

ABOUT THE PROOF OF THE FREDHOLM ALTERNATIVE THEOREMS

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ABSTRACT. In this short paper we review and extract some features of the Fredholm Alternative problem .

1. PRELIMINARIES

In mathematics, the Fredholm alternative, named after Ivar Fredholm, is one of Fredholm's theorems and is a result in Fredholm theory. It may be expressed in several ways, as a theorem of linear algebra, a theorem of integral equations, or as a theorem on Fredholm operators. Part of the result states that a non-zero complex number in the spectrum of a compact operator is an eigenvalue.[1]

1.1. Linear algebra. If V is an n -dimensional vector space and T is a linear transformation, then exactly one of the following holds

1-For each vector v in V there is a vector u in V so that $T(u) = v$. In other words: T is surjective (and so also bijective, since V is finite-dimensional).

2- $\dim(Ker(T)) > 0$

A more elementary formulation, in terms of matrices, is as follows. Given an mn matrix A and a $m1$ column vector b , exactly one of the following must hold

Either: $Ax = b$ has a solution x Or: $A^T y = 0$ has a solution y with $y^T b \neq 0$. In other words, $Ax = b$ has a solution if and only if for any $A^T y = 0, y^T b = 0$.

1.2. Integral equations. Let $K(x,y)$ be an integral kernel, and consider the homogeneous equation, the Fredholm integral equation

$$(1) \quad \lambda\phi(x) - \int_a^b K(x,y)\phi(y)dy = f(x)$$

,The Fredholm alternative states that, for any non-zero fixed complex number λ , either the Homogenous equation has a non-trivial solution, or the inhomogeneous equation has a solution for all $f(x)$. A sufficient condition for this theorem to hold is for $K(x,y)$ to be square integrable on the rectangle $[a, b] \times [a, b]$ (where a and/or b may be minus or plus infinity).

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1.3. Functional analysis. Results on the Fredholm operator generalize these results to vector spaces of infinite dimensions, Banach spaces.

1.4. Correspondence. Loosely speaking, the correspondence between the linear algebra version, and the integral equation version, is as follows: Let $T = \lambda - K$ or, in index notation,

$T(x, y) = \lambda\delta(x - y) - K(x, y)$ with $\delta(x - y)$ the Dirac delta function. Here, T can be seen to be a linear operator acting on a Banach space V of functions $f(x)$, so that $T : V \mapsto V$ is given by $\phi \mapsto \psi$ with ψ given by $\psi(x) = \int_a^b K(x, y)\phi(y)dy$

In this language, the integral equation alternatives are seen to correspond to the linear algebra alternatives

1.5. Alternative. In more precise terms, the Fredholm alternative only applies when K is a compact operator. From Fredholm theory, smooth integral kernels are compact operators. The Fredholm alternative may be restated in the following form: a nonzero λ is either an eigenvalue of K , or it lies in the domain of the resolvent[2],[3].

We recall that a bounded linear operator $T : V \rightarrow V$ on a Banach space V is called a *compact* operator if T maps the closed unit ball of V to a relatively compact subset of V . The following is a basic result about compact operators known as the *Fredholm Alternative*. Let T be a compact operator on a Banach space V and let \mathcal{I} denote the identity operator on V . Then either the operator $\mathcal{I} - T$ is invertible (i.e. has a bounded inverse), or there exists a nonzero vector $\zeta \in V$, $\zeta \neq 0$, such that $T\zeta = \zeta$.

2. SADDLE POINT'S VERSION OF FREDHOLM ALTERNATIVE THEOREM

Theorem : Let the operator $A : H \mapsto H$ be linear, compact and self-adjoint on the separable Hilbert space H . Let $f \in H$, $\lambda \neq 0 \in \mathbb{R}$. Then the problem

$$(2) \quad \lambda u - Au = f$$

has a solution iff

$$(3) \quad (u, f) = 0$$

for all $u \in H$. There is a beautiful theorem about it

Theorem : Let the operator $A : H \mapsto H$ be a linear, compact, symmetric and positive on the separable Hilbert space H and let $\lambda \neq 0 \in \mathbb{R}$, $\lambda_1 > \lambda > 0$. Then the Fredholm alternative for the operator A and the equation (2) is a consequence of the Saddle Point Theorem. For a good proof see the reference[4].In this reference it the author has been proved that the Fredholm alternative for such an operator is a consequence of the Saddle Point Theorem.Also see the reference [4] in it[5].

3. UNBOUNDED FREDHOLM OPERATORS

One can study the topology of the space of all (generally unbounded) self-adjoint Fredholm operators, and to put the notion of spectral flow for continuous paths of such operators on a firm mathematical footing. Also there is a parallel method due to Nicolaescu's approach which requires the continuity of the Riesz map and to achieve that additional properties of the families of boundary problems [6]. Operator curves on manifolds with boundary. In low-dimensional topology and quantum field theory, various examples of operator curves appear which take their departure in a symmetric elliptic differential operator of first order (usually an operator of Dirac type) on a fixed compact Riemannian smooth manifold M with boundary Σ . Posing a suitable well-posed boundary value problem provides for a nicely spaced discrete spectrum near 0. Then, varying the coefficients of the differential operator and the imposed boundary condition suggests the use of the powerful topological concept of spectral flow. We can show under which conditions the curves of the induced self-adjoint L^2 -extensions become continuous curves in the gap topology such that their spectral flow is well defined and truly homotopy invariant. But we remind it to another work.

4. SUMMARY

We review some aspects of the Fredholm theorem. We show that there is a relation between topology and this theorem.

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