nonsingular minimal hypersurface in a positive scalar curvature manifold admits itself a metric of positive scalar curvature. In connection with results from geometric measure theory that provide nonsingular minimal hypersurfaces representing codimension one homology classes in manifolds of dimension at most eight [41], this can inductively be used to exclude the existence of positive scalar curvature metrics on tori up to dimension eight, for instance. In higher dimensions the discussion of singularities on minimal hypersurfaces representing codimension one homology classes is a subtle topic and the subject of recent work of Lohkamp [8, 30, 31]. This theme, which has important connections to the positive mass theorem in general relativity, will not be pursued further in our paper.

The second, index theoretic, obstruction is both more restrictive as it requires a spin structure on the underlying manifold (or at least its universal cover), and less elementary as it is based on the Atiyah-Singer index theorem. In its most basic form it says that closed spin manifolds with non-vanishing $\hat{A}$-genus do not admit metrics of positive scalar curvature, the $\hat{A}$-genus being an integer (in the spin case) which depends on the rational Pontrjagin classes of the underlying manifold and its orientation class and hence only on its oriented homeomorphism type.

This obstruction has been refined by Hitchin [25] and Rosenberg [35] and in its most general form takes values in $KO_*(C^*_{\mathbb{R},max}\pi_1(M))$, the K-theory of the real maximal group C^*-algebra of the fundamental group of the underlying manifold. It therefore touches important questions in noncommutative geometry linked to the Baum-Connes and Novikov conjectures. The Gromov-Lawson-Rosenberg conjecture predicts that for closed spin manifolds of dimension at least five the vanishing of this index obstruction is not only necessary, but also sufficient for the existence of a positive scalar curvature metric. Despite the fact that this conjecture is wrong in general [38],

the index obstruction being surpassed by the minimal hypersurface obstruction in some cases, it is a remarkable fact that it holds for simply connected manifolds [42] and - in a stable sense - for all spin manifolds where the assembly map to the K-theory of the real group C^*-algebra of the fundamental group is injective [43], see Theorem 2.4 below. It is up to date unknown whether this conjecture in its original, unstable, form is true for spin manifolds with finite fundamental groups, although in this case the injectivity of the assembly map is known. The index theoretic obstruction to positive scalar curvature will be recalled in Section 2.

Gromov and Lawson used the index of the usual Dirac operator on closed spin manifolds twisted with bundles of small curvature to prove that some manifolds with vanishing $\hat{A}$-genus do not admit positive scalar curvature metrics. For this aim they introduced different kinds of largeness properties for Riemannian manifolds, the most important ones being the notion of enlargeability [16, 18] and infinite K-area [14]. These properties have an asymptotic character in that they require, for each $\epsilon > 0$, the existence of certain geometric structures attached to the underlying manifold which are ϵ-small in an appropriate sense. Precise definitions will be given in Section 2 below.

In light of the common index theoretic origin of these obstructions it is reasonable to expect that they are related to the Rosenberg index. In the papers [19, 20, 21] it is proved that the Rosenberg obstruction indeed subsumes the enlargeability obstruction in the sense that the former is non-zero for enlargeable spin manifolds. Moreover, it was shown in the cited papers that enlargeable manifolds are *essential*, i.e. the classifying maps of their universal covers map the homological fundamental classes to non-zero classes in the homology of the fundamental groups. This notion was introduced by Gromov in [13] in connection with the systolic inequality giving an upper bound of the length of the shortest noncontractible loop in a Riemannian manifold M in terms of the volume of M. In particular it follows that enlargeable manifolds obey Gromov's systolic inequality.

The methods introduced in [20, 21] were applied in [22] to prove some cases of the strong Novikov conjecture. This is implied by the Baum-Connes conjecture and predicts that for discrete groups G the rational assembly map

$$K_*(BG) \otimes \mathbb{Q} \to K_*(C^*_{max}G) \otimes \mathbb{Q}$$

is injective. In loc. cit. it is shown that this map is indeed non-zero on all classes in $K_*(BG) \otimes \mathbb{Q}$ which are detected by classes in the subring generated by $H^{\leq 2}(BG; \mathbb{Q})$. As a corollary higher signatures associated to elements in this subring of $H^*(BG; \mathbb{Q})$ are oriented homotopy invariants, a fact which had been proven first by Mathai [32].

It turns out that the methods of [20, 22] fit very nicely the concept of K-area introduced by Gromov in [14]. It is one purpose of the paper at hand to elaborate on this connection. Our main result, Theorem 3.9, states that K-homology classes of *infinite K-area* in closed manifolds M map nontrivially to $K_*(C^*_{max}\pi_1(M))$ under the assembly map. Generalizing the original concept of Gromov we call a K-homology class of *infinite K-area*, if it can be detected by bundles of finitely generated Hilbert A-modules equipped with holonomy representations which are arbitrarily close to the identity, where A is some C^*-algebra with unit. Precise definitions are given in Section 3 below, see in particular Definition 3.5.

From Theorem 3.9, the main results of the papers [19, 20, 21, 22] are derived fairly easily. More generally we will show that closed spin manifolds whose K-theoretic fundamental classes are of infinite K-area have non-vanishing Rosenberg index (Corollary 3.10) and oriented manifolds

with fundamental classes of infinite K-area are essential (Theorem 5.2). The first result solves a problem stated in the introduction of [28].

In [7] essentialness is discussed from a purely homological point of view. Among other things it is proved that the property of being enlargeable depends only on the image of the homological fundamental class of the underlying manifold in the rational homology of its fundamental group. This flexible formulation allows the construction of manifolds which are essential, but not enlargeable. We will briefly review these results in Section 5. We do not know whether a proof of Theorem 5.2 is feasible which avoids the "infinite product construction" laid out in Section 3. Also, we do not know an essential manifold whose fundamental class is not of infinite K-area, see Question 5.6.

This paper is intended on the one hand as a report on recent results on the positive scalar curvature question in the light of methods from index theory, K-theory and asymptotic geometry as obtained by the author and his coauthors. On the other hand it is meant to establish the point of view that the notion of infinite K-area may serve as a unifying principle for these results, which sometimes allows short and conceptual proofs.

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2. Index obstruction to positive scalar curvature

The Gauß-Bonnet formula implies that closed surfaces with nonpositive Euler characteristic do not admit positive scalar curvature metrics. These comprise all closed surfaces apart from the two sphere and the real projective plane. The mechanism behind this obstruction is the fact that a topological invariant, the Euler characteristic, may be expressed as an integral over a curvature quantity, the Gauß curvature.

In higher dimensions obstructions to positive scalar curvature metrics can be obtained in a more indirect way by use of the Atiyah-Singer index theorem. Let M be a closed smooth oriented manifold of dimension divisible by four. The $\hat{A}$-genus $\hat{A}(M) \in \mathbb{Q}$ of M is obtained by evaluating the $\hat{A}$-polynomial

$$\hat{\mathcal{A}}(M) = 1 - \frac{p_1(M)}{24} + \frac{-4p_2(M) + 7p_1^2(M)}{2^7 \cdot 3^2 \cdot 5} + \dots$$

in the Pontrjagin classes of M on the fundamental class of M. This is an invariant of the homeomorphism type of M by the topological invariance of rational Pontrjagin classes. It is an integer, if M is equipped with a spin structure. This is implied by the fact that in this case the Atiyah-Singer index theorem gives an equation

$$\hat{A}(M) = \mathrm{ind}(D_g^+) = \dim_{\mathbb{C}}(\ker D_g^+) - \dim_{\mathbb{C}}(\mathrm{coker}\, D_g^+)$$

where

$$D_g^{\pm} : \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$$

is the Dirac operator on the complex spinor bundle $S = S^+ \oplus S^- \to M$ of (M, g). Here g is an arbitrary Riemannian metric on M. Due to the appearance of g in the definition of D_g^+, the Atiyah-Singer index theorem relates topological to geometric properties of M. Detailed information on the definition of D_g^+ and spin geometry in general can be found in [26].

The Bochner-Lichnerowicz-Weitzenböck formula [27]

$$D_g^- \circ D_g^+ = \nabla^*\nabla + \frac{\mathrm{scal}_g}{4}$$

implies that if $\mathrm{scal}_g(M) > 0$, then the Dirac operator D_g^+ is invertible and hence $\mathrm{ind}(D_g^+) = 0$. From this we obtain the following fundamental result, see [27, Theorème 2].

Theorem 2.1. *Let M be a closed spin manifold with $\hat{A}(M) \neq 0$. Then M does not admit a metric of positive scalar curvature.*

However, the vanishing of this obstruction is not sufficient for the existence of positive scalar curvature metrics. For example, the $\hat{A}$-genus of the $4k$-dimensional torus T^{4k} vanishes for all $k > 0$, because these manifolds are parallelizable. The index theoretic approach explained above can be refined by considering the twisted Dirac operator

$$D_{g,E}^+ : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$$

where $E \to M$ is some finite dimensional Hermitian vector bundle equipped with a Hermitian connection, cf. [26, Prop. 5.10]. The Atiyah-Singer index theorem computes the index of this operator as

$$\mathrm{ind}(D_{g,E}^+) = \langle \mathcal{A}(M) \cup \mathrm{ch}(E), [M] \rangle \in \mathbb{Z}\,.$$

Due to the appearance of the Chern character $\mathrm{ch}(E) \in H^{ev}(M;\mathbb{Q})$ this number can be non-zero even though $\hat{A}(M)$ vanishes. Unfortunately the nonvanishing of $\mathrm{ind}(D_{g,E}^+)$ does not obstruct positive scalar curvature metrics on M as the following example shows.

Example 2.2. *Let $M^n = S^{4k+2}$. Because the Chern character defines an isomorphism*

$$\mathrm{ch} : K^0(M) \otimes \mathbb{Q} \cong H^{ev}(M;\mathbb{Q})\,,$$

there is a finite dimensional Hermitian bundle $E \to M$ with $\mathrm{ch}_{2k+1}(E) \neq 0 \in H^n(M;\mathbb{Q})$. Hence, for any connection on E and any choice of Riemannian metric g on M, we get $\mathrm{ind}(D_{g,E}^+) \neq 0$ although M admits a metric of positive scalar curvature.

The reason for this is that the Bochner-Lichnerowicz-Weitzenböck formula

$$D_{g,E}^- \circ D_{g,E}^+ = \nabla^*\nabla + \frac{\mathrm{scal}_g}{4} + R^E$$

contains an additional operator $R^E : \Gamma(S^\pm \otimes E) \to \Gamma(S^\pm \otimes E)$ of order 0 which depends on the curvature of the bundle E, cf. [26, Theorem 8.17], so that even in the case when $\mathrm{scal}_g > 0$, the operator $D_{g,E}^+$ may not be invertible.

Gromov and Lawson observed in [16] that this method does still lead to an effective obstruction to positive scalar curvature metrics on M in case that for all ϵ there is a twisting bundle $E \to M$ which satisfies $\|R^E\| < \epsilon$ and whose Chern character contributes nontrivially to $\mathrm{ind}(D_{g,E}^+)$. If in this case M carried a metric g satisfying $\mathrm{scal}_g > 0$ we would find a twisting bundle E with

$$\|R^E\| < \frac{\max_{p\in M} |\,\mathrm{scal}_g(p)|}{4}$$

and the Bochner-Lichnerowicz-Weitzenböck formula would then imply that $\mathrm{ind}\, D_{g,E}^+ = 0$, a contradiction.

For example this reasoning can be used to show that the tori T^n do not admit metrics of positive scalar curvature [16]. A general class of manifold where twisting bundles with the described

property can be found are *enlargeable* manifolds, which were introduced in loc. cit., and manifolds of infinite K-area in the sense of [14]. We will discuss these notions and put them in a general context in Section 3.

The index theoretic point of view was refined by Rosenberg [35, 36]. For any discrete group G the group C^*-algebra C^*G is constructed by completing the group algebra $\mathbb{C}[G]$ with respect to some pre-C^*-norm coming from unitary representations of G on a Hilbert space and taking the induced embedding of $\mathbb{C}[G]$ into the bounded operators on this Hilbert space. More specifically, if one starts with the regular representation of G on the space of square summable functions $\ell^2(G)$ this leads to the *reduced group C^*-algebra* $C^*_{red}G$ and taking all unitary representations of G into account one arrives at the *maximal group C^*-algebra* $C^*_{max}G$. For more details we refer to [3, 24, 44]. These C^*-algebras and their K-theories are in general different [24, Exercise 12.7.7], but the following construction works for both variants, and this is why we drop the subscript from our notation. Note that the left translation action of G on $\mathbb{C}[G]$ induces a left G-action on C^*G.

Let M be a closed spin manifold of even dimension. The Mishchenko-Fomenko bundle $E \to M$ is defined as

$$E = \widetilde{M} \times_{\pi_1(M)} C^*\pi_1(M)\,.$$

It is a locally trivial bundle of free right Hilbert $C^*\pi_1(M)$-modules of rank one in the sense of [39, 44]. The fibreweise inner product is induced by the canonical inner product

$$\begin{aligned} C^*\pi_1(M) \times C^*\pi_1(M) &\to C^*\pi_1(M) \\ (x,y) &\mapsto x^* \cdot y\,. \end{aligned}$$

By construction the bundle $E \to$M can be equipped with a flat connection. Depending on the choice of a metric g on M we obtain a twisted Dirac operator

$$D^+_{g,E} : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$$

with an index

$$\alpha(M) := \mathrm{ind}(D^+_{g,E}) = \text{“}\ker(D^+_{g,E})'' - \text{“}\mathrm{coker}\,(D^+_{g,E})'' \in K_0(C^*\pi_1(M))\,.$$

The group $K_0(C^*\pi_1(M))$ consists of formal differences of finitely generated projective $C^*\pi_1(M)$-modules, cf. [3]. For the infinite dimensional twisting bundle E the modules $\ker(D^+_{g,E})$ and $\mathrm{coker}\,(D^+_{g,E})$ are not in this class in general, but this holds after a $C^*\pi_1(M)$-compact perturbation of $D^+_{g,E}$ which makes this operator to a $C^*\pi_1(M)$-Fredholm operator. For precise formulations and more details on the involved theory we refer the reader to [33], in particular to Theorem 3.4.

It follows again from the Bochner-Lichnerowicz-Weitzenböck formula (which does not contain a curvature term R^E as E is flat) that the index $\alpha(M) \in K_0(C^*\pi_1(M))$ vanishes, if $\mathrm{scal}_g > 0$. Moreover the Mishchenko-Fomenko index theorem [33] implies that - similar to the invariant $\hat{A}(M)$ - the obstruction $\alpha(M)$ does not depend on the choice of a Riemannian metric on M, but only on the oriented homeomorphism type of M.

There is an alternative construction of $\alpha(M)$ based on analytic K-homology [3, 24]. As before let M be a closed spin manifold. We do no longer assume that $n := \dim M$ is even (this assumption was used to simplify the index theoretic considerations above).

In this setting $\alpha(M)$ is defined as the image of the K-theoretic fundamental class $[M]_K \in K_n(M)$, which is induced by the given spin structure, under the composition

$$K_n(M) = K_n^{\pi_1(M)}(\widetilde{M}) \to K_n^{\pi_1(M)}(\underline{E}\pi_1(M)) \xrightarrow{\mu} K_n(C^*\pi_1(M))\,.$$

Here the first map is induced by the $\pi_1(M)$-equivariant classifying map $\widetilde{M} \to \underline{E}\pi_1(M)$ from the universal cover of M to the universal contractible $\pi_1(M)$-space with finite isotropy groups and the second map is the Baum-Connes assembly map, cf. [3].

There is a real analog $\alpha_{\mathbb{R}}(M)$ of the index obstruction $\alpha(M)$ which, for simply connected manifolds, was introduced in the paper [25] and is defined as the image of the KO-theoretic fundamental class $[M]_{KO} \in KO_n(M)$ under the composition

$$KO_n(M) = KO_n^{\pi_1(M)}(\widetilde{M}) \to KO_n^{\pi_1(M)}(\underline{E}\pi_1(M)) \xrightarrow{\mu} KO_n(C^*\pi_1(M))\,.$$

The invariant $\alpha_{\mathbb{R}}(M)$ is more sensitive to differential topological properties of M than $\alpha(M)$. For example it is different from zero on some exotic spheres [25]. A refined variant of the Bochner-Lichnerowicz-Weitzenböck argument shows that $\alpha_{\mathbb{R}}(M) = 0$, if M admits a metric of positive scalar curvature.

In case we are dealing with the reduced group C^*-algebra $C^*_{red}\pi_1(M)$, the vanishing of the α-obstruction is closely linked to properties of the Baum-Connes assembly map

$$\mu_{\mathbb{R}} : KO_*^G(\underline{E}G) \to KO_*(C^*_{red}G)$$

and its complex analog

$$\mu_{\mathbb{C}} : K_*^G(\underline{E}G) \to K_*(C^*_{red}G)\,.$$

According to the Baum-Connes conjecture [3], a central open problem in noncommutative geometry, these two maps are isomorphisms for all discrete groups G.

The following conjecture has played a prominent role in the subject. It expresses the expectation that the Rosenberg obstruction is in some sense optimal.

Conjecture 2.3 (Gromov-Lawson-Rosenberg conjecture)**.** *Let M be a closed spin manifold of dimension at least five and with $\alpha_{\mathbb{R}}(M) = 0$. Then M admits a metric of positive scalar curvature.*

This is true, if M is simply connected [42], but wrong in general [38]. In dimensions two and three analogs of the Gromov-Lawson-Rosenberg conjecture are true [34], but in dimension four there are additional obstructions coming from Seiberg-Witten theory. However, the following stable version of the conjecture conditionally holds in the following sense.

Theorem 2.4 ([43])**.** *Assume that the real Baum-Connes assembly map $\mu_{\mathbb{R}}$ is injective for $\pi_1(M)$ and that $\alpha_{\mathbb{R}}(M) = 0$. Then some manifold of the form $M \times B^8 \times \ldots \times B^8$ admits a metric of positive scalar curvature, where B^8 is an arbitrary eight dimensional spin manifold with $\hat{A}(M) = 1$.*

This result is remarkable, because it is not understood how it can happen that a manifold N does not admit a positive scalar curvature metric, but $N \times B^8$ does. Notice that the vanishing or non-vanishing of $\alpha_{\mathbb{R}}(M)$ is not affected, when M is multiplied with copies of B^8. In this respect Theorem 2.4 establishes $\alpha_{\mathbb{R}}(M)$ as the universal stable index theoretic obstruction to positive scalar curvature metrics.

If the assembly map for the complex group C^*-algebra is injective, then also the rational assembly map

$$K_*^G(\underline{E}G) \otimes \mathbb{Q} = K_*(BG) \otimes \mathbb{Q} \to K_*(C^*_{max}G) \otimes \mathbb{Q}$$

is injective. The strong Novikov conjecture [3] states that here injectivity holds for all discrete groups G.

Therefore it makes sense to single out those manifolds M whose fundamental classes map non-trivially to $K_*(B\pi_1(M)) \otimes \mathbb{Q}$. This motivates the next definition.

Definition 2.5. *A closed* spin^c*-manifold* M^n *is called* (rationally) K-theoretic essential*, if the classifying map* $\phi : M \to B\pi_1(M)$ *for the universal cover of* M *satisfies*

$$\phi_*([M]_K) \neq 0 \in K_n(B\pi_1(M)) \otimes \mathbb{Q}\,,$$

where $[M]_K \in K_n(M)$ *is the* K*-theoretic fundamental class of* M*.*

Conjecture 2.6. *A* K*-theoretic essential spin manifold does not admit a metric of positive scalar curvature.*

It follows from the previous remarks that this conjecture holds, if the rational assembly map for the associated fundamental group is injective. An important consequence of Conjecture 2.6 is the following

Conjecture 2.7 ([16])**.** *Let* M *be a closed aspherical spin manifold. Then* M *does not admit a metric of positive scalar curvature.*

The following is a variation of Definition 2.5 for singular homology.

Definition 2.8 ([13])**.** *A closed oriented manifold* M^n *is called* (rationally) essential*, if the classifying map* $\phi : M \to B\pi_1(M)$ *satisfies*

$$\phi_*([M]_H) \neq 0 \in H_n(B\pi_1(M); \mathbb{Q})\,,$$

where $[M]_H$ *is the fundamental class of* M *in singular homology.*

Recall that the homological Chern character defines an isomorphism

$$\mathrm{ch} : K_{(*)}(M) \otimes \mathbb{Q} \cong H_{(*)}(M; \mathbb{Q})\,,$$

where the brackets in the subscripts indicate that we regard both theories as $\mathbb{Z}/2$-graded. Keeping in mind that for a closed spin^c-manifold M^n we have

$$\mathrm{ch}([M]_K) = [M]_H + c$$

where $c \in H_{<n}(M; \mathbb{Q})$ we see that essential spin^c-manifolds are also K-theoretic essential. Hence it makes sense to formulate the following conjecture.

Conjecture 2.9. *An essential manifold does not admit a metric of positive scalar curvature.*

This seems especially intriguing, if the universal cover of this manifold is not spin (so that index theoretic obstructions are not available). Evidence for the conjecture in this case is provided by the fact that sometimes essential manifolds satisfy a weak form of enlargeability [11, 12].

3. K-AREA FOR HILBERT MODULE BUNDLES

All manifolds in this section are closed, smooth and connected. We recall the following definition from [18].

Definition 3.1. *Let* (M^n, g) *be an orientable Riemannian manifold.*

- *We call* M enlargeable*, if for every* $\epsilon > 0$ *there is a Riemannian cover* $(\overline{M}, \overline{g})$ *of* (M, g) *together with an* ϵ*-Lipschitz map* $f_\epsilon : \overline{M} \to S^n$ *which is constant outside of a compact subset of* $\overline{M}$ *and of non-zero degree.*
- *We call* (M, g) area-enlargeable*, if for every* $\epsilon > 0$ *there is a Riemannian cover* $(\overline{M}, \overline{g})$ *of* (M, g) *together with a smooth map* $f_\epsilon : \overline{M} \to S^n$ *which is* ϵ*-contracting on* 2*-forms, constant outside of a compact subset of* $\overline{M}$ *and of nonzero degree.*

Because M is compact, all Riemannian metric on M are in bi-Lipschitz correspondence and hence both of the above properties are independent on the specific choice of the metric g on M. Enlargeability is therefore a purely topological property of M. Indeed, whether M is enlargeable depends only on the image of the fundamental class of M in the rational group homology of $\pi_1(M)$ under the classifying map, see [7, Corollary 3.5] and Theorem 5.3. We do not know, whether a similiar result holds for the property of being area-enlargeable.

Examples for enlargeable manifolds are manifolds which admit Riemannian metrics of nonpositive sectional curvature. This follows from the Cartan-Hadamard theorem.

Area-enlargeable spin manifolds allow the construction of finite dimensional Hermitian twisting bundles for the Dirac operator as described after Example 2.2. We remark that the index theoretic setting explained there needs to be slightly generalized (relative index theory on open manifolds, see [18]), if infinite covers of M are involved (this case is not excluded in Definition 3.1). These considerations lead to the following theorem.

Theorem 3.2 ([16, 18])**.** *Let M be an area-enlargeable spin manifold. Then M does not admit a metric of positive scalar curvature.*

At this point one might ask whether the enlargeability obstruction is reflected by the Rosenberg obstruction. The twisting bundles $E \to M$ of arbitrarily small curvature going into the obstruction expressed in Theorem 3.2 motivate the notion of *K-area*, see [14].

In this section we will introduce a related property for K-homology classes of M. Examples of such K-homology classes are K-theoretic fundamental classes of area-enlargeable spin manifolds, see Proposition 3.8. The main result in this section, Theorem 3.9, shows that classes in $K_0(M) \otimes \mathbb{Q}$ of infinite K-area are mapped to non-zero classes in $K_0(C^*_{max}\pi_1(M))$ under the assembly map. Together with Proposition 3.8 this implies that the Rosenberg obstruction subsumes the enlargeability obstruction of Gromov and Lawson:

Theorem 3.3 ([20, 21])**.** *Let M^n be an area-enlargeable spin manifold. Then the Rosenberg index $\alpha(M) \in K_n(C^*_{max}\pi_1(M))$ is different from zero.*

A convenient setting for our discussion is provided by Kasparov's KK-theory, cf. [3], which associates to any pair of separable C^*-algebras S and T an abelian group $KK(S,T)$. We work over the field of complex numbers and will restrict attention to the special cases $S = C(M)$, $T = \mathbb{C}$ and $S = \mathbb{C}$, $T = C(M) \otimes A$ for a seperable unital C^*-algebra A. Here we will work only with ungraded KK-groups.

According to the analytic description of K-homology [24] we have a canonical identification

$$KK(C(M), \mathbb{C}) \cong K_0(M)$$

the 0-th K-homology of M which, for example, can be defined homotopy theoretically as the homology theory dual to topological K-theory [1].

Elements in $KK(A,B)$ are represented by *Fredholm triples* (E, ϕ, F) where E is a countably generated graded Hilbert B-module, $\phi : A \to B(E)$ is a graded $*$-homomorphism ($B(E)$ is the graded C^*-algebra of graded adjointable bounded B-module homomorphisms $E \to E$) and $F \in B(E)$ is an operator of degree 1 such that the commutator $[F, \phi(a)]$ and the operators $(F^2 - \mathrm{id}_E)\phi(a)$ and $(F - F^*)\phi(a)$ are B-compact for all $a \in A$. In our context we will be dealing with Fredholm triples of very special forms which will be specified in a moment. The reader who is interested in more information on the notion of Hilbert modules and the construction of Kasparov KK-theory can consult the sources [3, 44].

A typical situation arises, if M is a spin manifold of even dimension equipped with a Riemannian metric g. The Dirac operator from Section 2

$$D_g : \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$$

is a symmetric graded first-order elliptic differential operator of order one. It therefore gives rise to an element $[D_g] \in KK(C(M), \mathbb{C})$ represented by the Fredholm triple $(L^2(S), \phi, F)$ where $L^2(S)$ is the space of L^2-sections of the bundle $S^+ \oplus S^-$, $\phi : C(M) \to B(L^2(S))$ is the standard representation as multiplication operators and $F \in B(E)$ is a bounded operator which is obtained from D_g by functional calculus.

The construction works more generally for symmetric graded elliptic differential operators on graded smooth Hermitian vector bundles over M, cf. [24, Theorem 10.6.5]. In this way we may think of elements in $KK(C(M), \mathbb{C}) = K_0(M)$ as a kind of generalized symmetric elliptic differential operators over M. In this picture, the index of a graded elliptic differential operator translates to the image of the KK-class represented by this operator under the map

$$K_0(M) \to K_0(*) = \mathbb{Z}$$

which is induced by the unique map $M \to *$.

If $E \to M$ is a (finite dimensional) Hermitian bundle with a Hermitian connection we obtain the twisted Dirac operator

$$D_{g,E} : \Gamma(S^{\pm} \otimes E) \to \Gamma(S^{\mp} \otimes E)$$

which is again a symmetric graded elliptic differential operator and hence has an index in $\mathbb{Z}$. The index of the twisted operator $D_{g,E}$ has the following description in KK-theory, cf. [3]. The bundle $E \to M$ represents a class $[E]$ in topological K-theory $K^0(M)$, which can be canonically identified with $KK(\mathbb{C}, C(M))$. The element $[E] \in KK(\mathbb{C}, C(M))$ is represented by the Fredholm triple $(\Gamma(E), \phi, 0)$ where $\Gamma(E)$ is the $C(M)$-module of continuous sections $M \to E$ equipped with the $C(M)$-valued inner product given by fibrewise application of the Hermitian inner product on E and $\phi : \mathbb{C} \hookrightarrow B(\Gamma(E))$ is the standard embedding.

Under the Kasparov product map [3]

$$KK(\mathbb{C}, C(M)) \times KK(C(M), \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$$

which in this case corresponds to the usual Kronecker product pairing of K-homology and topological K-theory (i.e. K-cohomology)

$$\begin{aligned} K^0(M) \times K_0(M) &\to \mathbb{Z} \\ (c, h) &\mapsto \langle c, h \rangle \end{aligned}$$

the pair $([E], [D_g])$ is sent to $\mathrm{ind}(D_{g,E}) \in \mathbb{Z}$.

This point of view may be generalized by allowing twisting bundles $E \to M$ which are locally trivial bundles of finitely generated right Hilbert A-modules where A is a unital C^*-algebra.

We recall [39, 44] that each finitely generated Hilbert A-module bundle $E \to M$ is isomorphic to an orthogonal direct summand of a trivial A-module bundle $M \times A^n \to M$ where A^n carries the canonical A-valued inner product

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle \mapsto a_1^* b_1 + \ldots + a_n^* b_n \,.$$

We can take this description as definition of finitely generated Hilbert A-module bundles.

Let $E \to M$ be a finitely generated Hilbert A-module bundle. We associate to $E \to M$ a KK-class $[E] \in KK(\mathbb{C}, C(M) \otimes A)$ as follows. First note that the space $\Gamma(E)$ of continuous

sections in E is a finitely generated Hilbert $(C(M) \otimes A)$-module and the identity $\Gamma(E) \to \Gamma(E)$ is a $(C(M) \otimes A)$-compact (indeed finite rank) operator by a partition of unity argument. Therefore the triple $(\Gamma(E), \phi, 0)$, where $\phi : \mathbb{C} \to B(\Gamma(E))$ is the standard embedding, defines an element in $KK(\mathbb{C}, C(M) \otimes A)$.

Using the Kasparov-product (which we again interprete as a Kronecker product pairing)

$$\begin{aligned} KK(\mathbb{C}, C(M) \otimes A) \times KK(C(M), \mathbb{C}) &\to KK(\mathbb{C}, A) = K_0(A) \\ (c, h) &\mapsto \langle c, h \rangle \end{aligned}$$

we have a pairing of generalized elliptic differential operators on M and finitely generated Hilbert A-module bundles.

If M is a spin manifold of even dimension, then the element in $K_0(A)$ obtained by evaluating the pair $([E], [D_g])$ can be interpreted as the index of the Dirac operator D_g twisted with the bundle E, cf. [3]. Hence, for the special case when $E \to M$ is the Mishchenko-Fomenko bundle, this class

$$\langle [E], [D_g] \rangle \in K_0(C^* \pi_1(M))$$

coincides with the Rosenberg index $\alpha(M)$ defined in Section 2.

We will now, for general M, single out those K-homology classes $h \in K_0(M)$ which can be detected by finitely generated Hilbert A-module bundles of arbitrary small curvature. In order to avoid the discussion of smooth bundles and curvature notions we express this condition in terms of holonomy representations of the path groupoid of M.

Recall that the *path groupoid* $\mathcal{P}_1(M)$ of M has as objects the points in M and as morphisms $\operatorname{Mor} \mathcal{P}_1(M)(x, y)$ the set of piecewise smooth paths $[0, 1] \to M$ connecting x and y. This is a topological category.

Let A be a unital C^*-algebra and let $E \to M$ be a finitely generated Hilbert A-module bundle. The *transport category* $\mathcal{T}(E \to M)$ has as objects the points in M and as set of morphisms

$$\mathcal{T}(E \to M)(x, y) := \operatorname{Iso}_A(E_x, E_y) .$$

This is again a topological category where the set of morphisms is topologized by choosing local trivializations in order to identify nearby fibres of $E \to M$ and the set of Hilbert A-module isomorphisms $\operatorname{Iso}_A(E_x, E_y)$ is topologized as a subset of the Banach space $\operatorname{Hom}_A(E_x, E_y)$.

A *holonomy representation* on $E \to M$ is a continous functor

$$\mathcal{H} : \mathcal{P}_1(M) \to \mathcal{T}(E \to M) .$$

It is called *ϵ-close to the identity*, if for each $x \in M$ and each closed contractible loop $\gamma \in \operatorname{Mor}(\mathcal{P}_1(M))$ based at $x \in M$ we have

$$\| \mathcal{H}(\gamma) - \operatorname{id}_{E_x} \| < \epsilon \cdot \ell(\gamma) .$$

Here we use the operator norm on the left hand side and denote by $\ell(\gamma)$ the length of γ.

The following proposition establishes a link to the notion to parallel transport in differential geometry.

Proposition 3.4. *For each $n \geq 0$, there is a real constant $C(n) > 0$, which depends only on $n = \dim M$, with the following property. Let $E \to M$ be a finite dimensional smooth Hermitian bundle of rank d equipped with a smooth Hermitian connection ∇ whose curvature $\eta \in \Omega^2(M; \mathfrak{u}(d))$ is norm bounded by ϵ. Then the parallel transport with respect to ∇ is $(C(n) \cdot \epsilon)$-close to the identity.*

Proof. We only sketch the argument, for more details see [20]. It is enough to prove the assertion, if $\gamma : [0,1] \to M$ is a piecewise smooth loop which is contained in a closed cooordinate neighbourhood $D \subset M$ which is diffeomorphic to the n-dimensional cube $D^n \subset \mathbb{R}^n$. Furthermore we can assume that the length $\ell(\gamma)$ of γ is bounded above by a fixed constant $\ell > 0$. We construct a trivialization of $E|_D \to M$ by choosing a diffeomorphism $E|_{(0,\ldots,0)} \cong \mathbb{C}^d$ and extending this trivialization inductively into each of the n coordinate directions by parallel transport. We denote the induced connection one form with respect to this trivialization by $\omega \in \Omega^1(D; \mathfrak{u}(d))$. Now [20, Lemma 2.3] shows the following bound for the norm of ω (measured with respect to the operator norm on $\mathfrak{u}(d)$ and the maximum norms on the unit sphere bundles of T^*M and $\Lambda^2 M$) :

$$\|\omega\| \leq n\|\eta\| \,.$$

Let $\phi : [0,1] \to E$ be a parallel vector field along γ. Using the above trivialization we consider ϕ as a smooth map $\phi : [0,1] \to \mathbb{C}^d$. It satisfies the differential equation

$$\phi'(t) + (\omega_{\gamma(t)}(\gamma'(t))) \cdot \phi(t) = 0$$

and it follows that

$$\|\phi(1) - \phi(0)\| \leq \exp(2\ell(\gamma)\|\omega\|) \cdot \|\phi(0)\| \,.$$

Because $\exp : \mathbb{C}^d \to \mathbb{C}^d$ is uniformly Lipschitz continuous on each bounded neighbourhood of 0, the claim follows. □

Definition 3.5. *Let $h \in K_0(M) \otimes \mathbb{Q}$. We say that h has* infinite K-area*, if for each $\epsilon > 0$ there is a unital C^*-algebra A and a finitely generated Hilbert A-module bundle $E \to M$ which carries a holonomy representation which is ϵ-close to the identity and satisfies*

$$\langle [E], h \rangle \neq 0 \in K_0(A) \otimes \mathbb{Q}$$

where $[E] \in KK(\mathbb{C}, C(M) \otimes A)$ is the element represented by $E \to M$. If h is not of infinite K-area, we say that it is of finite K-area*.*

A class $h \in H_{ev}(M; \mathbb{Q})$ is defined to be of infinite K-area, if the class $\mathrm{ch}^{-1}(h) \in K_0(M) \otimes \mathbb{Q}$ is of infinite K-area.

The notion of finitely generated Hilbert A-module bundles can be generalized to C^*-algebras without a unit. However, in the context of Definition 3.5 this does not lead to a wider class of K-homology classes of infinite K-area, since any finitely generated Hilbert A-module bundle is in a trivial way also a finitely generated Hilbert A^+-module bundle over the unitalization A^+ of A. This procedure does not change the property of $\langle [E], h \rangle$ being zero or not (in the rationalization of the K-homology of A and A^+ respectively).

Our Definition 3.5 is inspired by the preprint [28] where the property of finite K-area is investigated from a homological perspective. In contrast to the approach in loc. cit. and in the original source [14] we do not further quantify classes of finite K-area, since we will be concentrating on the property of infinite K-area as one instance of a largeness property besides enlargeability and essentialness. The discussion in [28] and other previous papers is restricted to finite dimensional smooth Hermitian vector bundles as twisting bundles $E \to M$ occuring in our Definition 3.5. Our more general setting is needed in connection with enlargeability questions and applications to the strong Novikov conjecture, see Section 4.

By a suspension procedure we can also define classes in $h \in K_1(M) \otimes \mathbb{Q}$ of infinite K-area by requiring that the class $h \times [S^1]_K \in K_0(M \times S^1) \otimes \mathbb{Q}$ be of infinite K-area, with an arbitrary choice of a K-theoretic fundamental class $[S^1]_K \in K_1(S^1)$. Note that with this definition the

class $[S^1]_K \in K_1(S^1) \otimes \mathbb{Q}$ is of infinite K-area. The following discussion can be extended to K-homology classes of odd degree, but we restrict our exposition to classes in $K_0(M) \otimes \mathbb{Q}$ for simplicity.

The following two facts are similiar to Propositions 2 and 3 in [28], cf. also Proposition 3.4. and Theorem 3.6 in [7].

Proposition 3.6. *The elements of finite K-area in $K_0(M) \otimes \mathbb{Q}$ form a rational vector subspace.*

Proof. If $h \in K_0(M) \otimes \mathbb{Q}$ is of infinite K-area, then the same is true for any nonzero rational multiple of h. This implies that the set of elements of finite K-area is closed under scalar multiplication. Now assume that $h + h'$ is of infinite K-area. It follows from Definition 3.5 that either h or h' are of infinite K-area (choose $\epsilon := \frac{1}{k}$ with $k = 1, 2, \ldots$). This shows that the set of elements of finite K-area is closed under addition. □

Proposition 3.7. *If $f : M \to M'$ is a continuous map, then $f_* : K_0(M) \otimes \mathbb{Q} \to K_0(M') \otimes \mathbb{Q}$ restricts to a map between vector subspaces consisting of elements of finite K-area. In particular, the vector subspace of elements of finite K-area in $K_0(M) \otimes \mathbb{Q}$ is an invariant of the homotopy type of M.*

We will return to homological aspects of largeness properties in Section 5. The notion of infinite K-area is illustrated by the following examples.

Assume that M is an oriented manifold of even dimension $2n$ which has infinite K-area in the sense of [14]. By definition this means that for each $\epsilon > 0$ there is a finite dimensional smooth Hermitian vector bundle $V \to M$ with a Hermitian connection whose curvature form in $\Omega^2(M; \mathfrak{u}(d))$ (here $d = \operatorname{rk} V$) has norm smaller than ϵ and with at least one nonvanishing Chern number.

Using linear combinations of tensor products and exterior products of V one can show that there is a Hermitian bundle $E \to M$ with Hermitian connection whose curvature has norm smaller than $C \cdot \epsilon$ (where C is a bound which depends only on $\dim M$) and which satisfies

$$\langle \operatorname{ch}(E), \operatorname{PD}(\mathcal{A}(M)) \rangle \neq 0 \in H_0(M; \mathbb{Q}) \,.$$

where $\operatorname{PD}(\mathcal{A}(M))$ is the Poincaré dual of the $\mathcal{A}$-polynomial of M in $H_{ev}(M; \mathbb{Q})$.

The precise argument is carried out in [10] where the following fact is shown: There is a number N depending only on $\dim M$ with the following property: Assume that $V \to M$ is a complex vector bundle and assume that all bundles $V' \to M$ which may be constructed out of V be at most N operations of the form direct sum, tensor product and exterior product satisfy

$$\langle \operatorname{ch}(V'), \operatorname{PD}(\mathcal{A}(M)) \rangle = 0 \in H_0(M; \mathbb{Q}) \,.$$

Then all Chern numbers of $V \to M$ are zero.

Considering Hermitian vector bundles as finitely generated projective Hilbert $\mathbb{C}$-module bundles this means in the language of Definition 3.5 that the class $\operatorname{PD}(\mathcal{A}(M)) \in H_{ev}(M; \mathbb{Q})$ has infinite K-area (here we use that the Chern character is compatible with the Kronecker pairing). If M is equipped with a spin^c-structure, this element is equal to the K-theoretic fundamental class $[M]_K$ of M and hence we have shown that under the stated assumption, the class $[M]_K$ has infinite K-area in our sense.

By a similar argument one shows that if M has infinite K-area in the sense of Gromov, then

$$[M]_H \in H_{2n}(M; \mathbb{Q})$$

has infinite K-area where $[M]_H \in H_{2n}(M; \mathbb{Q})$ is the homological fundamental class of M.

As a second example, cf. [20, Section 4], assume that M is area-enlargeable and that the covers $\overline{M} \to M$ in Definition 3.1 can always be assumed to be finite. By pulling back a suitable Hermitian bundle $V \to S^{2n}$ with connection to these covers along the maps $f_\epsilon : \overline{M} \to S^{2n}$ and wrapping these bundles up to get finite dimensional Hermitian bundles $E \to M$ with small curvature, one can show that the classes $[M]_H \in H_{2n}(M;\mathbb{Q})$ and $[M]_K \in K_0(M) \otimes \mathbb{Q}$ (if M is spin^c) have infinite K-area.

More generally assume that M^{2n} is area-enlargeable with no restriction on the covers $\overline{M} \to M$. Then [21, Proposition 1.5] implies that the classes $[M]_H$ and $[M]_K$ (if M is spin^c) have infinite K-area. In this case we get infinite dimensional bundles $E \to M$ which shows the usefulness of Definition 3.5 in the general context of Hilbert A-module bundles where A is a C^*-algebra different from $\mathbb{C}$.

For later reference we state the last observation seperately.

Proposition 3.8. *Let M be area-enlargeable and of even dimension. Then the K-area of $[M]_H$ is infinite. If M is spin^c then also the K-area of $[M]_K$ is infinite.*

We are now ready to state our main result. We denote by

$$\alpha : K_0(M) \to K_0(B\pi_1(M)) \xrightarrow{\mu} K_0(C^*_{max}\pi_1(M))$$

the composition of the map induced by the classifying map $M \to B\pi_1(M)$ and the assembly map. If M is a spin manifold of even dimension, note the equations

$$\alpha(M) = \alpha([M]_K)\,.$$

(the left hand side is the Rosenberg index) and

$$\alpha(h) = \langle [E], h\rangle \in K_0(C^*_{max}\pi_1(M)) \otimes \mathbb{Q}$$

for all $h \in K_0(M) \otimes \mathbb{Q}$ where $E \to M$ is the Mishchenko-Fomenko bundle for $C^*_{max}\pi_1(M)$.

The following is our main result.

Theorem 3.9. *Let M be a closed connected smooth manifold and let $h \in K_0(M) \otimes \mathbb{Q}$ be of infinite K-area. Then*

$$\alpha(h) \neq 0 \in K_0(C^*_{max}\pi_1(X)) \otimes \mathbb{Q}\,.$$

We note the following implication for the Rosenberg index.

Corollary 3.10. *Let M be a closed spin manifold of even dimension whose K-theoretic fundamental class has infinite K-area. Then*

$$\alpha(M) \neq 0 \in K_0(C^*_{max}\pi_1(M))\,.$$

In particular, closed even-dimensional spin manifolds of infinite K-area in the sense of Gromov [14] *have nonvanishing Rosenberg index. (A similar result holds, if M is odd dimensional.)*

The proof of Theorem 3.9 is based on the construction of "infinite product bundles" from [20]. We shall explain how this construction fits the setting of the paper at hand.

Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of finitely generated Hilbert (A_k)-module bundles over M, where $(A_k)_{k=1,2,\ldots}$ is a sequence of unital C^*-algebras. We assume that the fibre of E_k is isomorphic (as a Hilbert A_k-module) to q_kA_k where $q_k \in A_k$ is a (self adjoint) projection. This assumption is important for our construction. In general the fibre of E_k is of the form $q \cdot (A_k)^n$ for some n with a projection $q \in \mathrm{Mat}(A_k, n)$. In this case we use the same transition functions as for E_k to

construct a Hilbert $\mathrm{Mat}(A_k, n)$-module bundle of the required form. By Morita equivalence of A_k and $\mathrm{Mat}(A_k, n)$ this does not affect the K-theoretic considerations relevant for our discussion.

We consider the unital C^*-algebra A consisting of norm bounded sequences

$$(a_k)_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} A_k$$

and wish to construct a Hilbert A-module bundle $E \to M$ with fibre qA, where $q = (q_k)$ is the product of the projections q_k, by taking the "infinite product" of the bundles E_k. However, when taking the infinite product of the transition functions for the bundles E_k, this does not result in a continuous bundle over M in general. The following example indeed shows that an infinite product construction of this kind may be obstructed by topological properties of the bundles E_k.

Example 3.11. *Let $E_k \to S^2$ be the Hermitian line bundle with Chern number k. Assume we have a Hilbert A-module bundle $E \to S^2$ over the C^*-algebra $A = \prod_k \mathbb{C}$ (which is equal to the standard seperable Hilbert space) with typical fibre $V = \prod_k \mathbb{C}$ and Lipschitz continous transition functions in diagonal form so that the kth component of this bundle is isomorphic to E_k as a complex line bundle.*

Restricting the transition functions of E to the single factors leads to trivializations for the bundles $E_k \to S^2$ whose transition functions have uniformly (in k) bounded Lipschitz constants. This implies that the Euler numbers of the bundles E_k are bounded, contrary to our assumption.

This example indicates that we need to be able to choose Lipschitz trivializations of the bundles E_k so that the resulting transition functions have uniformly bounded Lipschitz constants. This can be achieved as follows.

Proposition 3.12. *Assume that each bundle $E_k \to M$ is equipped with a holonomy representation $\mathcal{H}_k$ so that $\mathcal{H}_k$ is ϵ-close to the identity for a constant ϵ which is independent of k. Then there is a finitely generated Hilbert A-module bundle $V \to M$ with transition functions in diagonal form and so that the kth component of this bundle is isomorphic to E_k as an A_k-Hilbert module bundle.*

Proof. Let (U_i) be a finite open cover of M where each U_i is diffeomorphic to the n-dimensional cube D^n. Over each U_i we choose a trivialization of E_k by an inductive application of parallel transport (prescribed by the holonomy representation $\mathcal{H}_k$) in the n coordinate directions, cf. the proof of Proposition 3.4. This leads to local trivializations of all the bundles E_k whose transition maps have uniformly bounded (in i and k) Lipschitz constants. Hence the product of these transition maps can be used to define the Hilbert A-module bundle $V \to M$ as required. □

We remark that the product bundle $V \to M$ is a bundle of finitely generated Hilbert A-modules isomorphic to qA by our assumption that E_k has typical fibre $q_k A_k$.

For the proof of Theorem 3.9 we assume that $h \in K_0(M) \otimes \mathbb{Q}$ and that (E_k) is a sequence of Hilbert A_k-module bundles with fibres $q_k A_k$ so that $\langle [E_k], h\rangle \neq 0 \in K_0(A_k) \otimes \mathbb{Q}$ for all k. Furthermore we assume that E_k is equipped with a holonomy representation $\mathcal{H}_k$ which is $1/k$-close to the identity.

We consider the Hilbert A-module bundle $V \to M$ constructed in Proposition 3.12.

Starting from V we can construct various other Hilbert module bundles over M as follows. There are canonical C^*-algebra morphisms

$$\psi_k : A \to A_k$$

given by projecting onto the kth component. Moreover we set

$$A' := \bigoplus_{k=1}^{\infty} A_k \subset A\,,$$

the closed two sided ideal consisting of sequences in A tending to zero and

$$Q = A/A'\,,$$

the quotient C^*-algebra. We denote by

$$\psi : A \to Q$$

the quotient map.

We obtain canonical Hilbert A_k-bundle isomorphism

$$E_k \cong V \otimes A_k$$

and a Hilbert Q-module bundle

$$W := V \otimes Q$$

with typical fibre qQ, where we identify $q \in A$ and its image in Q.

The following fact is crucial

Proposition 3.13. *The bundle W has local trivializations with locally constant transition maps. More precisely, it can be written as an associated bundle*

$$W = \widetilde{M} \times_{\pi_1(M)} Q$$

for some unitary representation $\pi_1(M) \to \mathrm{Hom}_Q(qQ, qQ)$.

Proof. The holonomy representations $\mathcal{H}_k$ induce a holonomy representation on W which is equal to the identity on each closed contractible loop in M. This follows because the holonomy representation $\mathcal{H}_k$ is $1/k$-close to the identity. Using this holonomy representation on W we construct the desired local trivializations of W. □

These facts in combination with naturality properties of Kasparov KK-theory allow us to show that $\alpha(h) \neq 0 \in K_0(C^*_{max}\pi_1(M)) \otimes \mathbb{Q}$. The holonomy representation for the bundle W induces an involutive map

$$\pi_1(M) \to \mathrm{Hom}_Q(qQ, qQ) = qQq$$

with values in the unitaries of the C^*-algebra qQq. Hence, by the universal property of $C^*_{max}\pi_1(M)$ we get an induced map of C^*-algebras

$$\phi : C^*_{max}\pi_1(M) \to qQq \hookrightarrow Q\,.$$

Note that this step is not possible in general, if we used the reduced C^*-algebra $C^*_{red}\pi_1(M)$ instead. Let $E = \widetilde{M} \times_{\pi_1(M)} C^*_{max}\pi_1(M) \to M$ be the Mishchenko-Fomenko bundle.

We study the commutative diagram

$$\begin{array}{ccccc} K_0(M) & \xrightarrow{\langle [E], - \rangle} & K_0(C^*_{max}\pi_1(M)) & \xrightarrow{\phi_*} & K_0(Q) \\ \downarrow = & & & & \downarrow = \\ K_0(M) & \xrightarrow{\langle [V], - \rangle} & K_0(A) & \xrightarrow{\psi_*} & K_0(Q) \end{array}$$

The composition

$$K_0(M) \stackrel{\langle [V]_1 - \rangle}{\to} K_0(A) \stackrel{(\psi_k)_*}{\to} K_0(A_k)$$

sends the element h to $\langle [E_k], h\rangle \in K_0(A_k)$ which is different from zero by assumption. This implies that under the map

$$\begin{aligned} \chi : K_0(A) &\to \prod_k K_0(A_k) \\ z &\mapsto ((\psi_k)_*(z))_{k=1,2,\ldots} \end{aligned}$$

the element $z := \langle [V], h\rangle$ is sent to a sequence all of whose components are different from zero. We will conclude from this that also $\psi_*(z) \neq 0$ finishing the proof of Theorem 3.9.

Consider the long exact sequence in K-theory induced by the short exact sequence

$$0 \to A' \to A \to Q \to 0\,.$$

Using the fact that K-theory commutes with direct limits we have a canonical isomorphism

$$K_0(A') \cong \bigoplus_k K_0(A_k)\,.$$

Assume that $\psi_*(z) = 0$. This implies that χ maps z to a sequence $(z_k) \in \prod_k K_0(A_k)$ with only finitely many nonzero entries. But this contradicts our calculation carried out before. Hence $\psi_*(z) \neq 0$.

4. The strong Novikov conjecture

The method presented in the previous paragraph can be used to prove a special case of the strong Novikov conjecture. Let G be a discrete group and let $\Lambda^*(G) \subset H^*(BG;\mathbb{Q})$ be the subring generated by $H^{\leq 2}(BG;\mathbb{Q})$

Theorem 4.1 ([22])**.** *Let $h \in K_0(BG)\otimes\mathbb{Q}$ be a K-homology class with the following property: There is a class $c \in \Lambda^*(G)$ so that $\langle c, \mathrm{ch}(h)\rangle \neq 0 \in H_0(BG;\mathbb{Q}) = \mathbb{Q}$. Then under the assembly map*

$$K_0(BG)\otimes\mathbb{Q} \to K_0(C^*_{max}G)\otimes\mathbb{Q}$$

the element h is sent to a a non-zero class.

As a corollary one obtains the following special case of the classical Novikov conjecture.

Corollary 4.2 ([9, 32])**.** *Let M be a connected closed oriented manifold, let G be a discrete group and let $f : M \to BG$ be a continuous map. Then for all $c \in \Lambda^*(c)$ the higher signature $\langle \mathcal{L}(M) \cup f^*(c), [M]\rangle$ is an oriented homotopy invariant, where $\mathcal{L}(M)$ denotes the Hirzebruch L-polynomial.*

We will establish Theorem 4.1 as a fairly straightforward cosequence of Theorem 3.9. It illustrates again the flexibility of the notion of infinite K-area in Definition 3.5 based on Hilbert module bundles. For simplicity we restrict to the case that there is a class $c \in H^2(BG;\mathbb{Q})$ with $\langle c, \mathrm{ch}(h)\rangle \neq 0$.

Using the description of K-homology due to Baum and Douglas [2] there is a closed spin manifold M of even dimension together with a finite dimensional complex vector bundle $V \to M$ and a continuous map $f : M \to BG$ so that

$$f_*([V] \cap [M]_K) = h\,.$$

Here we regard again $V \to M$ as an element in $K^0(M)$ and use the cap product pairing

$$\cap : K^0(M) \times K_0(M) \to K_0(M)\,.$$

We can assume that f induced an isomorphism of fundamental groups. In view of Theorem 3.9 we need to show that the class $[V] \cap [M]_K \in K_0(M)$ is of infinite K-area.

Let $L \to M$ be the complex line bundle classified by c. We pick a Hermitian connection on L and denote by $\eta \in \Omega^2(M; i\mathbb{R})$ the associated curvature form. Because the universal cover of BG is contractible, the pull back $\pi^*(L) \to \widetilde{M}$ of L to the universal cover $\pi : \widetilde{M} \to M$ is trivial. We fix a trivialization and denote the 1-form associated to the pull back connection by $\omega \in \Omega^1(\widetilde{M}; i\mathbb{R})$. The curvature form $\pi^*(\eta)$ is equal to $d\omega$, since $\mathrm{U}(1)$ is abelian. However, the connection 1-form ω is in general not invariant under the action of the deck transformation group on $\widetilde{M}$, because in this case the curvature of L would be trivial and hence $L \to M$ would be the trivial line bundle.

The assumption on c implies

$$\langle [L], [V] \cap [M]_K \rangle \neq 0 \in K_0(\mathbb{C}) = \mathbb{Z}\,.$$

So it remains to "flatten" the bundle $L \to M$ by scaling its curvature by a constant $0 < t < 1$. Unfortunately this is impossible, because the first Chern class of L would no longer be integral.

The following construction originating from [22] gives a solution to this problem by considering infinite dimensional bundles. First we consider the Hilbert space bundle

$$E = \widetilde{M} \times_G l^2(G) \to C$$

where $l^2(G)$ is the set of square summable complex valued functions on G and G acts on the left of $l^2(G)$ by the formula

$$(\gamma\psi)(x) = \psi(x\gamma)$$

and on the right of $\widetilde{M}$ by $(x, g) \mapsto g^{-1}x$. Let $0 < t < 1$. We consider the G-invariant connection 1-form on $\widetilde{M} \times l^2(G)$ which on the subbundle

$$\widetilde{M} \times \mathbb{C} \cdot 1_g \subset \widetilde{M} \times l^2(G)$$

concides with $(g^{-1})^*(t\omega)$. Here $1_g \in l^2(G)$ is the characteristic function of $g \in G$. Because this one form is G-invariant, we obtain an induced connection ∇^t on the Hilbert space bundle E whose curvature form is norm bounded by $t \cdot \|\eta\|$. Now we have a Hilbert space bundle with holonomy representations which are arbitrarily close to the identity, and it remains to show that it detects the K-homology class $[V] \cap [M]_K$. Because, by Kuiper's theorem, any Hilbert space bundle is trivial we shall at first reduce the structure group of E. This will result in finitely generated Hilbert A_t-module bundles $E_t \to M$ with appropriate unital C^*-algebras A_t, where $t \in (0, 1]$. The algebras A_t will depend on t.

We fix a base point $p \in M$ and choose a point $q \in \widetilde{M}$ above p. The fibre over p is then identified with the Hilbert space $l^2(G)$. Now we define

$$A_t \subset B(l^2(G))$$

as the norm-linear closure of all maps $l^2(G) \to l^2(G)$ aring from parallel transport with respect to ∇^t along piecewise smooth loops in M based at p. We furthermore define a bundle $E_t \to M$ whose fibre over $x \in M$ is given by the norm-linear closure in $\mathrm{Hom}(E|_p, E|_x)$ of all Hilbert space isomorphisms $E|_p \to E|_x$ aring from parallel transport with respect to ∇^t along piecewise smooth curves connecting p with x. In this way we obtain, for each $t \in (0, 1]$, a free Hilbert A_t-module

bundle of rank 1 where the A_t-module structure on each fibre is induced by precomposition with parallel transport along piecewise smooth loops based at p.

Now, on the one hand, parallel transport with respect to ∇^t induces a holonomy representation on $E_t \to M$ which, for small enough t, is arbitrarily closed to the identity.

On the other hand, each of the algebras A_t carries a canonical trace

$$\tau_t : A_t \to \mathbb{C}\,,\ \tau_t(\psi) = \langle \psi(1_e), 1_e \rangle$$

where $1_e \in l^2(G)$ is the characteristic function of the neutral element $e \in G$ and $\langle -, - \rangle$ is the inner product on $l^2(G)$. For details we refer to [22, Lemma 2.2]. Using the Chern Weil calculus from [39] we obtain

$$\tau_t(\langle [E_t], [V] \cap [M]_K \rangle) = \langle \exp(tc), \mathrm{ch}(h) \rangle \in \mathbb{R}[t]\,.$$

See also [22]. In particular, for infinitely many $k \in \mathbb{N}$ we have

$$\langle [E_{1/k}], [V] \cap [M]_K \rangle \neq 0 \in K_0(A_{1/k}) \otimes \mathbb{Q}$$

This implies that $[V] \cap [M]_K$ is a class of infinite K-area and together with Theorem 3.9 finishes the proof of Theorem 4.1.

5. Homological invariance of essentialness

Recall from Definition 2.8 that a closed oriented manifold M^n is called *essential,* if the classifying map $\phi : M \to B\pi_1(M)$ satisfies

$$\phi_*([M]_H) \neq 0 \in H_n(B\pi_1(M); \mathbb{Q})\,.$$

Essential manifolds obey Gromov's systolic inequality:

Theorem 5.1 ([13])**.** *Let M be an essential Riemannian manifold of dimension n. Then there is a noncontracible loop $\gamma : [0,1] \to M$ satisfying*

$$\ell(\gamma) \leq C(n) \cdot \mathrm{vol}(M)^{1/n}$$

where the constant $C(n)$ depends only on n.

We show the following implication.

Theorem 5.2. *Let M be an oriented manifold of even dimension $2n$. If the class $[M]_H \in H_{2n}(M; \mathbb{Q})$ has infinite K-area, then M is essential.*

Proof. Let $E \to M$ be the Mishchenko-Fomenko bundle. The proof of Theorem 5.2 is based on the commutative diagram

$$\begin{array}{ccccc}
K_0(M) \otimes \mathbb{Q} & & \xrightarrow{\quad\langle [E], - \rangle e\quad} & & K_0(C^*_{max}\pi_1(M)) \otimes \mathbb{Q} \\
\downarrow = & & & & \downarrow = \\
K_0(M) \otimes \mathbb{Q} & \xrightarrow{\phi_*} & K_0(B\pi_1(M)) \otimes \mathbb{Q} & \xrightarrow{\mu} & K_0(C^*_{max}\pi_1(M)) \otimes \mathbb{Q} \\
\downarrow \mathrm{ch} & & \downarrow \mathrm{ch} & & \\
H_{ev}(M; \mathbb{Q}) & \xrightarrow{\phi_*} & H_{ev}(B\pi_1(M), \mathbb{Q}) & &
\end{array}$$

Indeed, by Theorem 3.9 the image of $\mathrm{ch}^{-1}([M]_H)$ under the map in the first line is non-zero. □

This theorem implies

- Closed manifolds of infinite K-area in the sense of Gromov are essential.
- ([20, 21]) Area-enlargeable manifolds are essential (use Proposition 3.8).

The second implication can be obtained without reference to Kasparov theory. This is carried out in the paper [7], where several largeness properties of Riemannian manifolds are investigated from a purely homological point of view. The best results can be obtained for enlargeable manifolds where we have the following homological invariance result.

Theorem 5.3 ([7])**.** *Let G be a finitely presented group. Then there is a rational vector subspace*

$$H_*^{sm}(BG;\mathbb{Q}) \subset H_*(BG;\mathbb{Q})$$

with the following property: Let M be a closed oriented manifold of dimension n. Then M is enlargeable, if and only if under the classifying map $\phi : M \to B\pi_1(M)$ we have

$$\phi_*([M]) \notin H_n^{sm}(B\pi_1(M);\mathbb{Q})$$

This result indeed implies that enlargeable manifolds are essential, because $0 \in H_n(B\pi_1(M);\mathbb{Q})$ is contained in every vector subspace of $H_n(B\pi_1(M);\mathbb{Q})$.

Theorem 5.3 can be seen as a form of homological invariance of enlargeability. The proof is based on the following definition of enlargeable homology classes in simplicial complexes.

Definition 5.4 ([7])**.** *Let C be a connected simplicial complex with finitely generated fundamental group. A homology class $h \in H_n(C;\mathbb{Q})$ is called* enlargeable*, if the following holds: Let $S \subset C$ be a finite subcomplex carrying h and inducing a surjection on π_1. Then, for every $\epsilon > 0$, there is a cover $\overline{C} \to C$ and an ϵ-Lipschitz map $\overline{S} \to S^n$ which is constant outside a compact subset of $\overline{S}$ and sends the transfer $\mathrm{tr}(h) \in H_n^{lf}(\overline{S};\mathbb{Q})$ in the locally finite homology of $\overline{S}$ to a nonzero class in the reduced homology $\widetilde{H}_n(S^n;\mathbb{Q})$. Here $\overline{S}$ is the preimage of S under the covering map $\overline{C} \to C$.*

It is shown in [7] that the condition for c described in this definition is independent of the finite subcomplex $S \subset C$ carrying c and inducing a surjection on π_1. Using this property it is not difficult to prove the following fact, see [7, Prop. 3.4.].

Proposition 5.5. *Let $f : C \to D$ be a continuous map inducing an isomorphism of (finitely generated) fundamental groups. Then a class $h \in H_*(C;\mathbb{Q})$ is enlargeable, if and only if the class $f_*(h) \in H_*(D;\mathbb{Q})$ is enlargeable.*

From this Theorem 5.3 follows, if we define $H_n^{sm}(BG;\mathbb{Q})$ as the subset consisting of all homology classes which are not enlargeable.

Theorem 5.3 transforms the problem of determining enlargeable manifolds to a problem in group homology: Given a finitely generated group G, determine $H_*^{sm}(BG;\mathbb{Q})$, the "small" group homology of G. In light of Theorem 5.3 and the fact that the fundamental classes of enlargeable manifolds (of even dimension) are of infinite K-area (see Proposition 3.8) it is desirable to decide whether $H_*^{sm}(BG;\mathbb{Q})$ can be non-zero. This is answered in the positive in [7, Theorem 4.8] by use of the Higman 4-group [23]. Together with Theorem 5.3 this implies that there are essential manifolds which are not enlargeable, see [7, Theorem 1.5].

In contrast to these positive results we do not know, whether there are essential manifolds which are not area-enlargeable. These manifolds would exist, if the following question had an affirmative answer.

Question 5.6. *Is there an essential manifold whose fundamental class in singular homology $[M]_H$ is of finite K-area?*

6. Rosenberg index for the reduced group C^*-algebra

Let M^n be a closed spin manifold. The method of Section 2 can be used equally well to construct an index obstruction to positive scalar curvature

$$\alpha(M) \in K_n(C^*_{red}\pi_1(M)) \,.$$

The reduced group C^*-algebra does not share the universal property of the maximal group C^*-algebra which we used in the proof of Theorem 3.9.

Exploiting the connection of $C^*_{red}\pi_1(M)$ to coarse geometry [24] we have

Theorem 6.1 ([19])**.** *Let M^n be an enlargeable spin manifold. Then*

$$\alpha(M) \neq 0 \in K_n(C^*_{red}\pi_1(M)) \,.$$

Question 6.2. *Does Theorem 3.9 remain true for the reduced group C^*-algebra?*

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