TOPOLOGICAL CLASSIFICATION OF ZERO-DIMENSIONAL $\mathcal{M}_\omega$-GROUPS

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Abstract. A topological group G is called an $\mathcal{M}_\omega$-group if it admits a countable cover $\mathcal{K}$ by closed metrizable subspaces of G such that a subset U of G is open in G if and only if $U \cap K$ is open in K for every $K \in \mathcal{K}$.

It is shown that any two non-metrizable uncountable separable zero-dimenisional $\mathcal{M}_\omega$-groups are homeomorphic. Together with Zelenyuk's classification of countable k_ω-groups this implies that the topology of a non-metrizable zero-dimensional $\mathcal{M}_\omega$-group G is completely determined by its density and the compact scatteredness rank $r(G)$ which, by definition, is equal to the least upper bound of scatteredness indices of scattered compact subspaces of G.

In [Ze] (see also [PZ, §4.3]) E.Zelenyuk has proven that the topology of a countable topological k_ω-group G is completely determined by its compact scatteredness rank $r(G)$ which, by definition, is equal to the least upper bound of scatteredness indices of compact scattered subsets of G. In this note we extend this Zelenyuk's classification result onto the class of punctiform $\mathcal{M}_\omega$-groups.

Let us recall that a topological space X is *scattered* if every non-empty subset of X has an isolated point. For a scattered space X its scatteredness index $i(X)$ is defined as the smallest ordinal α such that the α-th derived set $X^{(\alpha)}$ of X is finite. Derived sets $X^{(\beta)}$ of X are defined by transfinite induction: $X^{(0)} = X$, $X^{(1)}$ is the set of all non-isolated points of X; $X^{(\beta+1)} = (X^{(\beta)})^{(1)}$ and $X^{(\beta)} = \bigcap_{\gamma<\beta} X^{(\gamma)}$ if β is a limit ordinal. It can be easily shown that $i(X) < \omega_1$ if X is a hereditarily Lindelöf scattered topological space (in particular, a countable compactum). For a topological space X let

$$r(X) = \sup\{i(K) : K \text{ is a compact scattered subset of } X\}$$

be the *compact scattered rank* of X.

A topological space X is defined to be a *k_ω-space* (resp. an *$\mathcal{M}_\omega$-space*) if X admits a countable cover $\mathcal{K}$ by compact Hausdorff subspaces (resp. by closed metrizable subspaces) of X such that a subset U of X is open in X if and only if $U \cap K$ is open in K for every $K \in \mathcal{K}$. A space X is called an *$\mathcal{MK}_\omega$-space* if X is both

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a k_ω-space and an $\mathcal{M}_\omega$-space. A topological group G is called a *k_ω-group* (resp. *$\mathcal{MK}_\omega$-group, $\mathcal{M}_\omega$-group*) if its underlying topological space is k_ω-space (resp. an $\mathcal{MK}_\omega$-space, an $\mathcal{M}_\omega$-space). Since each countable compactum is metrizable, we conclude that each countable k_ω-space is an $\mathcal{MK}_\omega$-space. On the other hand, according to Theorem 4 of [Ba], every non-metrizable $\mathcal{M}_\omega$-group is homeomorphic to the product $H \times D$, where H is an open $\mathcal{MK}_\omega$-subgroup in G and D is a discrete space.

Following [En_2, 1.4.3], we say that a topological space X is *punctiform* if it contains no connected compact subspace containing more than one point. Each punctiform σ-compact space is zero-dimensional [En_2, §1.4]. On the other hand, there exist strongly infinite-dimensional separable complete-metrizable punctiform spaces [En_2, 6.2.4]. Given a topological space X by $d(X)$ its density is denoted.

Main Theorem. *The topology of a non-metrizable punctiform $\mathcal{M}_\omega$-group is completely determined by its density and its compact scatteredness rank. In other words, two non-metrizable punctiform $\mathcal{M}_\omega$-groups G, H are homeomorphic if and only if $d(G) = d(H)$ and $r(G) = r(H)$.*

To prove this theorem we need to make first some preliminary work. We say that a topological space X carries the direct limit topology with respect to a tower $X_1 \subset X_2 \subset X_3 \subset \dots$ of subsets of X (this is denoted by $X = \varinjlim X_n$) if $X = \bigcup_{n=1}^\infty X_n$ and a subset $U \subset X$ is open if and only if $U \cap X_n$ is open in X_n for every $n \in \mathbb{N}$.

Since the union of any two compact (resp. closed metrizable) subspaces in a topological space is compact (resp. closed and metrizable, see [En_1, 4.4.19]), we get the following

Lemma 1. *A topological space X is an $\mathcal{M}_\omega$-space (an $\mathcal{MK}_\omega$-space) if and only if X carries the direct limit topology with respect to a tower $X_1 \subset X_2 \subset \dots$ of closed metrizable (compact) subsets of X.*

Under a *Cantor set* we understand a zero-dimensional metrizable compactum without isolated points.

Lemma 2 [Ke, 6.5]**.** *Each uncountable metrizable compactum contains a Cantor set.*

According to a classical theorem of Brouwer [Ke, 7.4], each Cantor set is homeomorphic to the Cantor cube $2^\omega = \{0,1\}^\omega$. It is well known that the Cantor cube is universal for the class of metrizable zero-dimensional compacta. In fact, it is universal is a stronger sense, see [vE], [Po].

Lemma 3. *Suppose A is a closed subset of a zero-dimensional metrizable compactum B. Every embedding $f : A \to 2^\omega$ such that $f(A)$ is nowhere dense in 2^ω extends to an embedding $\bar{f} : B \to 2^\omega$.*

Given a cardinal τ denote by $(2^\tau)^\infty = \varinjlim (2^\tau)^n$ the direct limit of the tower

$$2^\tau \subset (2^\tau)^2 \subset (2^\tau)^3 \subset \dots$$

consisting of finite powers of the Cantor discontinuum 2^τ (here $(2^\tau)^n$ is identified with the subspace $(2^\tau)^n \times \{*\}$ of $(2^\tau)^{n+1}$, where $*$ is any fixed point of 2^τ).

Using Lemma 3 by standard "back-and-forth" arguments (see [Sa]) one may prove

Lemma 4. *A space X is homeomorphic to $(2^\omega)^\infty$ if and only if X is a zero-dimensional $\mathcal{MK}_\omega$-space satisfying the following property:*

($\mathcal{SU}$) every embedding $f : B \to X$ of a closed subspace B of a zero-dimensional metrizable compactum A may be extended to an embedding $\bar{f} : A \to X$.

Now we are able to prove a "separable" version of Main Theorem.

Theorem. *Every non-metrizable uncountable separable punctiform $\mathcal{M}_\omega$-group is homeomorphic to $(2^\omega)^\infty$.*

Proof. Suppose G is a non-metrizable uncountable separable punctifurm $\mathcal{M}_\omega$-group. It follows from Theorem 4 of [Ba] that G is an $\mathcal{MK}_\omega$-group. Then G, being σ-compact and punctiform, is zero-dimensional, see [En_2, §1.4]. According to Lemma 4, to show that G is homeomorphic to $(2^\omega)^\infty$ it remains to verify the property ($\mathcal{SU}$) for the group G.

Fix any embedding $f : B \to G$ of a closed subspace of a metrizable zero-dimensional compactum A. By the continuity of the multiplication $*$ on G, the set $f(B)^{-1} * f(B) = \{f(b)^{-1} * f(b') : b, b' \in B\} \subset G$ is compact. It follows from Theorem 4 of [Ba] that there exists a sequence $(x_n)_{n=1}^\infty \subset G$ converging to the neutral element e of G and such that $x_n \notin f(B)^{-1} * f(B)$ for every $n \in \mathbb{N}$. This implies that $f(B)$ is a nowhere dense subset in the compactum $f(B) * S_0$, where $S_0 = \{e\} \cup \{x_n : n \in \mathbb{N}\}$. Next, since the $\mathcal{MK}_\omega$-group G is uncountable and σ-compact, it contains an uncountable metrizable compactum which in its turn, contains a Cantor set $C \subset G$ according to Lemma 2. Without loss of generality, $C \ni e$. It can be easily shown that the compactum $f(B) * S_0 * C$ has no isolated point and contains $f(B)$ as a nowhere dense subset. Since $f(B) * S_0 * C$ is a zero-dimensional metrizable compactum without isolated points, it is homeomorphic to the Cantor cube 2^ω, which allows us to apply Lemma 3 to produce an embedding $\bar{f} : A \to f(B) * S_0 * C \subset G$ extending the embedding f. Thus the space G satisfies the condition ($\mathcal{SU}$) and G is homeomorphic to $(2^\omega)^\infty$. □

Lemma 5. *If G is a non-metrizable $\mathcal{M}_\omega$-group, then $r(G) \le \omega_1$. Moreover, $r(G) = \omega_1$ if and only if G contains a Cantor set.*

Proof. Suppose G is a non-metrizable $\mathcal{M}_\omega$-group. Write $G = \varinjlim M_i$, where $M_1 \subset M_2 \subset \dots$ of a tower of closed metrizable subspaces of G with $G = \bigcup_{i=1}^\infty M_i$. It follows that each scattered compactum $K \subset G$ is contained in some M_i and being metrizable and scattered, is countable, see Lemma 2. Consequently, $r(K) < \omega_1$ for every such $K \subset G$. Hence $r(G) \le \omega_1$.

If G contains a Cantor set C, then $r(G) \ge r(C) \ge \omega_1$ because C, being universal in the class of zero-dimensional metrizable compacta, contains copies of all countable compacta (whose scatteredness indices run over all countable ordinals, see [Ke, 6.13]).

Assume finally that $r(G)=\omega_1$. According to Theorem 4 of [Ba], G is homeomorphic to the product $H\times D$ of an $\mathcal{KM}_\omega$-group $H\subset G$ and a discrete space D. Clearly, $\omega_1=r(G)=r(H\times D)=r(H)$. Write $H=\varinjlim K_i$, where $K_1\subset K_2\subset\dots$ is a tower of metrizable compacta in H. One of these compacta is uncountable (otherwise we would get $r(H)=\sup\{r(K_i):i\in\mathbb{N}\}<\omega_1$, a contradiction with $r(H)=\omega_1$). Consequently, the group H contains a Cantor set C, see Lemma 2. □

Proof of Main Theorem. Suppose G_1, G_2 are two non-metrizable $\mathcal{M}_\omega$-groups with $r(G_1)=r(G_2)$ and $d(G_1)=d(G_2)$. By Theorem 4 of [Ba], for every $i=1,2$ the space G_i is homeomorphic to the product $H_i\times D_i$, where $H_i\subset G_i$ is an $\mathcal{KM}_\omega$-group and D_i is a discrete space. Since $d(G_1)=d(G_2)$ and the spaces H_1,H_2 are separable, we may assume that $|D_1|=|D_2|$ (if $d(G_1)=d(G_2)$ is countable, then replacing H_i by G_i, we may assume that $|D_1|=|D_2|=1$). Thus to prove that the groups G_1 and G_2 are homeomorphic, it suffices to verify that the groups H_1 and H_2 are homeomorphic. Observe that $r(G_i)=r(H_i\times D_i)=r(H_i)$ for $i=1,2$ and hence $r(H_1)=r(H_2)$.

If $r(H_1)=r(H_2)<\omega_1$, then by Lemmas 2 and 6, the $\mathcal{KM}_\omega$-groups H_1 and H_2 are countable and by Zelenyuk's theorem [Ze], they are homeomorphic. If $r(H_1)=r(H_2)=\omega_1$, then we may apply Theorem and Lemmas 2, 5 to conclude that both groups H_1 and H_2 are homeomorphic to $(2^\omega)^\infty$. □

A topological space X is defened to be an *AE(0)-space* if every continuous map $f:B\to X$ from a closed subset of a zero-dimensional compact Hausdorff space A can be extended to a continuous map $\bar f:A\to X$.

Conjecture. *An uncountable zero-dimensional k_ω-group G is homeomorphic to $(2^\tau)^\infty\times 2^\kappa$ for some cardinals $\tau\le\kappa$ if and only if G is an AE(0)-space.*

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