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A FINITENESS PROPERTY OF GRADED SEQUENCES OF IDEALS

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Abstract. Given a graded sequence of ideals $(\mathfrak{a}_m)_{m\geq 1}$ on X, we show that if there are divisors E_m over X computing the log canonical threshold of $\mathfrak{a}_m$, and such that the log discrepancies of the divisors E_m are bounded, then the set $\{E_m \mid m\geq 1\}$ is finite.

1. Introduction

Let X be a smooth algebraic variety over an algebraically closed field k of characteristic zero. The log canonical threshold of a nonzero ideal $\mathfrak{a}$ on X is a fundamental invariant of the singularities of the subscheme defined by $\mathfrak{a}$. Originally known as the *complex singularity index*, it shows up in many contexts related to singularities, and it has found a plethora of applications in birational geometry (see [Kol] and [EM]).

In this note we will be interested in the behavior of this invariant in certain sequences of ideals. Let $\mathfrak{a}_\bullet=(\mathfrak{a}_m)_{m\geq 1}$ be a *graded sequence of ideals* on X, that is, a sequence of ideals that satisfies $\mathfrak{a}_\ell\cdot\mathfrak{a}_m\subseteq\mathfrak{a}_{\ell+m}$ for every $\ell,m\geq 1$. We always assume that, in addition, some ideal $\mathfrak{a}_m$ is nonzero. The main motivating example is the graded sequence $\mathfrak{a}_\bullet^L$ associated to a line bundle L of nonnegative Iitaka dimension on a smooth projective variety X: the ideal $\mathfrak{a}_m^L$ defines the base-locus of the linear system $|L^m|$. Note that in this case the behavior of $\mathfrak{a}_\bullet^L$ is easy to understand if the section ring $\oplus_m\Gamma(X,L^m)$ is finitely generated over k. Indeed, in this case there is a positive integer p such that $\mathfrak{a}_{mp}=\mathfrak{a}_p^m$ for all m. The study of $\mathfrak{a}_\bullet^L$ is useful precisely when the section ring is not finitely generated (or at least, when this property is not known *a priori*).

To a graded sequence $\mathfrak{a}_\bullet$ as above, one can associate an asymptotic version of the log canonical threshold, by putting

$$\mathrm{lct}(\mathfrak{a}_\bullet):=\sup_{m;\mathfrak{a}_m\neq(0)} m\cdot\mathrm{lct}(\mathfrak{a}_m).$$

This can be infinite: for example, if $\mathfrak{a}_\bullet=\mathfrak{a}_\bullet^L$ as above, with L big, then $\mathrm{lct}(\mathfrak{a}_\bullet)$ is infinite if and only if L is nef (see Remark 2.2 below).

We will be concerned with the divisors that compute the log canonical thresholds of the elements of a graded sequence. We denote by $A(\mathrm{ord}_E)$ the log discrepancy of a divisor E over X (see §2 for the relevant definitions). The following is our main result.

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It gives a positive answer to a question of Mihai Păun, related to Siu's approach to the finite generation of the canonical ring in [Siu].

Theorem A. Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on a smooth variety X such that $\mathrm{lct}(\mathfrak{a}_\bullet) < \infty$. If $I \subseteq \mathbf{Z}_{>0}$ is a subset such that for all $m \in I$ we have a divisor E_m over X that computes $\mathrm{lct}(\mathfrak{a}_m)$ such that $\{A(\mathrm{ord}_{E_m}) \mid m \in I\}$ is bounded, then the set $\{E_m \mid m \in I\}$ is finite.

Corollary B. Under the hypothesis in Theorem A, suppose that the set I is infinite. Then there is a divisor E over X that computes $\mathrm{lct}(\mathfrak{a}_m)$ for infinitely m. In particular, E computes $\mathrm{lct}(\mathfrak{a}_\bullet)$.

In fact, since our proof will require replacing X by a suitable blow-up, we will need to prove a stronger version of the above theorem, in which we replace the log canonical threshold by the possibly higher jumping numbers, in the sense of [ELSV] (see Theorem 4.1 below for the precise statement).

Here is a sketch of the proof. Let Z_m be the image of E_m on X, and let W be the Zariski closure of $\bigcup_{m\in I} Z_m$. We may assume that W is irreducible, and we first show that since $\mathrm{lct}(\mathfrak{a}_\bullet) < \infty$, the asymptotic order of vanishing $\mathrm{ord}_W(\mathfrak{a}_\bullet)$ is positive. In particular, W is a proper subset of X. If W has codimension at least two in X, then blowing-up X along W decreases the log discrepancies of the divisors E_m, and since these are bounded above, we reduce to the case when W is a hypersurface. In this case, we use the following result, which we believe is of independent interest.

Theorem C. Let H be a hypersurface in X, and $\mathfrak{a}$ a nonzero ideal. Suppose that E is a divisor over X that computes $\mathrm{lct}(\mathfrak{a})$. If the image Z of E on X is a proper subset of H, and if H is smooth at the generic point of Z, then the following inequality holds

$$\mathrm{ord}_Z(\mathfrak{a}) \geq \mathrm{ord}_H(\mathfrak{a}) \cdot \left(1 + \frac{\mathrm{ord}_E(I_Z)}{A(\mathrm{ord}_E)}\right),$$

where I_Z is the ideal defining Z.

Of course, as we have already mentioned, we need in fact a version of this result that applies also to higher jumping numbers (see Theorem 3.1 below for this more general version of the theorem). Using Theorem C, we show that if there were infinitely many Z_m that were properly contained in W, then the ideals in $\mathfrak{a}_\bullet$ would vanish along W more than they should. Therefore all but finitely many of the E_m are equal to W (note that at this point we are on some blow-up of our original variety).

In the following section we review some basic facts about log canonical thresholds and higher jumping numbers. The proofs of the stronger versions of Theorems C and A are given in §3, and respectively, §4.

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2. Jumping numbers and valuations

In this section we recall some definitions and results concerning the invariants of singularities that we will use, and set the notation for the rest of the paper. We work over a fixed algebraically closed field k of characteristic zero. Let X be a smooth variety over k (in particular, we assume that X is connected and separated). All ideal sheaves on X are assumed to be coherent.

By a divisor E over X we mean a prime divisor on a normal variety Y that has a proper birational morphism $\pi\colon Y \to X$. This induces a discrete valuation of the function field $K(Y) = K(X)$, that we denote by ord_E. As usual, we identify two such divisors if they induce the same valuation. In particular, it follows from Hironaka's theorem on resolution of singularities that we may assume that both Y and E are nonsingular. If we denote by $K_{Y/X}$ the relative canonical divisor, then the *log discrepancy* of ord_E is given by $A(\mathrm{ord}_E) := 1 + \mathrm{ord}_E(K_{Y/X})$. Note that this depends on the variety X, and whenever the variety is not clear from the context, we will write $A_X(\mathrm{ord}_E)$. The *center* of E on X is the image $c_X(E) := \pi(E)$ of E. We always consider on $c_X(E)$ the reduced scheme structure. If $\mathfrak{a}$ is a nonzero ideal sheaf on X, we put

$$\mathrm{ord}_E(\mathfrak{a}) := \min\{\mathrm{ord}_E(f) \mid f \in \mathcal{O}_{X,c_X(E)}\} \in \mathbf{R}_{\geq 0}.$$

If Z is the subscheme defined by $\mathfrak{a}$, we also denote this by $\mathrm{ord}_E(Z)$.

Given an irreducible closed subset Z of X, we define the order of vanishing along Z as follows. Consider the normalized blow-up of X along Z, and put $\mathrm{ord}_Z := \mathrm{ord}_E$, where E is the unique irreducible component of the exceptional divisor that dominates Z. It is clear that in this case $c_X(E) = Z$. Note also that $\mathrm{ord}_Z(\mathfrak{a}) = \min_{x\in Z} \mathrm{ord}_x(\mathfrak{a})$.

Let us recall the definition of multiplier ideals. For details and proofs we refer to [Laz, §9]. Suppose that $\mathfrak{a}$ is a nonzero ideal on X. Let $\mu\colon X' \to X$ be a log resolution of $(X, \mathfrak{a})$, that is, π is proper and birational, X' is nonsingular, $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for an effective divisor F, and $F + K_{X'/X}$ has simple normal crossings. For every $\lambda \in \mathbf{R}_{\geq 0}$, the multiplier ideal of $\mathfrak{a}$ of exponent λ is given by

$$\mathcal{J}(\mathfrak{a}^\lambda) := \pi_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor \lambda F \rfloor).$$

The definition is independent of the choice of log resolution.

It is clear from the above definition that if $\lambda < \lambda'$, then $\mathcal{J}(\mathfrak{a}^{\lambda'}) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)$. Furthermore, for every λ there is $\varepsilon > 0$ such that $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^t)$ for every $t \in [\lambda, \lambda + \varepsilon]$. One says that $\lambda > 0$ is a *jumping number* of $\mathfrak{a}$ if $\mathcal{J}(\mathfrak{a}^\lambda) \neq \mathcal{J}(\mathfrak{a}^{\lambda'})$ for every $\lambda' < \lambda$. It follows from the definition that if we write $F = \sum_i a_i E_i$, then for every jumping number λ there is i such that λa_i is an integer. In particular, the jumping numbers form a discrete set of rational numbers.

For basic properties of the jumping numbers and applications, we refer to [ELSV]. The most important jumping number is the smallest one, known as the *log canonical threshold* and denoted by $\mathrm{lct}(\mathfrak{a})$. This is the smallest λ such that $\mathcal{J}(\mathfrak{a}^\lambda) \neq \mathcal{O}_X$ (note that $\mathcal{J}(\mathfrak{a}^0) = \mathcal{O}_X$).

It is convenient to index the jumping numbers as follows (see [JM]). Let $\mathfrak{q}$ be a nonzero ideal on X. We put

$$\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}) := \min\{\lambda \mid \mathfrak{q} \not\subseteq \mathcal{J}(\mathfrak{a}^{\lambda})\}.$$

Note that $\operatorname{lct}^{\mathcal{O}_X}(\mathfrak{a})$ is the log canonical threshold $\operatorname{lct}(\mathfrak{a})$ of $\mathfrak{a}$. It follows from the definition that if $\mathfrak{a} \neq \mathcal{O}_X$, then $\bigcap_{\lambda\geq 0}\mathcal{J}(\mathfrak{a}^{\lambda}) = (0)$, hence $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a})$ is finite. When $\mathfrak{a} = \mathcal{O}_X$, we make the convention $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}) = \infty$. We will also use the notation $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}) := 1/\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a})$ (where Arn stands for *Arnold multiplicity*). It follows from the definition that we have

$$\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}) = \max_E \frac{\operatorname{ord}_E(\mathfrak{a})}{A(\operatorname{ord}_E) + \operatorname{ord}_E(\mathfrak{q})}, \tag{1}$$

where the maximum can be taken either over all divisors over X, or just over those lying on a log resolution of $(X, \mathfrak{a})$. We say that E computes $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a})$ (or $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a})$) if the maximum in (1) is achieved by E.

The most interesting of the jumping numbers is the log canonical threshold. However, as the following lemma shows, the other jumping numbers appear naturally when we consider higher birational models.

Proposition 2.1. *Let $\pi\colon X' \to X$ be a proper birational morphism, with X' smooth, and $\mathfrak{a}$ and $\mathfrak{q}$ nonzero ideals on X. If $\mathfrak{a}' = \mathfrak{a}\cdot\mathcal{O}_{X'}$, and $\mathfrak{q}' = \mathfrak{q}\cdot\mathcal{O}_{X'}(-K_{X'/X})$, then*

$$\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}) = \operatorname{lct}^{\mathfrak{q}'}(\mathfrak{a}').$$

Proof. This is an immediate consequence of (1), and of the fact that for every divisor E over X, we have $A_X(\operatorname{ord}_E) = A_{X'}(\operatorname{ord}_E) + \operatorname{ord}_E(K_{X'/X})$. □

Suppose now that $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X, and let $S = \{m \mid \mathfrak{a}_m \neq (0)\}$. Note that S is closed under addition. In this case we have the following asymptotic version of the jumping numbers:

$$\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) := \sup_{m\in S} m\cdot\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_m) = \lim_{m\to\infty, m\in S} m\cdot\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_m) \tag{2}$$

(see [JM, §2]). We put $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = 1/\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. When $\mathfrak{q} = \mathcal{O}_X$, we simply write $\operatorname{lct}(\mathfrak{a}_\bullet)$ and $\operatorname{Arn}(\mathfrak{a}_\bullet)$. Note that $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) \in \mathbf{R}_{>0} \cup \{\infty\}$. One can show that $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \infty$ if and only if $\operatorname{lct}(\mathfrak{a}_\bullet) = \infty$ (see [JM, Corollary 6.10]).

Remark 2.2. If X is a smooth projective variety, L is a big line bundle on X, and $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet^L$ is the graded sequence of ideals defining the base loci of the powers of L (see Introduction), then [ELMNP, Corollary 2.10] shows that $\operatorname{lct}(\mathfrak{a}_\bullet) = \infty$ if and only if L is nef.

If $\mathfrak{a}_\bullet$ is as above and E is a divisor over X, we will also consider the following asymptotic version of the order of vanishing along E:

$$\operatorname{ord}_E(\mathfrak{a}_\bullet) := \inf_m \frac{\operatorname{ord}_E(\mathfrak{a}_m)}{m} = \lim_{m\to\infty, m\in S}\frac{\operatorname{ord}_E(\mathfrak{a}_m)}{m}.$$

We have the following extension of (1)

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \sup_E \frac{\mathrm{ord}_E(\mathfrak{a}_\bullet)}{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})}. \tag{3}$$

For these facts, we refer to [JM, §2]. We say that E *computes* $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ if the supremum in (3) is achieved by E. Note however that unlike in the case of one ideal, there may be no divisor E that computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ (see [JM, Example 8.5]).

We will use the following Izumi-type estimate (see [Izu, ELS]).

Proposition 2.3. *If E is a divisor over X with $c_X(E) = Z$, then*

$$\mathrm{ord}_E(\mathfrak{a}) \leq A(\mathrm{ord}_E) \cdot \mathrm{ord}_Z(\mathfrak{a})$$

for every ideal sheaf $\mathfrak{a}$ on X.

Proof. We may replace X by an affine open subset of the generic point of Z, and therefore assume that X is affine. In this case we may assume that $\mathfrak{a}$ is principal. If $\mathrm{ord}_Z(\mathfrak{a}) = m$, then for a general $p \in Z$ we have $\mathrm{ord}_p(\mathfrak{a}) = m$. By [Kol, Lemma 8.10], there is an open neighborhood U of p such that $\mathrm{lct}(\mathfrak{a}|_U) \geq 1/m$, and we get the assertion in the proposition since $U \cap Z \neq \emptyset$ implies $\frac{A(\mathrm{ord}_E)}{\mathrm{ord}_E(\mathfrak{a})} \geq \mathrm{lct}(\mathfrak{a}|_U)$. □

3. An inequality between orders of vanishing

We keep the notation and the conventions from §2. The following is the main result in this section. Note that in the special case $\mathfrak{q} = \mathcal{O}_X$, this recovers Theorem C in the Introduction.

Theorem 3.1. *Let H be a hypersurface in X, and $\mathfrak{a}$, $\mathfrak{q}$ nonzero ideals on X. Suppose that E is a divisor over X that computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$. If the center Z of E on X is a proper subset of H, and if H is smooth at the generic point of Z, then the following inequality holds*

$$\mathrm{ord}_Z(\mathfrak{a}) \geq \mathrm{ord}_H(\mathfrak{a}) \cdot \left(1 + \frac{\mathrm{ord}_E(Z)}{A(\mathrm{ord}_E)(1 + \mathrm{ord}_H(\mathfrak{q}))}\right).$$

We start by recalling a basic estimate for the log discrepancy of a valuation. For a proof, see for example [Laz, p. 157].

Lemma 3.2. *Let E be a divisor over X with $c_X(E) = Z$, and let ξ be the generic point of Z. If $x_1, \ldots, x_r$ form a regular system of parameters of $\mathcal{O}_{X,\xi}$, then*

$$A(\mathrm{ord}_E) \geq \sum_{i=1}^{r} \mathrm{ord}_E(x_i).$$

Corollary 3.3. *If H is a hypersurface in X, and E is a divisor over X such that $Z := c_X(E)$ is a proper subset of H, and H is smooth at the generic point of Z, then*

$$A(\mathrm{ord}_E) \geq \mathrm{ord}_E(H) + \mathrm{ord}_E(Z). \tag{4}$$

Proof. Let ξ be the generic point of Z. Since H is smooth at ξ, we may choose a regular system of parameters $x_1, \ldots, x_r$ of $\mathcal{O}_{X,\xi}$ such that H is defined at ξ by (x_1). Note that by assumption $r \geq 2$. By definition, we have $\mathrm{ord}_E(Z) = \min_j \mathrm{ord}_E(x_j)$. Let i be such that $\mathrm{ord}_E(x_i) = \mathrm{ord}_E(Z)$. If $i \geq 2$, then by the lemma

$$A(\mathrm{ord}_E) \geq \mathrm{ord}_E(x_1) + \mathrm{ord}_E(x_i) = \mathrm{ord}_E(H) + \mathrm{ord}_E(Z).$$

On the other hand, if $i = 1$, then using again the lemma we get

$$A(\mathrm{ord}_E) \geq \mathrm{ord}_E(x_1) + \mathrm{ord}_E(x_2) \geq 2 \cdot \mathrm{ord}_E(x_1) = \mathrm{ord}_E(H) + \mathrm{ord}_E(Z).$$

□

Proof of Theorem 3.1. Let us put $v = \mathrm{ord}_E$, $m = \mathrm{ord}_H(\mathfrak{a})$, and $p = \mathrm{ord}_H(\mathfrak{q})$. We can write $\mathfrak{a} = \mathcal{O}_X(-H)^m \cdot \widetilde{\mathfrak{a}}$, and we get

$$v(\mathfrak{a}) = m \cdot v(H) + v(\widetilde{\mathfrak{a}}), \ \ \mathrm{ord}_Z(\mathfrak{a}) = m + \mathrm{ord}_Z(\widetilde{\mathfrak{a}}) \tag{5}$$

(note that $\mathrm{ord}_Z(H) = 1$ since H is smooth at the generic point of Z). Since E computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$, it follows from (1) that

$$\frac{v(\mathfrak{a})}{A(v) + v(\mathfrak{q})} \geq \frac{\mathrm{ord}_H(\mathfrak{a})}{A(\mathrm{ord}_H) + \mathrm{ord}_H(\mathfrak{q})} = \frac{m}{1+p}. \tag{6}$$

Corollary 3.3 gives $v(H) \leq A(v) - v(Z)$, and combining this with (6) we deduce

$$\begin{aligned} m \leq (1+p) \cdot \frac{v(\mathfrak{a})}{A(v) + v(\mathfrak{q})} &\leq \frac{v(\widetilde{\mathfrak{a}}) + m(A(v) - v(Z)) + p \cdot v(\mathfrak{a})}{A(v) + v(\mathfrak{q})} \\ &= m + \frac{v(\widetilde{\mathfrak{a}}) + p \cdot v(\mathfrak{a}) - m(v(\mathfrak{q}) + v(Z))}{A(v) + v(\mathfrak{q})}. \end{aligned} \tag{7}$$

Therefore $v(\widetilde{\mathfrak{a}}) \geq m(v(\mathfrak{q}) + v(Z)) - p \cdot v(\mathfrak{a})$. Using one more time the first equation in (5), this implies

$$(1+p) \cdot v(\widetilde{\mathfrak{a}}) \geq m(v(\mathfrak{q}) + v(Z)) - pm \cdot v(H). \tag{8}$$

On the other hand, by Proposition 2.3 we have $v(\widetilde{\mathfrak{a}}) \leq A(v) \cdot \mathrm{ord}_Z(\widetilde{\mathfrak{a}})$, while clearly $v(\mathfrak{q}) \geq p \cdot v(H)$. Putting these together with (8) gives

$$(1+p)A(v) \cdot \mathrm{ord}_Z(\widetilde{\mathfrak{a}}) \geq (1+p) \cdot v(\widetilde{\mathfrak{a}}) \geq m(v(\mathfrak{q}) + v(Z)) - pm \cdot v(H) \geq m \cdot v(Z).$$

Combining this with the second equality in (5), we obtain

$$\mathrm{ord}_Z(\mathfrak{a}) = m + \mathrm{ord}_Z(\widetilde{\mathfrak{a}}) \geq m \cdot \left(1 + \frac{v(Z)}{A(v)(1+p)}\right),$$

which completes the proof of the theorem. □

Remark 3.4. In Theorem 3.1 one can replace ord_E by any real valuation of $K(X)$, having center on X and computing $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$. The proof goes through if one uses the definition of $A(v)$ from [JM, §5]. In this case, the assertion in Lemma 3.2 follows from [JM, Corollary 5.4].

4. The main result

In this section we prove the generalized version of Theorem A in the Introduction. We work in the same setting as in §2.

Theorem 4.1. *Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X, and $\mathfrak{q}$ a nonzero ideal on X such that $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) < \infty$. If $I \subseteq \mathbf{Z}_{>0}$ is a subset such that for all $m \in I$ we have a divisor E_m over X that computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m)$ such that $\{A(\mathrm{ord}_{E_m}) \mid m \in I\}$ is bounded, then the set $\{E_m \mid m \in I\}$ is finite.*

Corollary 4.2. *Under the same hypothesis as in Theorem 4.1, suppose that the set I is infinite. Then there is a divisor E over X that computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m)$ for infinitely many m. In particular, E computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.*

Proof of Theorem 4.1. Note that the hypothesis implies, in particular, that $\mathfrak{a}_m$ is nonzero for every $m \in I$. We may assume that I is an infinite set, and that $E_i \neq E_j$ for all $i \neq j$ in I, and we aim to derive a contradiction. Let $Z_m = c_X(E_m)$. We argue by induction on $M := \max\{A(\mathrm{ord}_{E_i}) \mid i \in I\}$. This is finite by assumption. Note that M is a positive integer, and $M = 1$ if and only if all E_i are divisors on X. At several stages in the proof we will replace I by an infinite subset. Note that this can only decrease the value of M.

We start with the following lemma.

Lemma 4.3. *With the above notation, suppose that there is an infinite subset $J \subseteq I$ such that $W := \overline{\cup_{j \in J} Z_j}$ is irreducible, and $Z_j \neq W$ for all $j \in J$. In this case*

$$\mathrm{ord}_W(\mathfrak{a}_\bullet) \geq \mathrm{Arn}(\mathfrak{a}_\bullet) \geq \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0.$$

Proof. We only need to prove the first inequality. Let $C = \mathrm{Arn}(\mathfrak{a}_\bullet)$, so that $\mathrm{Arn}(\mathfrak{a}_m) \geq Cm$ for every m. If $j \in J$, then by Proposition 2.3 we have $\mathrm{Arn}(\mathfrak{a}_j) \leq \mathrm{ord}_{Z_j}(\mathfrak{a}_j)$.

We need to show that $\mathrm{ord}_W(\mathfrak{a}_m) \geq Cm$ for every $m \geq 1$. We may, of course, assume that $\mathfrak{a}_m$ is nonzero. By hypothesis, we can find $0 \leq \ell \leq m-1$ such that the set

$$\bigcup_{j \in J, j \equiv \ell (\mathrm{mod}\, m)} Z_j \tag{9}$$

is dense in W. Since all Z_j are proper subsets of W, this implies that if in (9) we only take the union over those $j \in J$ with $j \equiv \ell \pmod m$ and with $j \geq N$, for some N, then the union is still dense in W. Let us fix $j_0 \in J$ with $j_0 \equiv \ell \pmod m$, and let $C' := \max_{x \in W} \mathrm{ord}_x(\mathfrak{a}_{j_0}) < \infty$ (recall that $\mathfrak{a}_{j_0}$ is nonzero). If $mp + j_0 \in J$, then the inclusion $\mathfrak{a}_m^p \cdot \mathfrak{a}_{j_0} \subseteq \mathfrak{a}_{mp+j_0}$ implies

$$p \cdot \mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_m) + \mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_{j_0}) \geq \mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_{mp+j_0}) \geq \mathrm{Arn}(\mathfrak{a}_{mp+j_0}) \geq C(mp+j_0).$$

Therefore $\mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_m) \geq Cm - \frac{C'}{p}$. Since we have arbitrarily large such p, and since the union of the corresponding Z_{mp+j_0} is dense in W, we conclude that $\mathrm{ord}_W(\mathfrak{a}_m) \geq Cm$, as required. □

A first consequence of the lemma is that if W is the closure of $\cup_{i \in I} Z_i$, then $W \neq X$. In particular, this shows that when $M = 1$, we have a contradiction.

Arguing by Noetherian induction on W, we may assume that W is minimal in X with the property that there is an infinite family of divisors $(E_i)_{i\in I}$ as above, with $\max\{A(\mathrm{ord}_{E_i}) \mid i \in I\} \leq M$. This implies first that W is irreducible. Indeed, if we consider the irreducible decomposition $W = W_1 \cup \ldots \cup W_r$, then there is j such that $Z_i \subseteq W_j$ for infinitely many $i \in I$. Since we may replace I by $\{i \in I \mid Z_i \subseteq W_j\}$, it follows from the minimality assumption on W that $W = W_j$.

A second consequence of the minimality of W is that for every infinite subset $J \subseteq I$, the union $\cup_{j\in J} Z_j$ is dense in W. In particular, if U is an open subset of X that meets W, then there are infinitely many $i \in I$ such that U meets Z_i (and the union of these $Z_i \cap U$ is dense in $W \cap U$). Therefore in order to deduce a contradiction we may replace X by U and each $\mathfrak{a}_m$ by its restriction to U. We may thus assume that W is nonsingular.

We claim that the induction hypothesis on M implies that W is a hypersurface in X. Indeed, suppose that $c = \mathrm{codim}(W, X) \geq 2$, and let $\pi\colon X' \to X$ be the blow-up of X along W. If E is the exceptional divisor of π, then $K_{X'/X} = (c-1)E$. Since $c_X(E_i) \subseteq W$ for every $i \in I$, it follows that $c_{X'}(E_i) \subseteq E$, hence

$$A_{X'}(\mathrm{ord}_{E_i}) = A_X(\mathrm{ord}_{E_i}) - \mathrm{ord}_{E_i}(K_{X'/X}) \leq A_X(\mathrm{ord}_{E_i}) - (c-1).$$

If $\mathfrak{a}'_m = \mathfrak{a}_m \cdot \mathcal{O}_{X'}$ and $\mathfrak{q}' = \mathfrak{q} \cdot \mathcal{O}_{X'}(-K_{X'/X})$, then by Proposition 2.1 we have $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_i) = \mathrm{lct}^{\mathfrak{q}'}(\mathfrak{a}_i)$, and it follows from hypothesis and (1) that E_i computes $\mathrm{lct}^{\mathfrak{q}'}(\mathfrak{a}'_i)$ for every $i \in I$. Since $\max\{A_{X'}(\mathrm{ord}_{E_i}) \mid i \in I\} \leq M - 1$, we have a contradiction by induction on M.

Therefore W is a smooth hypersurface in X. If $Z_i = W$, then $E_i = W$, hence this can be the case for at most one i. After discarding this i, we may assume that each Z_i is a proper subset of W. In particular, we may apply Theorem 3.1 to get

$$\mathrm{ord}_{Z_i}(\mathfrak{a}_i) \geq \mathrm{ord}_W(\mathfrak{a}_i) \cdot \left(1 + \frac{\mathrm{ord}_{E_i}(Z_i)}{A(\mathrm{ord}_{E_i})(1 + \mathrm{ord}_W(\mathfrak{q}))}\right). \tag{10}$$

Note that $\mathrm{ord}_{E_i}(Z_i) \geq 1$ for all $i \in I$. Let $\alpha = \mathrm{ord}_W(\mathfrak{a}_\bullet)$. We have $\alpha > 0$ by Lemma 4.3. Let us fix $\varepsilon > 0$ with $\varepsilon < \frac{1}{M(1+\mathrm{ord}_W(\mathfrak{q}))}$. If we show that $\mathrm{ord}_W(\mathfrak{a}_m) \geq \alpha m(1+\varepsilon)$ for every $m \geq 1$, then $\alpha = \mathrm{ord}_W(\mathfrak{a}_\bullet) \geq \alpha(1+\varepsilon)$, a contradiction. We now argue as in the proof of Lemma 4.3. Let $0 \leq \ell \leq m-1$ be such that the set in (9) is dense in W. We fix $j_0 \in I$ such that $j_0 \equiv \ell \pmod{m}$, and let $C' := \max_{x\in W} \mathrm{ord}_x(\mathfrak{a}_{j_0})$. It follows from the inclusion $\mathfrak{a}_m^p \cdot \mathfrak{a}_{j_0} \subseteq \mathfrak{a}_{mp+j_0}$ and from (10) that for every p such that $mp + j_0 \in I$ we have

$$p \cdot \mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_m) \geq \mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_{mp+j_0}) - \mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_{j_0}) \geq \mathrm{ord}_W(\mathfrak{a}_{mp+j_0})(1+\varepsilon) - C'.$$

Therefore for every such p we have $\mathrm{ord}_{Z_{mp+j_0}}(\mathfrak{a}_m) \geq \alpha m(1+\varepsilon) - \frac{C'}{p}$. Since there are arbitrarily large such p, and the union of the corresponding Z_{mp+j_0} is dense in W, we conclude that $\mathrm{ord}_W(\mathfrak{a}_m) \geq \alpha m(1+\varepsilon)$. As we have seen, this leads to a contradiction, and thus completes the proof of the theorem. □

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