Lieb-Robinson Bounds and Quasi-locality for the Dynamics of Many-Body Quantum Systems

Robert Sims

Department of Mathematics, University of Arizona,
Tucson, AZ 85721, USA
**E-mail: rsims@math.arizona.edu*
http://math.arizona.edu/∼rsims/

We review a recently proven Lieb-Robinson bound for general, many-body quantum systems with bounded interactions. Several basic examples are discussed as well as the connection between commutator estimates and quasi-locality.

Keywords: Lieb-Robinson bounds, quasi-locality, quantum dynamics

1. Introduction

Much physical intuition is based on locality properties of the system under consideration. Objects (or particles) are associated with regions (or points) in space and non-trivial interactions typically occur over short distances. For systems governed by a relativistic dynamics, the time evolution preserves this notion of locality.

In condensed matter physics, however, many interesting physically relevant phenomena are modeled by quantum many-body systems, e.g., super-conductivity and Bose-Einstein condensation. More abstractly, the models of quantum computation and discrete versions of field theory are described in terms of quantum lattice systems. For these non-relativistic systems, defined e.g. by a Hamiltonian with nearest neighbor interactions, the associated dynamics does not preserve locality in the sense that there is no strict equivalent to a finite speed of light.

Remarkably, Lieb and Robinson, see [1], proved that an approximate form of locality, which we refer to as *quasi-locality*, does hold for the dynamics associated to certain quantum spin systems. This important result establishes the existence and proves a bound for an approximate light cone which limits the rate at which disturbances, as evolved by the dynamics, can propagate through the system. More concretely, they proved that a local observable evolved for a time $t > 0$ remains essentially localized to a region of space whose diameter is proportional to t. They dubbed their estimate as a bound on the system's group velocity, but we prefer to describe the analysis as a Lieb-Robinson bound and the resulting estimate: the Lieb-Robinson velocity.

After the initial result by Lieb and Robinson in 1972 and some calculations for specific models [2] a few years later, these locality estimates for quantum systems received relatively little attention. It was not until Hastings' impressive work of 2004, see [3], that a genuinely renewed interest in these bounds was established. Since then, a number of generalizations of the original result [4–11] and a wealth of interesting applications [3, 12–27] have demonstrated the importance of these bounds.

In this brief note, we introduce the general set-up for Lieb-Robinson bounds in the context of quantum systems with bounded interactions. We describe the correspondence between quasi-locality and the usual commutator estimates typically referred to as Lieb-Robinson bounds. The final section is a short list of further generalizations and a variety of applications.

Before we begin, we make the following useful observation. In quantum mechanics, one is often interested in a single quantum system, i.e., a specific Hilbert space $\mathcal{H}$ and a densely defined, self-adjoint operator H. For each normalized vector, or state $\psi \in \mathcal{H}$, the solution of the Schrödinger equation

$$\partial_t \psi = -iH\psi \tag{1}$$

governs the dynamics of this system. The solution, of course, is $\psi(t) = e^{-itH}\psi$. For particles in a domain $\Lambda \subset \mathbb{R}^\nu$, the typical Hilbert space is $\mathcal{H} = L^2(\Lambda)$ and it is common to have H a self-adjoint realization of the Laplacian. Corresponding to any normalized vector $\psi \in L^2(\Lambda)$ the solution $\psi(t)$ is called a wave function, and it is interesting to investigate the evolution of this function in space, i.e. Λ.

This is *not* the form of locality established by a Lieb-Robinson bound. Lieb-Robinson bounds are about collections of interacting quantum systems distributed in space. The bounds estimate the rate at which disturbances propagate through this collection.

2. The General Set-Up

As discussed above, Lieb-Robinson bounds estimate the rate at which disturbances propagate through a collection of quantum systems. The basic set-up is as follows.

2.1. *Collections of Quantum Systems*

Let Γ be a countable set, and consider a collection of quantum systems labeled by $x \in \Gamma$. By this, we mean that corresponding to each site $x \in \Gamma$ there is a Hilbert space $\mathcal{H}_x$ and a densely defined, self-adjoint operator H_x acting on $\mathcal{H}_x$. The operator H_x is typically referred to as the on-site Hamiltonian. For finite $\Lambda \subset \Gamma$, the Hilbert space of states corresponding to Λ is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \, , \tag{2}$$

and the algebra of observables is

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda) \, , \tag{3}$$

where $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators over the Hilbert space $\mathcal{H}$. Thus an observable $A \in \mathcal{A}_\Lambda$ depends only on those degrees of freedom in Λ. Of course, for any finite $X \subset \Lambda$, an observable $A \in \mathcal{A}_X$ can be uniquely identified with the observable $A \otimes \mathbb{1}_{\Lambda \setminus X} \in \mathcal{A}_\Lambda$, and therefore $\mathcal{A}_X \subset \mathcal{A}_\Lambda$.

In general, these collections of quantum systems are used to describe many interesting physical phenomena e.g., the moments associated with atoms in a magnetic material, a lattice of coupled oscillators, or an array of qubits in which quantum information is stored. Below we indicate two important types of examples.

Example 2.1. A *quantum spin system* over Γ is defined by associating a finite dimensional Hilbert space to each site $x \in \Gamma$, e.g., $\mathcal{H}_x = \mathbb{C}^{n_x}$ for some integer $n_x \geq 2$. The dimension of $\mathcal{H}_x$ is related to the spin at site x by $n_x = 2J_x + 1$, i.e. $n_x = 2$ corresponds to spin $J_x = 1/2$, $n_x = 3$ corresponds to spin $J_x = 1$, etc. As an on-site Hamiltonian, a common choice is to select a spin matrix in the n_x-dimensional irreducible representation of $su(2)$. When $n_x = 2$, these are just the Pauli spin matrices:

$$S^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad S^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Example 2.2. A *quantum oscillator system* over Γ corresponds to associating an L^2 space to each site of Γ, e.g. one can take $\mathcal{H}_x = L^2(\mathbb{R})$ for each $x \in \Gamma$. In contrast to the previous example, each single site Hilbert space is infinite dimensional, and moreover, the on-site Hamiltonians are typically functions of position q_x, the multiplication operator by q_x in $L^2(\mathbb{R}, dq_x)$, and momentum $p_x = -i\frac{d}{dq_x}$; both unbounded self-adjoint operators.

Despite the fact that these examples are quite different, the general techniques described below apply equally well to both cases.

2.2. *Interactions and Models*

The systems described above are of particular interest when they are allowed to interact. In general, a *bounded interaction* for such quantum systems is a mapping Φ from the set of finite subsets of Γ into the algebra of observables which satisfies

$$\Phi(X)^* = \Phi(X) \in \mathcal{A}_X \quad \text{for all finite } X \subset \Gamma. \tag{5}$$

A *model* is defined by the set Γ, the collection of quantum systems $\{(\mathcal{H}_x, H_x)\}_{x\in\Gamma}$, and an interaction Φ.

Associated to a given model there is a family of *local Hamiltonians*, $\{H_\Lambda\}$, parametrized by the finite subsets of Γ. In fact, to each finite $\Lambda \subset \Gamma$,

$$H_\Lambda = \sum_{x\in\Lambda} H_x + \sum_{X\subset\Lambda} \Phi(X) \tag{6}$$

is a densely defined, self-adjoint operator. Here the second sum is over all finite subsets of Λ, and is therefore finite. By Stone's theorem, the corresponding *Heisenberg* *dynamics*, τ_t^Λ, given by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \quad \text{for all } A \in \mathcal{A}_\Lambda \text{ and } t \in \mathbb{R}, \tag{7}$$

is a well-defined, one-parameter group of automorphisms on $\mathcal{A}_\Lambda$. Here are two typical models of interest.

Example 2.3. Fix an integer $\nu \geq 1$ and let $\Gamma = \mathbb{Z}^\nu$. Consider the quantum spin system obtained by setting $\mathcal{H}_x = \mathbb{C}^2$ for all $x \in \mathbb{Z}^\nu$. Take as on-site Hamiltonian $H_x = S^3$ using the notation from (4) above. Let Φ be the interaction defined by setting

$$\Phi(X) = \begin{cases} S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3 & \text{if } X = \{x, y\} \text{ and } |x-y| = 1, \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where for each $z \in \mathbb{Z}^\nu$, any $k \in \{1, 2, 3\}$, and each finite volume $\Lambda \subset \mathbb{Z}^\nu$, the quantity

$$S_z^k = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes S^k \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \tag{9}$$

where S^k, again from (4), appears in the z-th factor of $\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathbb{C}^2)$.

A nearest neighbor, spin 1/2 Heisenberg model on $\mathbb{Z}^\nu$ corresponds to

$$\begin{aligned} H_\Lambda &= h \sum_{x \in \Lambda} H_x + J \sum_{X \subset \Lambda} \Phi(X) \\ &= \sum_{x \in \Lambda} h S_x^3 + \sum_{\substack{x,y \in \Lambda: \\ |x-y|=1}} J \left(S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3 \right) \end{aligned} \tag{10}$$

for all finite subsets $\Lambda \subset \mathbb{Z}^\nu$. Here h and J are real-valued parameters of the model.

Example 2.4. Fix an integer $\nu \geq 1$ and let $\Gamma = \mathbb{Z}^\nu$. Consider the quantum oscillator system obtained by setting $\mathcal{H}_x = L^2(\mathbb{R})$ for all $x \in \mathbb{Z}^\nu$. A nearest neighbor, anharmonic model on $\mathbb{Z}^\nu$ is defined analogously, e.g. with

$$H_\Lambda = \sum_{x \in \Lambda} p_x^2 + V(q_x) + \sum_{\substack{x,y \in \Lambda: \\ |x-y|=1}} \Phi(q_x - q_y) \tag{11}$$

for all finite subsets $\Lambda \subset \mathbb{Z}^\nu$. The parameters of this model are V and Φ. Of course, V must be chosen so that the on-site Hamiltonian $H_x = p_x^2 + V(q_x)$ is self-adjoint, and Φ is assumed to be in $L^\infty(\mathbb{R})$.

2.3. *Observables and Support*

The support of an observable plays a crucial role in Lieb-Robinson bounds. We introduce this notion here. Let Γ be a countable set and $\{(\mathcal{H}_x, H_x)\}_{x \in \Gamma}$ a collection of quantum systems. As we have seen above, for any two finite sets $\Lambda_0 \subset \Lambda \subset \Gamma$, each $A \in \mathcal{A}_{\Lambda_0}$ can be identified with a unique element $A \otimes \mathbb{1}_{\Lambda \setminus \Lambda_0} \in \mathcal{A}_\Lambda$. For this reason, $\mathcal{A}_{\Lambda_0} \subset \mathcal{A}_\Lambda$ for any $\Lambda_0 \subset \Lambda$.

Given an observable $A \in \mathcal{A}_\Lambda$, we say that A is *supported* in $X \subset \Lambda$ if A can be written as $A = \tilde{A} \otimes \mathbb{1}_{\Lambda \setminus X}$ with $\tilde{A} \in \mathcal{A}_X$. The *support* of an observable A is then the minimal set X such that A is supported in X. We will denote the support of an observable A by $\mathrm{supp}(A)$.

Due to the fact that we are considering non-relativistic systems, i.e., models for which there is no strict equivalent to a finite speed of light, the following observation is generally true. Let $X \subset \Lambda \subset \Gamma$. Consider a local Hamiltonian H_Λ defined in terms of a non-trivial interaction; assume e.g. the interaction is nearest neighbor. Then, for general $A \in \mathcal{A}_X$, $\mathrm{supp}(\tau_0(A)) = \mathrm{supp}(A) \subset X$, however, $\mathrm{supp}(\tau_t^\Lambda(A)) = \Lambda$ for all $t \neq 0$. Hence, a strict notion of *locality*, implicitly defined here in terms of the support of an observable, is not generally preserved by the Heisenberg dynamics.

Lieb-Robinson bounds address the following simple question: Does the Heisenberg dynamics corresponding to, e.g. short range interactions, satisfy some weaker form of locality?

2.4. *From Locality to Commutators*

Lieb-Robinson bounds are often expressed in terms of commutator estimates. The relationship between these estimates and the support of local observables is due mainly to the tensor product structure of the observable algebras. We briefly discuss this fact in this section.

Let Γ be countable set and $\{(\mathcal{H}_x, H_x)\}_{x\in\Gamma}$ denote a collection of quantum systems. Consider two finite sets $X, Y \subset \Gamma$. If $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $X \cap Y = \emptyset$, then for any finite set $\Lambda \subset \Gamma$ for which $X \cup Y \subset \Lambda$ we can regard $A, B \in \mathcal{A}_\Lambda$ and as such $[A, B] = 0$ due to the structure of the tensor product. In words, observables with disjoint supports commute.

Conversely, Schur's lemma demonstrates the following. If $A \in \mathcal{A}_\Lambda$ and

$$[A, \mathbb{1}_{\Lambda\setminus Y} \otimes B] = 0 \quad \text{for all } B \in \mathcal{A}_Y \,, \tag{12}$$

then $\mathrm{supp}(A) \subset \Lambda \setminus Y$. In fact, a more general statement is true. If $A \in \mathcal{A}_\Lambda$ almost commutes with all $B \in \mathcal{A}_Y$, then A is approximately supported in $\Lambda \setminus Y$. The following lemma appears in [28].

Lemma 2.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Suppose there exists $\epsilon \geq 0$ for which*

$$\|[A, \mathbb{1}_1 \otimes B]\| \leq \epsilon \|B\| \qquad \textit{for all } B \in \mathcal{B}(\mathcal{H}_2). \tag{13}$$

Then, there exists $A' \in \mathcal{B}(\mathcal{H}_1)$, such that

$$\|A' \otimes \mathbb{1}_2 - A\| \leq \epsilon. \tag{14}$$

Equipped with this lemma, we see that uniform estimates on commutators provide approximate information on the support of observables. Thus, for any $A \in \mathcal{A}_X$, we can approximate $\mathrm{supp}(\tau_t^\Lambda(A))$ by bounding $\|[\tau_t^\Lambda(A), B]\|$ for all B with $\mathrm{supp}(B) \subset Y$. Here the estimates will, of course, depend on the distance between X and Y in Γ and the time t for which the observable has been evolved. This is the basic idea of a Lieb-Robinson bound.

3. Lieb-Robinson Bounds

In this section, we describe in detail assumptions on the set Γ and the interactions Φ under which one can prove a Lieb-Robinson bound. We also present a precise statement of the estimate and discuss several relevant consequences. For a proof in this setting, we refer the interested reader to [9].

3.1. *On the Geometry of* Γ

For many models, e.g both Example 2.3 and 2.4, the set $\Gamma = \mathbb{Z}^\nu$. In general, though, the lattice structure of $\mathbb{Z}^\nu$ is not necessary to prove a Lieb-Robinson bound. The following assumptions are sufficient. Let Γ be a set equipped with a metric d. If Γ has infinite cardinality, we further assume that there is a non-increasing function $F : [0, \infty) \to (0, \infty)$ which satisfies two conditions. First, we assume that F is *uniformly integrable* over Γ, i.e.,

$$\|F\| = \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) < \infty \,. \tag{15}$$

Next, we assume there exists $C > 0$ such that the following *convolution condition* is satisfied: for all $x, y \in \Gamma$,

$$\sum_{z \in \Gamma} F(d(x,z)) F(d(z,y)) \, \leq \, C \, F(d(x,y)). \tag{16}$$

The inequality (16) is quite useful in the iteration scheme which is at the heart of proving a Lieb-Robinson bound.

Here is an important observation. Let Γ be a set with a metric and F satisfy the properties mentioned above with respect to Γ. In this case, the function F_a defined by setting $F_a(r) = e^{-ar}F(r)$ for any $a \geq 0$ also satisfies (15) and (16) above with $\|F_a\| \leq \|F\|$ and $C_a \leq C$. The choice of an exponential weight here is convenient, but not necessary. In fact, $G = wF$ satisfies (15) and (16) for any positive, non-increasing, *logarithmically super-additive* weight w, i.e. a function w for which $w(x + y) \geq w(x)w(y)$.

Example 3.1. Consider the case of $\Gamma = \mathbb{Z}^\nu$. For any $\epsilon > 0$, the function $F(r) = (1 + r)^{-(\nu+\epsilon)}$ is positive, non-increasing, and

$$\|F\| = \sum_{x \in \mathbb{Z}^\nu} \frac{1}{(1 + |x|)^{\nu+\epsilon}} < \infty. \tag{17}$$

A short calculation shows that the convolution constant for this F satisfies $C \leq 2^{\nu+\epsilon+1}\|F\|$. Thus, such functions do exist. As a final remark, we note that the exponential function *does not* satisfy the convolution condition (16) on $\mathbb{Z}^\nu$, however, $F_a(r) = e^{-ar}/(1 + r)^{\nu+\epsilon}$ certainly does.

3.2. *Assumptions on the Interaction* Φ

Locality estimates are valid when the interactions are sufficiently short range. For general sets Γ, a sufficient decay assumption can be made precise in terms of the F function introduced in the previous sub-section.

Let Γ be a set with a metric d and a function F as in Section 3.1. For any $a \geq 0$, denote by $\mathcal{B}_a(\Gamma)$ the set of all those interactions Φ for which

$$\|\Phi\|_a \,=\, \sup_{x,y \in \Gamma} \frac{1}{F_a(d(x,y))} \sum_{\substack{X \subset \Gamma: \\ x,y \in X}} \|\Phi(X)\| \,<\, \infty. \tag{18}$$

Example 3.2. Many interesting models have finite range interactions, see e.g. Example 2.3 and 2.4. An interaction Φ is said to be of finite range if there exists a number $R > 0$ for which $\Phi(X) = 0$ whenever the diameter of X exceeds R. In the case of $\Gamma = \mathbb{Z}^\nu$ and $F(r) = (1+r)^{-\nu-\epsilon}$, it is easy to see that all uniformly bounded, finite range interactions satisfy $\|\Phi\|_a < \infty$ for all $a \geq 0$.

Example 3.3. Another important class of models involve pair interactions. An interaction Φ is called a pair interaction if $\Phi(X) = 0$ unless $X = \{x, y\}$ for some points $x, y \in \Gamma$. In the case of $\Gamma = \mathbb{Z}^\nu$ and $F(r) = (1+r)^{-\nu-\epsilon}$, it is easy to see that all uniformly bounded, pair interactions that decay exponentially in $|x - y|$ satisfy $\|\Phi\|_a < \infty$ for some $a > 0$. In fact, if the pair interactions decay faster than an appropriate inverse polynomial, then $\|\Phi\|_0 < \infty$, and this is sufficient for a decay estimate on the relevant commutators.

3.3. *The Main Result*

We can now state a Lieb-Robinson bound, proven in [9], for the systems introduced above.

Theorem 3.1. *Let Γ be a set with a metric d and a function F as described in Section 3.1. Fix a collection of quantum systems $\{(\mathcal{H}_x, H_x)\}_{x\in\Gamma}$ over Γ, and for any $a \geq 0$, let $\Phi \in \mathcal{B}_a(\Gamma)$. Then, the model defined by Γ, $\{(\mathcal{H}_x, H_x)\}_{x\in\Gamma}$, and Φ satisfies a Lieb-Robinson bound. In fact, for each fixed finite subsets $X, Y \subset \Gamma$, and any finite $\Lambda \subset \Gamma$ with $X \cup Y \subset \Lambda$, the estimate*

$$\|[\tau_t^\Lambda(A), B]\| \leq 2\|A\|\|B\| \min \left[1, g_a(t) \sum_{x \in X, y \in Y} F_a\left(d(x,y)\right) \right] \tag{19}$$

holds for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$. Here the function g_a is given by

$$g_a(t) = \begin{cases} C_a^{-1}\left(e^{2\|\Phi\|_a C_a |t|} - 1\right) & \text{if } X \cap Y = \emptyset, \\ C_a^{-1} e^{2\|\Phi\|_a C_a |t|} & \text{otherwise.} \end{cases} \tag{20}$$

As a corollary, a more familiar form of the Lieb-Robinson bound can be expressed in terms of $d(X, Y) = \min_{x\in X, y\in Y} d(x, y)$, namely

Corollary 3.1. *Given the assumptions of Theorem 3.1 and $a > 0$, the estimate*

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|\|B\|\|F\|}{C_a} \min[|X|, |Y|] e^{-a(d(X,Y) - v_\Phi(a)|t|)} \tag{21}$$

is valid. Here $|X|$ denotes the cardinality of X and the quantity $v_\Phi(a)$ is given by

$$v_\Phi(a) = \frac{2\|\Phi\|_a C_a}{a} \, . \tag{22}$$

Let us make a few remarks to help interpret these bounds. As we indicated in Section 2.2, the Heisenberg dynamics, τ_t^Λ, forms a one-parameter group of automorphisms on $\mathcal{A}_\Lambda$ and so the estimate $\|[\tau_t^\Lambda(A), B]\| \leq 2\|A\|\|B\|$ is always true. What we see from (19), and more directly in (21), is that if A and B have disjoint supports X and Y respectively, then $[\tau_0(A), B] = [A, B] = 0$ and the estimate on $[\tau_t(A), B]$ is small in the distance $d(X, Y)$ for times

$$|t| \leq \frac{d(X, Y)}{v_\Phi(a)} \, . \tag{23}$$

For this reason, the quantity $v_\Phi(a)$ is called a Lieb-Robinson velocity for the model under consideration. In fact, using Lemma 2.1, we see that for each $A \in \mathcal{A}_X$ the time evolution $\tau_t^\Lambda(A)$ is approximately supported in a ball of radius $v_\Phi(a)|t|$ about X. Thus the dynamics of the system remain essentially confined to a light cone defined by this Lieb-Robinson velocity. Moreover, the velocity $v_\Phi(a)$, see (22), which governs the rate at which disturbances propagate through the system, depends only on the interaction Φ and the geometry of Γ; specifically, it is independent of the on-site Hamiltonians.

Another crucial fact about these Lieb-Robinson bounds is that the explicit estimates, in particular the velocity, do not depend on the finite volume Λ on which the dynamics is defined. This suggests, and can be proven in this setting see e.g. [27], that a thermodynamic limit for the dynamics exists. It too satisfies the same Lieb-Robinson bound.

As has been useful in a variety of applications, it is interesting to note the dependence of these bounds on the support of the corresponding observables. Since only the minimum cardinality appears, one of two the observables could be allowed to be volume, i.e. Λ, dependent without sacrificing the bound. In fact, a more detailed analysis shows that only the minimum cardinality of the boundary of the supports of the observables is relevant, see [9] for a precise statement.

When $\Phi \in \mathcal{B}_a(\Gamma)$ for some $a > 0$, the Lieb-Robinson bounds decay exponentially in the distance between the supports of the observables. The rate of this exponential decay, here the number $a > 0$, is usually of little consequence. For this reason, if $\Phi \in \mathcal{B}_a(\Gamma)$ for all $a \in (\alpha, \beta)$, the optimal Lieb-Robinson velocity is given by

$$\inf_{a \in (\alpha, \beta)} v_\Phi(a) = \inf_{a \in (\alpha, \beta)} \frac{2\|\Phi\|_a C_a}{a} \, . \tag{24}$$

We now estimate this optimum velocity for the previously mentioned examples.

Example 3.4. Let $\Gamma = \mathbb{Z}^\nu$, $F(r) = (1+r)^{-\nu-\epsilon}$, and consider the spin 1/2 Heisenberg model introduced in Example 2.3. Clearly, for any $a > 0$,

$$\|\Phi\|_a = e^a 2^{\nu+\epsilon} 3J < \infty \,, \tag{25}$$

and therefore, a bound on the optimal velocity of this model is given by

$$3Je2^{2(\nu+\epsilon+1)} \sum_{x\in\mathbb{Z}^\nu} \frac{1}{(1+|x|)^{\nu+\epsilon}} \,. \tag{26}$$

As we observed above, this estimate on the velocity is independent of the on-site parameter h.

Example 3.5. Let $\Gamma = \mathbb{Z}^\nu$, $F(r) = (1+r)^{-\nu-\epsilon}$, and consider the anharmonic Hamiltonian introduced in Example 2.4. A similar calculation gives a bound on the optimal velocity of

$$\|\Phi\|_\infty e 2^{2(\nu+\epsilon+1)} \sum_{x\in\mathbb{Z}^\nu} \frac{1}{(1+|x|)^{\nu+\epsilon}} \,, \tag{27}$$

which is independent of the on-site function V, so long as the self-adjointness assumption is satisfied.

4. Some Words on Generalizations and Applications

Over the past few years applications have driven a number of interesting generalizations of the original Lieb-Robinson bounds. Several review articles have been devoted to many of these specific applications, see [8, 29], and some lecture notes from schools on topics concerning locality are now available [30, 31]. In this short note, we make no attempt to give an exhaustive list of generalizations and applications, but rather we list many relevant works to give the interested reader a reasonable starting point to further investigate this active area of research.

4.1. *On Generalizations*

The Lieb-Robinson bound stated in Theorem 3.1, and proven in [9], already includes several generalizations of the original result. Most importantly, it applies to quantum systems with infinite dimensional, single site Hilbert spaces. In addition, no assumption on the lattice structure of Γ is necessary, and the dependence of the bound on the support of the observables has been refined.

Recently, Lieb-Robinson bounds have been proven for time-dependent interactions, see [28]. Moreover, Poulin demonstrated in [11] that these estimates also hold for an irreversible, semi-group dynamics generated by Lindblad operators.

Quite some time ago, it was proven in [32] that the analogue of Lieb-Robinson bounds hold for the non-relativistic dynamics corresponding to classical Hamiltonian systems. In the past few years, further work in this direction has appeared in [33] and [34].

An important open question is: To what extent do Lieb-Robinson bounds apply in the case of unbounded interactions? For certain simple systems, there has been some progress on this issue. Lieb-Robinson bounds for general harmonic systems first appeared in [7]. It was proven in [9], see also [21, 27], that these estimates also hold for anharmonic systems if the perturbation is sufficiently weak. A recent result in [35] suggests that such bounds apply much more generally. Finally, a Lieb-Robinson estimate for commutator bounded operators appeared in [10].

4.2. *On Applications*

Many of the generalizations mentioned above came about by pursuing concrete applications. As discussed in the main text, the resurgence of interest in Lieb-Robinson bounds was mainly motivated by Hastings' 2004 paper [3] on a proof of the multi-dimensional Lieb-Schultz-Mattis theorem. In this incredibly influential paper, Hastings discussed generalized Lieb-Robinson bounds, an Exponential Clustering theorem, and pioneered his notion of a quasi-adiabatic evolution. This single work inspired a flurry of activity which continues to this day.

The Lieb-Schultz-Mattis theorem, see [36], concerns the spectral gap between the ground state energy and that of the first excited state for the nearest-neighbor, spin 1/2 Heisenberg model in one dimension. They proved that for a finite volume of size L, if the ground state is unique, then the gap is bounded by C/L, for some constant C. Further generalizations, to models with arbitrary half-integer spin and to a statement valid in the thermodynamic limit appeared in [37]. Hastings paper [3] developed a multi-dimensional analogue of this result. In fact, his argument yields a gap estimate applicable in a great deal of generality, see [15] for a precise statement. Recent reviews of these results appear in [8] and [30].

The Exponential Clustering theorem is a proof that the ground state expectations of gapped systems decay exponentially in space. Proofs of this result first appeared in [4] and [5]. A refinement of the dependence of the estimates on the support of the observables was proven in [8], and this fact was later used by Matsui in [26] to investigate a split property for quantum spin chains.

It is well known that, for quantum spin systems, a Lieb-Robinson bound may be used to establish the existence of a thermodynamic limit for the Heisenberg dynamics, see e.g. [38]. Improved estimates allowed for this result to be generalized, e.g., the case of polynomially decaying interactions was covered in [6] and the existence of the dynamics for the general systems considered here was proven in [27]. For perturbations of the harmonic system, the existence of the thermodynamic limit has been proven with two distinct methods, see [21] and [27].

An area law for gapped one-dimensional systems was proven by Hastings in [14]. In general, the area law conjecture states that the von Neumann entropy of the restriction of gapped ground states to a finite volume of size Λ grows no faster than a quantity proportional to the surface area of Λ. Certain aspects of Hastings' argument generalize to the multi-dimensional setting, e.g. a factorization property of gapped ground states was proven in [19], but a proof of the area law for general gapped systems in arbitrary dimension remains an important open question. Some progress on a class of unfrustrated spin Hamiltonians appears in [22]. A review of these topics is contained in [29], see also [31].

Quantization of the Hall conductance for a general class of interacting fermions was recently proven in [20]. This intriguing result makes crucial use of improved Lieb-Robinson bounds and the methods associated with Hastings' quasi-adiabtic evolution. A detailed analysis of this technique, with specific regards to its implications for perturbation theory, is the main topic of [28].

Finally, stability of topological order was addressed in [23, 24]. There the authors consider a class of Hamiltonians that are the sum of commuting short-range terms, such as the toric code model developed by Kitaev in [39], and proved that the topological order of the ground states is stable under arbitrary, small short-range perturbations.

Developing a better understanding of quantum dynamics and its perturbation theory will be crucial in providing new insight into complex physical phenomena. As indicated by the number of recent generalizations and applications, the analysis of Lieb-Robinson bounds is a thriving area of active research which attempts to address this very issue.

Acknowledgments

I would like to thank my collaborator Bruno Nachtergaele for many inspiring and insightful discussions which have greatly enriched my understanding of this field and strongly influenced the better parts of this presentation. The work reported in this paper was supported by the National Science Foundation under grants #DMS-0757581 and #DMS-0757424.

References

[1] E. Lieb and D. Robinson, *Commun. Math. Phys.* **28**, 251–257, (1972).
[2] C. Radin, *Commun. Math. Phys.* **62**, 159-166, (1978).
[3] M. Hastings, *Phys. Rev. B* **69**, 104431, (2004).
[4] B. Nachtergaele and R. Sims, *Commun. Math. Phys.* **265**, 119–130, (2006).
[5] M. Hastings and T. Koma, *Commun. Math. Phys.* **265**, 781–804, (2006).
[6] B. Nachtergaele, Y. Ogata, and R. Sims, *J. Stat. Phys.* **124**, no.1, 1–13, (2006).
[7] M. Cramer, A. Serafini, and J. Eisert, in *Quantum information and many body quantum systems.* (CRM Series, Pisa, 2008).
[8] B. Nachtergaele and R. Sims, in *New trends in mathematical physics. Selected contributions of the XVth international congress on mathematical physics.* (Springer, New York, 2009).
[9] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, *Commun. Math. Phys.* **286**, 1073–1098, (2009).
[10] I. Prémont-Schwarz, A. Hamma, I. Klich, and F. Markopoulou-Kalamara, *Phys. Rev. A.* **81**, no 4, (2010).
[11] D. Poulin, *Phys. Rev. Lett.* **104**, 190401, (2010).

[12] M. Hastings and X. Wen, *Phys. Rev. B.* **72**, 045141, (2005).
[13] S. Bravyi, M.B. Hastings, and F. Verstraete, *Phys. Rev. Lett.* **97**, 050401, (2006).
[14] M. Hastings, *JSTAT* P08024, (2007).
[15] B. Nachtergaele and R. Sims, *Commun. Math. Phys.* **276**, 437–472, (2007).
[16] C. Burrell and T. Osborne, *Phys. Rev. Lett.* **99**, 167201, (2007).
[17] T. Osborne, *J. Phys. A.* **75**, 032321, (2007).
[18] L. Erdos and B. Schlein, *J. Statist. Phys.* **134**, no.5-6, 859–870, (2009).
[19] E. Hamza, S. Michalakis, B. Nachtergaele, and R. Sims, *J. Math. Phys.* **50**, 095213, (2009).
[20] M. Hastings and S. Michalakis, *Quantization of Hall conductance for interacting electrons without averaging assumptions.*, arXiv:0911.4706.
[21] L. Amour, P. Levy-Bruhl, and J. Nourrigat, *Colloq. Math.* **118**, no. 2, 609–648, (2010).
[22] N. de Beaudrap, T. Osborne, and J. Eisert, *New J. Phys.* **12**, 095007, (2010).
[23] S. Bravyi, M. Hastings, and S. Michalakis, *J. Math. Phys.* **51**, 093512, (2010).
[24] S. Bravyi and M. Hastings, *A short proof of stability of topological order under local perturbations.*, arXiv:1001.4363.
[25] M. Hastings, *Quasi-adiabatic continuation for disordered systems: Applications to correlations, Lieb-Schultz-Mattis, and Hall conductance.*, arXiv:1001.5280.
[26] T. Matsui, *J. Math. Phys.* **51**, 015216, (2010).
[27] B. Nachtergaele, B. Schlein, R. Sims, S. Starr, and V. Zagrebnov, *Rev. Math. Phys.* **22**, issue 2, 207– 231, (2010).
[28] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, In preparation.
[29] J. Eisert, M. Cramer, and M. Plenio, *Rev. Mod. Phys.* **82**, 277 (2010).
[30] M. Hastings, *Locality in Quantum Systems.*, arXiv:1008.5137.
[31] B. Nachtergaele and R. Sims, *Contemp. Math.* **529** 141–176, (2010).
[32] C. Marchioro, A. Pellegrinotti, M. Pulvirenti, and L. Triolo, *J. Stat. Phys.* **19**, no. 5, 499–510, (1978).
[33] P. Buttà, E. Caglioti, S. Di Ruzza, and C. Marchioro. *J. Stat. Phys.* **127**, 313–325, (2007).
[34] H. Raz and R. Sims. *J. Stat. Phys.* **137**, 79–108, (2009).
[35] S. Harrison, N. Schuch, T. Osborne, and J. Eisert, *Information propagation for interacting particle systems.*, arXiv:1010.4576.
[36] E. Lieb, T. Schultz, and D. Mattis. *Ann. Phys.* **16**, 407–466, (1961).
[37] I. Affleck and E. Lieb. *Lett. Math. Phys.* **12**, 57–69, (1986).
[38] O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, 2nd Edition (Springer-Verlag, New York, 1997).
[39] A. Kitaev. *Ann. Phys.* **303**, 2, (2003).