

Product of real spectral triples

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Abstract

We construct the product of real spectral triples of arbitrary finite dimension (and arbitrary parity) taking into account the fact that in the even case there are two possible real structures, in the odd case there are two inequivalent representations of the gamma matrices (Clifford algebra), and in the even-even case there are two natural candidates for the Dirac operator of the product triple.

1 Introduction

In noncommutative geometry à la Connes a Riemannian spin manifold is encoded in terms of a spectral triple which satisfies seven additional properties [5], [8], [6]. These further properties have been formulated for noncommutative spectral triples [5]. They are satisfied e.g by the noncommutative torus, and part of them holds for sundry quantum groups.

The composition of not necessarily commutative spectral triples corresponding to the Cartesian product of manifolds is of relevance for construction of a would-be tensor category, but also bears interest for some applications in theoretical physics. For instance the almost commutative spectral triple corresponding to the standard model of particle physics [3] is a tensor product of a canonical commutative spectral triple with a finite dimensional noncommutative one. Moreover the tensor product with a spectral triple of complex dimension is used in the Connes-Marcolli treatment [6] of dimensional renormalization of quantum fields.

In this paper we study the behaviour under tensor product of one of the additional properties, namely the reality axiom. This axiom is much more important in the noncommutative case as it is there employed in the formulation of few of the other axioms.

We carefully analyse all the possibilities and note that in even dimensions there are always two real structure operators J , that differ by multiplication by the grading operator. None of them should be preferred as they are perfectly on the same

footing. This leads to a richer table of their possible tensor products, which we study systematically, completing the results of [12] (see also [11]) obtained for the even-even case and the even-odd (or odd-even) case. We construct also the tensor product of two odd real spectral triples (that requires a doubling of the tensor product of the Hilbert spaces). When dealing with odd spectral triples we are careful about the two inequivalent representations of gamma matrices (Clifford algebra).

Composing two even-dimensional Dirac operators we consider two choices, which differ by using the chirality operator (grading) either of the first or of the second space. The two operators thus obtained are unitarily equivalent if no other requirements are imposed, but this is no longer the case when boundaries are present ([2]). Moreover, the first expression is relevant for the composition of an even dimensional space with an odd dimensional one, while the second expression is relevant for the composition of an odd dimensional space with an even dimensional one.

For concreteness, we provide the explicit formulae for the eigenvalues and eigenvectors in terms of those of the individual components. Furthermore, we also analyse few of the additional conditions (axioms) like dimension, regularity, first order and orientation.

2 Commutative real spectral triple

In the following, the symbol \mathbb{Z}_+ will denote the set of strictly positive integers and the symbol \mathbb{N} will denote the set of non-negative integers.

2.1 Gamma matrices

For each $n = \mathbb{Z}_+$ consider the irreducible (complex) representations of the Clifford algebra $\mathcal{C}(\mathbb{R}^n)$ of Euclidean space \mathbb{R}^n with negative-definite metric. For even (resp. odd) n , let us denote by $\Gamma_{(n)}$ (resp. $\Gamma_{(n,+)}$ and $\Gamma_{(n,-)}$) a possible choice of sets of complex matrices generating the only irreducible representation (respectively the only two irreducible representations) of $\mathcal{C}(\mathbb{R}^n)$, given by:

$$\begin{aligned}
\Gamma_{(1,+)} &= \{\gamma_{(1,+)}^1\} \\
\Gamma_{(1,-)} &= \{\gamma_{(1,-)}^1\} \\
\Gamma_{(2m)} &= \{\gamma_{(2m)}^1, \dots, \gamma_{(2m)}^{2m}\} \\
\Gamma_{(2m+1,+)} &= \{\gamma_{(2m+1,+)}^1, \dots, \gamma_{(2m+1,+)}^{2m}, \gamma_{(2m+1,+)}^{2m+1}\} \\
\Gamma_{(2m+1,-)} &= \{\gamma_{(2m+1,-)}^1, \dots, \gamma_{(2m+1,-)}^{2m}, \gamma_{(2m+1,-)}^{2m+1}\},
\end{aligned} \tag{1}$$

where $\gamma_{(1,\pm)}^1 = \pm i$ and for $n = 2m$ ($m \in \mathbb{Z}_+$), $j = 1, \dots, m$, each $\gamma_{(n)}^\mu$ ($\mu = 1, \dots, n$) is a $2^m \times 2^m$ complex matrix given by

$$\begin{aligned}\gamma_{(n)}^j &= i \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{m-j} \otimes \sigma_1 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1}, \\ \gamma_{(n)}^{m+j} &= i \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{m-j} \otimes \sigma_2 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1},\end{aligned}\tag{2}$$

while for $n = 2m + 1$ ($m \in \mathbb{Z}_+$), $k = 1, \dots, n - 1$ we put

$$\begin{aligned}\gamma_{(n,\pm)}^k &= \gamma_{(n-1)}^k, \\ \gamma_{(n,\pm)}^n &= \pm i \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_m,\end{aligned}\tag{3}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\tag{4}$$

Indeed, the gamma matrices defined above anti-commute with each other and square to -1 . Note that they are anti-hermitean. Moreover we have chosen them so that for a given n the first m are imaginary, the next m are real and $\gamma_{(2m+1,\pm)}^{2m+1}$ is imaginary. The properties discussed in the sequel do not depend on this choice up to unitary equivalence of matrices.

Note also that

$$\gamma_{(2m+1,\pm)}^{2m+1} = \pm i (i)^{(m \bmod 2)} \gamma_{(2m)}^1 \gamma_{(2m)}^2 \cdots \gamma_{(2m)}^{2m}\tag{5}$$

For even n we define the grading operator $\chi_{(n)} := \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n/2}$. For odd n we define

also $\chi_{(n,\pm)} := \pm \mathbf{1}$. (In the sequel we shall often omit the lower indices to simplify notation). Note that, for any $n \in \mathbb{Z}_+$,

$$\chi = \alpha_n \gamma^1 \cdots \gamma^n,\tag{6}$$

where $\alpha_n = 1, -i, i, 1$ if $n = 0, 1, 2, 3 \bmod 4$ respectively.

2.2 Dirac operator

The (free) Dirac operator on \mathbb{R}^n is given by the formula

$$D = \sum_{\mu=1}^n \gamma^\mu \frac{\partial}{\partial x^\mu}.\tag{7}$$

where the gammas are as above. Note that, for even n ,

$$D\chi + \chi D = 0.$$

Note also that, for odd n , changing the representation $\Gamma_{(2m+1,+)}$ to $\Gamma_{(2m+1,-)}$ is equivalent to changing the orientation of the manifold.

The ‘minimal coupling’ interaction with gauge fields, notably the electromagnetic potential A_μ , amounts to the substitution of the usual derivatives by the covariant ones,

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} + ieA_\mu,$$

where e is the charge.

As well known, the Dirac operator can be defined for a flat metric of arbitrary signature, and generalized to (pseudo) Riemannian spin manifolds with the help of covariant derivative given by the Levi-Civita (spin) connection. The elliptic or Riemannian case is extremely important and well studied in mathematics. In theoretical physics the Lorentzian case describes the evolution of spinor fields (fermions), and is also useful in connection with general relativity, modern versions of Kaluza-Klein theories, and (super) string theory. Recently, A. Connes made the Dirac operator a fundamental ingredient of a ‘spectral triple’ and of the notion of noncommutative (spin) manifold.

2.3 Charge conjugation

In physics, the charge conjugation J of spinors exchanges the Dirac operators corresponding to charge e and $-e$, keeping invariant the other physical quantities. However, we allow a possibility that J either commutes or anticommutes with the ‘neutral’ D given by (7). We shall indicate by a subscript \pm these two options, which amount to demanding that

$$J_\pm \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + ieA_\mu \right) = \pm \gamma^\mu \left(\frac{\partial}{\partial x^\mu} - ieA_\mu \right) J_\pm. \quad (8)$$

The operators J_\pm have to be \mathbb{C} -antilinear, given by a composition of the complex conjugation with a constant matrix C_\pm , satisfying $C_\pm \bar{\gamma}^\mu = \pm \gamma^\mu C_\pm$. Hence, C_+ should anti-commute with γ^μ for $\mu \leq m$ which are imaginary and commute with γ^μ for $m < \mu \leq 2m$ which are real. By the uniqueness and anti-commutativity of gamma matrices, such C_+ is proportional to $\gamma^1 \gamma^2 \dots \gamma^m$ if m is even and to $\gamma^{m+1} \gamma^{m+2} \dots \gamma^{2m}$ if m is odd. It is just the other way for C_- .

For $\Gamma_{(2m)}$, this fixes the two solutions J_\pm (up to a scalar multiple). Moreover, J_- is obtained by multiplying J_+ with $\gamma^1 \dots \gamma^{2m}$ (up to a scalar multiple).

For $\Gamma_{(2m+1,+)}$ and for $\Gamma_{(2m+1,-)}$ we have to consider in addition the matrix $\gamma_{(2m+1,\pm)}^{2m+1}$, which is imaginary. Then the above (anti)-commutativity requirement selects J_+ as

the only solution if $m = 1, 3 \pmod 4$ (i.e. $n = 3, 7 \pmod 8$) and J_- if $m = 0, 2 \pmod 4$ (i.e. $n = 1, 5 \pmod 8$).

It can be checked that, with respect to the standard Hermitean scalar product \langle , \rangle on \mathbb{C}^{2^m} , the charge conjugation is a \mathbb{C} -antilinear isometry, that is $JJ^\dagger = 1 = J^\dagger J$, where the adjoint of a \mathbb{C} -antilinear operator is defined by $\langle \phi, J^\dagger \psi \rangle = \langle \psi, J\phi \rangle$. This reduces the ambiguity of J_\pm to be a scalar of modulus 1.

The commutation relation of J_\pm with D is by construction $DJ_\pm = \varepsilon' J_\pm D$, with $\varepsilon' = +1$ for J_+ and $\varepsilon' = -1$ for J_- . Next, the commutation relation with χ (if n is even) is governed by $\varepsilon'' = i^n = (-1)^{n/2}$. A straightforward computation gives $(J_\pm)^2 = \varepsilon \mathbb{I}$, where ε (together with $\varepsilon', \varepsilon''$) is given by table 1 below.

Table 1: Connes' selection in [6] is marked by \bullet

n	0	2	4	6	0	2	4	6	1	3	5	7
ε	+	-	-	+	+	+	-	-	+	-	-	+
ε'	+	+	+	+	-	-	-	-	-	+	-	+
ε''	+	-	+	-	+	-	+	-				
	\bullet	\bullet	\bullet	\bullet					\bullet	\bullet	\bullet	\bullet

Notice that altogether there are twelve different possibilities, which can be labeled by the so-called KO-dimension $n \in \mathbb{Z}_8$ with the additional index ε' if n is even (so for example the case $(\varepsilon, \varepsilon', \varepsilon'') = (+, -, -)$ is labelled by 2_-). We find it notationally convenient to place this additional index also in the case of odd n , though it is redundant there. (For pseudo-euclidean spaces the periodicity modulo 8 holds for the signature $p - q$ of the metric).

The geometrical significance of the charge conjugation J_\pm is that it governs the reduction of a spin^c structure to a spin structure (the Lie algebra $\text{spin}(n)$ is generated by $\gamma_\mu \gamma_\nu$ with $\mu < \nu$, which commute with J and so are invariant under Ad_J , while $\text{spin}^c(n)$ is generated by $\text{spin}(n)$ and one more matrix $i\mathbb{I}$, which anti-commutes with J).

The operator in (7) is a first-order partial differential operator with matrix coefficients. It acts on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^{2^m})$. After completion to $L^2(\mathbb{R}^n, \mathbb{C}^{2^m})$, D becomes an unbounded self-adjoint operator. The $*$ -algebra of smooth complex-valued functions on \mathbb{R}^n (with pointwise operations) is represented on $L^2(\mathbb{R}^n, \mathbb{C}^{2^m})$ as multiplication operators.

2.4 Dirac operator on Cartesian product

Let $n = n_1 + n_2$ with $n_1, n_2 \in \mathbb{Z}_+$. It is straightforward to see that the Dirac operator (7) on \mathbb{R}^n decomposes (up to unitary equivalences of matrices and a suitable renaming

of coordinates) into Dirac operators D_1 on \mathbb{R}^{n_1} and D_2 on \mathbb{R}^{n_2} as follows ($\mathbf{1}$ and χ denote the relevant identity and grading matrices):

- if $n_1 = 2m_1$ and $n_2 = 2m_2 + 1$ ($m_1 \in \mathbb{Z}_+$, $m_2 \in \mathbb{N}$)

$$D = \sum_{\mu=1}^{n_1} \gamma_{(n_1)}^\mu \otimes \mathbf{1} \frac{\partial}{\partial x^\mu} + \sum_{\nu=1}^{n_2} \chi_{(n_1)} \otimes \gamma_{(n_2, \pm)}^\nu \frac{\partial}{\partial x^{n_1+\nu}}, \quad (9)$$

(using (5) it is not difficult to see that the tensor product gamma matrices appearing in (9) belong to the representation $\Gamma_{(2m_1+2m_2+1, \pm)}$, with the index \pm identical to the one of $\gamma_{(n_2, \pm)}^\nu$, belonging to $\Gamma_{(2m_2+1, \pm)}$; in other words, the \pm -type of the irreducible representation is preserved);

- if $n_1 = 2m_1 + 1$ and $n_2 = 2m_2$ ($m_1 \in \mathbb{N}$, $m_2 \in \mathbb{Z}_+$)

$$D = \sum_{\mu=1}^{n_1} \gamma_{(n_1, \pm)}^\mu \otimes \chi_{(n_2)} \frac{\partial}{\partial x^{n_2+\mu}} + \sum_{\nu=1}^{n_2} \mathbf{1} \otimes \gamma_{(n_2)}^\nu \frac{\partial}{\partial x^\nu}, \quad (10)$$

(again the \pm -type of the irreducible representation is preserved);

- if both $n_1 = 2m_1$ and $n_2 = 2m_2$ ($m_1, m_2 \in \mathbb{Z}_+$) are even, both formulae 9 and 10 hold and are related by a unitary matrix;
- if both $n_1 = 2m_1 + 1$ and $n_2 = 2m_2 + 1$ ($m_1, m_2 \in \mathbb{N}$) are odd then

$$D = \sum_{\mu=1}^{n_1} \gamma_{(n_1)}^\mu \otimes \mathbf{1} \otimes \sigma_1 \frac{\partial}{\partial x^\mu} + \sum_{\nu=1}^{n_2} \mathbf{1} \otimes \gamma_{(n_2)}^\nu \otimes \sigma_2 \frac{\partial}{\partial x^{n_1+\nu}}. \quad (11)$$

Moreover, we can take $\chi := \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3$ as grading.

3 Definition of a real spectral triple

The classical setting presented in section 2 was generalized by Connes to the non-commutative case, which we now recall and supplement by keeping all the twelve possibilities for the reality structure. We recall from [6]

Definition 1. A spectral triple (A, \mathcal{H}, D) is given by an involutive unital algebra A (over \mathbb{R} or \mathbb{C}) faithfully represented as bounded operators on a complex separable Hilbert space \mathcal{H} and by a self-adjoint operator D with compact resolvent such that for each $a \in A$ the commutator¹ $[D, a]$ has bounded extension.

¹We assume $\text{adom}D \subset \text{dom}D$ for each $a \in A$, so that $[D, a]$ is defined on $\text{dom}D$.

A spectral triple is called even if the Hilbert space \mathcal{H} is endowed with a nontrivial \mathbb{Z}_2 -grading χ which² commutes with any $a \in A$ and anticommutes with D . Otherwise it is called odd.

The following definition is a modification of Definition 1.124 in [6] in order to cover all the twelve possibilities as discussed in the previous section.

Definition 2. A real structure of KO -dimension $n \in \mathbb{Z}_8$ on a spectral triple (A, \mathcal{H}, D) is an antilinear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$, with the property that

$$J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad \text{and, if } (A, \mathcal{H}, D) \text{ is even, } J\chi = \epsilon''\chi J. \quad (12)$$

Given n , the possibilities for arrays of numbers $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ are given by the tables in section 1. Moreover, the action of A satisfies the commutation rule

$$[a, Jb^*J^{-1}] = 0, \quad \forall a, b \in A \quad (13)$$

and the operator D satisfies the order one condition

$$[[D, a], Jb^*J^{-1}] = 0, \quad \forall a, b \in A. \quad (14)$$

A spectral triple (A, \mathcal{H}, D) endowed with a real structure J is called a real spectral triple.

Remarks.

- usually we will omit to indicate the $*$ -representation map $\rho : A \rightarrow \mathcal{B}(\mathcal{H})$ for simplicity;
- we recall that an antiunitary operator J is antilinear, bijective and $(Ju|Jv) = (v|u)$;
- the map $b \mapsto Jb^*J^{-1}$ is a representation of the opposite algebra A° on $\mathcal{B}(\mathcal{H})$;
- equation (13) which says that Ad_J sends A to its commutant is sometimes called the “zero order condition”;
- note that putting:

$$\vec{\epsilon}_\pm(n) := (\epsilon_\pm(n), \epsilon'_\pm(n), \epsilon''_\pm(n)), \quad (15)$$

(where $n \in \mathbb{Z}_8$) we have the relation:

$$\vec{\epsilon}_-(n) = -\vec{\epsilon}_+(n+2). \quad (16)$$

²By definition χ is a self-adjoint unitary such that $\chi^2 = \text{id}_{\mathcal{H}}$ and $\chi \neq \pm \text{id}_{\mathcal{H}}$. The Hilbert space \mathcal{H} can then be split into its eigenspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$; by requesting $[\chi, a] = 0$ for each $a \in A$ this splitting is invariant under the action of A on \mathcal{H} .

4 Product of real spectral triples

Following what happens in the commutative case, we shall produce a real spectral triple of dimension $n_1 + n_2$ out of two triples of dimensions n_1 and n_2 respectively. The new algebra is the tensor product algebra $A := A_1 \otimes A_2$, where \otimes is the algebraic tensor product³ and the involution is defined component-wise: $(a \otimes b)^* := a^* \otimes b^*$. It turns out that the other ingredients of the resulting spectral triple depend on the parity of the two given triples.

4.1 Even-even case

As the Hilbert space carrying the $*$ -representation of A we take the Hilbert tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and as the $*$ -representation we take the tensor product representation:

$$\rho_1 \otimes \rho_2: A_1 \otimes A_2 \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2). \quad (17)$$

The representation $\rho_1 \otimes \rho_2$ is faithful whenever ρ_1 and ρ_2 are. The grading operator is given by $\chi := \chi_1 \otimes \chi_2$ (it is easy to check that it is unitary, squares to $\text{id}_{\mathcal{H}_1 \otimes \mathcal{H}_2}$ and commutes with every element of the product algebra A). As for the Dirac operator, using χ_1 or χ_2 we take the following operators⁴:

$$\begin{aligned} \mathcal{D} &:= D_1 \otimes \text{id}_{\mathcal{H}_2} + \chi_1 \otimes D_2, \\ \tilde{\mathcal{D}} &:= D_1 \otimes \chi_2 + \text{id}_{\mathcal{H}_1} \otimes D_2, \end{aligned} \quad (18)$$

both defined on the dense domain $\text{dom}D_1 \otimes \text{dom}D_2$. They are unitarily equivalent:

$$\tilde{\mathcal{D}} = U\mathcal{D}U^\dagger, \quad (19)$$

where (see [12])

$$U = \frac{1}{2}(\text{id}_{\mathcal{H}_1} \otimes \text{id}_{\mathcal{H}_2} + \chi_1 \otimes \text{id}_{\mathcal{H}_2} + \text{id}_{\mathcal{H}_1} \otimes \chi_2 - \chi_1 \otimes \chi_2). \quad (20)$$

We now show that \mathcal{D} is essentially self-adjoint. This is immediate if one Hilbert space is finite dimensional, so we assume both Hilbert spaces to be infinite dimensional.

From the general theory of (linear unbounded) operators on a complex separable Hilbert space we know that each D_i has pure point spectrum consisting of countably many real eigenvalues, each with finite multiplicity, and the only limit point of their absolute values is $+\infty$. Let $\{v_{\lambda, m_\lambda}, v_{j_\pm} \mid \lambda \in \sigma(D_1)/\{0\}, m_\lambda = 1, \dots, M_\lambda, j_\pm = 1, \dots, K_\pm \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of D_1 , where $\sigma(D_1)$ is the

³Over \mathbb{R} if at least one of the two algebras is real, over \mathbb{C} if both algebras are complex.

⁴The simplest choice $D = D_1 \otimes \text{id}_{\mathcal{H}_2} + \text{id}_{\mathcal{H}_1} \otimes D_2$ has non-compact resolvent in general (e.g. $\ker D$ is infinite dimensional if $D_2 = -D_1$ with $\mathcal{H}_1 = \mathcal{H}_2$ infinite dimensional).

spectrum of D_1 , M_λ is the multiplicity of $\lambda \in \sigma(D_1)/\{0\}$, and finally $\{v_{j_\pm} \mid j_\pm = 1, \dots, K_\pm\}$ is a basis of $\ker D_1$ consisting of eigenvectors of χ_1 such that $\chi_1 v_{j_\pm} = \pm v_{j_\pm}$. Note that since D_1 and χ_1 anticommute, and χ_1 is unitary, then $\sigma(D_1)$ is symmetric and $M_\lambda = M_{-\lambda}$. Let $\{w_{\mu, n_\mu} \mid \mu \in \sigma(D_2), n_\mu = 1, \dots, N_\mu \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of D_2 . Finally, let us consider the following vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\begin{aligned} u_{\lambda, m_\lambda, \mu, n_\mu}^+ &:= \cos \theta_{\lambda\mu} (v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu}) + \sin \theta_{\lambda\mu} (\chi_1 v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu}), \\ u_{\lambda, m_\lambda, \mu, n_\mu}^- &:= -\sin \theta_{\lambda\mu} (v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu}) + \cos \theta_{\lambda\mu} (\chi_1 v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu}), \\ u_{j_\pm, \mu, n_\mu} &:= v_{j_\pm} \otimes w_{\mu, n_\mu}, \end{aligned} \quad (21)$$

where $\lambda \in \sigma(D_1) \cap \mathbb{R}_{>0}$, $\mu \in \sigma(D_2)$, $\theta_{\lambda\mu} := \frac{1}{2} \arctan \frac{\mu}{\lambda} \in (-\pi/4, \pi/4)$ and $j_\pm = 1, \dots, K_\pm$. Then the set

$$\{u_{\lambda, m_\lambda, \mu, n_\mu}^+, u_{\lambda, m_\lambda, \mu, n_\mu}^-, u_{j_\pm, \mu, n_\mu} \mid \lambda \in \sigma(D_1) \cap \mathbb{R}_{>0}, \mu \in \sigma(D_2), j_\pm = 1, \dots, K_\pm\} \quad (22)$$

is an orthonormal basis of eigenvectors of \mathcal{D} , with corresponding eigenvalues given by:

$$\begin{aligned} \mathcal{D}(u_{\lambda, m_\lambda, \mu, n_\mu}^\pm) &= \pm \sqrt{\lambda^2 + \mu^2} u_{\lambda, m_\lambda, \mu, n_\mu}^\pm, \\ \mathcal{D}(u_{j_\pm, \mu, n_\mu}) &= \pm \mu u_{j_\pm, \mu, n_\mu}. \end{aligned} \quad (23)$$

It can be easily seen that $\ker \mathcal{D} = \ker D_1 \otimes \ker D_2$. From the existence of a basis of eigenvectors for \mathcal{D} we can promptly conclude that $D := \overline{\mathcal{D}}$ (the closure of \mathcal{D}) is self-adjoint. From the analysis above it is clear that D is a self-adjoint operator with pure point spectrum consisting of countably many eigenvalues, each with finite multiplicity, and the only limit point of their absolute values is $+\infty$. By the general theory, we conclude that D has compact resolvent.

By unitary equivalence, the basis for \mathcal{D} gives a basis of eigenvectors of \tilde{D} with the same eigenvalues and multiplicities as those of \mathcal{D} , so the analysis above goes through and we conclude that $\tilde{D} := \overline{\tilde{\mathcal{D}}}$ is also self-adjoint with compact resolvent.

It is easy to check that $[D, a]$ extends to a bounded operator for each $a \in A_1 \otimes A_2$ (using the condition $[\chi, a] = 0$). Analogously for \tilde{D} .

From J_1 and J_2 we can construct $J = J_1 \otimes J_2$, which is easily seen to be antiunitary on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover we have

$$\begin{aligned} [a_1 \otimes a_2, J(b_1 \otimes b_2)^* J^{-1}] &= [a_1 \otimes a_2, (J_1 \otimes J_2)(b_1^* \otimes b_2^*)(J_1^{-1} \otimes J_2^{-1})] \\ &= [a_1, J_1 b_1^* J_1^{-1}] \otimes a_2 J_2 b_2^* J_2^{-1} + \\ &\quad + J_1 b_1^* J_1^{-1} a_1 \otimes [a_2, J_2 b_2^* J_2^{-1}] \\ &= 0. \end{aligned} \quad (24)$$

Labeling the ϵ -triples with n_+ or n_- according to the KO-dimension and the J involved in the product, we get the following tables for the KO-dimension and reality structure of the resulting triple (we distinguish the two cases for the total Dirac operator, D or \tilde{D}):

Table 2: D

1 \ 2	0_+	2_+	4_+	6_+	0_-	2_-	4_-	6_-
0_+	0_+	2_+	4_+	6_+				
2_+					2_+	4_+	6_+	0_+
4_+	4_+	6_+	0_+	2_+				
6_+					6_+	0_+	2_+	4_+
0_-					0_-	2_-	4_-	6_-
2_-	2_-	4_-	6_-	0_-				
4_-					4_-	6_-	0_-	2_-
6_-	6_-	0_-	2_-	4_-				

Table 3: \tilde{D}

1 \ 2	0_+	2_+	4_+	6_+	0_-	2_-	4_-	6_-
0_+	0_+		4_+			2_-		6_-
2_+	2_+		6_+			4_-		0_-
4_+	4_+		0_+			6_-		2_-
6_+	6_+		2_+			0_-		4_-
0_-		2_+		6_+	0_-		4_-	
2_-		4_+		0_+	2_-		6_-	
4_-		6_+		2_+	4_-		0_-	
6_-		0_+		4_+	6_-		2_-	

Remark. The two top blocks in table 2 correspond to the even-even cases covered by Vanhecke's paper [12].

4.2 Even-odd case

The Hilbert space \mathcal{H} , the $*$ -representation of A on $\mathcal{B}(\mathcal{H})$ and the reality structure J are the same as in the even-even case. Now we have only one nontrivial grading operator though: we then choose D or \tilde{D} from the previous construction, according to whether the even triple is the first one or the second one, respectively. The basis of eigenvectors is again given by (22) (or by the analogous construction using χ_2 instead of χ_1). The argument for proving self-adjointness and compactness of resolvent goes through exactly as before. For the KO-dimension and reality structure of the resulting triple we obtain the following tables:

Table 4: D

	2	1 ₋	3 ₊	5 ₋	7 ₊
1					
0 ₊			3 ₊		7 ₊
2 ₊		3 ₊		7 ₊	
4 ₊			7 ₊		3 ₊
6 ₊		7 ₊		3 ₊	
0 ₋		1 ₋		5 ₋	
2 ₋			5 ₋		1 ₋
4 ₋		5 ₋		1 ₋	
6 ₋			1 ₋		5 ₋

Table 5: \tilde{D}

	2	0 ₊	2 ₊	4 ₊	6 ₊	0 ₋	2 ₋	4 ₋	6 ₋
1									
1 ₋			3 ₊		7 ₊	1 ₋		5 ₋	
3 ₊		3 ₊		7 ₊			5 ₋		1 ₋
5 ₋			7 ₊		3 ₊	5 ₋		1 ₋	
7 ₊		7 ₊		3 ₊			1 ₋		5 ₋

Remark. Table 4 corresponds to the even-odd cases covered by Vanhecke's paper [12].

4.3 Odd-odd case

In this case we have no nontrivial grading operator available. In order to overcome this, motivated by the commutative situation, we consider the following construction:

$$\begin{aligned}
A &:= A_1 \otimes A_2, \\
\mathcal{H} &:= (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathbb{C}^2, \\
\mathcal{D} &:= D_1 \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_1 + \text{id}_{\mathcal{H}_1} \otimes D_2 \otimes \sigma_2, \\
J^\pm &:= J_1 \otimes J_2 \otimes M^\pm K, \\
\chi &:= \text{id}_{\mathcal{H}_1} \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_3,
\end{aligned} \tag{25}$$

where the σ s are the Pauli matrices, M^\pm are two complex matrices specified by the table below and K is the complex conjugation operator defined for the canonical basis of \mathbb{C}^2 (i.e., if (e_1, e_2) is the canonical basis, we have $K(\lambda e_i) = \bar{\lambda} e_i$ for every $\lambda \in \mathbb{C}$). The representation is understood to be trivial on the \mathbb{C}^2 factor, i.e. $\rho(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2) \otimes \text{id}_{\mathbb{C}^2}$.

Table 6: Odd-odd case

1 \ 2	1 ₋	3 ₊	5 ₋	7 ₊
1 ₋	σ_2, σ_1	σ_3, σ_0	σ_2, σ_1	σ_3, σ_0
3 ₊	σ_0, σ_3	σ_1, σ_2	σ_0, σ_3	σ_1, σ_2
5 ₋	σ_2, σ_1	σ_3, σ_0	σ_2, σ_1	σ_3, σ_0
7 ₊	σ_0, σ_3	σ_1, σ_2	σ_0, σ_3	σ_1, σ_2

Remarks. The entries in table 6 stand for the pair M^+, M^- . For convenience, the identity matrix is called σ_0 . Note that this construction still works under any permutation of the Pauli matrices (e.g., one can take $\mathcal{D} := D_1 \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_1 + \text{id}_{\mathcal{H}_1} \otimes D_2 \otimes \sigma_3$ and $\chi := \text{id}_{\mathcal{H}_1} \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_2$). The table obtained considering only the first element in each entry (i.e. M^+) corresponds to the odd-odd cases covered by Sitarz's notes [11].

Calling $n_1 = 2m_1 + 1$ and $n_2 = 2m_2 + 1$ the dimensions of the two triples involved, we have:

$$\begin{aligned}
M^+(n_1, n_2) &= \sigma_j, \quad j = \frac{1}{2} (5 + (-1)^{m_2+1}) + 2m_1 \pmod{4}, \\
M^-(n_1, n_2) &= \sigma_k, \quad k = \frac{1}{2} (1 + (-1)^{m_2}) + 2m_1 \pmod{4}.
\end{aligned} \tag{26}$$

Self-adjointness of $D := \overline{D}$ and compactness of its resolvent can be proven by the same argument of section 4.1 with suitable changes. In particular the eigenvectors u^\pm

are given by:

$$\begin{aligned}
u_{\lambda, m_\lambda, \mu, n_\mu}^+ &:= \frac{1}{\sqrt{2}} \cos \theta_{\lambda\mu} v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \sin \theta_{\lambda\mu} v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu} \otimes \begin{pmatrix} -i \\ i \end{pmatrix}, \\
u_{\lambda, m_\lambda, \mu, n_\mu}^- &:= -\frac{1}{\sqrt{2}} \sin \theta_{\lambda\mu} v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \cos \theta_{\lambda\mu} v_{\lambda, m_\lambda} \otimes w_{\mu, n_\mu} \otimes \begin{pmatrix} -i \\ i \end{pmatrix}.
\end{aligned} \tag{27}$$

5 Further properties and their preservation under products

- **Dimension** (“condition 1” on p. 481 in [8]): Assume $(A_i, \mathcal{H}_i, D_i, J_i, (\chi_i))_{i=1,2}$ are two real spectral triples of dimensions n_1 and n_2 respectively. Independently of the parities of the triples, the eigenvalues of D^2 are given by the sum of the eigenvalues of D_1^2 and D_2^2 , and this implies that the dimension of the product triple is $n_1 + n_2$ (see p. 486 in [8] for details).
- **Regularity** (“condition 2” on p. 482 in [8]): For even spectral triples the result that the product of two regular triples is regular is contained in [10], which uses the existence of an algebra of generalized differential operators; this works also when at least one of the two triples is even and for the odd-odd case the argument still can be carried over.
- **Reality** (“condition 4” on p. 483 in [8]): This condition is automatically preserved, by construction of the product triple.
- **First order** (“condition 5” on p. 484 in [8]): Assume $(A_i, \mathcal{H}_i, D_i, J_i, \chi_i)_{i=1,2}$ are two real spectral triples of even dimension satisfying this property. Then the product $(A, \mathcal{H}, D, J, \chi)$ satisfies it as well: indeed, taking $a, b \in A$, where $a = a_1 \otimes a_2$, $b = b_1 \otimes b_2$ we compute:

$$\begin{aligned}
[[D, a], b^\circ] &= [[D_1 \otimes \text{id}_{\mathcal{H}_2} + \chi_1 \otimes D_2, a], b^\circ] \\
&= [[D_1 \otimes \text{id}_{\mathcal{H}_2}, a] + [\chi_1 \otimes D_2, a], b^\circ] \\
&= [[D_1, a_1] \otimes a_2 + \chi_1 a_1 \otimes [D_2, a_2], b^\circ] \\
&= [[D_1, a_1] \otimes a_2, b^\circ] + [\chi_1 a_1 \otimes [D_2, a_2], b^\circ] \\
&= [[D_1, a_1], b_1^\circ] \otimes a_2 b_2^\circ + \chi_1 a_1 b_1^\circ \otimes [[D_2, a_2], b_2^\circ] \\
&= 0,
\end{aligned} \tag{28}$$

where $b^\circ := Jb^*J^{-1}$, where we have used the facts that the representations of A_i and A_i° commute (this is part of the content of condition 5) and also that

χ_i commutes with the representation of A_i . Analogously one can prove that the representations of A and A° commute, i.e. $[a, b^\circ] = 0$. The same computation also applies for \tilde{D} , and this concludes the proof for the even-even case. For the even-odd case the computations are analogous. For the odd-odd case we compute:

$$\begin{aligned}
[[D, a], b^\circ] &= [[D_1 \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_1 + \text{id}_{\mathcal{H}_1} \otimes D_2 \otimes \sigma_2, a], b^\circ] \\
&= [[D_1 \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_1, a_1 \otimes a_2 \otimes \text{id}_{\mathbb{C}^2}], b_1^\circ \otimes b_2^\circ \otimes \text{id}_{\mathbb{C}^2}] + \\
&\quad + [[\text{id}_{\mathcal{H}_1} \otimes D_2 \otimes \sigma_2, a_1 \otimes a_2 \otimes \text{id}_{\mathbb{C}^2}], b_1^\circ \otimes b_2^\circ \otimes \text{id}_{\mathbb{C}^2}] \\
&= [[D_1, a_1] \otimes a_2 \otimes \sigma_1, b_1^\circ \otimes b_2^\circ \otimes \text{id}_{\mathbb{C}^2}] + \\
&\quad + [a_1 \otimes [D_2, a_2] \otimes \sigma_2, b_1^\circ \otimes b_2^\circ \otimes \text{id}_{\mathbb{C}^2}] \\
&= [[D_1, a_1], b_1^\circ] \otimes a_2 b_2^\circ \otimes \sigma_1 + \\
&\quad + a_1 b_1^\circ \otimes [[D_2, a_2], b_2^\circ] \otimes \sigma_2 \\
&= 0,
\end{aligned} \tag{29}$$

where we used the fact that the reps of A_i and A_i° commute. As before, checking that the reps of A and A° commute is entirely analogous.

- **Orientation** (“condition 6” on p. 484 in [8]): Given two Hochschild cycles $a_i \in Z_{n_i}(A_i, A_i \otimes A_i^\circ)$, $i = 1, 2$, where n_i is the dimension of the algebra A_i according to “condition 1”, one can construct the Hochschild cycle $a \in Z_n(A, A \otimes A^\circ)$ (where $n = n_1 + n_2$ and $A = A_1 \otimes A_2$) using the shuffle product (see [9], section 4.2). Let us provide some details for the construction, following [9]. First define⁵ the shuffle product $\times : C_{n_1}(A_1) \otimes C_{n_2}(A_2) \rightarrow C_{n_1+n_2}(A_1 \otimes A_2)$ of two chains as follows:

$$\begin{aligned}
(a_0^1, a_1^1, \dots, a_p^1) \times (a_0^2, a_1^2, \dots, a_q^2) &:= \\
\sum_{\sigma} (-1)^\sigma \sigma \cdot (a_0^1 \otimes a_0^2, a_1^1 \otimes 1, \dots, a_p^1 \otimes 1, 1 \otimes a_1^2, \dots, 1 \otimes a_q^2), &\tag{30}
\end{aligned}$$

where

$$\sigma \cdot (a_0, a_1, \dots, a_n) := (a_0, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}) \tag{31}$$

and the sum is over all (p, q) -shuffles, i.e. permutations of $\{1, \dots, p+q\}$ preserving the order of $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$ separately. Then for the Hochschild boundary map ∂ the following formula holds:

$$\partial(x \times y) = \partial(x) \times y + (-1)^{|x|} x \times \partial(y). \tag{32}$$

From this formula it follows at once that the shuffle product of two Hochschild cycles is again a Hochschild cycle. The orientation condition for a triple now

⁵For the sake of generality, here we can take the first coefficient a_0 to be in some module over A ; we will be interested in the case where this module is $A \otimes A^\circ$.

states that there is a Hochschild cycle c satisfying the following formula:

$$\pi_D(c) = \chi, \quad (33)$$

where the map π_D is defined as:

$$\pi_D(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \tau_J(a_0)[D, \rho(a_1)] \cdots [D, \rho(a_p)], \quad (34)$$

where $a_0 \in A \otimes A^\circ$ and $\tau_J(a \otimes b) := \rho(a)J\rho(b^*)J^{-1}$. Given two real spectral triples of dimensions n_1 and n_2 respectively, we claim that if c_j ($j = 1, 2$) are Hochschild cycles satisfying $\pi_{D_j}(c_j) = \chi_j$ then the analogous cycle on the product triple is given by

$$c := \frac{1}{r} c_1 \times c_2, \quad (35)$$

where

$$r := \begin{cases} \nu_{n_1+n_2}, & \text{when } n_1 n_2 \text{ is even} \\ i\nu_{n_1+n_2}, & \text{when } n_1 n_2 \text{ is odd,} \end{cases} \quad (36)$$

$$\nu_n := \frac{1}{2}(n-1)n,$$

where $\nu(n) := \frac{1}{2}(n-1)n$. In order to check formula (33) on the product triple with c given by equation (35) we distinguish three cases depending on the parities involved:

Even-even. A simple computation shows that

$$\begin{aligned} [D, \rho_1(a) \otimes \text{id}_{\mathcal{H}_2}] &= [D_1, \rho_1(a)] \otimes \text{id}_{\mathcal{H}_2} \\ [D, \text{id}_{\mathcal{H}_1} \otimes \rho_2(b)] &= \chi_1 \otimes [D_2, \rho_2(b)] \end{aligned} \quad (37)$$

from which it follows that

$$\begin{aligned} &\pi_D(\sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \dots, a_{n_1} \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_{n_2})) = \\ &= \Pi \sigma \cdot (\tau_{J_1}(a_0), [D_1, \rho_1(a_1)], \dots, [D_1, \rho_1(a_{n_1})], \chi_1, \dots, \chi_1) \otimes \\ &\quad \otimes \Pi(\tau_{J_2}(b_0), [D_2, \rho_2(b_1)], \dots, [D_2, \rho_2(b_{n_2})]), \end{aligned} \quad (38)$$

where Π means algebra product of all the elements in the ordered list. Since χ_1 anti-commutes with D_1 and commutes with $\rho_1(a)$ for each $a \in A_1$, rearranging all the n_2 operators χ_1 side by side produces a $(-1)^\sigma$ sign which cancels the same sign from the shuffle product; moreover, since n_2 is even, we have $\chi_1^{n_2} = \text{id}_{\mathcal{H}_1}$; therefore we are left with a sum of $\frac{1}{2}(n_1+n_2-1)(n_1+n_2)$ identical terms:

$$\begin{aligned} \pi_D(c_1 \times c_2) &= \nu_{n_1+n_2} \pi_{D_1}(c_1) \otimes \pi_{D_2}(c_2) \\ &= \nu_{n_1+n_2} \chi_1 \otimes \chi_2. \end{aligned} \quad (39)$$

The same reasoning applies to \tilde{D} with obvious modifications.

Even-odd. The previous argument carries over unaltered, but this time we have $\chi_1^{n_2} = \chi_1$ since n_2 is odd. Then we get

$$\begin{aligned}
\pi_D(c_1 \times c_2) &= \nu_{n_1+n_2} \chi_1 \pi_{D_1}(c_1) \otimes \pi_{D_2}(c_2) \\
&= \nu_{n_1+n_2} \chi_1^2 \otimes \chi_2 \\
&= \nu_{n_1+n_2} \text{id}_{\mathcal{H}_1} \otimes (\pm \text{id}_{\mathcal{H}_2}) \\
&= \pm \nu_{n_1+n_2} \text{id}_{\mathcal{H}_1} \otimes \text{id}_{\mathcal{H}_2} ,
\end{aligned} \tag{40}$$

as expected. The same reasoning applies to the case where n_1 is odd and n_2 is even, with obvious modifications.

Odd-odd. In this case a simple computation shows that

$$\begin{aligned}
[D, \rho_1(a) \otimes \text{id}_{\mathcal{H}_2} \otimes \text{id}_{\mathbb{C}^2}] &= [D_1, \rho_1(a)] \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_1 \\
[D, \text{id}_{\mathcal{H}_1} \otimes \rho_2(b) \otimes \text{id}_{\mathbb{C}^2}] &= \text{id}_{\mathcal{H}_1} \otimes [D_2, \rho_2(b)] \otimes \sigma_2
\end{aligned} \tag{41}$$

from which it follows that

$$\begin{aligned}
&\pi_D(\sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \dots, a_{n_1} \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_{n_2})) = \\
&= \tau_{J_1}(a_0) [D_1, \rho_1(a_1)] \cdots [D_1, \rho_1(a_{n_1})] \otimes \\
&\quad \otimes \tau_{J_2}(b_0) [D_2, \rho_2(b_1)] \cdots [D_2, \rho_2(b_{n_2})] \otimes \\
&\quad \otimes \Pi \sigma \cdot (1, \underbrace{\sigma_1, \dots, \sigma_1}_{n_1 \text{ times}}, \underbrace{\sigma_2, \dots, \sigma_2}_{n_2 \text{ times}})
\end{aligned} \tag{42}$$

Since $\sigma_1 \sigma_2 = -\sigma_2 \sigma_1$ we can rearrange the σ_i s with all σ_1 s on the left and all σ_2 s on the right, producing a $(-1)^\sigma$ sign which cancels the same sign from the shuffle product; moreover since n_1 and n_2 are both odd we get $\sigma_1^{n_1} \sigma_2^{n_2} = \sigma_1 \sigma_2 = i \sigma_3$, so we end up with

$$\begin{aligned}
\pi_D(c_1 \times c_2) &= \nu_{n_1+n_2} \pi_{D_1}(c_1) \otimes \pi_{D_2}(c_2) \otimes i \sigma_3 \\
&= i \nu_{n_1+n_2} \text{id}_{\mathcal{H}_1} \otimes \text{id}_{\mathcal{H}_2} \otimes \sigma_3 .
\end{aligned} \tag{43}$$

Remark. The orientation axiom is consistent with the observation made in section 2.2 for the classical setting, namely that changing the representation $\Gamma_{(2m+1,+)}$ to $\Gamma_{(2m+1,-)}$ is equivalent to changing the orientation of the manifold. In the noncommutative setting, this translates into changing the sign of the Hochschild cycle c in (33).

6 Final comments

In this paper we are concerned with unital spectral triples but the canonical Dirac operator on \mathbb{R}^n (cf. section 2.2) is not of that type. The definition of a nonunital spectral triple is slightly different, as well as the additional axioms for it. In order to

remain in the realm of unital spectral triples, as a commutative compact case study we should take rather the flat torus with the trivial spin structure. This however does not change the form (7) of the canonical Dirac operator, but just supplements it with periodic boundary conditions.

It is worth mentioning that in our setup the metric and KO dimensions need not be equal modulo 8. This is the case in some of the recent examples of spectral triples [7], [3, 6], see also [1].

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